Understanding the π DA and improving the predictions for the $\pi - \gamma$ transition form factor

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Mapping Parton Distribution Amplitudes and Functions
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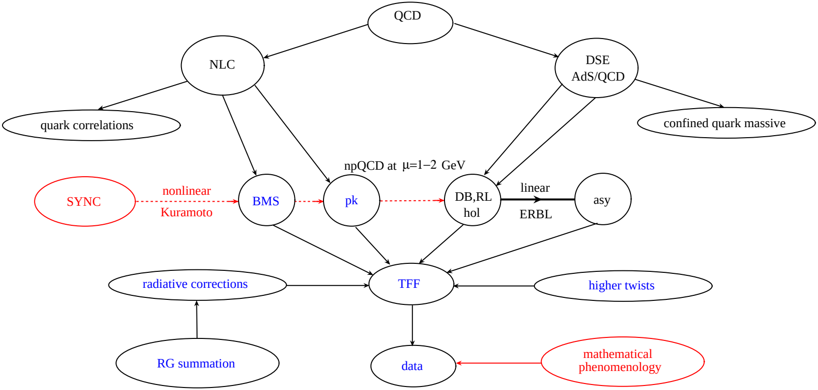
Overview


QCD
NLC
DSE
AdS/QCD
quark correlations
confined quark massive
npQCD at μ=1–2 GeV
SYNC
nonlinear
Kuramoto
BMS
pk
DB,RL
hol
linear
ERBL
asy
radiative corrections
TFF
higher twists
RG summation
data
mathematical
phenomenology

Outline

- Pion distribution amplitude (DA) of twist two $\varphi_\pi^{(2)}$ for valence $q\bar{q}$ pair
- Derivation: QCD sum rules with (non)local condensates, Dyson-Schwinger equations (DSE), AdS/QCD, chiral quark model, ...
- πDA models (Asy, BMS, pk, DSE, flat-like)
- TFF predictions with LCSRs at NNLO$_{\beta_0}$ level
- TFF predictions with upgraded LCSRs within Fixed Order Perturbation Theory (FOPT)
- LCSRs framework with Renormalization-group summation (RGS)
- RGS-LCSRs amount to *calibrated* Fractional Analytic Perturbation Theory (FAPT) via imposition of process-dependent boundary conditions to preserve QCD asymptotic limit of considered TFF
- Advantage: Improved TFF predictions for $F^{\gamma^*\gamma\pi^0}(Q^2)$ in low-momentum regime via RG summation of radiative corrections
- Conclusions

Pion distribution amplitude of twist two for $\pi \to u + d$:

- $\langle 0|\bar{q}(z)\gamma_\mu\gamma_5[z,0]q(0)|\pi(P)\rangle\Big|_{z^2=0} = iP_\mu f_\pi \int dx \exp[ix(z\cdot P)]\varphi_\pi^{(2)}(x;\mu^2)$

Gauge link: $[z,0] = \mathcal{P}\exp\left[ig\int_0^z A_\mu(\tau)d\tau^\mu\right] \qquad (A^+ = 0)$

Q^2 dependence known in pQCD by solving one-loop ERBL evolution equation:

- $\varphi_\pi^{(2)}(x;Q^2) = x(1-x)\sum_{n=0}^{\infty} a_n C_n^{3/2}(2x-1)\left(\ln Q^2/\Lambda^2_{\text{QCD}}\right)^{-\gamma_n/2\beta_0}$

DAs are **nonperturbative** but not directly measurable quantities to be derived from

- **QCD SRs [Chernyak, Zhitnitsky (CZ) 1982, 1984]**
- **"Nonlocal" QCD SRs [Mikhailov, Radyushkin (1986-1991)]; Bakulev, Mikhailov, NGS (BMS) 2001-2004]**
- **Instanton-vacuum, [Polyakov ... 1998; Dorokhov ... 2000; Nam, Kim 2006]**
- **Light-front quark model [Choi, Ji 2015-2017]**
- **DSE [Roberts ... 2013-2015]**
- **AdS/QCD [Brodsky, Cao, de Téramond 2011; Ahmady ... 2017-2018]**
- **Lattice QCD, [Braun ... 2006, 2015; Donnellan ... 2007; Arthur ... 2010; Segovia ... 2013; Bali ... 2017-2018]**

.....

(Re)construction of pion DA

Conformal expansion

- Expand $\varphi_\pi^{(2)}(x,\mu^2)$ over eigenfunctions of ERBL Eq.: $\{\psi_n(x)\}$ on $x \in [0,1]$

$$\varphi_\pi^{(2)}(x,\mu^2) = \sum_{n=0,2,4,\ldots}^{\infty} a_n(\mu^2)\psi_n(x)\,;\;\; \psi_n(x) = 6x\bar{x}C_n^{(3/2)}(2x{-}1)\,;\;\; \varphi_\pi^{\mathrm{asy}}(x) = 6x\bar{x}$$

