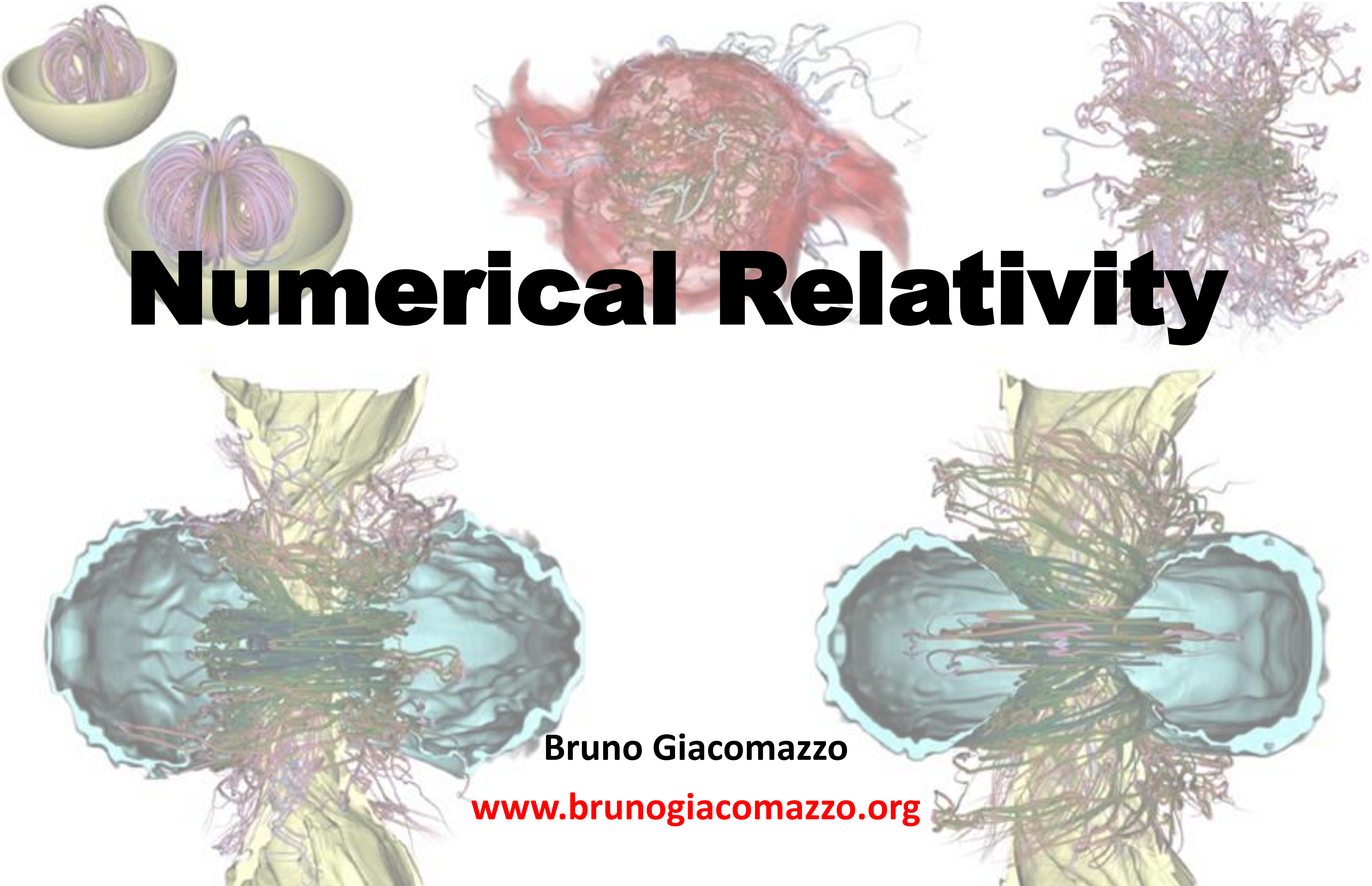
Numerical Relativity



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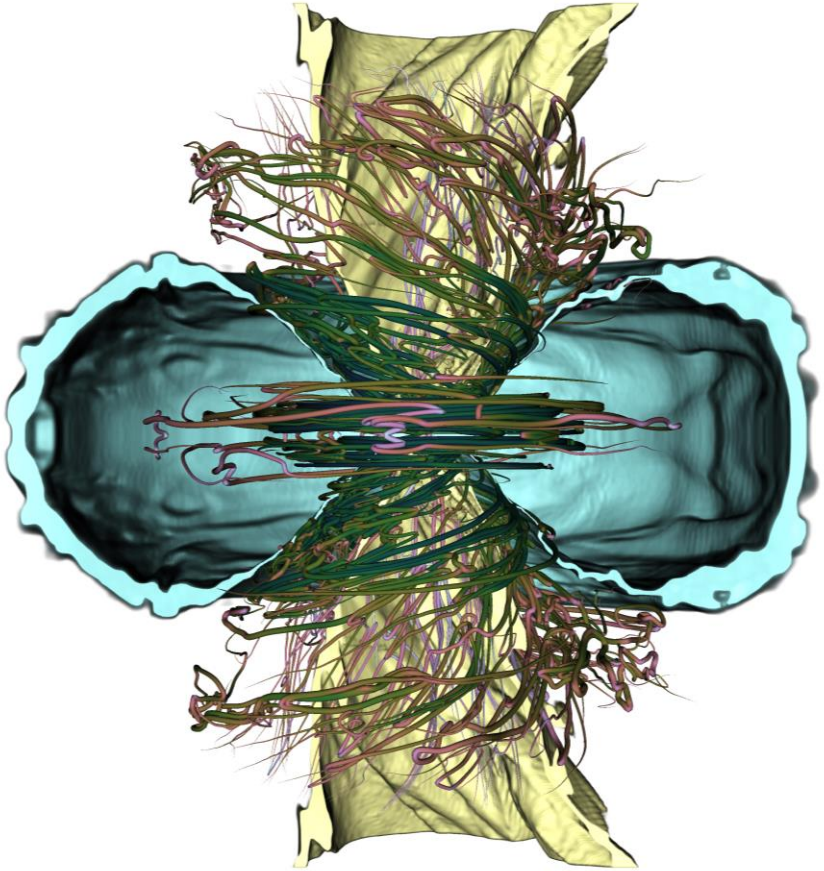
General Relativity and Astrophysics

- Binary Black Hole Mergers
- Binary Neutron Star Mergers
- Neutron Star – Black Hole Mergers
- Supernovae
- Accretion Disks
- Cosmology

