General Relativistic Hydrodynamics

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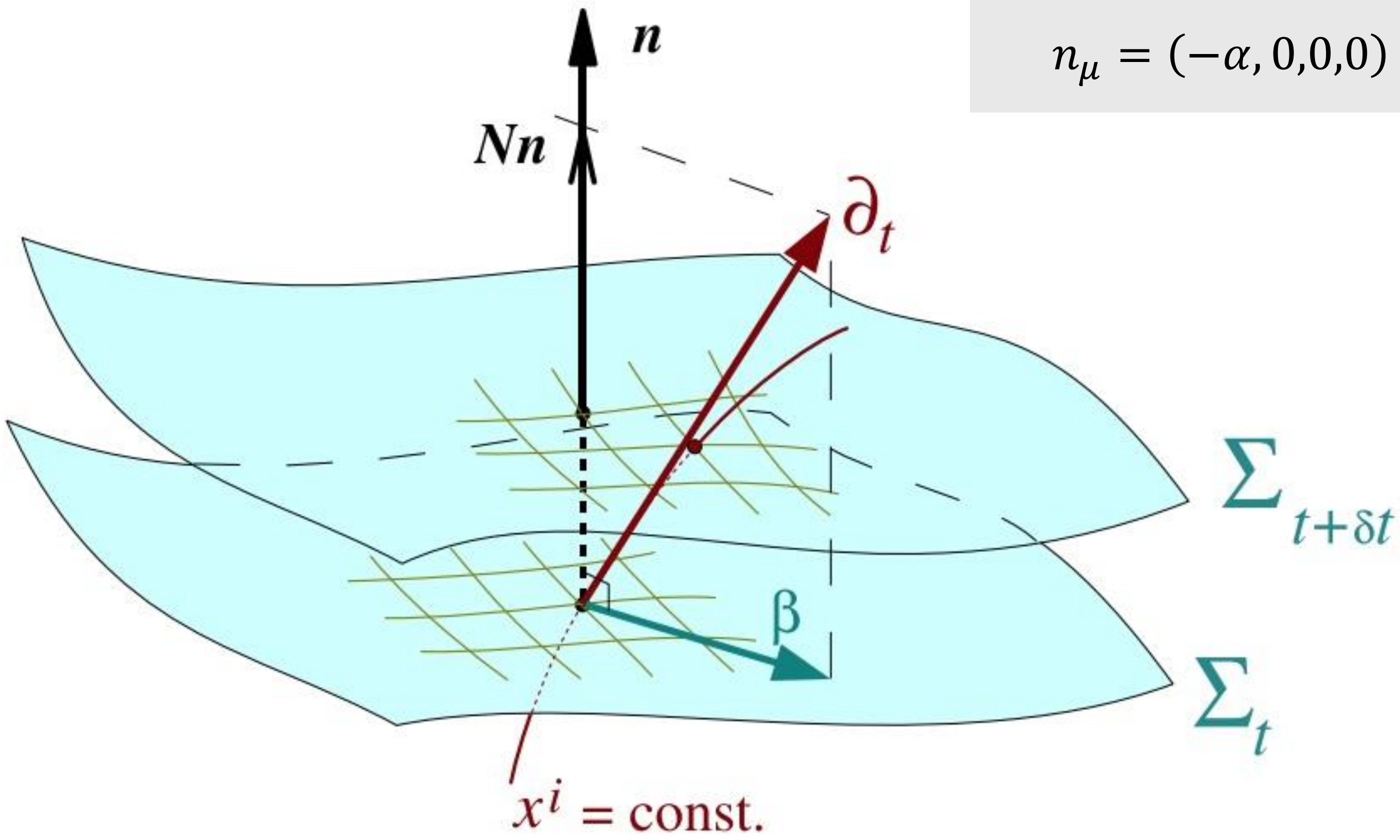
The metric in the 3+1 form

$G = c = 1$

$n_\mu = (-\alpha, 0,0,0) \quad n^\mu = \frac{1}{\alpha}\left(1, -\beta^i\right)$



https://arxiv.org/abs/gr-qc/0703035

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

Equations

Einstein Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

Hydro Equations

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\nabla_\mu J^\mu = 0$$

$$P = P(\rho, \epsilon)$$

$$J^\mu = \rho u^\mu$$

$$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

$$h \equiv 1 + \epsilon + P/\rho$$

Eulerian Observer

- It moves with 4-velocity $\boldsymbol{n}$
- u^μ is the four-velocity of the fluid
- $u^\mu \equiv \frac{dx^\mu}{d\tau}$ and the velocity is $v^i = \frac{dx^i}{dt} = \frac{dx^i}{d\tau}\frac{d\tau}{dt} = \frac{u^i}{u^t}$

- In 3+1 GR, the Eulerian observer will measure the following velocity:

$$v^i \equiv \frac{\gamma^i_\mu u^\mu}{W}$$

$$\gamma^i_\mu \equiv g^i_\mu + n^i n_\mu$$

where $\mathrm{W} = \alpha u^t$ is the Lorentz factor, i.e., $W = \frac{1}{\sqrt{1 - v^i v_i}} = \frac{1}{\sqrt{1 - v^2}}$

