Equation of State for Astrophysical Applications

DTP/TALENT 2024 — Nuclear Theory for Astrophysics

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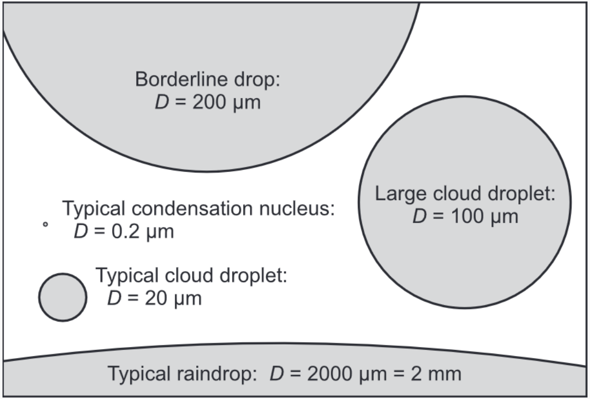
Introduction

Microphysics

Physics on scales much smaller than the dynamical scales of the problem of interest.



Clouds span km in size. Water droplets and drops span μm to mm.
Figure from Houze, Cloud Microphysics (2014).

Core-collapse Supernovae

Macrophysics:

- Iron core of a massive star $R_{\text{Fe core}} \sim 10^5$ km
- Collapses onto a PNS $R_{\text{PNS}} \lesssim 10^2$ km

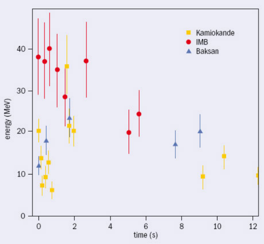
Microphysics:

- Nuclear interactions ($r_n \sim 1$ fm);
- Neutrino physics ($\lambda_\nu \sim 100$ fm);
- Turbulence ($l \sim 10$ m?);
- Photons;
- Electrons;
- ...

Goal: constrain subatomic physics from astrophysical observations.

Basic Interactions from Astrophysical Observations

No direct method to determine basic interactions from observations!



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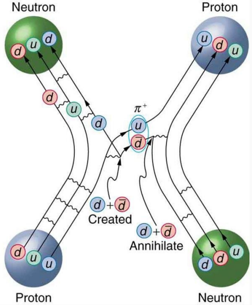

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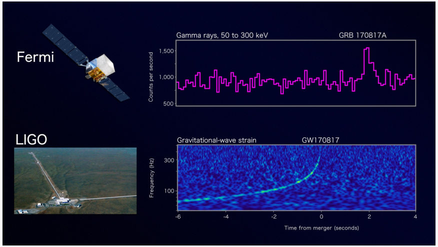
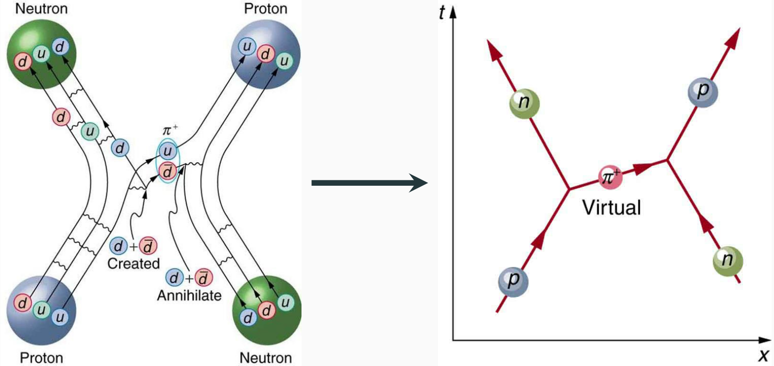


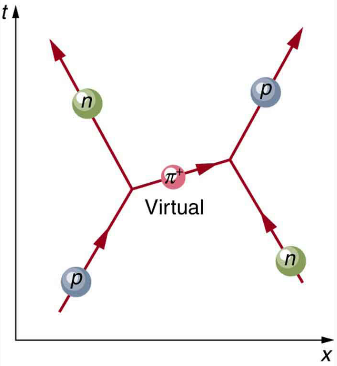
Figures by Orduña, Kamiokande/IMB/Baskan and NASA GSFC & Caltech/MIT/LIGO Lab.

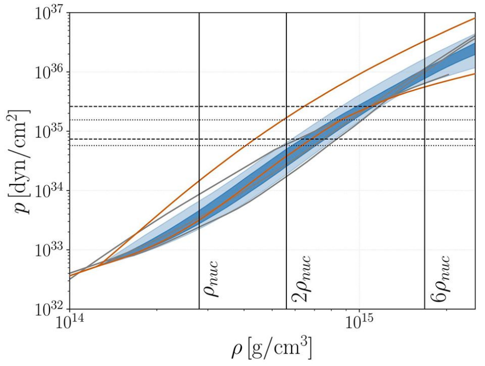
QCD not solved! Use an Effective Model



Figures by Orduña and LIGO/Virgo.

Can't solve the effective model over entire domain of the system.
Instead, we use an Equation of State!





Figures by Orduña and LIGO/Virgo.

What is an Equation of State?

An **EOS** is any thermodynamic equation relating properties of matter (state variables).

$$\boxed{F = f(\rho, T, X_i)\,, \quad P = f(\rho, T, X_i)\,, \quad s = f(\rho, T, X_i)\,, \quad \ldots}$$

Relates the microscopic description of matter to its macroscopic properties.

Equations of State

EOS is known or well constrained in some cases.

- Ideal gas: $P_i(\rho_i, T) = \rho_i T$;

$$P_{\text{tot}} = \sum_i P_i(\rho, T)$$

- Photon gas:

$$P_\gamma = \frac{1}{3}U = \frac{4}{3}\sigma T^4$$

- High-T and/or low-μ QCD.

- Dense plasmas (WDs, NS crusts):
  - $P = P_e + P_{\text{ions}}$.
  - Degenerate non-relativistic electrons:

$$P_e \propto \rho_e^{5/3}, \quad \text{if } E_e \gg T.$$

  - Degenerate relativistic electrons:

$$P_e \propto \rho_e^{4/3}, \quad \text{if } E_e \gg T.$$

Equations of State

For CCSNe and NS mergers after contact: EOS is **NOT** well-constrained!

- High density:

$$10^{14} \lesssim \rho \lesssim 10^{16}\,\mathrm{g\,cm^{-3}}$$

- Large isospin asymmetry:

$$\delta = \frac{n_n - n_p}{n_n + n_p} \gtrsim 0.5$$

- High temperatures:

$$T \gtrsim \mathrm{few MeV}$$

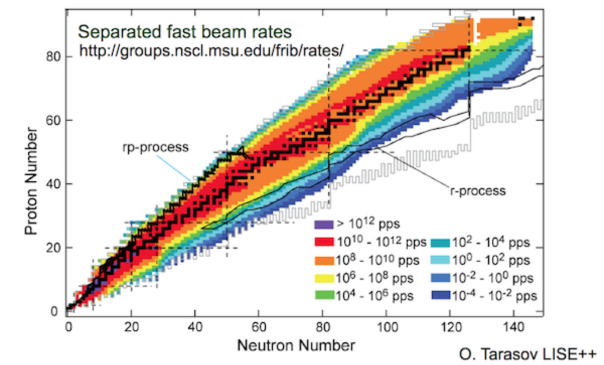
- Conditions cannot be reproduced in laboratories.

- Strongly interacting many-body systems can be very difficult to model.