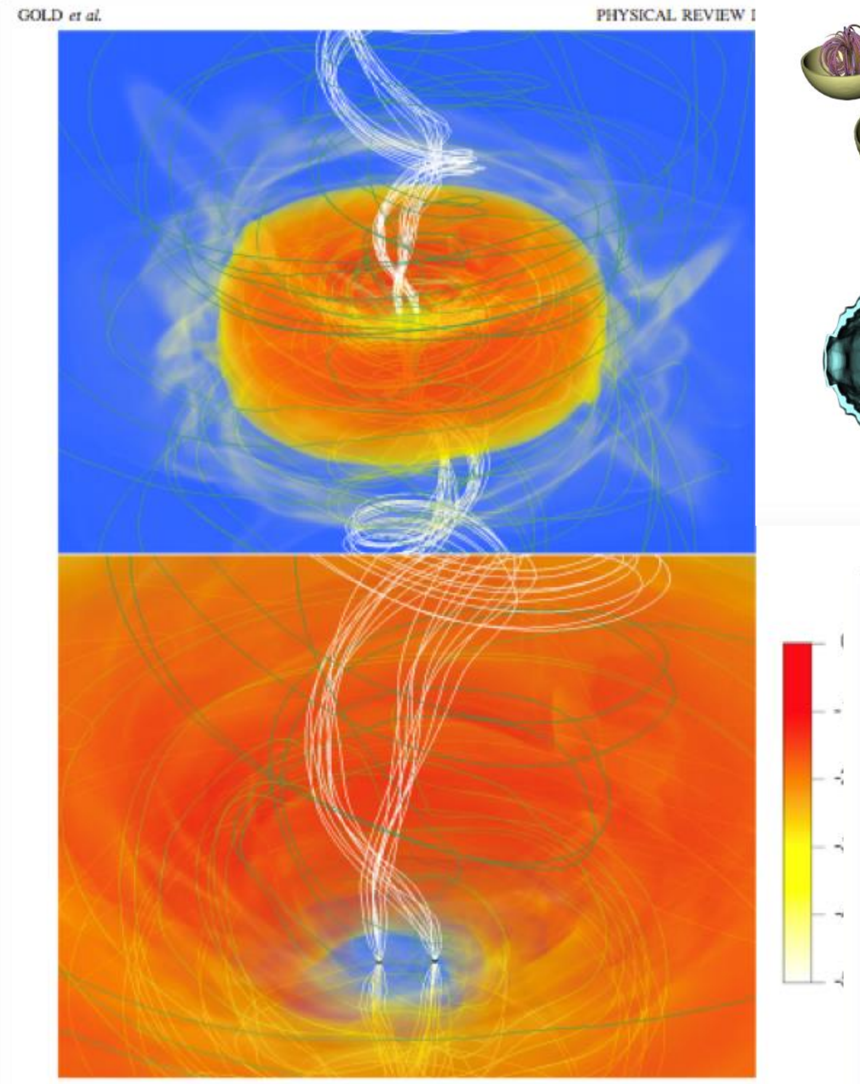


Kawamura et al 2016

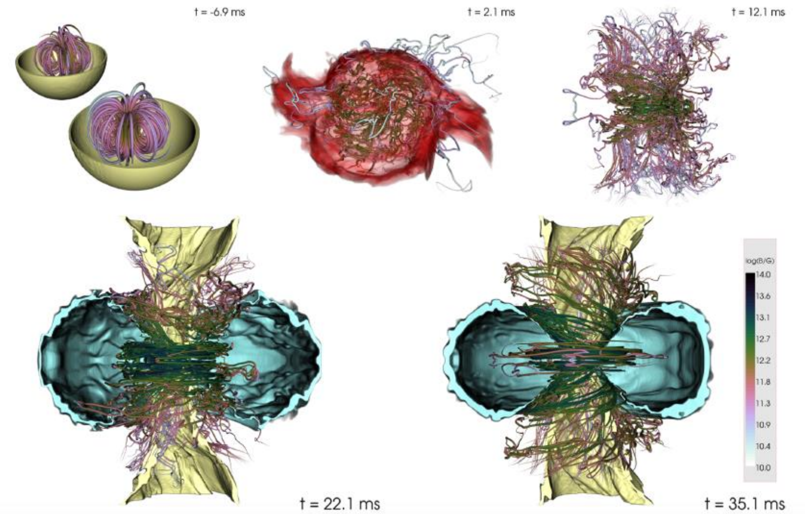
In all these scenarios general relativity plays a fundamental role.

