Born-Oppenheimer RG for high energy evolution

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Motivation

Two prong: issues with NLO BFKL/JIMWLK and the need to account for DGLAP logs, especially in view of EIC.

From the beginning of time ($\sim$ 27 years) it has been known that frequency is a more suitable evolution parameter than the longitudinal momentum. Salam (1998), Sabio Vera (2005), Altinoluk et.al. (2014), Ducloe et. al. (2019)

In NLO BFKL switching to frequency (k^-) evolution eliminates higher poles in the characteristic function at $\gamma = 1$, and makes the kernel better behaved. Seems to have a similar effect in NLO dipole model (beyond linear limit).

This is in the eikonal limit - i.e. assuming (multiple) soft scatterings off the target. But not only. For DGLAP frequency also is the right evolution parameter: increasing Q^2 increases the frequency of the fluctuations in the target resolved by the hard scattering:

DGLAP collinear splittings

$(k^+, k_\perp \sim 0) \to (p^+ \sim k^+, p_\perp \gg k_\perp) + (k^+ - p^+ \sim k^+, -p_\perp)$ increase the frequency of the relevant modes $\frac{p_\perp^2}{2p^+} \gg \frac{k_\perp^2}{2k^+}$.

Frequency increase $\leftrightarrow$ Born-Oppenheimer physics

The physics principle for frequency evolution is exactly the same as for the famous Born-Oppenheimer approximation. As the external frequency with which we probe the system is increased (be it the total energy E or the transverse resolution scale Q^2), faster modes participate in the process.

Thus to understand the evolution, we need to solve for "fast" modes (higher frequency) on the background of the slower modes (lower frequency).

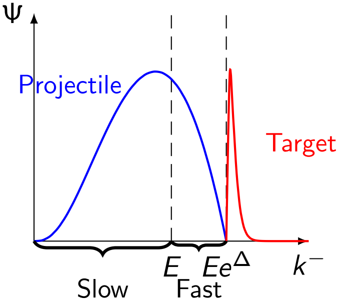
Of course as we go higher and higher in the resolution (energy, Q^2 ...) we need to include faster and faster modes. We call this procedure "the Born-Oppenheimer RG". The BO RG should unify the BFKL-type and the DGLAP-type evolution in a single framework.

The setup I

As always, we assume factorization between the projectile and target degrees of freedom, so that the wave function before scattering

$$|\Psi_{in}\rangle = |\Psi_P\rangle \otimes |\Psi_T\rangle$$

The projectile wave function contains modes with frequencies below those of the target:



The setup II

We neglect quarks. Also we assume dilute limit, i.e. projectile fields are small.

The slow and fast modes of the projectile interact via H^I_{SF}.
The LCWF of the fast modes is found by calculating

$$U(0,\tau) = \mathcal{T}\exp\{i\int_0^\tau dx^+ H^I_{SF}(x^+)\}$$

The operator that diagonalizes the fast modes (in the fixed slow background) is

$$\Omega = \lim_{\tau\to\infty} U(0,\tau)$$

The LCWF of the projectile is then

$$|\Psi_P\rangle = \Omega\,|0\rangle_F \otimes |\psi_0\rangle_S$$

We follow this procedure through perturbatively to leading order.

The interaction Hamiltonian I

This is straightforward - just staring at the QCD Hamiltonian and separating out the highest frequency mode:

$$\mathcal{H}_I(p) = -ig \int_{max(k^-,(k-p)^-)<p^-} A_i^a(k^+,\mathbf{k}) f^{abc} \times$$
$$\times \left\{ \left[\delta_{ki}\delta_{jl} \left(\frac{2k^+}{p^+} - 1 \right) + \epsilon_{ki}\epsilon_{jl} \right] \boldsymbol{P}_j^{bd} A_l^{\dagger d}(p^+,\boldsymbol{p}) A_k^{\dagger c}(k^+ - p^+, \boldsymbol{k} - \boldsymbol{p}) + \right.$$
$$\left. + \left[\delta_{ki}\delta_{jl} \left(\frac{2k^+}{k^+ - p^+} - 1 \right) + \epsilon_{ki}\epsilon_{jl} \right] (\boldsymbol{K} - \boldsymbol{P})_j^{bd} A_l^{\dagger d}(k^+ - p^+, \boldsymbol{k} - \boldsymbol{p}) A_k^{\dagger c}(p^+,\boldsymbol{p}) \right\} + h$$