How to constrain the EOS on Earth

EOS constraints come from many sources:

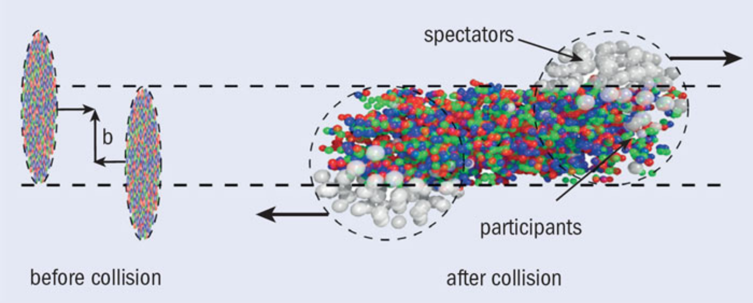
- Nuclear properties, nuclear reactions and decays



How to constrain the EOS on Earth

EOS constraints come from many sources:

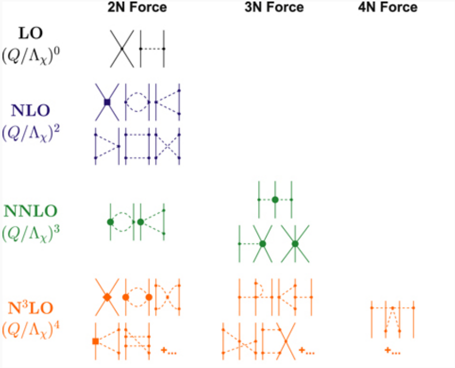
- Heavy ion collision experiments and simulations

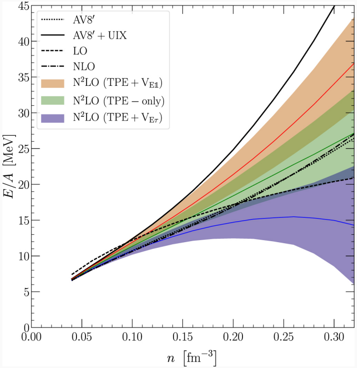


How to constrain the EOS on Earth

EOS constraints come from many sources:

- Many-body calculations

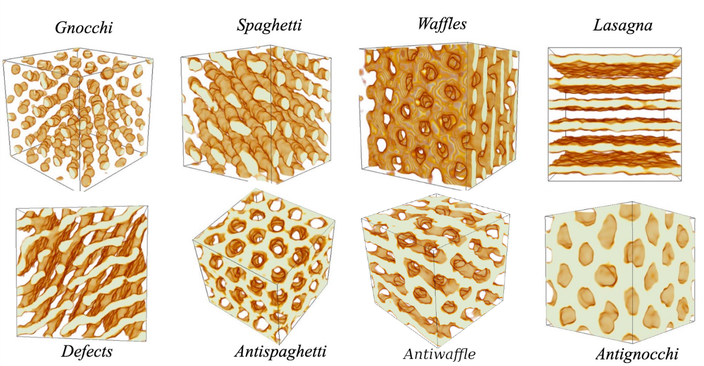




How to constrain the EOS on Earth

EOS constraints come from many sources:

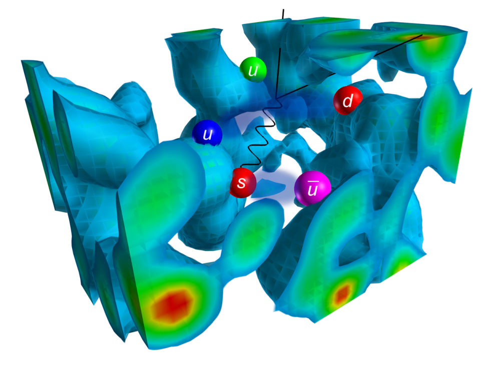
- Simulations of dense matter



How to constrain the EOS on Earth

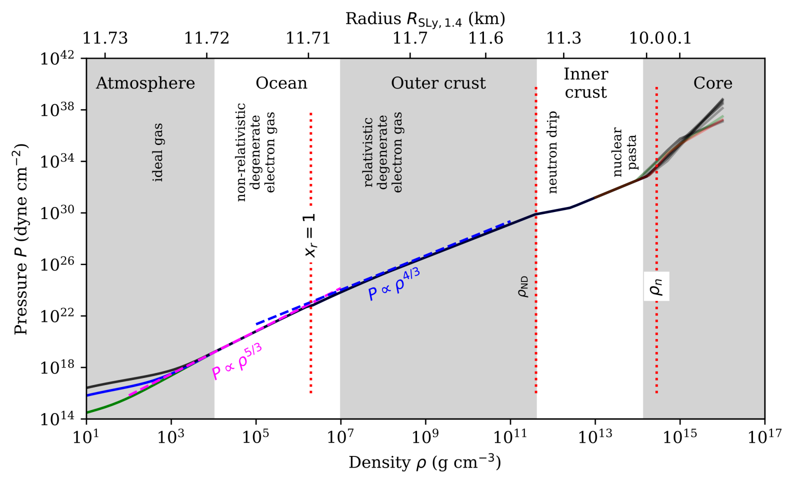
EOS constraints come from many sources:

- Simulations of even denser matter



EOS Constraints far from Earth

Zero-temperature beta-equilibrated EOS



NS EOS by Nättilä & Kajava. EOS unconstrained for $\rho \gtrsim 10^{14}\,\mathrm{g\,cm^{-3}}$.

Pressure of Dense Matter

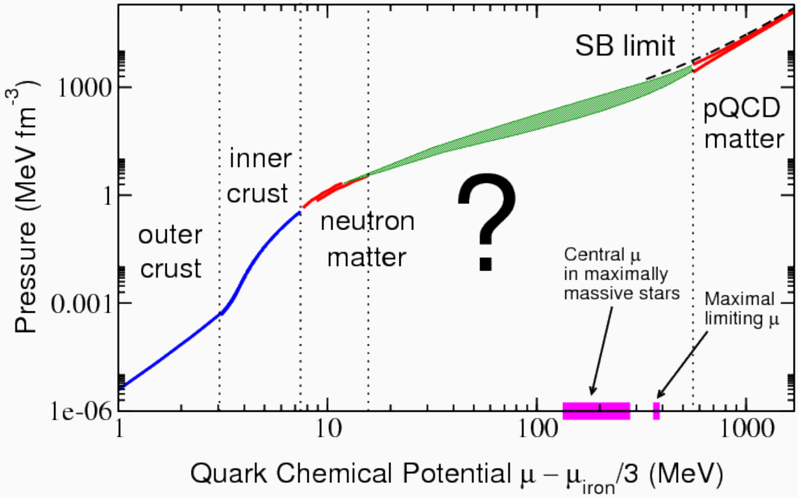


Figure from Fraga *et al.* (2015).

Tolmann-Oppenheimer-Volkov Equations

Hydrostatic equilibrium in GR:

$$\frac{dm}{dr} = 4\pi r^2 \epsilon\,, \qquad \frac{dP}{dr} = -(\epsilon + P)\frac{m + 4\pi r^3 P}{r(r-2m)}\,, \qquad \frac{d\Phi}{dr} = -\frac{1}{\epsilon + P}\frac{dP}{dr}\,.$$

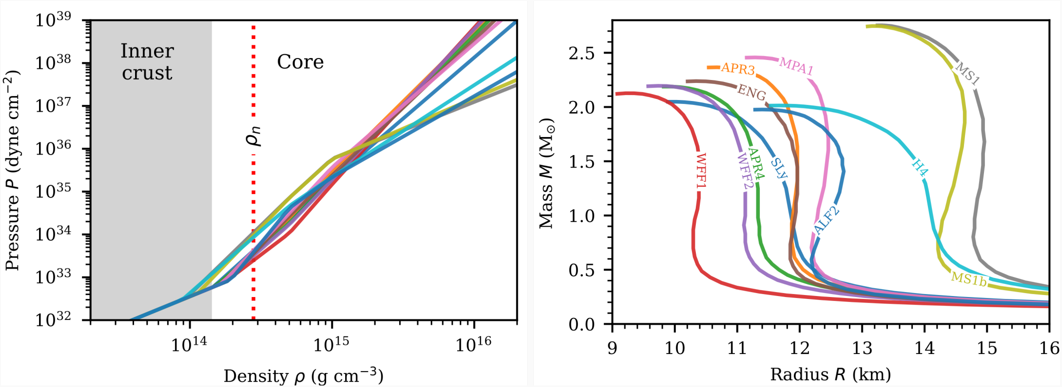
- Three equations and four unknowns: $\epsilon(r)$, $P(r)$, $m(r)$, $\Phi(r)$.
- EOS is the fourth equation $P \equiv P(\epsilon)$.