Almost all scenarios require numerical solutions -> numerical relativity

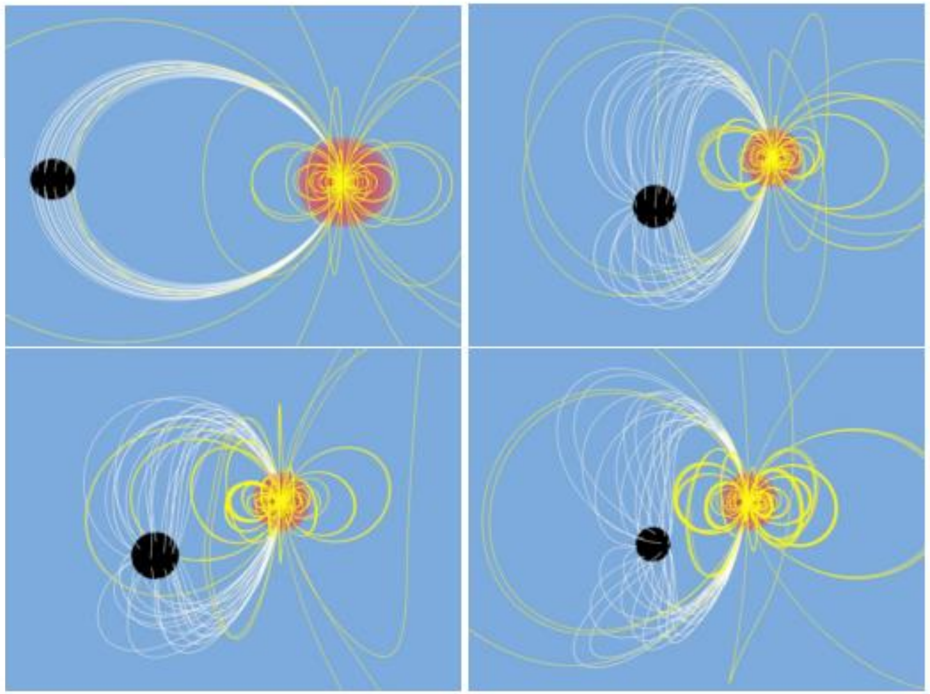
APPLICATIONS



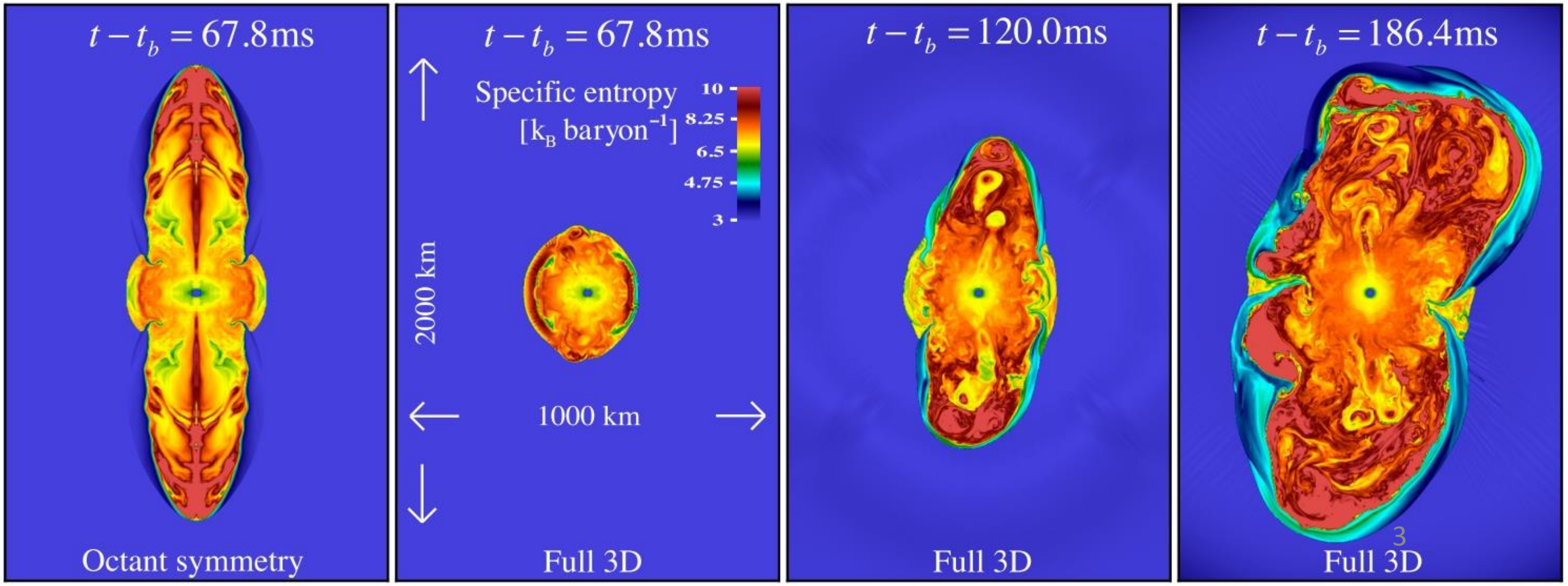
Gold et al 2014



Kawamura et al 2016

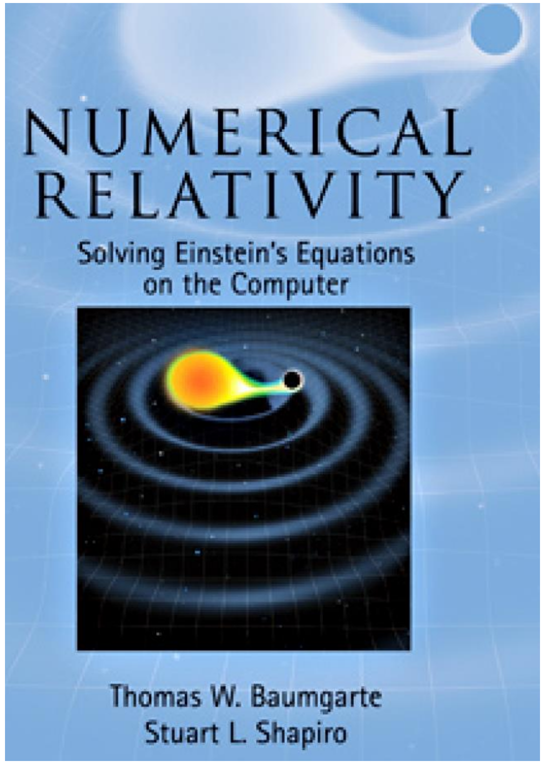


Paschalidis et al 2013



Moesta et al 2014

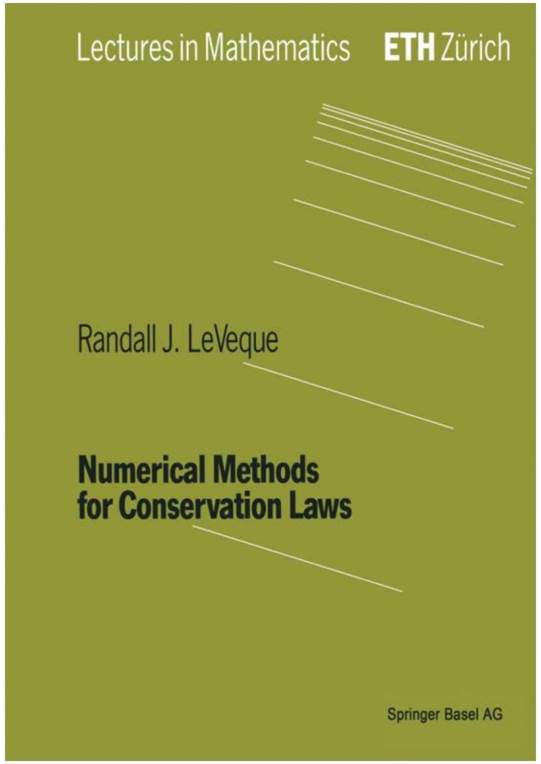
Useful Textbooks



NUMERICAL RELATIVITY

Solving Einstein's Equations on the Computer

Thomas W. Baumgarte
Stuart L. Shapiro

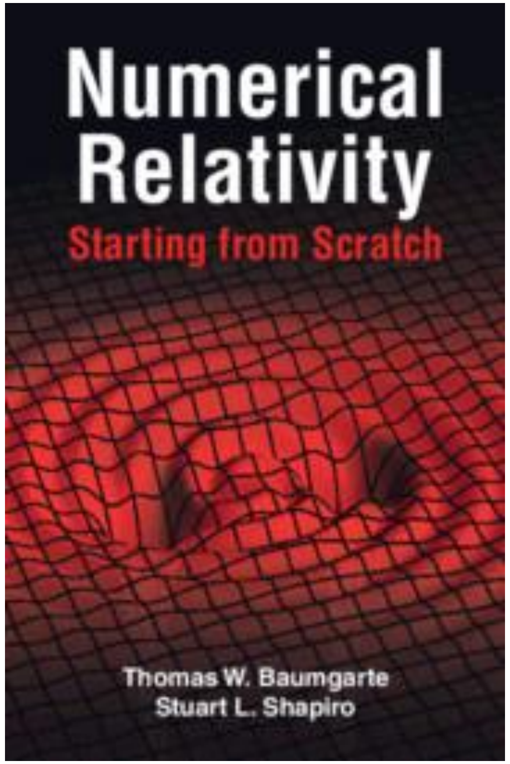


Lectures in Mathematics ETH Zürich

Randall J. LeVeque

Numerical Methods for Conservation Laws

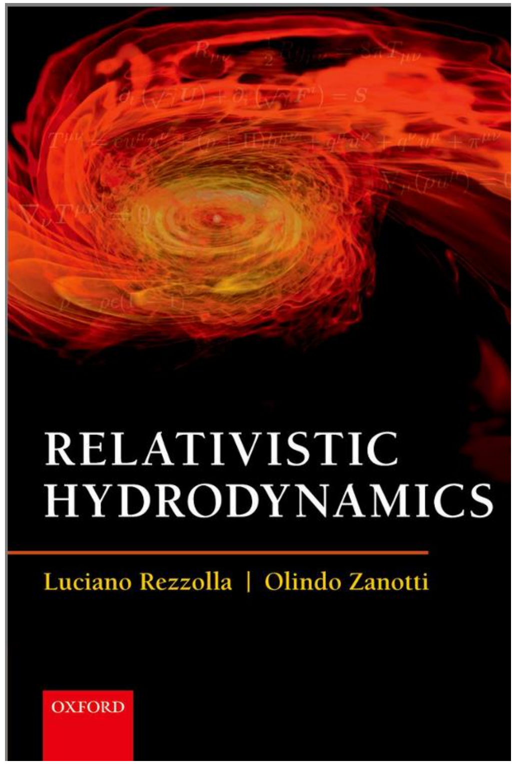
Springer Basel AG



Numerical Relativity

Starting from Scratch

Thomas W. Baumgarte
Stuart L. Shapiro



RELATIVISTIC HYDRODYNAMICS

Luciano Rezzolla | Olindo Zanotti

OXFORD

History of Numerical Relativity
(see also https://link.springer.com/article/10.1007/lrr-2015-1)