Here

$$P_i^{ab} \equiv \boldsymbol{p}_i \delta^{ab} + igf^{abc} \int_{k^+ \ll p^+; k^- \ll p^-} \left[\alpha_i^{\dagger c}(k^+,\boldsymbol{k}) + \alpha_i^c(k^+,-\boldsymbol{k}) \right]$$

α^i - are *very slow and soft* fields with all components of momentum small - "soft fields" in the SCET language. This interaction is outside of either BFKL or DGLAP framework. The soft fields have to be dealt with in the dense limit. So far we have only considered dilute limit, and so we close our eyes and hearts to their existence.

The interaction Hamiltonian II

The no-soft - fields Hamiltonian is

$$
\begin{aligned}
\mathcal{H}_I(p) = -ig \int_{max(k^-,(k-p)^-)<p^-} & A^a_i(k^+,\boldsymbol{k}) f^{abc} \times \\
\times \quad \Bigg\{ \Bigg[\delta_{ki}\delta_{jl} \left(\frac{2k^+}{p^+} - 1 \right) + \epsilon_{ki}\epsilon_{jl} \Bigg] & \boldsymbol{p}_j A^{\dagger b}_l(p^+,\boldsymbol{p}) A^{\dagger c}_k(k^+ - p^+, \boldsymbol{k}-\boldsymbol{p}) \\
- \quad \Bigg[\delta_{ki}\delta_{jl} \left(\frac{2k^+}{k^+ - p^+} - 1 \right) + \epsilon_{ki}\epsilon_{jl} \Bigg] & \boldsymbol{p}_j A^{\dagger b}_l(k^+ - p^+, \boldsymbol{k}-\boldsymbol{p}) A^{\dagger c}_k(p^+,\boldsymbol{p}) \Bigg\} + h.c.
\end{aligned}
$$

To find the LO wave function we need the energy denominator. For $k \to p, k-p$ it is

$$
D^{-1} \equiv k^- - p^- - (k-p)^- = \frac{\boldsymbol{k}^2}{2k^+} - \frac{\boldsymbol{p}^2}{2p^+} - \frac{(\boldsymbol{k}-\boldsymbol{p})^2}{2(k^+ - p^+)} \approx -\frac{\boldsymbol{p}^2 k^+}{2p^+(k^+ - p^+)}
$$

The LCWF for a fast mode p

We then find the perturbative LCWF

$$\Omega_p = 1 + iG(p^+, \boldsymbol{p}) \approx e^{iG(p^+, \boldsymbol{p})}; \qquad G(p^+, \boldsymbol{p}) = \mathcal{H}_I(p)D$$

where

$$G(p^+, \boldsymbol{p}) = A_i^\dagger(p^+, \boldsymbol{p})C_i(p^+, \boldsymbol{p}) + A_i(p^+, \boldsymbol{p})C_i^\dagger(p^+, \boldsymbol{p})$$

$$C_i^a(p^+, \boldsymbol{p}) = g\int_{k^- < p^-; \quad (k-p)^- < p^-} F^i_{lk}(k, p)A_l^\dagger(k^+ - p^+, \boldsymbol{k} - \boldsymbol{p})T^a A_k(k^+, \boldsymbol{k})$$

$$F^i_{lk}(k, p) = \frac{4p^+(k^+ - p^+)}{k^+}\left\{\delta_{kl}\delta_{ji}\frac{k^+}{p^+} + \delta_{ki}\delta_{jl}\frac{k^+}{k^+ - p^+} - \delta_{kj}\delta_{il}\right\}\frac{\boldsymbol{p}_j}{\boldsymbol{p}^2}$$

No matter the details: C_i is the "classical field" (Ω is a coherent operator) (in the eikonal limit reduces to the usual $\frac{\boldsymbol{p}_i}{\boldsymbol{p}^2}\rho^a(\boldsymbol{p})$).