Pressure as a function of stellar radius is set by ($\epsilon = \rho + \varepsilon$):

$$\frac{dP}{dr} = -\frac{G\rho(r)m(r)}{r^2}\left(1 + \frac{\varepsilon(r) + P(r)}{\rho(r)c^2}\right)\left(1 + \frac{4\pi r^3 P(r)}{m(r)c^2}\right)\left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1}$$

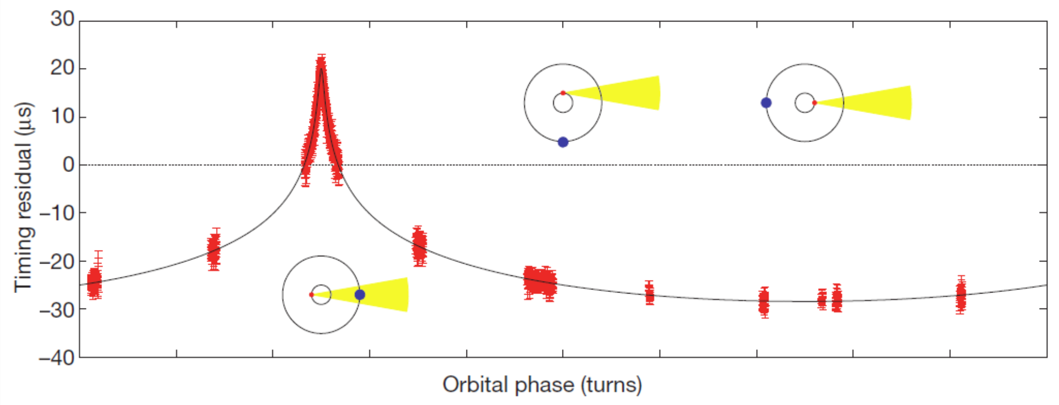
Newtonian version: $dP/dr = -G\rho m/r^2$.

TOV solutions for NSs



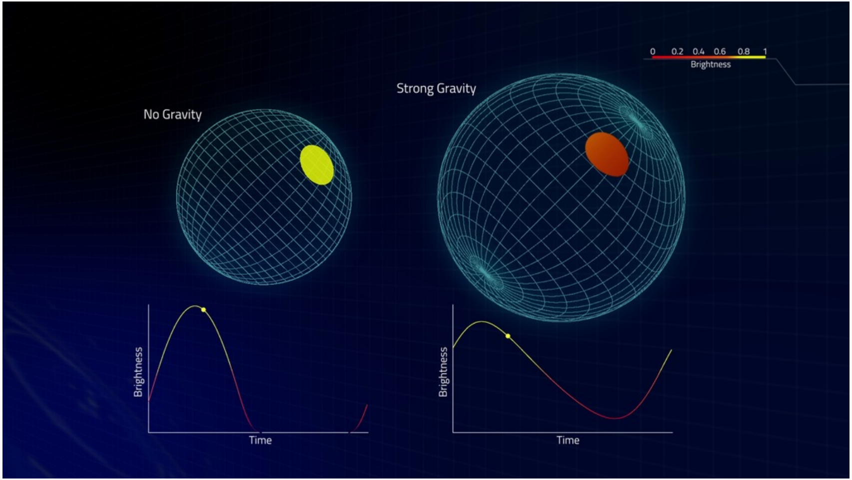
Mass-radius relationships obtained from TOV equation solutions for different EOS models.
Figure by Nättilä & Kajava

NS Mass Measurements



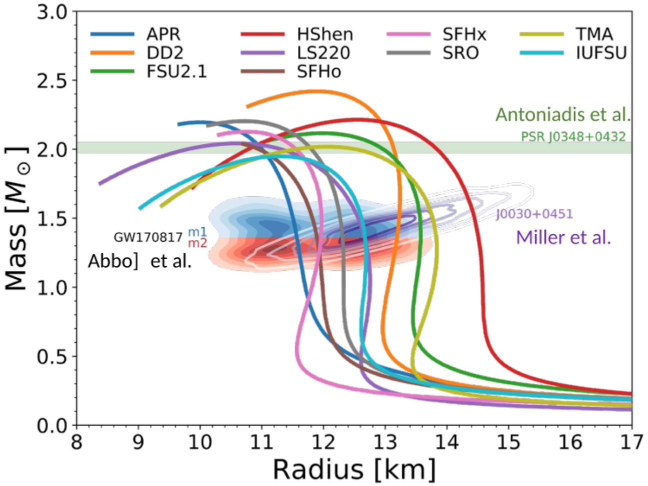
NS mass measurement using *Shapiro delay*.
Figure from Dermorest *et al.* (2010).

NS mass-radius measurements



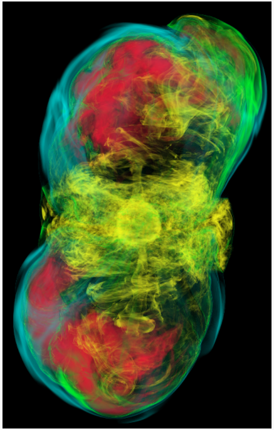
Spacetime curvature allows measurements of NS mass and radius simultaneously.
Figure from Neutron Star Interior Composition Explorer Mission (NICER).

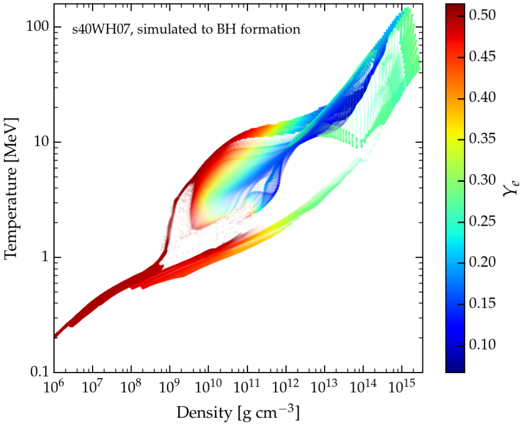
TOV solutions for NSs



From Evan O'Connor.

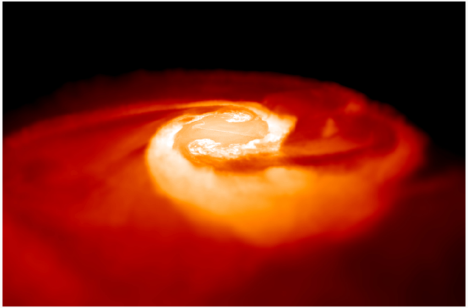
Typical CCSNe Conditions

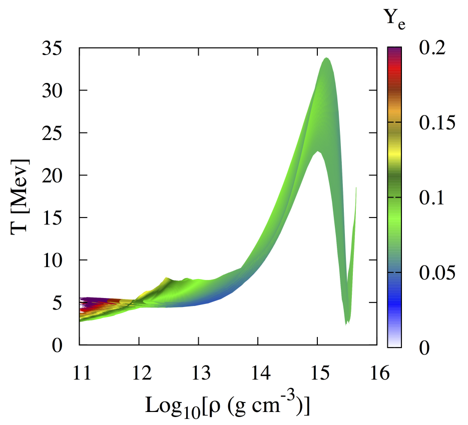




Figures from Ott (left) and Kiuchi (right).

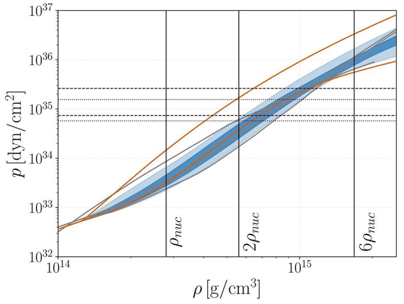
CCSNe and NS merger Conditions



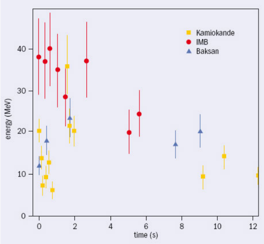


Figures from Rosswog (left) and Kiuchi (right).

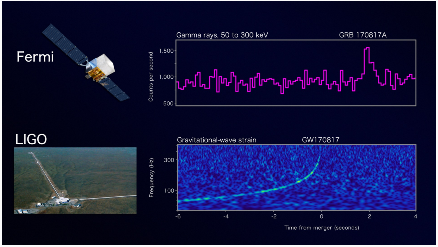
Check if EOS is consistent with observations.
If not consistent: discard EOS!



consistent?



consistent?