- Determine conformal coefficients $a_n(\mu^2)$ via moments

$$\langle \xi^N \rangle_\pi \equiv \int_0^1 dx(2x-1)^N \varphi_\pi^{(2)}(x,\mu^2)$$

at typical hadronic scale $\mu^2 \gtrsim 1$ GeV^2 with $\bar{x} = 1 - x$; $\xi = 2x - 1 = x - \bar{x}$:

$$a_2 = \frac{7}{12}\left(5\left\langle \xi^2 \right\rangle - 1\right)\;; \quad a_4 = \frac{77}{8}\left(\left\langle \xi^4 \right\rangle - \frac{2}{3}\left\langle \xi^2 \right\rangle + \frac{1}{21}\right)$$

$$a_6 = \frac{5}{64}\left(429\langle \xi^6 \rangle - 495\langle \xi^4 \rangle + 135\langle \xi^2 \rangle - 5\right) \quad \ldots$$

- Conformal coefficients $a_n(Q^2 > \mu^2)$ to be computed by ERBL evolution

(Re)construction of pion DA

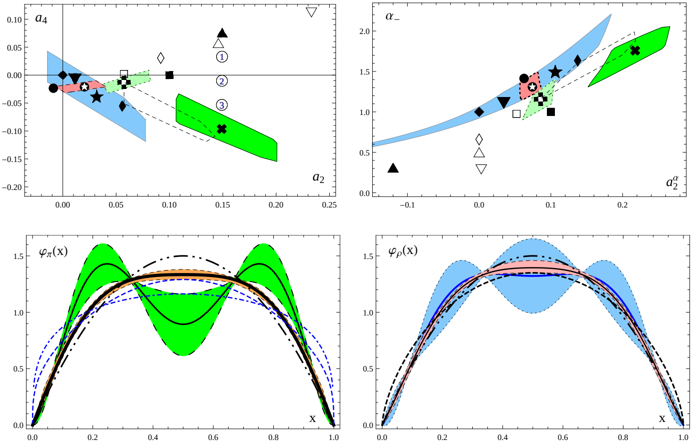
Gegenbauer-α representation

Chang et al., PRL110 (2013) 132001, Gao et al., PRD90 (2014) 014011

- $$\varphi_{\pi}^{(2)}(x,\mu^2) = f(\{\alpha, a_2^{\alpha}, ..., a_{j_s}^{\alpha}\}, x) = \psi_0^{(\alpha)}(x) \quad + \sum_{j=2,4,\ldots}^{j_s} a_j^{\alpha}(\mu^2)\psi_n^{(\alpha)}(x)$$
- Basis functions
  $$\psi_n^{(\alpha)}(x) = N_{\alpha}(x\bar{x})^{\alpha_-}\, C_n^{(\alpha)}(2x-1) \quad \text{[in general } \alpha \neq 3/2]$$
- $N_{\alpha} = 1/B(\alpha+1/2, \alpha+1/2)$; $\alpha_- = \alpha - 1/2$; $[B(x,y)$ Euler beta function]
  ★ **Disadvantage**: Set $\{\psi_n^{(\alpha)}(x)\}$ **NOT eigenfunctions of one-loop ERBL Eq.**
  To evolve $\varphi_{\pi}^{(2)}(x,\mu^2)$ to $Q^2 > \mu^2$, one has to project it first onto conformal basis $\{\psi_n(x)\}$ and then determine α_- and a_j^{α} at the new scale
  ★ **Advantage**: Sufficient to include only one coefficient: a_2^{α}; fast convergence
- Pion DA expressible only in terms of two parameters: a_2^{α} and α_-
  $$\varphi_{\pi}^{(\alpha)}(x,\mu^2) = N_{\alpha}(x\bar{x})^{\alpha_-}[1 + a_2^{\alpha}\, C_2^{(\alpha)}(x-\bar{x})]$$
- We present results for the pion DA in both forms: (a_2, a_4) and $\left(N_{\alpha}, \alpha_-, a_2^{\alpha}\right)$

Model DAs for π and $\rho_{\parallel}$

Graphics from NGS, Pimikov, NPA945(2015)248



- Large green band denotes BMS pion DAs [Bakulev, Mikhailov, NGS (2001)]
- Small green strip shows platykurtic regions (π and $\rho_{\parallel}$ DAs in orange strips)
- Blue band analogous results for $\rho_{\parallel}$ [NGS, Pimikov NPA945 (2015)]

Functional details of various model DAs for π

- QCD sum rules with nonlocal condensates

$$\varphi_\pi^{\text{BMS/pk}}(x,\mu^2 \gtrsim 1\ \text{GeV}^2) = 6x\bar{x}\left[1 + a_2 C_2^{(3/2)}(x-\bar{x}) + a_4 C_4^{(3/2)}(x-\bar{x})\right]$$

$a_2^{\text{BMS}}(x) = 0.2,\ a_4^{\text{BMS}} = -0.14,\ \lambda_q^2 = 0.4\ \text{GeV}^2$ [Bakulev, Mikhailov, NGS, (2001)]
$a_2^{\text{pk}}(x) = 0.08,\ a_4^{\text{pk}} = -0.019,\ \lambda_q^2 = 0.45\ \text{GeV}^2$ [NGS, PLB738 (20014) 483]

- Dyson-Schwinger equations [Chang et al., PRL110 (2013) 132001]

$$\varphi_\pi^{(\alpha)}(x,\mu^2) = N_\alpha (x\bar{x})^{\alpha_-}[1 + a_2^\alpha C_2^{(\alpha_-)}(x-\bar{x})] \quad \alpha_- = \alpha - 1/2$$