The evolved LCWF

Now that we have diagonalized the Hamiltonian for a single fast mode, we write the wave function evolved over a finite range of frequency

$$|\Psi_P\rangle_E = \mathcal{P}\exp\left\{i\int_{E_0}^{E} dp^-\mathcal{G}(p^-)\right\}|\Psi_P\rangle_{E_0}$$

where

$$\mathcal{G}(p^-) \equiv \int_p \delta(p^- - \frac{\boldsymbol{p}^2}{2p^+})G(p^+,\boldsymbol{p}^2);$$

$$G(p^+,\boldsymbol{p}) = A_i^\dagger(p^+,\boldsymbol{p})C_i(p^+,\boldsymbol{p}) + A_i(p^+,\boldsymbol{p})C_i^\dagger(p^+,\boldsymbol{p})$$

Given the wave function we can discuss evolution!

On to the evolution

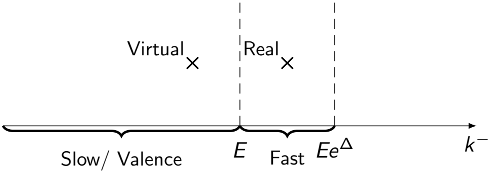
For any operator $\hat{O}$ in the projectile Hilbert space

$$\langle \hat{O} \rangle_E = \langle \Psi_P | \hat{O} | \Psi_P \rangle_E$$

The evolution equation follows from

$$\frac{d}{\eta} \langle \hat{O} \rangle = \lim_{\Delta \to 0} \frac{1}{\Delta} \left[\langle \hat{O} \rangle_{Ee^\Delta} - \langle \hat{O} \rangle_E \right]; \qquad \eta \equiv \ln E/E_0$$

Quite generally for an arbitrary $\hat{O}$ there are two types of contributions to the evolution: virtual (or Lindblad) - due to gluonic degrees of freedom in $\hat{O}$ that live below E, and real - due to those that live between E and Ee^Δ.



For $\langle \rho(\boldsymbol{x}) \rho(\boldsymbol{y}) \rangle$ - both contributions are present (BFKL) - more difficult.
For TMD $T(p^+, \boldsymbol{p})$: $\frac{\boldsymbol{p}^2}{2p^+} < E$ only the Lindblad term contributes (CSS) - easier.

Evolution of gluon TMD

Consider

$$\hat{T}(k) \equiv a_i^{\dagger a}(k) a_i^a(k); \quad \mathcal{T}(k) \equiv \langle \Psi_P | \hat{T}(k) | \Psi_P \rangle_E; \quad k^- < E$$

Direct calculation:

$$\delta\mathcal{T}(k) = \delta\mathcal{T}^L(k) + \delta\mathcal{T}^{NL}(k)$$

with

$$\delta\mathcal{T}^L(k) = \frac{g^2 N_c}{2} \int_{E<p^-<Ee^{\Delta}} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^+} \Big[\quad \frac{1}{4k^+(k^+ + p^+)} F^l_{st}(k+p,p) F^l_{st}(k+p,p) \mathcal{T}(k+p) \quad \text{gain}$$
$$- \frac{1}{4k^+(k^+ - p^+)} F^l_{st}(k,p) F^l_{st}(k,p) \mathcal{T}(k) \Big] \quad \text{loss} \tag{1}$$

and

$$\delta\mathcal{T}^{NL}(k) = \frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3l}{(2\pi)^3} \frac{1}{2p^+} \times$$
$$\Big[F^i_{lk}(l,p) F^i_{nm}(k+p,p) \langle : A^\dagger_l(l^+ - p^+, \boldsymbol{l} - \boldsymbol{p}) T^a A_k(l^+, \boldsymbol{l}) A^\dagger_m(k^+ + p^+, \boldsymbol{k} + \boldsymbol{p}) T^a A_n(k^+, \boldsymbol{k}) : \rangle$$
$$- F^i_{lk}(l,p) F^i_{nm}(k,p) \langle : A^\dagger_l(l^+ - p^+, \boldsymbol{l} - \boldsymbol{p}) T^a A_k(l^+, \boldsymbol{l}) A^\dagger_m(k^+, \boldsymbol{k}) T^a A_n(k^+ - p^+, \boldsymbol{k} - \boldsymbol{p}) : \rangle \Big] + h.c.$$

These are long but make sense

TMD and CSS

First:

$$F^i_{ln}(k,p)F^i_{ln}(k,p) = 32(k^+)^2\zeta(1-\zeta)\left[\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}+\zeta(1-\zeta)\right]\frac{1}{\boldsymbol{p}^2}; \quad \zeta \equiv \frac{p^+}{k^+}$$