Remember: for a normal observer $d\tau = \alpha dt$

Eulerian Observer

- Therefore, $v^i = \frac{1}{W}\left(g^i_\mu + n^i n_\mu\right)u^\mu = \frac{1}{W}\left(u^i + \frac{\beta^i}{\alpha}\alpha u^t\right) = \frac{u^i}{W} + \frac{\beta^i}{\alpha}$

$$v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha}$$

$$v_i = \frac{u_i}{W}$$

Remember: $n_\mu = (-\alpha, 0,0,0)$, $n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right)$

Conservation of Rest Mass

$$\nabla_\mu J^\mu = 0 \ \rightarrow \nabla_\mu(\rho u^\mu) = 0 \ \rightarrow$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\ \rho \mathrm{u}^\mu) = 0$$

$$\partial_t(\alpha\sqrt{\gamma}\rho u^t) + \partial_i\left(\alpha\sqrt{\gamma}\rho u^i\right) = 0$$

$$u^i = \left(v^i - \frac{\beta^i}{\alpha}\right)W$$

$$D \equiv \sqrt{\gamma}\rho\alpha u^t = \sqrt{\gamma}\rho W$$

$$\partial_t(D) + \partial_i\left[\sqrt{\gamma}\left(\alpha v^i - \beta^i\right)W\rho\right] = 0$$

$$\partial_t(D) + \partial_i\left[D\left(\alpha v^i - \beta^i\right)\right] = 0$$

Conservation of Energy and Momentum

$$\nabla_\mu T^{\mu\nu} = 0$$

$$g^{\nu\lambda}\left[\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}T^{\mu}_{\lambda}\right) - \frac{1}{2}T^{\alpha\beta}\partial_\lambda g_{\alpha\beta}\right] = 0$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}T^{\mu}_{\lambda}\right) = \frac{1}{2}T^{\alpha\beta}\partial_\lambda g_{\alpha\beta}$$

$$\partial_t\left(\sqrt{\gamma}\alpha T^{0}_{\lambda}\right) + \partial_i\left(\sqrt{\gamma}\alpha T^{i}_{\lambda}\right) = \frac{\sqrt{-g}}{2}T^{\alpha\beta}\partial_\lambda g_{\alpha\beta}$$

GRHD Equations

The system of equations is now written in a flux-conservative form (**Valencia formulation**, Banyuls et al 1997, Anton et al 2006):

$$\partial_t \boldsymbol{U} + \partial_i \boldsymbol{F}^i = \boldsymbol{S}$$

where $\boldsymbol{U}$ is the vector of conserved variables, $\boldsymbol{F}^i$ the fluxes, and $\boldsymbol{S}$ the source terms.

For example, let's take the conservation of rest mass:

$$\partial_t(D) + \partial_i\left[D\left(\alpha v^i - \beta^i\right)\right] = 0$$

$\tilde{v}^i \equiv v^i - \beta^i/\alpha$

then $U = D = \sqrt{\gamma}\rho W, F^i = D\left(\alpha v^i - \beta^i\right) = \alpha D \tilde{v}^i, S = 0.$

GRHD Equations

$$\boldsymbol{U} = (D, S_j, \tau)$$

$$D = \sqrt{\gamma}\rho W$$
$$S_j = \sqrt{\gamma}\left(\rho h W^2 v_j\right)$$
$$\tau = \sqrt{\gamma}(\rho h W^2 - P) - D$$

In the non-relativistic case, $D \to \rho, S_j \to \rho v_j, \tau \to \rho\epsilon$

GRHD Equations

$$F^i = \alpha \times \begin{bmatrix} D\tilde{v}^i \\ S_j\tilde{v}^i + \sqrt{\gamma}P\delta_j^i \\ \tau\tilde{v}^i + \sqrt{\gamma}Pv^i \end{bmatrix}$$

$$S = \alpha\sqrt{\gamma} \times \begin{bmatrix} 0 \\ T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu} - \Gamma_{\mu\nu}^\lambda g_{\lambda j}\right) \\ \alpha\left(T^{\mu 0}\frac{\partial \ln \alpha}{\partial x^\mu} - T^{\mu\nu}\Gamma_{\mu\nu}^0\right) \end{bmatrix}$$