DSE-DB ▲ (- - -): $(N_\alpha = 0.181, \alpha_- = 0.31, a_2^\alpha = -0.12)$
DSE-RL ▽ (- . -): $(N_\alpha = 0.174, \alpha_- = 0.29, a_2^\alpha = 0.0029)$

- AdS/QCD △ [Brodsky, de Teramond, PRD77 (2008) 056007]:
$\varphi_\pi^{\text{AdS/QCD}}(x,\mu^2 = 1\ \text{GeV}^2) = (8/\pi)(x\bar{x})^{1/2}$
- Asymptotic DA ♦ (- . . -): $\varphi^{\text{asy}}(x,\mu^2 \to \infty) = 6x\bar{x}$
  - BMS (bimodal), platykurtic (unimodal) both have endpoints suppressed
  - Light-front based DA [Choi, Ji, PRD91 (2015) 014018, improved AdS/QCD model DA [Ahmady et al., (2017)] both close to platykurtic DA
  - DSE, AdS/QCD (unimodal) both have endpoints enhanced

Pion-photon TFF in LCSRs

- TFF $F^{\gamma^*\gamma^*\pi^0}(Q^2, q^2)$ with $Q^2 = -q_1^2$, $q^2 = -q_2^2$ defined as current correlator

$$\int d^4x e^{-iq_1\cdot x}\langle\pi^0(P)|T\{j_\mu(x)j_\nu(0)\}|0\rangle = i\epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta F^{\gamma^*\gamma^*\pi^0}(Q^2, q^2)$$

- TFF for $q^2 = 0$ expressed in dispersive form via a LCSR [Khodjamirian, EPJC6 (1999), see also Balitsky, Braun, Kolesnichenko, NPB 312 (1989)]

$$Q^2F^{\gamma^*\gamma\pi}(Q^2) = \frac{\sqrt{2}}{3}f_\pi\left[\frac{Q^2}{m_\rho^2}\int_{x_0}^1 \exp\left(\frac{m_\rho^2 - Q^2\bar{x}/x}{M^2}\right)\bar{\rho}(Q^2, x)\frac{dx}{x} + \int_0^{x_0}\bar{\rho}(Q^2, x)\frac{dx}{\bar{x}}\right]$$

$x_0 = \frac{Q^2}{Q^2+s_0}$, $s_0 \approx 1.5$ GeV^2: effective threshold, M^2: Borel parameter, $m_\rho = 770$ MeV

- Main ingredient of LCSR is spectral density $\bar{\rho}(Q^2, x) = (Q^2 + s)\rho^{\text{pert}}(Q^2, s)$

$$\rho^{\text{pert}}(Q^2, s) = \frac{1}{\pi}\mathbf{Im}F^{\gamma^*\gamma^*\pi^0}(Q^2, -s, -i\varepsilon) = \rho_{\text{tw-2}} + \rho_{\text{tw-4}} + \rho_{\text{tw-6}} + \ldots$$

- Twist two contribution:

$$\rho_{\text{tw-2}} \sim \frac{1}{\pi}\mathbf{Im}[T_{\text{LO}} + T_{\text{NLO}} + T_{\text{NNLO}}\ldots] \otimes \varphi_\pi^{\text{tw-2}}(x, \mu^2)$$

Details in Mikhailov, Pimikov, NGS, PRD93 (2016) 114018

Spectral density

Spectral density calculable within perturbative QCD:

$$\rho(Q^2,s) = \rho^{(0)}(Q^2,s) + \frac{\alpha_s}{4\pi}\rho^{(1)}(Q^2,s) + \left(\frac{\alpha_s}{4\pi}\right)^2 \rho^{(2)}(Q^2,s)$$

- **NLO** spectral density computed for $\psi_0(x)$ by Schmedding, Yakovlev PRD 62 (2000) 116002; for any $\psi_n(x)$ in [Mikhailov, NGS, NPB821 (2009) 291], corrected by Agaev et al. in PRD 83 (2011) 054020 ($x = Q^2/(s+Q^2)$):

$$\rho^{(1)}(Q^2,s) = \frac{\mathbf{Im}}{\pi}\left[(T_1 \otimes \varphi_\pi)(Q^2, -s - i\epsilon)\right]\ ,\ s \geq 0$$

- **NNLO**$_{\beta_0}$ spectral density calculated by Mikhailov, NGS, (2009) for any Gegenbauer harmonic $\psi_n(x)$:

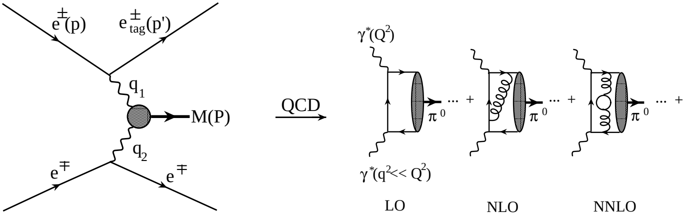
$$\rho^{(2,\beta_0)}(Q^2,s) = \beta_0 \frac{\mathbf{Im}}{\pi}\left[\left(T_2^{\beta_0} \otimes \varphi_\pi\right)(Q^2, -s - i\epsilon)\right]\ ,\ s \geq 0$$