Figures by Orduña, Kamiokande/IMB/Baskan and NASA GSFC & Caltech/MIT/LIGO Lab.

Microphysics of CCSNe

Core-collapse Supernovae Review

- Massive star Fe/Ni core collapses under its own weight:
  - Central density: $\rho_c \sim 10^{9-10}\mathrm{g\,cm^{-3}}$
  - Temperature: $T \sim 1$ MeV
  - Proton fraction: $y_e \sim 0.50$
  - Radius: $\sim 10^9$ cm.
  - Mass: $\sim M_\odot$
- Collapse:
  - time-scale: a few 100 ms.
  - electron capture and neutrino emission changes composition;
  - nuclei are photo-dissociated.

$\Rightarrow$ in $\sim 3 \times 10^{55}$ nuclei decay into a PNS.

- Core becomes a protoneutron star (PNS):
  - Central density: $\rho_c \sim 10^{14-15}\mathrm{g\,cm^{-3}}$
  - Temperature: $T \sim 10 - 100$ MeV
  - Proton fraction: $y_e \lesssim 0.25$
  - Radius: $\sim 3 \times 10^6$ cm
  - Mass: $\sim M_\odot$
- Outer stellar shells in freefall bounce off PNS surface $\Rightarrow$ *shock wave* forms:
  - shock propagates outwards
  - loses energy and *stalls*!
  - if shock wave revived $\Rightarrow$ *successful SN.*
  - if shock wave not revived $\Rightarrow$ *failed SN.*

Role of the EOS in CCSNe

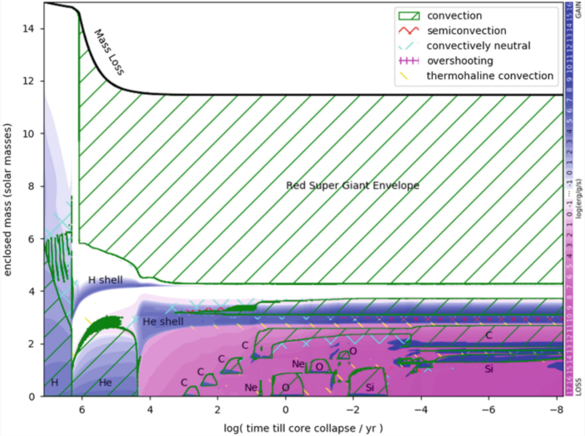
EOS affects:

- When the collapse starts, $p + e^- \rightarrow n + \nu_e$;
- When core-bounce occurs;
- Heat is produced during collapse;
- PNS mass, size, temperature, composition;
- Energy is transferred to the shock;
- PNS cooling rate;
- Neutrino production rate and their spectra;
- Gravitational wave signal;
- If/when there is a second collapse to a quark star or BH;
- Final compact star mass, radius, and composition or BH mass.
- ...

EOS Composition

Evolve composition directly:

- No thermodynamic equilibrium!
- No chemical equilibrium!
- Set initial composition and initial conditions.
- Evolve network of reactions.
- How stellar evolution from ZAMS to core-collapse is simulated.
- Resources increase very fast with increasing number of nuclei and reactions.
- Not recommended for CCSNe.



Kippenhahn diagram by Thielemann *et al.* (2018) for the evolution of a non-rotating 15 $M_\odot$ with solar metalicity using the GENEC code.

EOS Composition

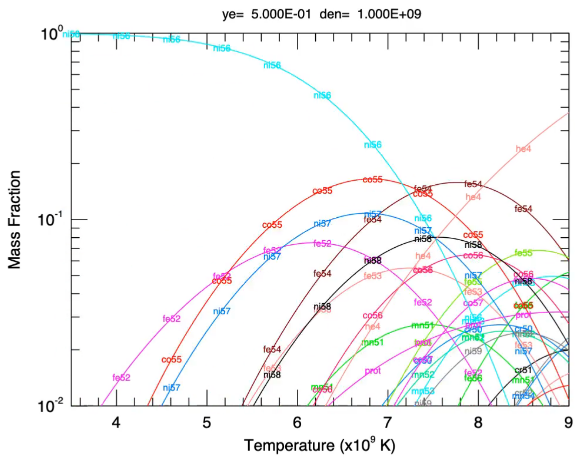
Evolve thermodynamic conditions:

- Assume chemical equilibrium.
- Assume ensemble of nuclei in Nuclear Statistical Equilibrium (NSE). Approximately correct for

$$\rho \gtrsim 10^7\,\mathrm{g\,cm^{-3}}$$
$$T \gtrsim 0.5\,\mathrm{MeV}\,.$$

- Does not assume β-equilibrium:

$$p + e^- \rightleftharpoons n + \nu_e$$



Composition of Symmetric Nuclear Matter (SNM) at $\rho = 10^9\,\mathrm{g\,cm^{-3}}$. Figure by F.X. Timmes.

EOS Model for Astrophysical Applications

To evolve a CCSN, need composition and thermodynamic properties at:

$$10^3\,\mathrm{g\,cm}^{-3} \lesssim \rho \lesssim 10^{15}\,\mathrm{g\,cm}^{-3}$$
$$10^{-2}\,\mathrm{MeV} \lesssim T \lesssim 10^{2}\,\mathrm{MeV}$$
$$0 \lesssim y \lesssim 0.60$$

Over this large range, matter may

- behave like a gas, a liquid, or a solid;
- be in its ground state or in a highly excited state;
- be degenerate or non-degenerate;
- be uniform or non-uniform (nuclei or "pasta phase");
- be isospin symmetric, proton rich or very neutron rich;

Computing the EOS may be time consuming and computationally expensive $\Rightarrow$ use pre-computed tables and interpolate them.

EOS Model for Astrophysical Applications

Low density:

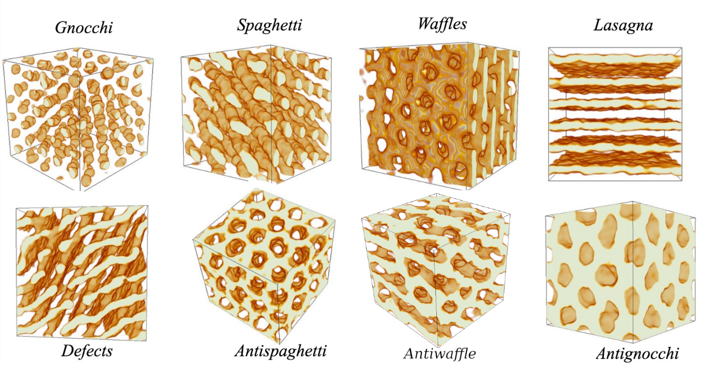
- Matter is made of nucleons;
- Immersed in lepton + photon gas;
- Nucleons may cluster to form nuclei:
  - Single nucleus approximation plus representative light nucleus (αs).
  - or ensemble of nuclei in NSE.
- Interactions:
  - Coulomb;
  - Surface;
  - Excluded volume;
  - Excitation energies;

High density:

- Uniform matter;
- Immersed in lepton + photon gas;
- Nucleons near saturation density;
- Interactions:
  - **ab initio models**: χEFT
  - **phenomenological models**: Skyrme or Relativistic Mean Field.
- At higher densities, may include:
  - heavy baryons (*e.g.* Λs, hyperons);
  - pions, kaons, ...;
  - quarks.