The importance of flux-conservative Form

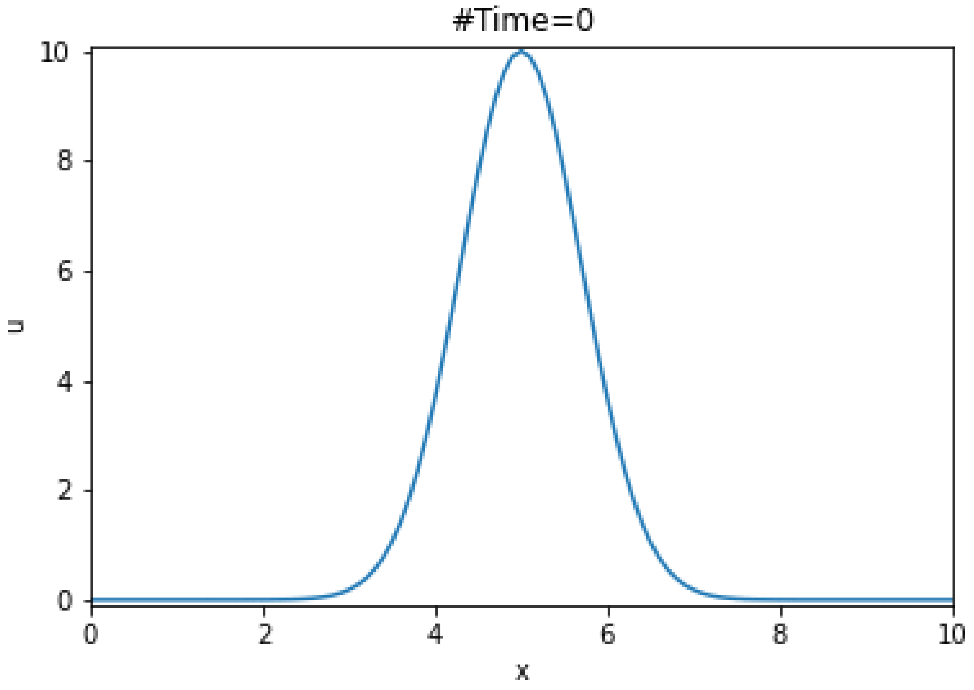
- **Lax-Wendroff Theorem** (1960): If a consistent numerical method written in a flux conservative form converges to a function u(x,t) for dx that goes to zero, then u(x,t) is a solution of the conservation law*.

- **Hou-LeFlock Theorem** (1994): non-conservative schemes do not converge to the correct solution if a shock wave is present in the flow.

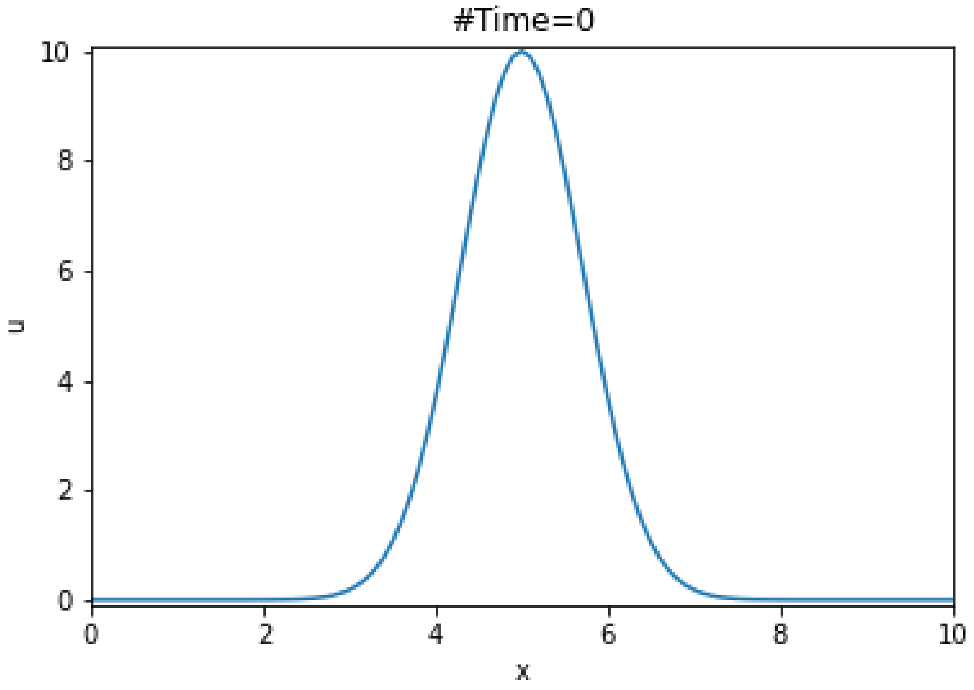
*note that the proper formulation of the Lax-Wendroff theorem is slightly different from what reported here (but for our purposes it is OK).