- **Tw-6** contribution computed by Agaev et al., PRD83 (2011) 077504]; independently verified in [Mikhailov et al., PRD93 (2016) 114018] [$x \equiv Q^2/(Q^2+s)$]:

$$\rho^{\mathrm{Tw}-6}(Q^2,x) = 8\pi C_{\mathrm{F}} \alpha_s \frac{\langle \bar{q}q\rangle^2}{N_c f_\pi^2} \frac{x^2}{Q^6}\left[2x\ln(x\bar{x}) - x + 2\delta(\bar{x}) - \left[\frac{1}{1-x}\right]_+\right]$$

Pion-photon transition form factor in QCD and experiment

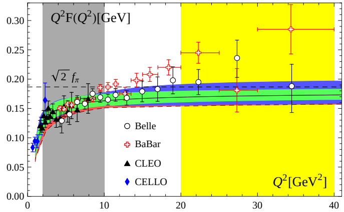
Left: Generic experimental setup for $e^+e^- \to e^+e^-\pi^0$ two-photon production process
Right: QCD description within FOPT

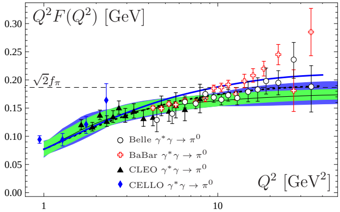


Measurements of differential cross sections in different "single-tagged" experiments for $q_2^2 \approx 0$ giving access to $F^{\gamma^*\gamma^*\pi^0}(q_1^2, q_2^2 \approx 0)$:

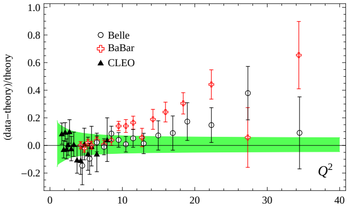
CELLO (1991): 0.70 $\div$ 2.20 GeV^2 H. J. Behrend et al., Z Phys C49 (1991) 401
CLEO (1998): 1.64 $\div$ 7.90 GeV^2 J. Gronberg et al., PRD57 (1998) 33
BaBar (2009): 4.24 $\div$ 34.36 GeV^2 B. Aubert et al., PRD80 (2009) 052002
Belle (2012): 4.46 $\div$ 34.46 GeV^2 S. Uehara et al., PRD86 (2012) 092007

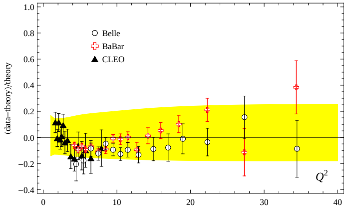
Predictions for pion-photon TFF using LCSRs





- Left. a) Green band BMS DAs, b) Blue strips are errors from NNLO_{β_0} $\oplus$ tw-6 terms, c) Red curve pk DA [NGS, PLB738 (2014)]
- Right. a) Blue strips show errors due to a_6 b) Blue curve DSE-DB DA, c) Dotted line AdS/QCD DA [Mikhailov et al., Few-Body Syst. 55 (2014) 367]





- Δ=(Data-Theory)/Theory vs CLEO, *BABAR*, Belle data — BMS DAs (left)
- Δ for DSE DAs in range [DSE-DB, DSE-RL] using $(a_1, a_2, \ldots, a_{10})$ (right)

Upgraded LCSRs for pion-photon TFF using FOPT

LCSRs at $\mathbf{NNLO}_{\beta_0}$ level

[Bakulev et al., PRD84 (2011); PRD86 (2012); NGS et al., PRD87 (2013)]

$$\text{TFF} = (\text{LO+NLO}) \otimes \varphi_\pi^{(2)} + \text{Tw-4} + \Delta(= \text{uncertainties})$$

$$\Delta = [\text{DA range}] + [\text{NNLO}_{\beta_0} \otimes \varphi_\pi^{(2)} + \text{Tw-6}] + \Delta\text{Tw-4}$$

Upgraded LCSRs

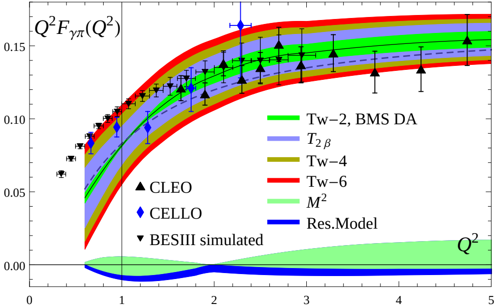
[Mikhailov et al., PRD93 (2016) 114018]

- $\text{TFF} = \text{Tw-2} + \text{Tw-4} + \text{Tw-6} + \tilde{\Delta}$
- $\text{Tw-2} = (\text{LO} + \text{NLO} + \text{NNLO}_{\beta_0} + \underbrace{\text{NNLO}_{\Delta\text{V}}}_{\text{new term}}) \otimes \varphi_\pi^{(2)}$

- $\tilde{\Delta} = [\text{DA range}] + [\overbrace{\text{T}_c}^{\text{unknown}} \underbrace{\simeq \beta_0 \mathcal{T}_\beta^{(2)}}_{\in \text{T}_{\text{NNLO}}}] \otimes \varphi_\pi^{(2)} + \Delta\text{Tw-4} + \Delta\text{Tw-6}$