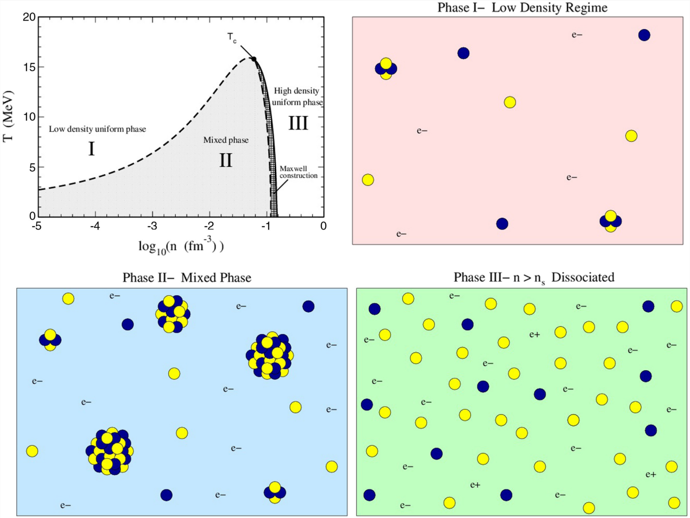
EOS Model for Astrophysical Applications

In the transition from low density (nuclei) to high density (uniform nuclear matter) matter may cluster in exotic shapes known as *nuclear pasta*.



Non-homogeneous nuclear pasta phases obtained by Semi-classical Molecular Dynamics simulations.
Adapted from Schneider *et al.* (2013, 2014), Caplan & Horowitz (2016), Lin *et al.* (2020).

EOS Model for Astrophysical Applications



Expected phases of nucleonic matter for a range of density (horizontal axis) and temperatures (vertical axis). Figure from Lattimer and Prakash (2016).

General Purpose EOSs

How to compute a general-purpose EOS?

Decisions:

- Range in ρ, y, T.
- Relativistic vs Non-relativistic.
- Realistic vs Effective potentials.
- SNA vs NSE vs reaction networks.
- Particles to include: nucleons, muons, pions, hyperons, quarks, ...

May combine different approaches:

- SNA over entire phase space;
- NSE over entire phase space;
- NSE with transition to SNA at high ρ;
- Reaction networks that transition to NSE.

General Purpose EOSs

Only a “few” finite temperature EOSs produced over the years.

Authors	Year	Model	DoF	Nuclei	Tables
Lattimer & Swesty	1991	Skyrme	$np\alpha A$	SNA	3+
H. Shen *et al.*	1998	RMF	$np\alpha A$	SNA	1
Ishizuka *et al.*	2008	RMF	$np\alpha A\Lambda(\pi)$	SNA	8
H. Shen *et al.*	2011	RMF	$np\alpha A(\Lambda)$	SNA	2
Togashi *et al.*	2017	RMF	$np\alpha A$	NSE	1
Furusawa *et al.*	2017	RMF	$np\alpha A$	NSE	2
G. Shen *et al.*	2010	RMF	$np\alpha A$	SNA+NSE	3
Hempel *et al.*	2013	RMF	$np\alpha A$	NSE	6
Steiner *et al.*	2014	RMF	$np\alpha A$	NSE	2
Banik *et al.*	2014	RMF	$np\alpha A\Lambda(\phi)$	NSE	2
Schneider *et al.*	2017	Skyrme	$np\alpha A$	SNA(NSE)	12x2+
Schneider *et al.*	2019	Skyrme APR	$np\alpha A$	SNA(NSE)	4+
Du *et al.*	2022	RMF	$np\alpha A$	NSE	9+

Some of the general purpose EOSs publicly available.

See compose.obspm.fr and stellarcollapse.org

Empirical Parameters

Specific energy density of nuclear matter:

$$\epsilon(n,y) = \epsilon_{\rm is}(x) + \delta^2 \epsilon_{\rm iv}(x) + \mathcal{O}(\delta^4)$$

where $\delta = 1 - 2y$, $y = \dfrac{n_p}{n_p + n_n}$, and $x = \dfrac{n - n_{\rm sat}}{3 n_{\rm sat}}$.

$$\epsilon_{\rm is}(x) = \epsilon_{\rm sat} + \frac{1}{2} K_{\rm sat} x^2 + \frac{1}{3!} Q_{\rm sat} x^3 + \ldots ,$$

$$\epsilon_{\rm iv}(x) = \epsilon_{\rm sym} + L_{\rm sym} x + \frac{1}{2} K_{\rm sym} x^2 + \frac{1}{3!} Q_{\rm sym} x^3 + \ldots .$$

Empirical parameters may be constrained by experiments, theory, and observations.

Quantity	Exp./Theor.	Units
$n_{\rm sat}$	0.155 ± 0.005	fm^{3}
$\epsilon_{\rm sat}$	-15.8 ± 0.3	MeV
$\epsilon_{\rm sym}$	32 ± 2	MeV
$L_{\rm sym}$	60 ± 15	MeV
$K_{\rm sat}$	230 ± 20	MeV
$K_{\rm sym}$	-100 ± 100	MeV
$Q_{\rm sat}$	300 ± 400	MeV
$Q_{\rm sym}$	0 ± 400	MeV

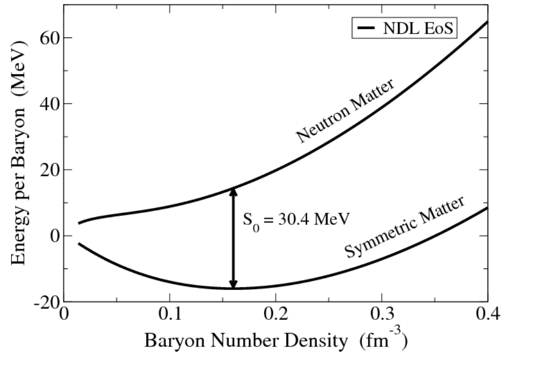
Constraints from Margueron *et al.* (2018).

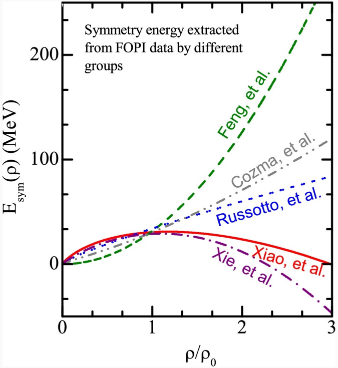
Symmetry Energy

$\delta = 0 \Rightarrow n_n = n_p$: Symmetric nuclear matter (SNM)

$\delta = 1 \Rightarrow n_n = n$ and $n_p = 0$: Pure neutron matter (PNM)

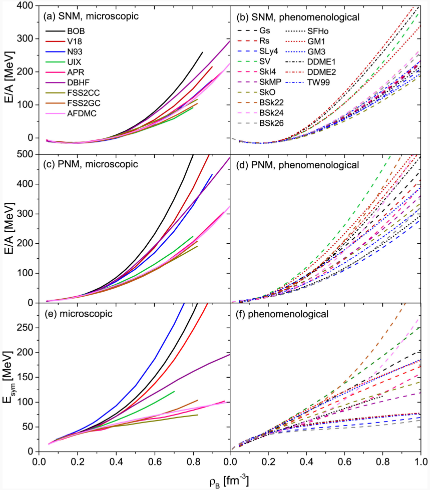
Symmetry energy: $\epsilon_{\text{sym}} = \epsilon_{\text{iv}} = \epsilon_{\text{PNM}} - \epsilon_{\text{SNM}}$





Meixner *et al.* (2013) and Guo *et al.* (2014)

Empirical Parameters



Burgio *et al.* (2021)

- Top: SNM ($y = 0.5$ or $\delta = 0$)
- Middle: PNM ($y = 0$ or $\delta = 1$)
- Bottom: Symmetry Energy
- Left: Microscopic Models
- Right: Phenomenological Models

Microscopic Models:

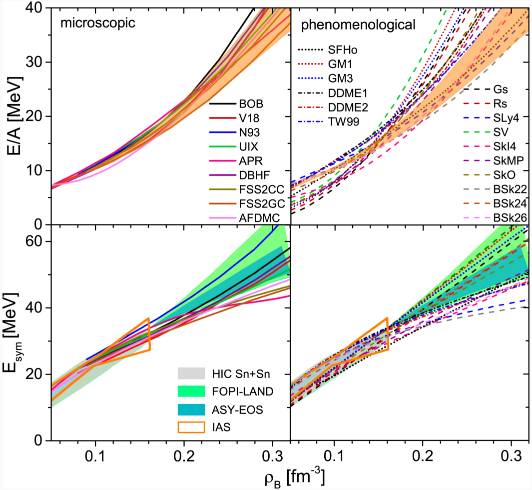
good at low-ρ & for PNM.