Burgers' Equation

FC



NFC



$$\frac{\partial u}{\partial t} + \frac{\partial\left(\frac{1}{2}u^2\right)}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

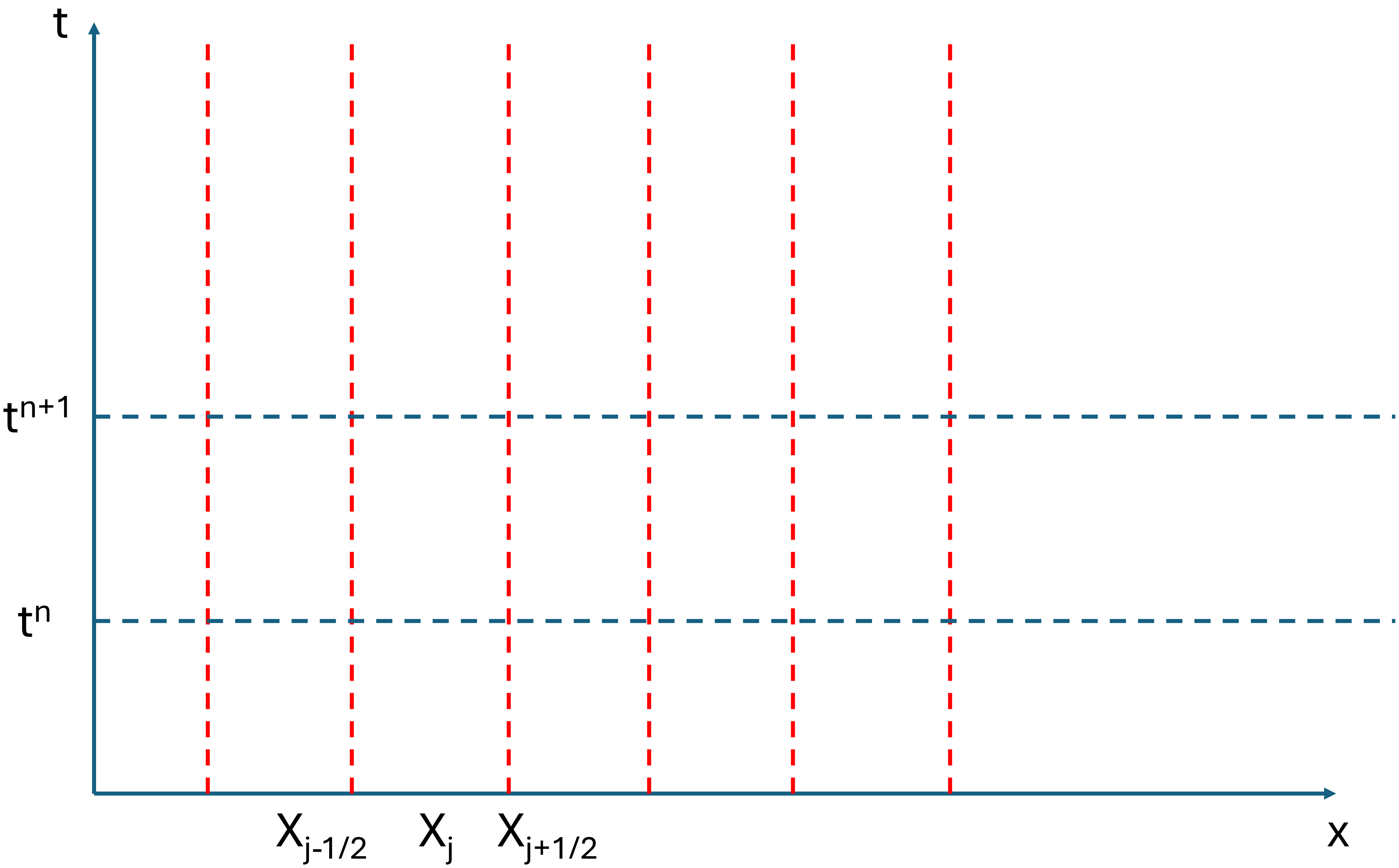
WHAT IS A FLUX-CONSERVATIVE FORM?

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

Let's solve it on a numerical grid

$$x_j = j \times \Delta x \ , j = 0, \cdots , J-1$$

$$t^n = n \times \Delta t \ , n = 0, \cdots , N-1$$


t
t^{n+1}
t^n
$X_{j-1/2}$
X_j
$X_{j+1/2}$
x

We now take the integral in t and x

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \int_{t^n}^{t^{n+1}} \frac{\partial u}{\partial t} dx\, dt + \int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial f(u)}{\partial x} dx\, dt = 0$$

$$\downarrow$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \left[u(x, t^{n+1}) - u(x, t^n)\right] dx + \int_{t^n}^{t^{n+1}} \left[f\left(u(x_{j+1/2}, t)\right) - f\left(u(x_{j-1/2}, t)\right)\right] dt = 0$$

We then divide by Δx

$$
\begin{aligned}
\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^{n+1})\,dx \quad &= \quad \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^{n})\,dx \\
&- \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2},t)\right)dt - \int_{t^n}^{t^{n+1}} f\left(u(x_{j-1/2},t)\right)dt\right] = 0
\end{aligned}
$$

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^{n+1})\,dx = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^{n})\,dx - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2},t)\right)dt - \int_{t^n}^{t^{n+1}} f\left(u(x_{j-1/2},t)\right)dt\right] = 0$$