- 1962 Arnowitt, Deser and Misner (ADM) 3+1 formulation
- 1964 Hahn and Lindquist first attempt at head-on collision of wormholes
- 1966 May and White first 1D GR simulation of collapse to BH
- 1975 Smarr and Eppley first head-on collision of BH in axisymmetry
- 1985 Stark and Piran extract GWs from a simulation of rotating collapse to a BH in NR.
- 1992 Bona and Massó “1+log” slicing (gauge) condition
- 1994 “Binary Black Hole Grand Challenge Project” is launched in the USA
- 1995-1998 BSSN formulation
- 1996 Brügmann mesh refinement simulation of BHs
- 1997 Cactus 1.0 is released

History of Numerical Relativity
(see also https://link.springer.com/article/10.1007/lrr-2015-1)

- 2000 Brandt et al. simulate the first grazing collisions of BHs using a revised version of the Grand Challenge Alliance code
- 2000 Shibata and Uryū first NS-NS merger simulation in GR
- 2003 Schnetter et al “Carpet” AMR driver for Cactus
- 2005 Pretorius first simulation of BH-BH inspiral and merger
- 2006 Shibata and Uryū first NS-BH merger simulation
- 2008 Anderson et al first GRMHD simulation of an NS-NS merger
- 2010 Chawla et al first GRMHD simulation of an NS-BH merger
- 2010 The first release (code name "Bohr") of the **Einstein Toolkit** is announced

Einstein Equations

Notation:
We assume G=c=1, metric signature (-,+,+,+), $\mu \in [0,3]$

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

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$R \equiv R^{\mu}_{\mu}$ Ricci scalar

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$R_{\mu\nu} \equiv R^{\rho}_{\mu\rho\nu}$ Ricci tensor

$R^{\sigma}_{\mu\rho\nu} \equiv \partial_\rho \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} + \Gamma^{\sigma}_{\tau\rho} \Gamma^{\tau}_{\mu\nu} - \Gamma^{\sigma}_{\tau\nu} \Gamma^{\tau}_{\mu\rho}$ Riemann tensor

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We assume G=c=1, metric signature (-,+,+,+), $\mu \in [0,3]$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

$R \equiv R^{\mu}_{\mu}$ Ricci scalar

$R_{\mu\nu} \equiv R^{\rho}_{\mu\rho\nu}$ Ricci tensor

$R^{\sigma}_{\mu\rho\nu} \equiv \partial_\rho \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} + \Gamma^{\sigma}_{\tau\rho}\Gamma^{\tau}_{\mu\nu} - \Gamma^{\sigma}_{\tau\nu}\Gamma^{\tau}_{\mu\rho}$ Riemann tensor

$$\Gamma^{\sigma}_{\mu\rho} \equiv \frac{1}{2} g^{\sigma\tau}\left(\partial_\mu g_{\rho\tau} + \partial_\rho g_{\mu\tau} - \partial_\tau g_{\mu\rho}\right)$$

Types of PDEs

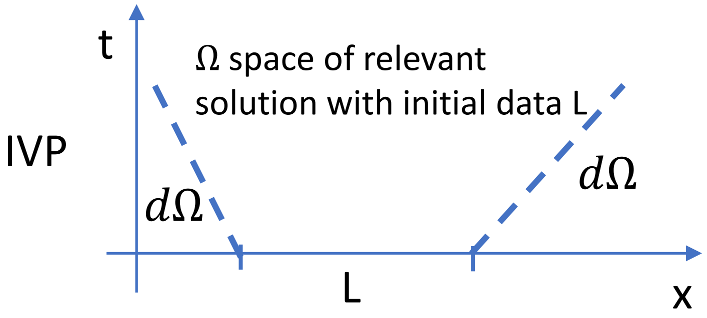
- $A\partial_\xi^2\phi + 2B\partial_\xi\partial_\eta\phi + C\partial_\eta^2\phi = \rho(\xi, \eta, \phi, \partial_\xi\phi, \partial_\eta\phi)$
- A, B, and C are real and do not vanish simultaneously
- $AC - B^2 > 0$ → Elliptic
- $AC - B^2 = 0$ → Parabolic
- $AC - B^2 < 0$ → Hyperbolic

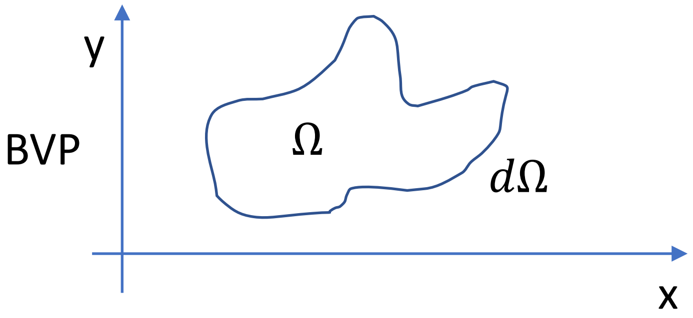
Types of PDEs

Examples

- Elliptic: $\partial_x^2\phi + \partial_y^2\phi = \rho$ (Poisson's equation)
- Parabolic: $\partial_t\phi - k\partial_x^2\phi = 0$ (Heat diffusion equation)
- Hyperbolic: $\partial_t^2\phi - c^2\partial_x^2\phi = 0$ (wave equation)

- Both parabolic and hyperbolic eqs constitute Initial Value Problems (IVP)
- Elliptic eqs constitute Boundary Value Problems (BVP)





Wave Equation

General Solution

$\phi(x,t) = g(x+ct) + h(x-ct)$

$$\partial_t^2 \phi - c^2 \partial_x^2 \phi = 0$$

$$k \equiv -\partial_t \phi$$
$$l \equiv \partial_x \phi$$

$$\left\{ \begin{array}{c} \partial_t \phi = -k \\ \partial_t k + c^2 \partial_x l = 0 \\ \partial_t l + \partial_x k = 0 \end{array} \right.$$

Wave Equation

- In a more compact notation

$$\partial_t \boldsymbol{u} + \boldsymbol{A} \cdot \partial_x \boldsymbol{u} = \boldsymbol{S}$$

where

- $\boldsymbol{u} \equiv (\phi, k, l)$ is the solution vector
- $\boldsymbol{S} \equiv (-k, 0{,}0)$ is the source vector
- $\boldsymbol{A} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c^2 \\ 0 & 1 & 0 \end{pmatrix}$ is the velocity matrix

Wave Equation

- $\boldsymbol{A}$ admits 3 eigenvalues $(c, -c, 0)$ and these correspond to the characteristic speeds
- $\boldsymbol{A}$ can be diagonalized into $\mathbf{D} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & -c \end{pmatrix}$ via a matrix $\boldsymbol{\Lambda}$ such that

  $\boldsymbol{\Lambda}^{-\mathbf{1}}\mathbf{A}\boldsymbol{\Lambda} = \mathbf{D}$

- Let's apply $\boldsymbol{\Lambda}$ to our equation: $\partial_t \boldsymbol{u} + \boldsymbol{A} \cdot \partial_x \boldsymbol{u} = \boldsymbol{S}$

$$\boldsymbol{\Lambda}^{-\mathbf{1}}\partial_t \boldsymbol{u} + \boldsymbol{\Lambda}^{-\mathbf{1}}\boldsymbol{A} \cdot \boldsymbol{\Lambda}\, \boldsymbol{\Lambda}^{-\mathbf{1}}\partial_x \boldsymbol{u} = \boldsymbol{\Lambda}^{-\mathbf{1}}\boldsymbol{S}$$

$$\partial_t \boldsymbol{w} + \boldsymbol{D} \cdot \partial_x \boldsymbol{w} = \boldsymbol{\Lambda}^{-\mathbf{1}}\boldsymbol{S} \text{ where } \boldsymbol{w} \equiv \boldsymbol{\Lambda}^{-\mathbf{1}}\boldsymbol{u}$$

- and these are essentially 3 advection equations, including one with a solution propagating toward the right and one toward the left at speed c.