- $\text{NNLO}_{\Delta\text{V}} \ll \text{NNLO}_{\beta_0}$ Melić et al., PRD68 (2003) 014013

Upgraded TFF results in [1-5] GeV^2 range



Source	Uncertainty (%) at $Q^2 = 3$ GeV^2
Unknown NNLO term $\mathcal{T}_c^2$	∓ 4.8
Range of Tw-2 BMS DAs	$-3.4 \div 4.1$
Tw-4 coupling $\delta^2 = [0.152 - 0.228]$ GeV^2	± 3.0
Tw-6 parameter variation	$-2.4 \div 3.0$
Total	$-13.6 \div 14.9$
Borel parameter $M^2 \in [0.7 - 1.5]$ GeV^2	$-1.6 \div 7.2$
Resonance description δ vs. BW	$-3.6 \div 0$
Small virtuality of quasireal photon	$-5.4 \div 0$

[Mikhailov et al., PRD93 (2016) 114018]

Pion-photon TFF with RG improvement

Hard process $\gamma^*(-Q^2)\gamma^*(-q^2) \to \pi^0$, $Q^2 > m_\rho^2, q^2 > m_\rho^2$ at twist level **two** described by TFF $[a_s(\mu^2) = \alpha_s(\mu^2)/4\pi;\ \bar{a}_s(y) \equiv \bar{a}_s(q^2\bar{y} + Q^2 y);\ N_{\rm T} = \sqrt{2}f_\pi/3;\ f_\pi = 132\ \text{MeV}]$

$$F^{(\text{tw}=2)}(Q^2, q^2) = N_{\rm T}\, T_0(y) \underset{y}{\otimes} \left\{ \left[\mathbb{1} + \bar{a}_s(y)\mathcal{T}^{(1)}(y,x) + \bar{a}_s^2(y)\mathcal{T}^{(2)}(y,x) + \ldots \right] \underset{x}{\otimes} \exp\left[-\int_{a_s}^{\bar{a}_s(y)} d\alpha \, \frac{V(\alpha; x, z)}{\beta(\alpha)} \right] \right\} \underset{z}{\otimes} \varphi_\pi^{(2)}(z, \mu^2)$$

- $T_0(y) \equiv T_0(Q^2, q^2; y) = 1/(q^2\bar{y} + Q^2 y)$: **Born term** of HSA
- $\mathbb{1} = \delta(x - y)$
- $\mathcal{T}^{(i)}$ coefficient function of quark-gluon subprocess at loop order i
- $V(a_s) = a_s V_0 + a_s^2 V_1 + \ldots$ evolution kernel of ERBL evolution equation
- Gegenbauer expansion $\varphi_\pi^{(2)}(x, \mu^2) = \psi_0(x) + \sum_{n=2,4,\ldots}^{\infty} a_n(\mu^2)\psi_n(x)$ yields
- $F^{(\text{tw}=2)}(Q^2, q^2) = F_0^{(\text{tw}=2)}(Q^2, q^2) + \sum_{n=2,4,\ldots}^{\infty} a_n(\mu^2) F_n^{(\text{tw}=2)}(Q^2, q^2)$

[Ayala, Mikhailov, NGS, 1806.07790]

Radiative corrections in dispersive representation

Key idea: Combination of causality, encoded in dispersion relations of LCSRs, with RG invariance, induces analyticity of perturbative expansion, transferring power-series expansion of pion-photon TFF in terms of usual couplings with ghost singularities into functional expansion over special analytic couplings that preserve the UV asymptotics of this observable. [Ayala et al., 1806.07790]

Conformal expansion of RG-improved TFF

$$F^{(\text{tw}=2)}_{(\text{1-loop})n} = N_{\text{T}}\, T_0(y) \underset{y}{\otimes} \left\{ \left[1\!\!1 + \bar{a}_s(y)\mathcal{T}^{(1)}(y,x) \right] \left(\frac{\bar{a}_s(y)}{a_s(\mu^2)} \right)^{\nu_n} \right\} \underset{x}{\otimes} \psi_n(x)$$

$$\nu_n = \frac{1}{2}\frac{\gamma_0(n)}{\beta_0}; \quad \varphi^{(2)}_{\pi}(x,\mu^2) = \psi_0(x) + \sum_{n=2,4,\ldots}^{\infty} a_n(\mu^2)\psi_n(x)$$

Zeroth-order harmonic TFF

$$F^{(\text{tw}=2)}_{n=0}(Q^2,q^2) = N_{\text{T}}\, T_0(y) \underset{y}{\otimes} \left[1\!\!1 + \bar{a}_s(y)\mathcal{T}^{(1)}(y,x) \right] \underset{x}{\otimes} \psi_0(x)$$

Analytic Perturbation Theory (**APT**) [Shirkov, Solovtsov, PRL79 (1997) 1209; Theor. Math. Phys. 150 (2007) 132; Shirkov, *ibid.* 127 (2001) 409]
Fractional APT (**FAPT**) [Bakulev, Mikhailov, NGS, PRD72 (2005) 074014, PRD75 (2007) 056005; Karanikas, NGS, PLB504 (2001) 225; Bakulev, Phys. Part. Nucl.40 (2009) 715, NGS, *ibid.* 44 (2013) 494]