We now define

$$\tilde{u}_j^n \equiv \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^n)\,dx$$

$$\tilde{u}_j^{n+1} = \tilde{u}_j^n - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2}, t)\right) dt - \int_{t^n}^{t^{n+1}} f\left(u(x_{j-1/2}, t)\right) dt\right]$$

$$\tilde{u}_j^{n+1} = \tilde{u}_j^n - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2}, t)\right) dt - \int_{t^n}^{t^{n+1}} f\left(u(x_{j-1/2}, t)\right) dt\right]$$

Let’s also define

$$f_{j+1/2}^n \equiv \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2}, t)\right) dt$$

And our equation reduces to:

$$\tilde{u}_j^{n+1} = \tilde{u}_j^n - \frac{\Delta t}{\Delta x}\left(f_{j+1/2}^n - f_{j-1/2}^n\right)$$

$$\tilde{u}_j^n \equiv \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n)\, dx$$

$$f_{j+1/2}^n \equiv \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2}, t)\right) dt$$

a numerical method written in this way is said to be in **flux conservative form**.

Methods written in this form conserve $\tilde{u}$, indeed by summing over j

$$\Delta x \, \Sigma_{j=0}^{J-1} \, \tilde{u}_j^{n+1} = \Delta x \, \Sigma_{j=0}^{J-1} \, \tilde{u}_j^{n} - \Delta t \left(f_{J-1/2}^{n} - f_{-1/2}^{n} \right)$$

so $\tilde{u}$ is conserved except for fluxes at the boundaries of the numerical domain.

How do we compute the flux?

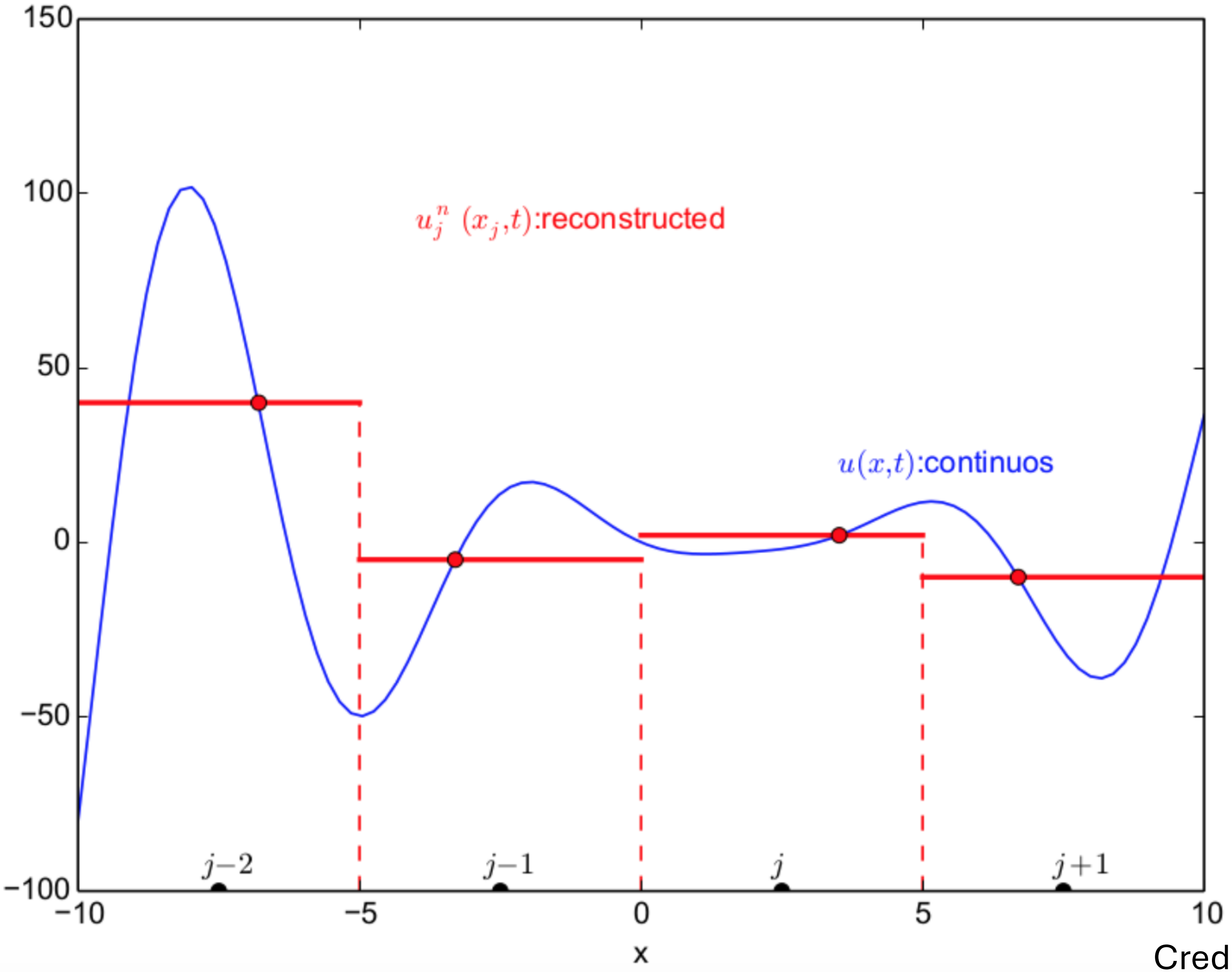
A very simple choice could be

$$f^n_{j+1/2} = \frac{1}{2}\left[f(\tilde{u}^n_j) + f(\tilde{u}^n_{j+1})\right]$$