Wave Equation

We could have obtained the diagonalized version directly by using these variables:

$$\partial_t^2 \phi - c^2 \partial_x^2 \phi = 0$$

$$\begin{gathered} w_1 \equiv \phi \\ w_2 \equiv (\partial_t - c\partial_x)\phi \\ w_3 \equiv (\partial_t + c\partial_x)\phi \end{gathered}$$

$$\begin{gathered} \partial_t w_1 = (w_2 + w_3)/2 \\ \partial_t w_2 + c\partial_x w_2 = 0 \\ \partial_t w_3 - c\partial_x w_3 = 0 \end{gathered}$$

Notion of Hyperbolicity

- Hyperbolic PDEs can be written as $\partial_t \boldsymbol{u} + \boldsymbol{A} \cdot \partial_x \boldsymbol{u} = \boldsymbol{S}$
- In more than 1 spatial dimension we have:

$$\partial_t \boldsymbol{u} + \boldsymbol{A}^i \cdot \partial_i \boldsymbol{u} = \boldsymbol{S}$$

- if $\boldsymbol{u}$ has n components each $\boldsymbol{A}^i$ has nxn components
- For simplicity we ignore the source vector (e.g., Einstein eqs in vacuum)

Notion of Hyperbolicity

- Definition: We call a problem **well-posed** if we can define some norm $\|\ldots\|$ so that the norm of the solution vector satisfies for all times $t \geq 0$

$$\left\|\boldsymbol{u}\left(t, x^i\right)\right\| \leq k e^{\alpha t}\left\|\boldsymbol{u}\left(0, x^i\right)\right\|$$

- Note: Not all hyperbolic systems guarantee this property.

Notion of Hyperbolicity

- Let’s consider an arbitrary unit vector n^i
- $\boldsymbol{P} = \boldsymbol{A}^i n_i$ is the principal symbol or characteristic matrix of the system

We call the system:

- **Strongly Hyperbolic** if, for all unit vectors n^i, $\boldsymbol{P}$ has real eigenvalues and a complete set of eigenvectors
- **Weakly Hyperbolic** if $\boldsymbol{P}$ has real eigenvalues, but not a complete set of eigenvectors

Notion of Hyperbolicity

- Theorem: Strongly hyperbolic systems are well-posed. Weakly hyperbolic systems are not
  (for the proof, see chapter 2 of Kreiss & Lorentz 1989, "Initial Boundary Value Problems and the Navier-Stokes Equations")

- It is crucial to write hyperbolic PDEs in a strongly hyperbolic form.

- Note: from a numerical point of view, well-posedness is a necessary, but not sufficient condition. Well-posed problems can indeed have exponentially growing modes and these may crash a numerical simulation.

Numerical Relativity: 3+1 Formulation

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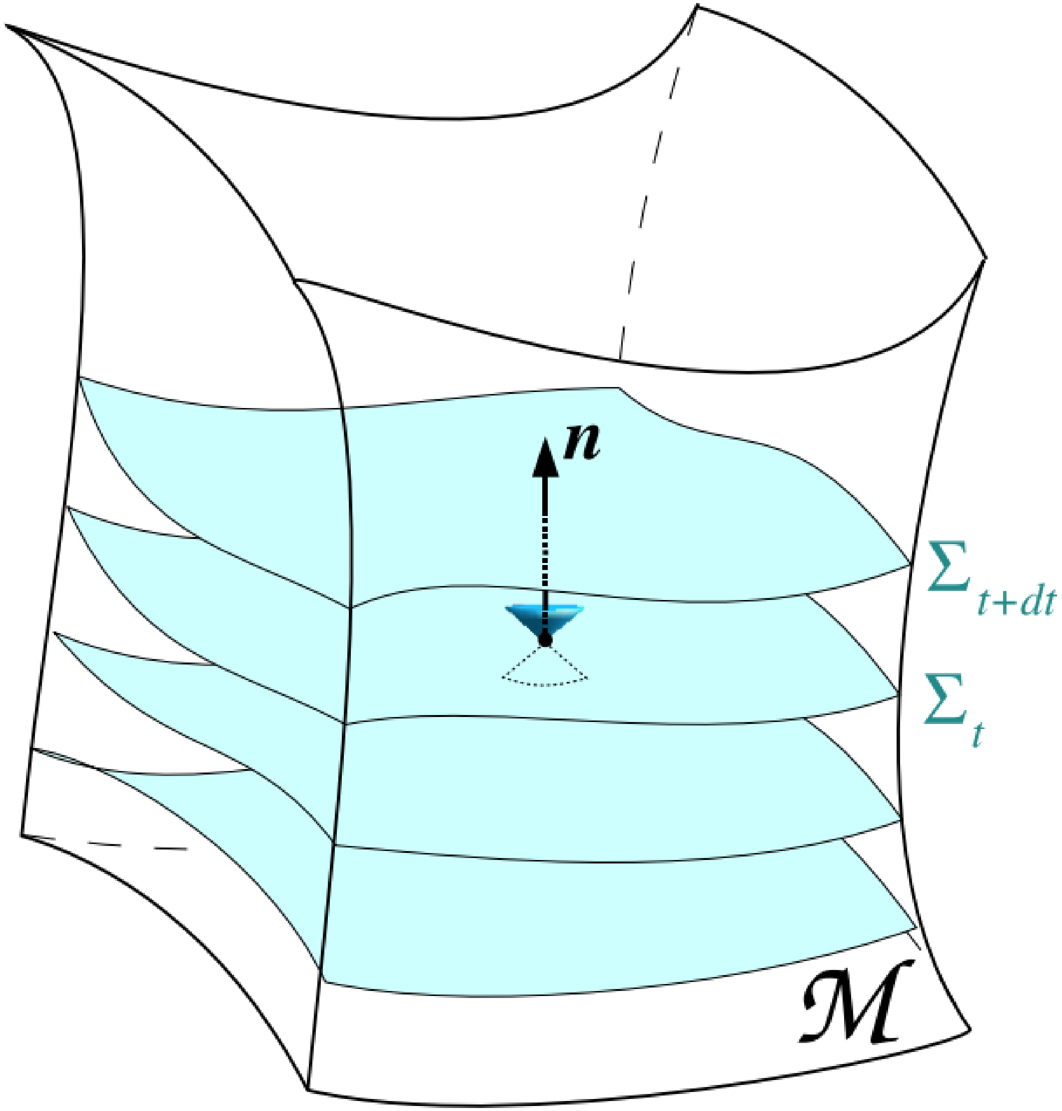
Space-Time Foliation