Pion-photon TFF in QCD FOPT

In QCD FOPT we get

$$F_{\text{FOPT}}^{(\text{tw}=2)}(Q^2,q^2) = N_{\text{T}}\left(T_{\text{LO}} + a_s\, T_{\text{NLO}} + a_s^2\, T_{\text{NNLO}} + \dots\right) \otimes \varphi_\pi^{(2)}$$

Radiative corrections $[L = L(y) = \ln\left[(q^2\bar{y} + Q^2 y)/\mu^2\right]]$ given by

$$\begin{aligned}
T_{\text{LO}}\,, &= a_s^0\; T_0(x)\\
a_s\, T_{\text{NLO}} &= a_s^1\; T_0(y) \otimes \left[\mathcal{T}^{(1)} + \underline{L\;V_0}\right](y,x)\,,\\
a_s^2\, T_{\text{NNLO}} &= a_s^2\; T_0(y) \otimes \Big[\mathcal{T}^{(2)} - \underline{L\;\mathcal{T}^{(1)}\beta_0} + \underline{L\;\mathcal{T}^{(1)} \otimes V_0} - \underline{\frac{L^2}{2}\;\beta_0 V_0}\\
&\quad + \underline{\frac{L^2}{2}\;V_0 \otimes V_0} + \underline{\underline{L\;V_1}}\Big](y,x)
\end{aligned}$$

- Plain terms $\Longleftrightarrow$ one-loop, $\mathcal{T}^{(1)}$, and two-loop, $\mathcal{T}^{(2)}$, corrections
- Underlined terms due to $\Longleftrightarrow$ $\bar{a}_s(y)$ and ERBL factor
- Double-Underlined term $\Longleftrightarrow$ first contribution of higher two-loop corrections

Dispersive form of TFF in FAPT

General expression for $F^{\gamma^*\pi}_{\text{FAPT}}(Q^2, q^2; m^2)$:

$$\nu(n=0)=0;\;\; F^{\gamma^*\pi}_{\text{FAPT},0}(Q^2, q^2; m^2) = N_{\text{T}}\, T_0(Q^2, q^2; y) \underset{y}{\otimes} \Big\{ \mathbb{1} + \mathbb{A}_1(m^2, y)\mathcal{T}^{(1)}(y,x) \Big\} \underset{x}{\otimes} \psi_0(x)$$

$$\nu(n\neq 0)\neq 0;\;\; F^{\gamma^*\pi}_{\text{FAPT},\text{n}}(Q^2, q^2; m^2) = \frac{N_{\text{T}}}{a_s^{\nu_n}(\mu^2)} T_0(Q^2, q^2; y) \underset{y}{\otimes} \Big\{ \mathbb{A}_{\nu_n}(m^2, y)\mathbb{1} + \mathbb{A}_{1+\nu_n}(m^2, y)\mathcal{T}^{(1)}(y,x) \Big\} \underset{x}{\otimes} \psi_n(x)$$

Definition of effective analytic couplings

★ $\mathbb{A}_\nu(m^2, y) = \underbrace{\mathcal{I}_\nu(m^2, Q(y))}_{\text{generalized coupling}} - \mathfrak{A}_\nu(m^2); \quad \mathbb{A}_\nu(0, y) = \mathcal{A}_\nu(Q(y)) - \mathcal{A}_\nu(0)$

$$\mathcal{I}_\nu(Y, X) \stackrel{def}{=} \int_Y^\infty \frac{d\sigma}{\sigma + X} \rho_\nu^{(l)}(\sigma)$$

Special cases

$$\mathcal{A}_\nu(X) = \mathcal{I}_\nu(Y\to 0, X),\; \mathfrak{A}_\nu(Y) = \mathcal{I}_\nu(Y, X\to 0),\; \mathcal{A}_1(0) = \mathfrak{A}_1(0) = \mathcal{I}_1(Y\to 0, X\to 0)$$

TFF in modified FAPT — Calibration procedure

$F^{\gamma\pi}_{\text{FAPT}}(Q^2; m^2)$ in the limits $q^2 \to 0$, $Q(y) \to yQ^2$ and $m^2 \geqslant 0$:

$\nu(n=0)=0$

$$Q^2 F^{\gamma\pi}_{\text{FAPT},0} \equiv F_0(Q^2; m^2) = N_{\text{T}} \left\{ \int_0^1 \frac{\psi_0(x)}{x}\, dx + \left(\frac{\mathbb{A}_1(m^2, y)}{y} \right) \underset{y}{\otimes} \mathcal{T}^{(1)}(y, x) \underset{x}{\otimes} \psi_0(x) \right\}$$

$\nu(n \neq 0) \neq 0$

$$Q^2 F^{\gamma\pi}_{\text{FAPT},n} \equiv F_n(Q^2; m^2) =$$

$$\frac{N_{\text{T}}}{a_s^{\nu_n}(\mu^2)} \left\{ \left(\frac{\mathbb{A}_{\nu_n}(m^2, y)}{y} \right) \underset{y}{\otimes} \psi_n(y) + \left(\frac{\mathbb{A}_{1+\nu_n}(m^2, y)}{y} \right) \underset{y}{\otimes} \mathcal{T}^{(1)}(y, x) \underset{x}{\otimes} \psi_n(x) \right\}$$

These equations can be related to $F_n^{(\text{tw}=2)}(Q^2, q^2)$ via ★. Hence, UV behavior of TFF related to IR behavior of FAPT couplings $\mathfrak{A}_\nu(0) = \mathcal{A}_\nu(0)$ for $m^2 = 0$.