$$\begin{aligned}
\tilde{u}^{n+1}_j &= \tilde{u}^n_j - \frac{\Delta t}{2\Delta x}\left[f(\tilde{u}^n_j) + f(\tilde{u}^n_{j+1}) - f(\tilde{u}^n_{j-1}) - f(\tilde{u}^n_j)\right] \\
&= \tilde{u}^n_j - \frac{\Delta t}{2\Delta x}\left[f(\tilde{u}^n_{j+1}) - f(\tilde{u}^n_{j-1})\right]
\end{aligned}$$

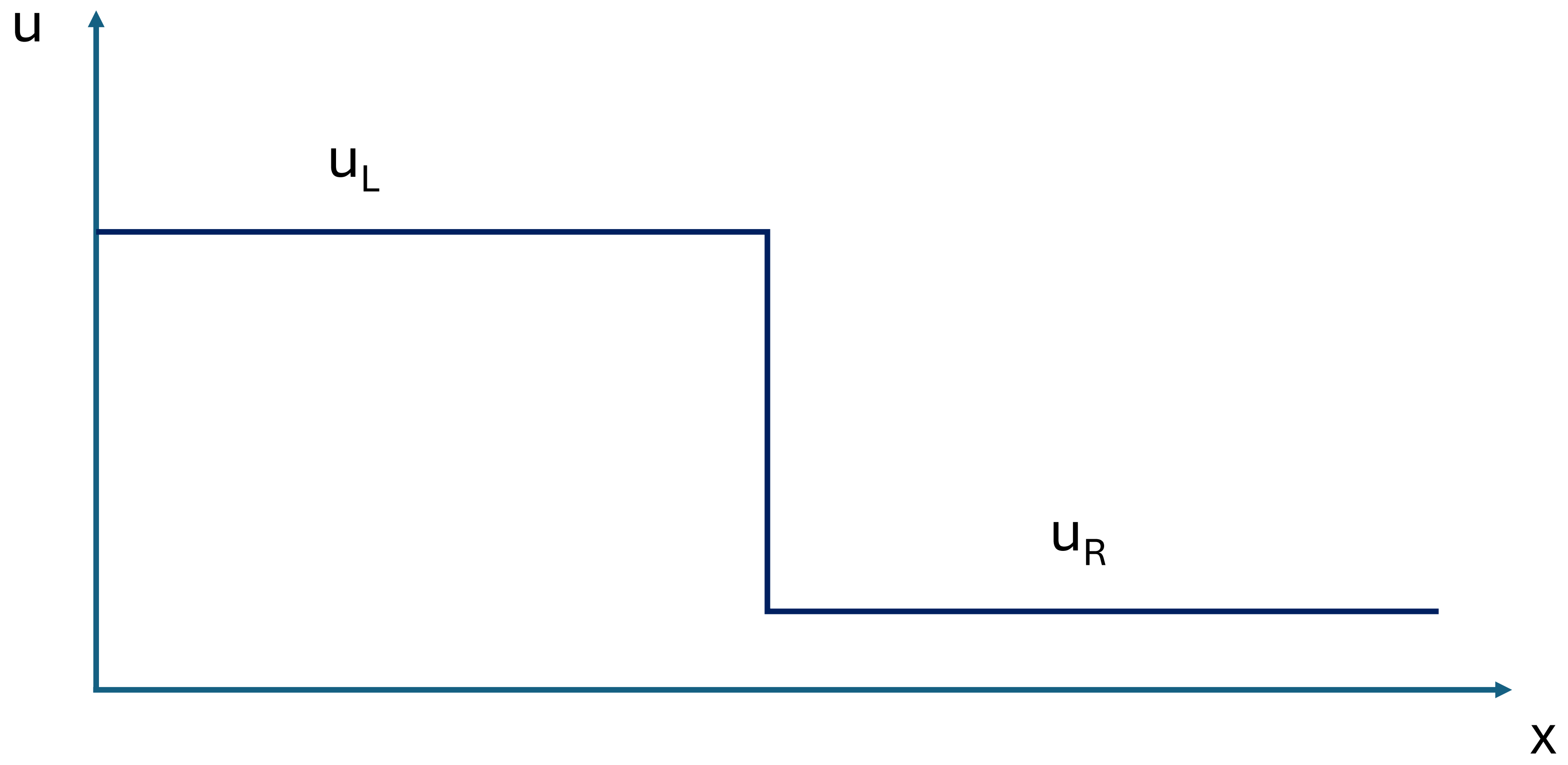
This method is known as FTCS and it is known to be unfortunately unstable...