The "loss term""

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^+}\frac{1}{4k^+(k^+-p^+)}F^l_{st}(k,p)F^l_{st}(k,p)\mathcal{T}(\boldsymbol{k},x) =$$
$$\frac{\Delta}{2\pi^2}\int_0^{1/2} d\zeta\left[\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}+\zeta(1-\zeta)\right]\mathcal{T}(\boldsymbol{k},x)$$

The "gain term" regulates the above at $\zeta \approx k^-/E$
So that

$$\frac{\partial}{\partial\eta}\mathcal{T}(\boldsymbol{k},x) \;=\; -\frac{g^2N_c}{4\pi^2}\int_{\frac{k^-}{E}}^{1/2} d\zeta\left[\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}+\zeta(1-\zeta)\right]\mathcal{T}(\boldsymbol{k},x)+\mathrm{NL}$$

Looks a lot like CSS equation, but only a single equation, and the evolution parameter is $\ln k^-$.

BO and CSS I

Recall CSS

$$\frac{\partial T(k^+,\boldsymbol{k}^2;\mu^2;\xi)}{\partial \ln \mu^2} = -\frac{\alpha_s}{2\pi} N_c \int_{\xi/k^+}^{1-\xi/k^+} d\zeta \left[\frac{1-\zeta}{\zeta} + \frac{\zeta}{1-\zeta} + \zeta(1-\zeta)\right] T\left(k^+,\boldsymbol{k}^2;\mu^2;\xi\right)$$

$$\frac{\partial T(k^+,\boldsymbol{k}^2;\mu^2;\xi)}{\partial \ln \frac{1}{\xi}} = -\frac{\alpha_s}{2\pi} 2N_c \int_{\boldsymbol{k}^2}^{\mu^2} \frac{dp^2}{p^2} T\left(k^+,\boldsymbol{k}^2;\mu^2;\xi\right)$$

The CSS and BO cascades are different: for an initial gluon with momentum k

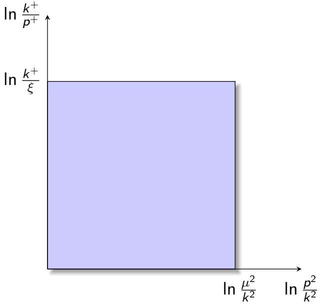


Figure: The phase space of the DGLAP/CSS cascade with resolution scales μ^2 and ξ. Only gluons with momenta $(p^+,\boldsymbol{p}^2)$ inside the blue rectangle are allowed in the wave function.

BO and CSS II

BO evolution - a single evolution parameter

$$\frac{\partial}{\partial\eta}\mathcal{T}(\boldsymbol{k},x) = -\frac{g^2N_c}{4\pi^2}\int_{\frac{k^-}{E}}^{1/2}d\zeta\left[\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}+\zeta(1-\zeta)\right]\mathcal{T}(\boldsymbol{k},x)+\mathrm{NL}$$

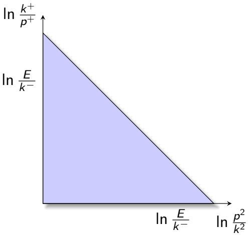


Figure: The phase space of the BO cascade. Only gluons with $p^- < E$, $p^+ < k^+$ and $\boldsymbol{p}^2 > \boldsymbol{k}^2$ are present in the wave function.

Nevertheless the two cascades are exactly equivalent (as far as the evolution of TMD is concerned) if we identify

$$\mathcal{T}_{BO}(\boldsymbol{k},k^+;E) = T_{CSS}(k^+,\boldsymbol{k};\mu^2(E);\xi(E))$$

$$\ln\frac{\mu^2(E)}{\boldsymbol{k}^2} = \ln\frac{E}{k^-}; \qquad \ln\frac{k^+}{\xi(E)}-\frac{11}{12} = \frac{1}{2}\left[\ln\frac{E}{k^-}-\frac{11}{12}\right]$$

The CSS evolution equations then collapse onto the BO evolution.