https://arxiv.org/abs/gr-qc/0703035

$$n_\mu \equiv -\alpha \nabla_\mu t = (-\alpha, 0,0,0)$$

$$n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right)$$

$$\gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$$

$$\gamma^{\mu\nu} \equiv g^{\mu\nu} + n^\mu n^\nu$$

$$t^\mu = \alpha n^\mu + \beta^\mu$$

$$\beta^\mu \equiv (0, \beta^i)$$

Spatial and Time Projections

- **Spatial Projection Operator**: $\gamma_{\nu}^{\mu} = g^{\mu\alpha}\gamma_{\alpha\nu} = g^{\mu\alpha}(g_{\alpha\nu} + n_{\alpha}n_{\nu}) =$

$$= g_{\nu}^{\mu} + n^{\mu}n_{\nu} = \delta_{\nu}^{\mu} + n^{\mu}n_{\nu}$$

- **Time Projection Operator**: $N_{\nu}^{\mu} \equiv -n^{\mu}n_{\nu}$

Spatial and Time Projections

- The two projectors are orthogonal to each other, indeed

$$\gamma^{\alpha}_{\mu} N^{\mu}_{\nu} = \left(\delta^{\alpha}_{\mu} + n^{\alpha} n_{\mu}\right)(-n^{\mu} n_{\nu}) = -n^{\alpha} n_{\nu} + n^{\alpha} n_{\nu} = 0$$

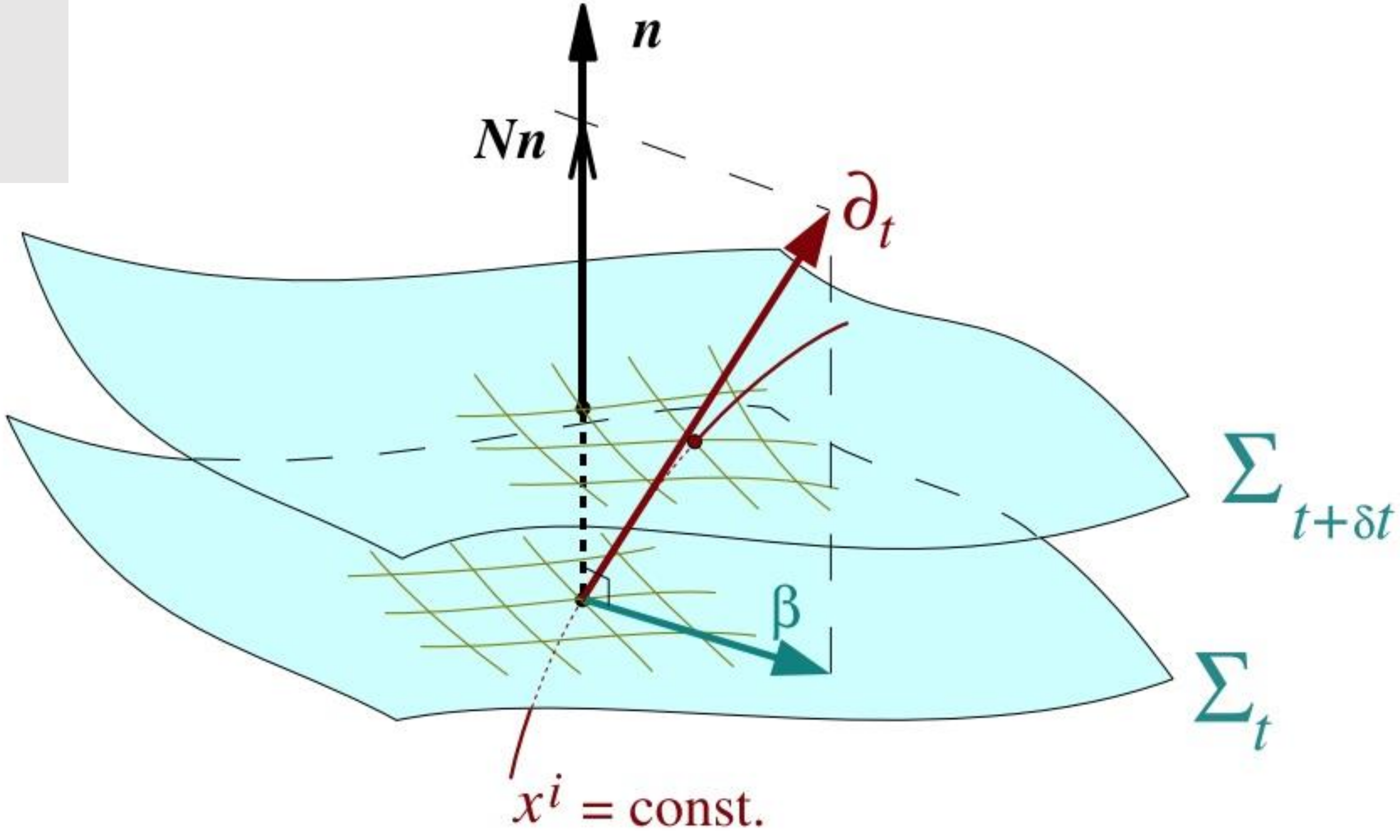
- Therefore a generic 4-vector $\boldsymbol{U}$ can be decomposed as

$$U^{\mu} = \underbrace{\gamma^{\mu}_{\nu} U^{\nu}}_{\text{spatial part}} + \underbrace{N^{\mu}_{\nu} U^{\nu}}_{\text{time part}}$$