Phenomenological Models:

fit at $n \sim n_{sat}$ and $y \sim 0.5$

$\Rightarrow$ do not do well for PNM.

Empirical Parameters



Burgio *et al.* (2021)

- Top: PNM ($y = 0$ or $\delta = 1$)
  Orange: PNM EOS from χEFT.
- Bottom: Symmetry Energy
  Green: constraints from HIC.
- Left: Microscopic Models
- Right: Phenomenological Models

Microscopic Models:
good at low-ρ & for PNM.

Phenomenological Models:
fit at $n \sim n_{sat}$ and $y \sim 0.5$
$\Rightarrow$ do not do well for PNM.

Skyrme Parametrization of Nuclear Forces

Used in the Lattimer & Swesty (1991) EOS:

- Nuclear matter is made of nucleons (protons and neutrons); Non-relativistic & immersed in lepton + photon gas;
- Nuclear matter has surface tension $\Rightarrow$ nucleons cluster to form nuclei.
- The specific energy density of nucleons is:

$$\epsilon(n,y,T) = \epsilon_{\mathrm{kin}}(n,y,T) + \epsilon_{\mathrm{pot}}(n,y)\,.$$

- Temperature dependence only in kinetic term ϵ_{kin}.
- Potential term ϵ_{pot} depends solely density n and composition y.

The SRO EOS[a] is based on the Lattimer & Swesty EOS.

[a] exercises this afternoon

Kinetic Energy Term

$$\epsilon_{\text{kin}}(n,y,T) = \frac{1}{n}\left(\frac{\hbar^2\tau_n}{2m_n^\star} + \frac{\hbar^2\tau_p}{2m_p^\star}\right)$$

- Nucleon kinetic energy density:

$$\tau_t = \frac{1}{2\pi^2}\left(\frac{2m_t^\star T}{\hbar^2}\right)^{\frac{5}{2}} F_{3/2}(\eta_t)$$

  where

$$\lim_{T\to 0} \tau_t = \frac{3}{5}\left(3\pi^2 n_t\right)^{2/3} n_t \,.$$

- $F_{3/2}(\eta_t)$ is a Fermi integral of η_t.
- η_t is the nucleon degeneracy parameter.

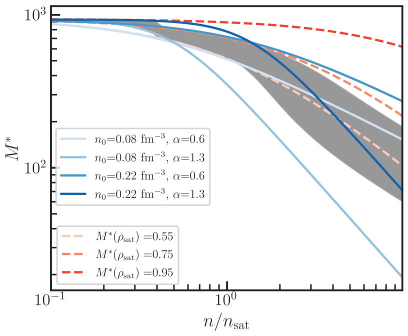
- Simple density dependent nucleon effective mass $m_t^\star$ (observable):

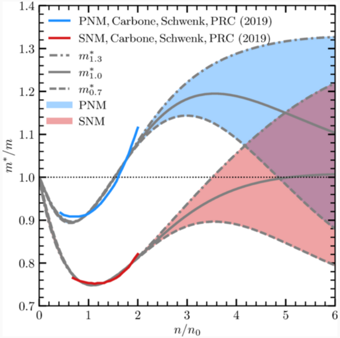
$$\frac{\hbar^2}{2m_t^\star} = \frac{\hbar^2}{2m_t} + \alpha_1 n_t + \alpha_2 n_{-t} \,.$$

- α_i are parameters of the model.
- If $t = n \Rightarrow -t = p$ and vice-versa.
- Nucleon vacuum masses m_n and m_p.

Nucleon Effective Mass

The $m_t^\star$ affect the temperature dependence of the EOS!
Possible complicated behavior not captured by simple Skyrme model.





Left: effective mass model from Raithel *et al.* (2023) and Schneider *et al.* (2020).
Right: effective mass model from Huth *et al.* (2020) based on results from Carbone & Schwenk (2019).

The Potential Energy Term

$$\epsilon_{\mathrm{pot}}(n,y) = \frac{1}{n}\sum_{i=0}^{N}\left[a_i + 4b_i y(1-y)\right] n^{\delta_i+1}.$$

N, a_i, b_i, and δ_i are parameters set by nuclear physics observables.

In the Lattimer & Swesty EOS:

- Set $\alpha_1 = \alpha_2 = 0 \Leftrightarrow m_t^\star = m_t$.
- Set $N = 1$, $\delta_0 = 1$, $b_1 = 0$.
- Compute a_0, a_1, b_0, δ_1 from

$$n_{\mathrm{sat}} = 0.155\ \mathrm{fm}^3$$

$$\epsilon_{\mathrm{sat}} = -16\,\mathrm{MeV/baryon}$$

$$\epsilon_{\mathrm{sym}} = 29.3\,\mathrm{MeV/baryon}$$

$$K_{\mathrm{sat}} = 180, 220, 375\,\mathrm{MeV/baryon}$$

Watershed Moment

The Lattimer & Swesty EOS (1991):

- Open-source Fortran code!
- First widely available tables for astrophysical applications.
- Before: each group had their EOS for their own usage.
- Phenomenological Skyrme parametrization of the nuclear forces.
- Contains:
    - protons and neutrons;
    - alpha particles;
    - electrons and positrons;
    - photons.
- Leptons and photons form a background gas.
- Three tables available: LS180, LS220, and LS375.
- LS220 still widely used in simulations of CCSNe and NS mergers.

The SROEOS code

The SRO EOS code

- by Schneider, Roberts, and Ott (2017).
- Based on the Lattimer & Swesty EOS.
- Open-source and (somewhat) modular.
- See also APR EOS by Schneider *et al.* (2019).

Working on v2 with Finia Jost and Almudena Arcones.

Improvements over other works:

- "Easy" to produce new tables.
- Allows studies with systematic variations in the EOS.
- Transitions from SNA to NSE.

Limitations:

- Limited to nucleons;
- Non-relativistic interactions;
- Crude SNA to NSE transition.

The SROEOS code

By setting $m_t^\star$ in two different places of parameter space we can obtain α_i.

- Effective mass for SNM ($y = 1/2$ or $\delta = 0$) at n_{sat}:

$$m^\star = m_n^\star(n = n_{\text{sat}}, y = 1/2) \simeq m_p^\star(n = n_{\text{sat}}, y = 1/2) = 0.75(10)\, m_n$$

- Effective mass splitting for PNM ($y = 0$ or $\delta = 1$) at n_{sat}:

$$\Delta m^\star = m_n(n = n_{\text{sat}}, y = 0) - m_p(n = n_{\text{sat}}, y = 0) = 0.10(10)$$

The SROEOS code

Set Empirical Parameters $\Rightarrow$ obtain Skyrme Parametrization

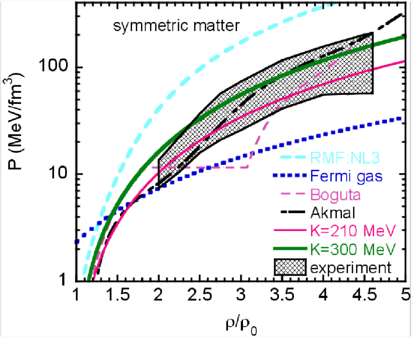
- Set $N = 3$ in the potential energy summation.
- Fix exponents: $\delta_0 = 1$, $\delta_1 = 4/3$, $\delta_2 = 2$, and $\delta_3 = 7/3$.
- Obtain a_i, b_i from 8 nuclear physics constraints.

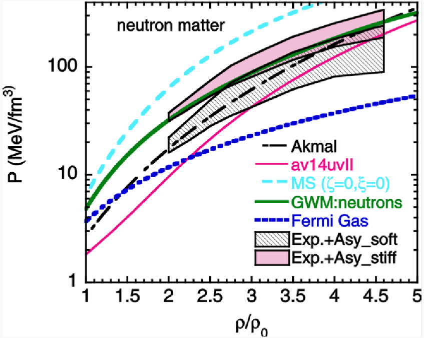
Constraints:

- Nuclear Saturation Density: $n_{\text{sat}} = 0.155(5)\ \text{fm}^3$
- Binding energy at n_{sat} $\epsilon_{\text{sat}} = -15.8(3)$ MeV/baryon
- Symmetry energy at n_{sat}: $\epsilon_{\text{sym}} = 32(2)$ MeV/baryon
- Slope of the symmetry energy $\epsilon_{\text{sym}} = 32(2)$ MeV/baryon
- Isoscalar incompressibility: $K_{\text{sat}} = 230(20)$ MeV/baryon
- Isovector incompressibility: $K_{\text{sym}} = -100(100)$ MeV/baryon

Higher order parameters from $\epsilon(n,y)$ are not well determined

Use other constraints that limit the EOS at high density!