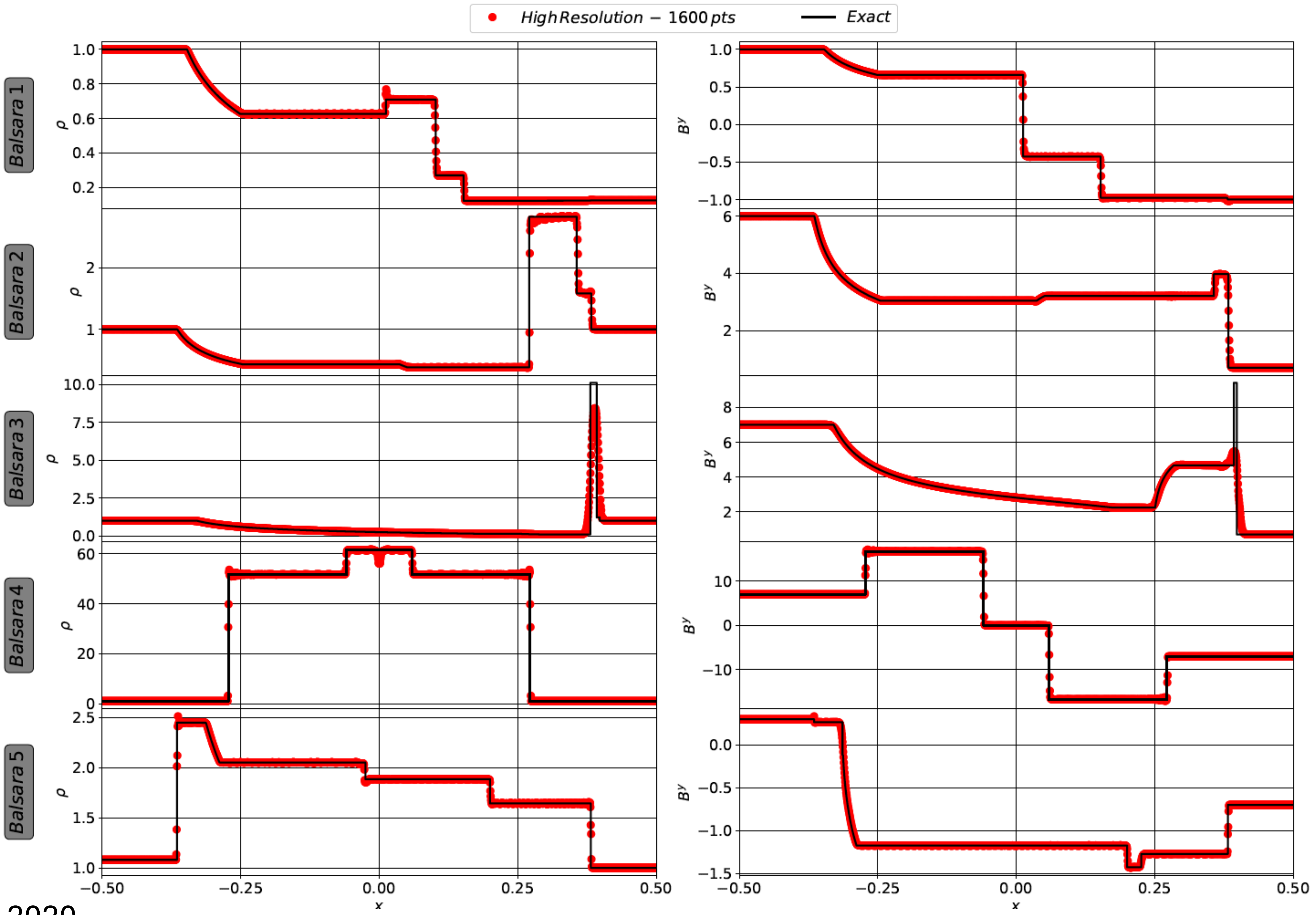
Godunov Method



Credit: A. Endrizzi

RIEMANN PROBLEM





Cipolletta et al 2020

RIEMANN PROBLEM

- By solving the Riemann problem one can compute

$$f^n_{j+1/2} \equiv \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2}, t)\right) dt$$

- My open-source exact RMHD Riemann solver can be downloaded here: https://github.com/bgiacoma/Exact_Riemann_Solver

- More computationally convenient to use approximate Riemann solvers, e.g., HLLE

HIGH RESOLUTION SHOCK-CAPTURING METHODS

- To increase the order, instead of assuming a step function one could use a piecewise linear function:

$$\tilde{u}(x,t^n) = \tilde{u}_j^n + \sigma_j^n (x - x_j) \quad \text{for} \quad x_{j-1/2} < x < x_{j+1/2}$$

$$\sigma_j^n = \text{minmod}\left(\frac{\tilde{u}_j^n - \tilde{u}_{j-1}^n}{\Delta x}, \frac{\tilde{u}_{j+1}^n - \tilde{u}_j^n}{\Delta x}\right)$$

$$\text{minmod}(a,b) \equiv \begin{cases} a \text{ if } |a|<|b| \text{ and } ab>0 \\ b \text{ if } |b|<|a| \text{ and } ab>0 \\ 0 \text{ if } ab<0 \end{cases}$$

or higher orders functions (e.g., PPM).