The Nonlinear terms - the stimulated emission I

Recall that we got nonlinear contributions to the evolution

$$\delta\mathcal{T}^{NL}(k) = \frac{g^2}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3l}{(2\pi)^3}\frac{1}{2p^+}\times$$

$$\Big[F^i_{lk}(l,p)F^i_{nm}(k+p,p)\langle: A^\dagger_l(l^+-p^+,\boldsymbol{l}-\boldsymbol{p})T^aA_k(l^+,\boldsymbol{l})A^\dagger_m(k^++p^+,\boldsymbol{k}+\boldsymbol{p})T^aA_n(k^+$$

$$-F^i_{lk}(l,p)F^i_{nm}(k,p)\langle: A^\dagger_l(l^+-p^+,\boldsymbol{l}-\boldsymbol{p})T^aA_k(l^+,\boldsymbol{l})A^\dagger_m(k^+,\boldsymbol{k})T^aA_n(k^+-p^+,\boldsymbol{k}-\boldsymbol{p})$$

Those look complicated, but have a simple interpretation in "dilute approximation".

Essentially we assume that spectators don't matter, and that the two particles that scatter here must be in the same final state as in the initial state in order to contribute to the forward scattering amplitude.

The Stimulate emission II

The things then simplify

$$\frac{\partial}{\partial\eta}[x\mathcal{T}(\boldsymbol{k},x)] = -\frac{g^2N_c}{4\pi^2}\Bigg[\int_{\boldsymbol{k}^2/Q^2}^{1/2} d\zeta \left[\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}+\zeta(1-\zeta)\right] + \frac{1}{4\pi(N_c^2-1)}\frac{1}{Q^2S_\perp}\int_{\boldsymbol{p}^2=\boldsymbol{k}^2}^{\boldsymbol{p}^2=Q^2}\frac{d^2\boldsymbol{p}}{(2\pi)^2}\,[x\mathcal{T}(\boldsymbol{p},x)]\Bigg][x\mathcal{T}(\boldsymbol{k},x)]$$

Here transverse resolution $Q^2 = 2Ek^+$ - the highest transverse momentum allowed in the BO wave function for particles with longitudinal momentum k^+.
The nonlinear term is just a stimulated emission: the probability of splitting $(k) \to (p) + (k-p)$ is enhanced if there is already a particle with momentum p in the wave function!

It is a nonlinear (higher twist) effect that depletes the number of particles at k. It has nothing to do with low x physics - indeed our "dilute approximation" in the low x regime is hardly justified.

PDF and DGLAP

Where there is TMD, there is PDF.

For PDF we need to include the real corrections due to the gluons in the "window" between E and Ee^{Δ}. This does not present difficulties. We find that at moderate x the DGLAP equation is reproduced where $Q^2 = 2Ek^+$.

$$\frac{\partial}{\partial \ln Q^2} \left[xG(x, Q^2)\right]^L = \frac{\alpha_s}{2\pi} \int_x^1 d\zeta P_{gg}(\zeta) \left[\frac{x}{\zeta} G\left(\frac{x}{\zeta}, \frac{Q^2}{\zeta}\right)\right] \tag{2}$$

At low x this deviates from the standard BFKL, as the resolution scale on the RHS becomes different from Q^2.

In addition the stimulated emission corrections also contribute, and are dominated by their virtual terms.

$$\frac{\partial}{\partial \ln Q^2}[xG(Q^2, x)]^{NL} = -\frac{\alpha_s N_c}{4\pi} \frac{1}{(2\pi)^3} \frac{1}{N_c^2 - 1} \frac{1}{Q^2 S_\perp}[xG(Q^2, x)]^2$$

A higher twist (obviously), but not the GLR-MQ term! No $\ln x$, but leading in α_s. A completely different physical effect, but also leads to slowing down of the evolution.

Future ...

Our ultimate goal is of course to unify DGLAP with BFKL. We need to consider the multiple soft scatterings, a.k.a. Wilson lines. That is still some way off.

The next step is to look at soft scattering off a dilute target, meaning consider the evolution of $\langle \rho^a(\boldsymbol{x}) \rho^a(\boldsymbol{y}) \rangle$. We thought we had it, but we don't - still working on it. It is not straightforward, but we are slowly progressing, and we will do it.

Then there is the question of soft fields, which we have set aside for now. Physically these describe rescattering of emitted gluons on the fields of the projectile when the projectile is dense. Are they important? Probably. One thing we know for sure - they contribute to part of the large transverse logs in NLO BFKL.