- The same can be done with any tensor

The metric in the 3+1 form

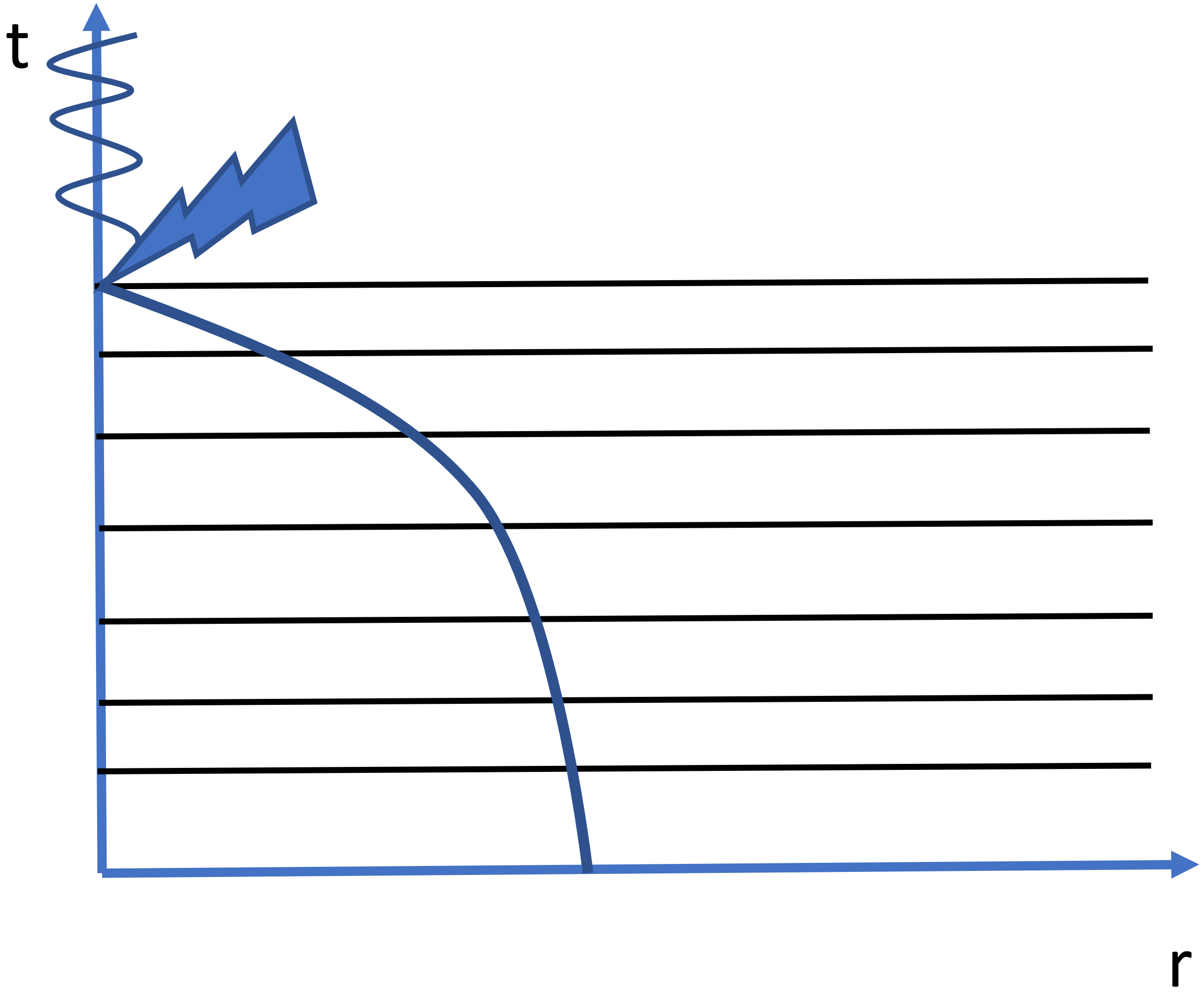
$$n_\mu = (-\alpha, 0,0,0)$$
$$n^\mu = \frac{1}{\alpha}(1, -\beta^i)$$



https://arxiv.org/abs/gr-qc/0703035

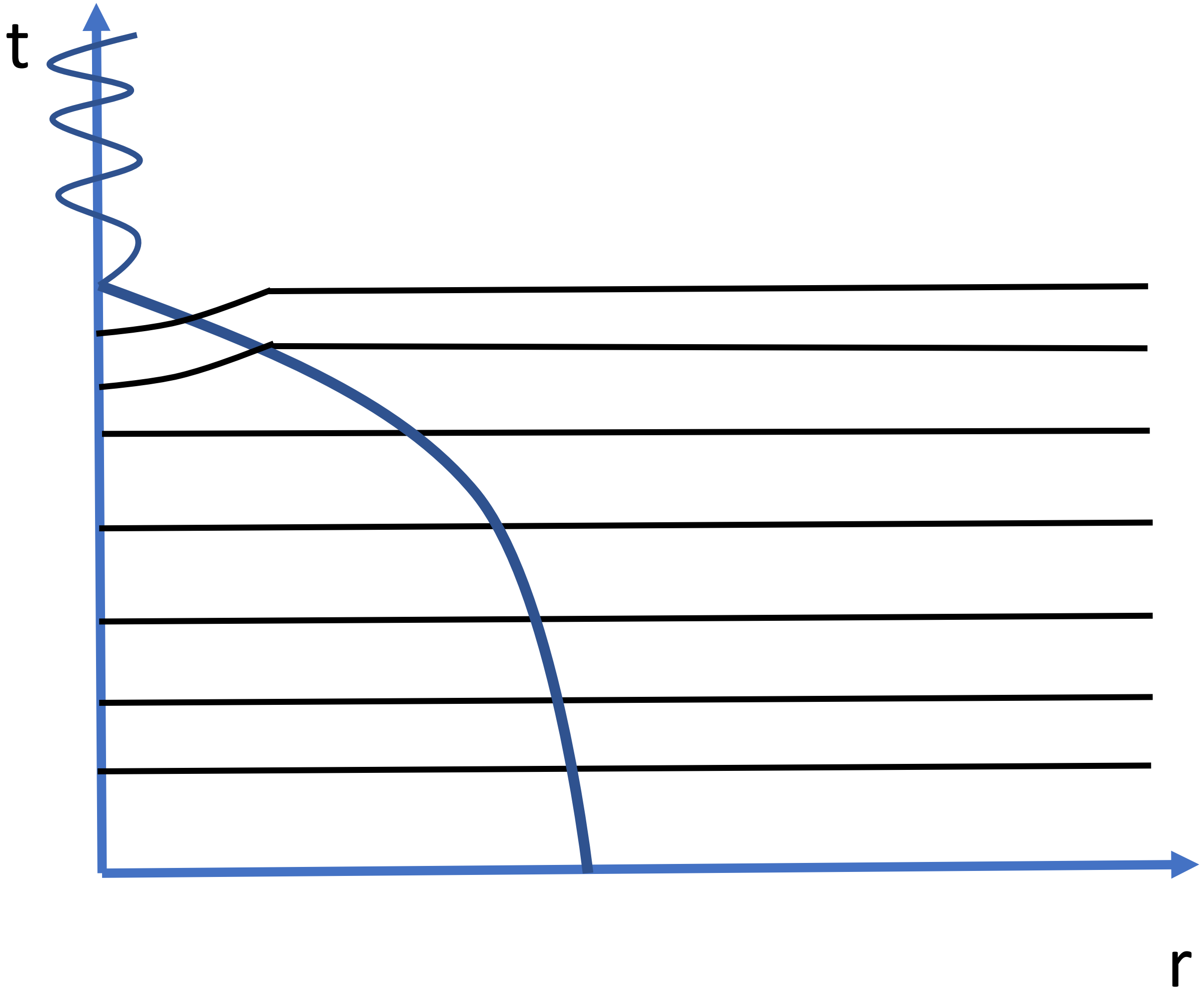
$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

Choice of Foliation: geodesic slicing



The simplest choice could be to just set the lapse to be constant ($\alpha = 1$) and the shift to zero.

Choice of Foliation: singularity-avoding slicing



Better choices use evolution equations for lapse and shift such that the singularity can be avoided.

Numerical Relativity: ADM Formulation

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ADM formulation

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

We assume to know $T_{\mu\nu}$ (later we will see how to compute it).

We use the 3+1 formulation to get a set of PDEs following what done by Arnowitt, Deser & Misner (1962).

ADM Equations

In the 3+1 formulation the metric is written as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij} \left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

And α and β^i can be chosen freely.

So to get $g_{\mu\nu}$ we "only" need γ_{ij}.