- Problem: The values $\mathcal{A}_1^1(0) = \mathfrak{A}_1^1(0) = 1/\beta_0$ yield to scaled TFF that violates asymptotic limit.
- Calibration of analytic couplings at the origin eliminates constant artifact $\Delta = -\left(\frac{\mathcal{A}_1(0)}{y} \right) \underset{y}{\otimes} \mathcal{T}^{(1)}(y, x) \underset{x}{\otimes} \psi_0(x)$ in TFF at $Q^2 \to \infty$:

$$\mathcal{A}_\nu^{(1)}(0) = \mathfrak{A}_\nu^{(1)}(0) = 0, \text{ for } 0 < \nu \leqslant 1$$

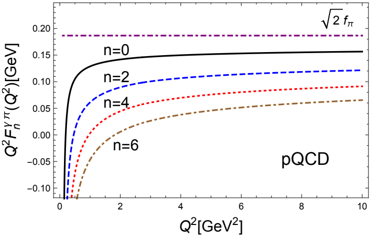
Dispersive form of $F^{\gamma\pi}_{\text{FAPT}}(Q^2, m^2)$ for $q^2 \to 0$

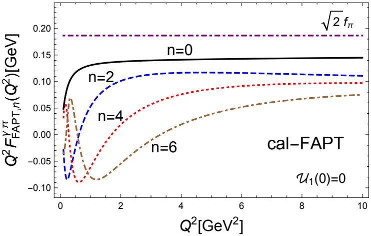
Results for $F^{\gamma\pi}_{\text{FAPT}}(Q^2, m^2)$ in the limit $q^2 \to 0$ using *calibrated* analytic couplings

$$\mathcal{A}^{(1)}_{\nu}(0) = \mathfrak{A}^{(1)}_{\nu}(0) = 0, \ \text{for } 0 < \nu \leqslant 1$$

$$\nu(n=0) = 0; \quad Q^2 F^{\gamma\pi}_{\text{FAPT},0}(Q^2; m^2) = N_{\text{T}} \left\{ \int_0^1 \frac{\psi_0(x)}{x}\, dx + \left(\frac{\mathbb{A}_1(m^2, y)}{y} \right) \underset{y}{\otimes} \mathcal{T}^{(1)}(y,x) \underset{x}{\otimes} \psi_0(x) \right\}$$

$$\nu(n \neq 0) \neq 0; \quad Q^2 F^{\gamma\pi}_{\text{FAPT},n}(Q^2; m^2) = \frac{N_{\text{T}}}{a_s^{\nu_n}(\mu^2)} \left\{ \left(\frac{\mathbb{A}_{\nu_n}(m^2, y)}{y} \right) \underset{y}{\otimes} \psi_n(y) + \left(\frac{\mathbb{A}_{1+\nu_n}(m^2, y)}{y} \right) \underset{y}{\otimes} \mathcal{T}^{(1)}(y,x) \underset{x}{\otimes} \psi_n(x) \right\}$$





Improved form of $F^{\gamma\pi}_{\rm LCSR}$ using cal-FAPT

$$Q^2 F^{\gamma\pi}_{\rm LCSR}\left(Q^2\right) \quad = \quad N_{\rm T}\left[Q^2 F^{\rm H}_{\rm FAPT}\left(Q^2\right) + Q^2 F^{\rm S}_{\rm FAPT}\left(Q^2\right) + \text{twist-4}\right]$$

Hard part

$$Q^2 F^{\rm H}_{{\rm FAPT},0}(Q^2; m^2, s_0) = N_{\rm T}\left\{\int_{x_0}^{1}\bar{\rho}_0(Q^2,\bar{x})\frac{dx}{x} + \left(\frac{\mathbb{A}_1(m^2, s_0; x)}{x}\right)\underset{x}{\otimes}\mathcal{T}^{(1)}(x,y)\underset{y}{\otimes}\psi_0(y)\right\}$$

$$Q^2 F^{\rm H}_{{\rm FAPT},n}(Q^2; m^2, s_0) = \frac{N_{\rm T}}{a_s^{\nu_n}(\mu^2)}\left\{\left(\frac{\mathbb{A}_{\nu_n}(m^2, s_0; x)}{x}\right)\underset{x}{\otimes}\mathbb{1} + \left(\frac{\mathbb{A}_{1+\nu_n}(m^2, s_0; x)}{x}\right)\underset{x}{\otimes}\mathcal{T}^{(1)}(x,y)\right\}\underset{y}{\otimes}\psi_n(y)$$

$$\mathbb{A}_\nu(m^2, s_0; y) = \theta(y \geqslant y_0)\mathcal{I}_\nu(m^2, Q(y)) - \mathfrak{A}_\nu(m^2)] + \theta(y < y_0)[\mathcal{I}_\nu(s_0(y), Q(y)) - \mathfrak{A}_\nu(s_0(y))]$$

$$s_0(y) = s_0\bar{y} - Q^2 y, \quad y_0 = (s_0 - m^2)/(s_0 + Q^2)$$ scale s_0 induced by LCSRs