Pressure of SNM (left) and of PNM (right) obtained from a heavy ion collision analysis of Danielewicz *et al.* (2002).

SRO EOSs

Schneider *et al.* (2019). See also Yasin *et al.* (2020).

Study EOS effects in 4 sets of 25 EOSs:

Analyze $\pm 1\sigma$ and $\pm 2\sigma$ changes in two quantities at a time.

Set	Quantity	x	Exp/Theory	Schneider+2019	Units
s_M	$m_n^\star(n_{sat}, 1/2)$	0	0.75 ± 0.10	0.75 ± 0.10	m_n
	$\Delta m^\star(n_{sat}, 0)$	0	0.10 ± 0.10	0.10 ± 0.10	m_n
-	n_{sat}	0	0.155 ± 0.005	0.155	fm^{-3}
	ϵ_{sat}	0	-15.8 ± 0.3	-15.8	MeV baryon^{-1}
s_S	ϵ_{sym}	0	32 ± 2	32 ± 2	MeV baryon^{-1}
	L_{sym}	0	60 ± 15	45 ± 7.5	MeV baryon^{-1}
s_K	K_{sat}	0	230 ± 20	230 ± 15	MeV baryon^{-1}
	K_{sym}	0	-100 ± 100	-100 ± 100	MeV baryon^{-1}
s_P	$P_{SNM}^{(4)}$	1	100 ± 50	125 ± 12.5	MeV fm^{-3}
	$P_{PNM}^{(4)}$	1	160 ± 80	200 ± 20	MeV fm^{-3}

EOS constraints from Margueron *et al.* (2018) and Danielewicz *et al.* (2002).

Helmholtz Free Energy

Once parametrization has ben chosen:
$\Rightarrow$ Minimize Helmholtz Free Energy

$$F(n,y,T) = F_o + F_h + F_\alpha + F_e + F_\gamma$$

- o: nucleon gas (outside)
- h: nucleons in heavy nuclei
- α: alpha particles
- e: leptons (e^- & e^+)
- γ: photons

Heavy nuclei free energy:

$$F_h = F_i + F_S + F_C + F_T$$

- $F_i \equiv$ nucleons inside heavy nuclei
- $F_S \equiv$ surface free energy
- $F_C \equiv$ coulomb free energy
- $F_T \equiv$ translational free energy

Heavy Nuclei Free Energy

Model of each component:

- Electrons, positrons and photons:
  - background uniform gas;
  - solve exactly
- Nucleons:
  - Skyrme-type interaction.
  - Free gas and/or heavy nuclei.
- Alpha particles ⇒ hard spheres.

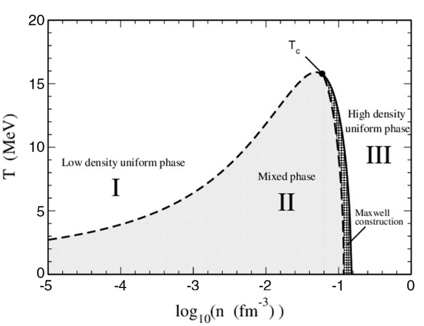


Figure from Lattimer and Prakash (2016).

Solve system of equations for uniform (I and III) and non-uniform (II) phases and either use the one with lowest free energy or, if both solutions exist, combine the two (Maxwell construction).

Helmholtz Free Energy

Uniform System

Minimize $F(n,y,T)$ w.r.t.

- free proton density n_{po};
- free neutron density n_{no};
- α particle density n_α.

Solve one equilibrium equation:

- Equilibrium between μ_α, μ_n, & μ_p.

Eq. bound by two constraints:

- Total density (n) is sum of all densities;
- Total charge density (ny) is sum of all charges.

Solve for Phase I $\Rightarrow$ Phase III recovered when $n_\alpha \to 0$.

Helmholtz Free Energy

Non-Uniform System

Minimize $F(n, y, T)$ w.r.t.

- free proton density n_{po};
- free neutron density n_{no};
- α particle density n_α;
- neutrons in heavy nuclei n_{ni};
- protons in heavy nuclei n_{pi};
- heavy nucleus size r;
- unit cell size R or occupied volume by heavy nuclei u.

Solve three equilibrium equations relating

- Pressure,
- chemical potential of protons, and
- chemical potential of neutrons,

inside and outside heavy nuclei.

Eqs. bound by four constraints:

- Equilibrium between μ_α, μ_n, & μ_p.
- Total density (n) is sum of all densities;
- Total charge density (ny) is sum of all charges.
- Nuclei size r and u set by virial theorem.

Surface Tension

For nuclei in the SNA, we need to know surface tension of nuclear matter.

Two semi-infinite slabs of nuclear matter separated by a surface:

High density:

- n_{ni}
- n_{pi}

Low density:

- n_{no}
- n_{po}

Set T and y_i and solve equilibrium equations:

$$P_i = P_o \,,$$

$$\mu_{ni} = \mu_{no} \,,$$

$$\mu_{pi} = \mu_{po} \,,$$

$$y_i = \frac{n_{pi}}{n_{ni} + n_{pi}}$$

This only determines equilibrium between two different densities!
We still need to determine the shape of the nuclear surface.

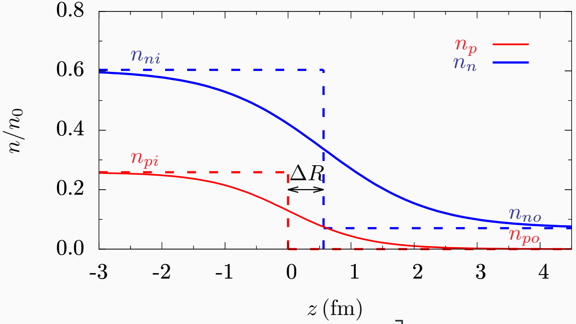
Surface Tension

Assume density has the form

$$n_t(z) = n_{to} + \frac{n_{ti} - n_{to}}{1 + \exp\left((z - z_t)/a_t\right)}$$

(Woods-Saxon form for Nuclear potential!)

Set $z_p = 0$ and solve for z_n, a_n, a_p that minimize the surface tension:



$$\sigma(y_i, T) = \int_{-\infty}^{+\infty} \left[F_B(z) + E_S(z) + P_o - \mu_{no} n_n(z) - \mu_{po} n_p(z) \right] dz\,.$$

$$E_S(z) = \frac{1}{2}\left[q_{nn}\left(\nabla n_n\right)^2 + q_{np}\nabla n_n \cdot \nabla n_p + q_{pn}\nabla n_p \cdot \nabla n_n + q_{pp}\left(\nabla n_p\right)^2 \right]$$

Surface Tension

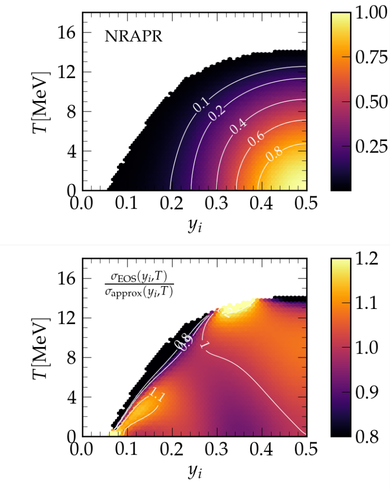
Surface tension can be fit by

$$\sigma(y_i, T) = \sigma_s h(y_i, T) \frac{2 \cdot 2^{\lambda} + q}{y_i^{-\lambda} + q + (1 - y_i)^{-\lambda}},$$

where $\sigma_s \equiv \sigma(0.5, 0)$ and

$$h(y_i, T) = \begin{cases} \left[1 - \left[\frac{T}{T_c(y_i)}\right]^2\right]^p, & T \leqslant T_c(y_i)\,; \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Could use observables (*e.g.* neutron skin thickness) to compute surface tension.