As in the wave equation, to reduce the time derivative to first order we introduce a new variable, the "extrinsic curvature"

$$K_{ij} \equiv -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\gamma_{ij} = -\frac{1}{2\alpha}\left(\partial_t - \mathcal{L}_{\boldsymbol{\beta}}\right)\gamma_{ij}$$

ADM Equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\partial_t K_{ij} = -D_i D_j \alpha + \left(\beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k\right) +$$

$$\alpha\left({}^{(3)}R_{ij} + K K_{ij} - 2K_{ik}K_j^k\right) + 4\pi\alpha\left[\gamma_{ij}(S - E) - 2S_{ij}\right]$$

$${}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi E$$

$$D_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi S^i$$

$$S_{\mu\nu} \equiv \gamma_\mu^\sigma \gamma_\nu^\tau T_{\sigma\tau} \qquad S_\mu \equiv -\gamma_\mu^\sigma n^\tau T_{\sigma\tau} \qquad S \equiv S_\mu^\mu \qquad E \equiv n^\sigma n^\tau T_{\sigma\tau}$$

plus a (free) choice for the lapse function α and the shift vector $\boldsymbol{\beta}$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

Numerical Relativity: BSSN Formulation

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Conformal Traceless Formulation

(Nakamura et al 1987, Shibata & Nakamura 1995, Baumgarte & Shapiro 1999)

- Conformal transformation: $\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}$
- $\phi = \frac{1}{12}\ln\left(\frac{\gamma}{\eta}\right)$ so that $\tilde{\gamma} = \eta = 1$ (in cartesian coordinates)

- Trace-Free Extrinsic Curvature $A_{ij} \equiv K_{ij} - \frac{1}{3}\gamma_{ij}K$
- Conformal transformation: $\tilde{A}_{ij} = e^{-4\phi}A_{ij}$; $\tilde{A}^{ij} = e^{4\phi}A^{ij}$
- Note: $\tilde{A}_{ij}\,\tilde{A}^{ij} = A_{ij}A^{ij}$

BSSN Equations

$$K_{ij} = e^{4\phi}\tilde{A}_{ij} + \frac{1}{3}\gamma_{ij}K$$

$$\gamma_{ij} \equiv e^{4\phi}\tilde{\gamma}_{ij}$$

$$\tilde{\Gamma}^i \equiv \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} = \partial_j\tilde{\gamma}^{ij}$$

$\partial_t\gamma_{ij}$

$$\partial_t\phi = -\frac{1}{6}\alpha K + \frac{1}{6}\partial_i\beta^i + \beta^i\partial_i\phi$$

$$\partial_t\,\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \beta^k\partial_k\tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k$$

$\partial_t K_{ij}$

$$\partial_t K = -D^iD_i\alpha + \alpha\left(\tilde{A}_{ij}\,\tilde{A}^{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha(E+S) + \beta^iD_iK$$

$$\partial_t\tilde{A}_{ij} = e^{-4\phi}\left[-\left(D_iD_j\alpha\right)^{TF} + \alpha\left({}^{(3)}R^{TF}_{ij} - 8\pi S^{TF}_{ij}\right)\right]$$

$$+\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) + \beta^k\partial_k\tilde{A}_{ij} + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k$$

$$\partial_t\tilde{\Gamma}^i = -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_jK - 8\pi\tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\right)$$

$$+\beta^j\partial_j\tilde{\Gamma}^i - \tilde{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\tilde{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\tilde{\gamma}^{li}\partial_l\partial_j\beta^j + \tilde{\gamma}^{lj}\partial_j\partial_l\beta^i$$

Numerical Relativity: Gauge Conditions

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Choosing the right slicing condition

1. If singularities are present, these should be avoided (“singularity-avoiding slicing conditions”)
2. If coordinate distortions take place, these should be counteracted
3. The gauge conditions should not be computationally expensive

Hyperbolic K-Driver Slicing Condition

$$\boxed{\left(\partial_t - \beta^i \partial_i\right)\alpha = -f(\alpha)\alpha^2(K - K_0)}$$

- $f(\alpha) = 1$ → harmonic slicing condition
- $f(\alpha) = \frac{q}{\alpha}$ → “1+log” slicing condition
- Most used choice $f(\alpha) = \frac{2}{\alpha}$

Gamma-Driver Shift Condition

$$\partial_t \beta^i - \beta^j \partial_j \beta^i = \frac{3}{4} B^i$$

$$\partial_t B^i - \beta^j \partial_j B^i = \partial_t \tilde{\Gamma}^i - \beta^j \partial_j \tilde{\Gamma}^i - \eta B^i$$

typical choice is $\eta = \frac{1}{2M}$

Computing GWs in Simulations

Spin-Weighted Spherical Harmonics

- GWs are usually decomposed in their different “modes”

$$h(t,\boldsymbol{x}) \equiv h_+ - i h_\times = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} h_{lm}(t,r)\ {}_{(-2)}Y_{lm}(\theta,\phi)$$

- Where ${}_sY_{lm}(\theta,\phi)$ are the spin-weighted spherical harmonics (s=0 corresponds to the “standard” spherical harmonics)
- h_{20} is for example the dominant mode for an axisymmetric collapse
- h_{22} is the dominant one for a typical inspiral signal

Moncrief Formalism

- Gauge invariant wavefunctions $Q_{lm}^{\times}$ and Q_{lm}^{+} are computed on spherical surfaces (see thorn Extract in the Einstein Toolkit, https://ui.adsabs.harvard.edu/abs/2012CQGra..29k5001L)
- It assumes the background metric to be Schwarzschild
- One can then compute the GW signal:

$$\begin{aligned} h &= h_{+} - ih_{\times} \\ &= \frac{1}{\sqrt{2}r} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left(Q_{lm}^{+} - i \int_{-\infty}^{t} Q_{lm}^{\times}(t')\, dt' \right) {}_{(-2)}Y_{lm}(\theta, \phi) \end{aligned}$$

Weyl Scalar

- A more accurate and general method uses the Weyl scalar Ψ_4 (see thorn WeylScal4 in the Einstein Toolkit):

$$\Psi_4 = R_{ijkl} n^i \bar{m}^j n^k \bar{m}^l + 2R_{0jkl}\left(n^0 \bar{m}^j n^k \bar{m}^l - \bar{m}^0 n^j n^k \bar{m}^l\right)$$

$$+R_{0j0l}(n^0 \bar{m}^j n^0 \bar{m}^l + \bar{m}^0 n^j \bar{m}^0 n^l - 2n^0 \bar{m}^j \bar{m}^0 n^l)$$

where $l^\mu \equiv \frac{1}{\sqrt{2}}(u^\mu + \tilde{r}^\mu), n^\mu \equiv \frac{1}{\sqrt{2}}(u^\mu - \tilde{r}^\mu), m^\mu \equiv \frac{1}{\sqrt{2}}(\tilde{\theta}^\mu + i\tilde{\phi}^\mu)$, u^μ is the unit normal to the hypersurface, and

$$\tilde{r}^\mu = \left\{0, x^i\right\}, \tilde{\phi}^\mu = \{0, -y, x, 0\}, \tilde{\theta}^\mu = \left\{0, \sqrt{\gamma}\,\gamma^{ik}\epsilon_{klm}\phi^l r^m\right\}$$

Weyl Scalar

- One can then compute the GW signal:

$$h = h_+ - ih_\times = -\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} \Psi_4 \, dt''$$

- This integration is usually done in Fourier space for more accurate results

  (see Reisswig & Pollney 2011, https://ui.adsabs.harvard.edu/abs/2011CQGra..28s5015R)

- The Python Kuibit library already implements the necessary tools