GRMHD equations

Maxwell Equations

It is useful to recap Maxwell Equations in a (special) Relativistic formulation (cgs units):

$$\boxed{\nabla_\nu F^{\mu\nu} = I^\mu} \rightarrow \nabla \cdot \vec{E} = 4\pi\rho_*, \qquad \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B} - 4\pi\vec{I}$$

$$\boxed{\nabla_\nu \ {}^*F^{\mu\nu} = 0} \rightarrow \nabla \cdot \vec{B} = 0, \qquad \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\boxed{\nabla_\nu I^\nu = 0} \rightarrow \frac{\partial \rho_*}{\partial t} + \nabla \cdot \vec{I} = 0 \text{ (charge conservation)}$$

Ideal MHD

- (special) Relativistic Ohm Law: $\vec{I} = \sigma W\left[\vec{E} + \vec{v} \times \vec{B} - \left(\vec{E} \cdot \vec{v}\right)\vec{v}\right] + \rho_* \vec{v}$
- In many astrophysical scenarios $\sigma \to \infty$ and this implies $\vec{E} = -\vec{v} \times \vec{B}$
- This is called the ideal MHD limit
- In this case we do not need the evolve the electric field explicitly
- Maxwell equations reduce to $\nabla_\nu \; {}^*F^{\mu\nu} = 0$

Equations

Einstein Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

Hydro Equations

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\nabla_\mu J^\mu = 0$$

$$P = P(\rho, \epsilon)$$

$$J^\mu = \rho u^\mu$$

$$T^{\mu\nu} = (\rho h + b^2) u^\mu u^\nu + \left(p + \frac{b^2}{2} \right) g^{\mu\nu} - b^\mu b^\nu$$

Maxwell Equations

$$\nabla_\nu {}^*F^{\mu\nu} = 0$$

GRMHD equations

The system of equations is written in a conservative form (Valencia formulation, Anton et al 2006):

$$\left.\begin{array}{rcl}\nabla_\mu(\rho u^\mu) & = & 0 \\ \nabla_\mu T^{\mu\nu} & = & 0\end{array}\right\} \Rightarrow \frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{U}}{\partial t}+\frac{\partial\sqrt{-g}\mathbf{F}^i}{\partial x^i}\right)=\mathbf{S}$$

where **U** is the vector of conserved variables, $\mathbf{F}^\mathbf{i}$ the fluxes, and **S** the source terms. They can then be solved using HRSC methods using approximate Riemann solvers.

To these one has to add the equations for the evolution of the magnetic field:

$$\frac{\partial}{\partial t}\left(\sqrt{\gamma}\vec{B}\right)=\nabla\times\left[\left(\alpha\vec{v}-\vec{\beta}\right)\times\left(\sqrt{\gamma}\vec{B}\right)\right]$$

$$\nabla\cdot\left(\sqrt{\gamma}\vec{B}\right)=0$$

GRMHD equations

$$\mathbf{U} \equiv \left[D, S_j, \tau, B^k\right]$$

$$D \equiv \rho W,$$

$$S_j \equiv (\rho h + b^2) W^2 v_j - \alpha b^0 b_j,$$

$$\tau \equiv (\rho h + b^2) W^2 - \left(p + \frac{b^2}{2}\right) - \alpha^2 (b^0)^2 - D.$$

GRMHD equations

$\tilde{v}^i \equiv \alpha v^i - \beta^i$

$$\mathbf{F}^i = \begin{pmatrix} D\tilde{v}^i/\alpha \\ S_j\tilde{v}^i/\alpha + (p + b^2/2)\delta^i_j - b_j B^i/W \\ \tau\tilde{v}^i/\alpha + (p + b^2/2)v^i - \alpha b^0 B^i/W \\ B^k\tilde{v}^i/\alpha - B^i\tilde{v}^k/\alpha \end{pmatrix},$$

$$\mathbf{S} = \begin{pmatrix} 0 \\ T^{\mu\nu}\left(\partial_\mu g_{\nu j} - \Gamma^\delta_{\nu\mu} g_{\delta j}\right) \\ \alpha\left(T^{\mu 0}\partial_\mu \ln\alpha - T^{\mu\nu}\Gamma^0_{\nu\mu}\right) \\ 0^k \end{pmatrix}$$

THE DIV(B)=0 PROBLEM

Several numerical techniques available to keep the magnetic field divergence-less:

1. Constrained Transport
   (Yee 1966, Evans & Hawley 1988, Balsara & Spicer 1999)

2. Hyperbolic Divergence Cleaning
   (Dedner et al 2002)

3. Vector Potential Evolution with Modified Lorenz Gauge
   (Etienne et al 2012, Farris et al 2012)

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