Pasta Phases

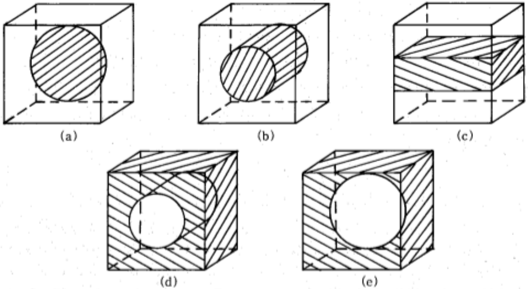


Fig. 1. Candidates for nuclear shapes. Protons are confined in the hatched regions, which we call nuclei. Then the shapes are, (a) sphere, (b) cylinder, (c) board or plank, (d) cylindrical hole and (e) spherical hole. Note that many cells of the same shape and orientation are piled up to form the whole space, and thereby the nuclei are joined to each other except for the spherical nuclei (a).

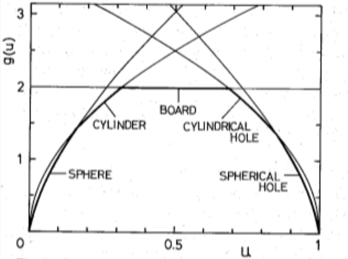


Fig. 2. Relative surface area $g(u)$. The parts giving the minimum area are drawn with thick lines.

Figura 1: PTP 71 320 (1984): dimensionality of nuclei depends on density.

Pasta Phases

Including the shape of nuclei:

$$F_S = \frac{3s(u)}{r}\sigma(y_i, T)$$
$$F_C = \frac{4\pi\alpha}{5}(y_i n_i r)^2 c(u).$$

- $s(u)$: surface shape function
- $c(u)$: Coulomb shape function
- r: generalized nuclear size
- $\sigma(y_i, T)$: surface tension per unit area

Shape function depend on topology of matter!

Using the nuclear virial Theorem: $F_S = 2F_C$

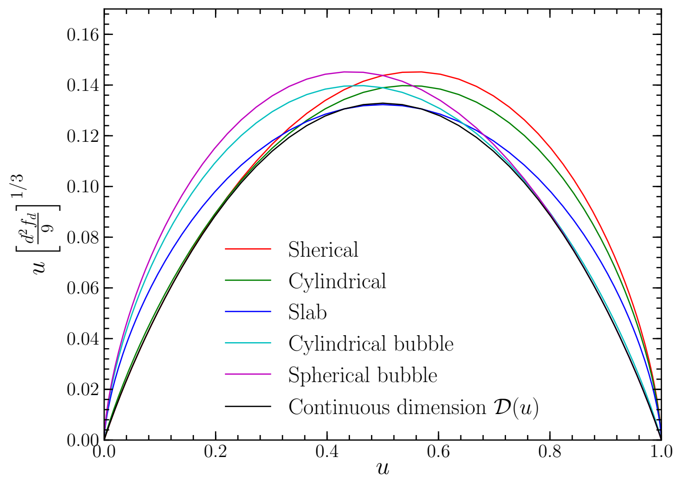
$$r = \frac{9\sigma}{2\beta}\left[\frac{s(u)}{c(u)}\right]^{1/3} \quad \text{where} \quad \beta = 9\left[\frac{\pi\alpha}{15}\right]^{1/3}(y_i n_i \sigma)^{2/3}$$

we can combine surface and coulomb terms:

$$F_S + F_C = \beta\left[c(u)s(u)^2\right]^{1/3} \equiv \beta\mathcal{D}(u).$$

where $\mathcal{D}(u)$ depends on the topology of nuclei.

Pasta Phases



Total energy as a function of volume occupied by dense matter for different topologies.

Pasta Phases

Lattimer & Swesty interpolate $\mathcal{D}(u)$ with

$$\mathcal{D}(u) = u(1-u)\frac{(1-u)D(u)^{1/3} + uD(1-u)^{1/3}}{u^2 + (1-u)^2 + 0.6u^2(1-u)^2}$$

where $D(u) = 1 - \frac{3}{2}u^{1/3} + \frac{1}{2}u$.

- $u \to 0$ reproduces free energy of spherical nuclei
- $u \to 1$ reproduces free energy of "bubble nuclei"
- intermediate u reproduces free energy of other phases: cylinders, slabs, and cylindrical holes.

Nuclear Statistical Equilibrium

As $u \to$ nuclei are so far apart that they may be consider point particles.

Consider ensemble of nuclei i immersed in a dilute gas of nucleons.

Chemical equilibrium results in:

$$\mu_i = m_i + E_{c,i} + T \log \left[\frac{n_i}{g_i} \left(\frac{2\pi}{m_i T} \right)^{3/2} \right] = Z_i \mu_p + (A_i - Z_i) \mu_n \,.$$

Solve for μ_n and μ_p to obtain total (1) density n and (2) charge density ny.

Just two equations to solve, but need to compute μ_i for all nuclei.
Partition functions of nuclei g_i difficult to obtain.

NSE or SNA

Why NSE?

- SNA properties can differ from actual properties of nuclei;
- No pairing nor shell closure;
- Neglects many-body effects;
- ...

Why SNA?

- NSE breaks down close to nuclear saturation density;
- Very large and/or very neutron rich nuclei not determined.
- No nuclear inversion (pasta phase).
- ...

Combining NSE and SNA

Use ad-hoc procedure to mix NSE and SNA free energies:

$$f_{\mathrm{MIX}} = \chi(n) f_{\mathrm{SNA}} + [1 - \chi(n)] f_{\mathrm{NSE}} .$$

Chose $\chi(n)$ such that:

$$\chi(n) \to 0, \text{ if } n \ll n_0$$

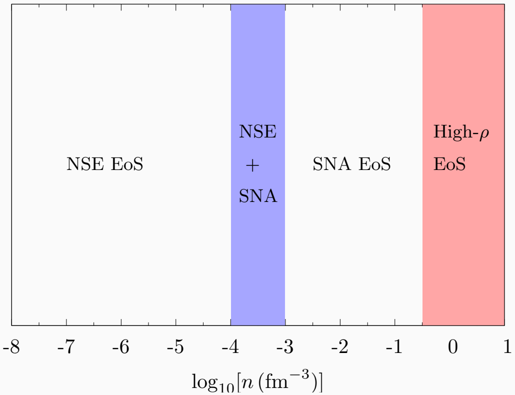
$$\chi(n) \to 1, \text{ if } n \gtrsim n_0/10$$

Corrections to thermodynamic quantities, *e.g.*

$$P_{\mathrm{MIX}} = n^2 \left. \frac{\partial (f_{\mathrm{MIX}})}{\partial n} \right|_{T,y} = \chi(n) P_{\mathrm{SNA}} + [1 - \chi(n)] P_{\mathrm{NSE}} + n^2 \frac{\partial \chi(n)}{\partial n} \left(f_{\mathrm{SNA}} - f_{\mathrm{NSE}} \right) .$$

EOS is self consistent!

Final Nuclear Equation of State



Components of Nuclear EOS combining NSE and SNA.

Once the Nuclear EOS has been obtained:

- Add lepton EOS;
- Add photon EOS;
- Write desired quantities to a table;
- Compute neutrino scattering table for EOS;
- Use tables in simulations!

Limitations

Limitations of our approach:

- Non-relativistic Skyrme model;
- Hadron sector only considers nucleons;
- Lepton sector only considers electrons;
- Simple effective mass behavior;
- “Pasta” has no shell and no pairing effects;
- “Pasta” depend on interpolation function;
- Only alphas as representative light nucleus;
- Alphas in a hard sphere approximation;
- Some properties of NSE nuclei unknown;
- Partition functions difficult to obtain;
- Free nucleons do not interact in NSE;
- SNA transition to NSE not entirely consistent;
- ...
- Code comments and documentation limited;
- Replacing modules not straightforward.

Next version of the code will (hopefully) fix some of these issues.