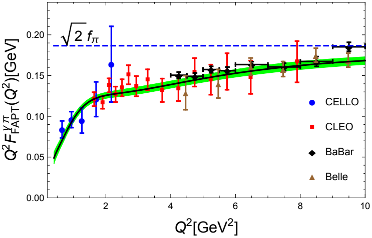
Soft part

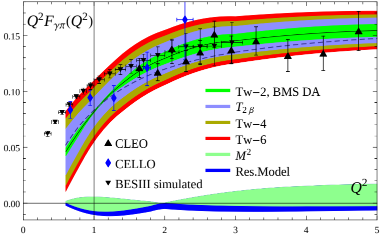
$$F^{\rm S}_{\rm FAPT}\left(Q^2\right) \quad = \quad \frac{1}{m_\rho^2}\exp\left(\frac{m_\rho^2}{M^2}\right)\hat{B}_{q^2\to M^2}\left[F^{\gamma^*\pi}_{\rm FAPT}(Q^2, q^2; m^2) - F^{\gamma^*\pi}_{\rm FAPT}(Q^2, q^2; s_0)\right]$$

Effect of RG summation on $F^{\gamma\pi}_{\mathrm{LCSR}}$ using cal-FAPT in LCSRs

Final expression for TFF in LCSRs endowed with RG summation

$$F^{\gamma\pi}_{\mathrm{LCSR}}\left(Q^2\right) = F^{\gamma\pi}_{\mathrm{LCSR};0}\left(Q^2\right) + \sum_{n=2,4,\ldots} a_n(\mu^2)\, F^{\gamma\pi}_{\mathrm{LCSR};n}\left(Q^2\right)$$





Left. Green strip indicates region of BMS DAs [Ayala, Mikhailov, NGS, 1806.07790]
Right. Detailed estimates for TFF with theoretical uncertainties [Mikhailov, Pimikov, NGS, PRD93 (2016)]

TFF parts for $n=0$ and $n\neq 0$ in $F^{\gamma\pi}_{\text{LCSR}}$ (supplementary)

Case $n=0$

$$Q^2F^{\gamma\pi}_{\text{LCSR};0}(Q^2) = N_{\text{T}}\Bigg\{\int_0^{\bar{x}_0}\bar{\rho}_0(Q^2,x)\frac{dx}{\bar{x}}+\frac{Q^2}{m_\rho^2}\int_{\bar{x}_0}^1\exp\left(\frac{m_\rho^2}{M^2}-\frac{Q^2}{M^2}\frac{\bar{x}}{x}\right)\bar{\rho}_0(Q^2,x)\frac{dx}{x}+$$
$$\left(\frac{\mathbb{A}_1(0,s_0;x)}{x}\right)\underset{x}{\otimes}\mathcal{T}^{(1)}(x,y)\underset{y}{\otimes}\psi_0(y)+$$
$$\frac{Q^2}{m_\rho^2}\int_{\bar{x}_0}^1\exp\left(\frac{m_\rho^2}{M^2}-\frac{Q^2}{M^2}\frac{\bar{x}}{x}\right)\frac{dx}{x}\Delta_1(0,\bar{x})\mathcal{T}^{(1)}(\bar{x},y)\otimes\psi_0(y)+O(\mathbb{A}_2)\Bigg\}$$

Case $n\neq 0$

$$Q^2F^{\gamma\pi}_{\text{LCSR};n}(Q^2) = \frac{N_{\text{T}}}{a_s^{\nu_n}(\mu^2)}\Bigg\{\left(\frac{\mathbb{A}_{\nu_n}(0,s_0;x)}{x}\right)\underset{x}{\otimes}\psi_n(x)+\left(\frac{\mathbb{A}_{1+\nu_n}(0,s_0;x)}{x}\right)\underset{x}{\otimes}$$
$$\mathcal{T}^{(1)}(x,y)\underset{y}{\otimes}\psi_n(y)+$$
$$\frac{Q^2}{m_\rho^2}\int_{\bar{x}_0}^1\exp\left(\frac{m_\rho^2}{M^2}-\frac{Q^2}{M^2}\frac{\bar{x}}{x}\right)\frac{dx}{x}\Big[\Delta_{\nu_n}(0,\bar{x})\psi_n(x)+$$
$$\Delta_{1+\nu_n}(0,\bar{x})\mathcal{T}^{(1)}(\bar{x},y)\otimes\psi_n(y)\Big]+O(\mathbb{A}_2)\Bigg\}$$

$$\mathbb{A}_\nu(m^2;y)-\mathbb{A}_\nu(m^2,s_0;y) = \theta(y<y_0)\,\Delta_\nu(m^2,y)$$
$$\mathbb{A}_\nu(0;x)-\mathbb{A}_\nu(0,s_0;x) = \theta(x<x_0)\,\Delta_\nu(0,x)$$

Conclusions

- LCSRs provide selfconsistent method to calculate TFF in systematic way on the basis of collinear factorization and twist expansion
- Various pion DAs can be used in convolution scheme including ERBL evolution at LO and NLO level
- Hard-scattering amplitude computed within FOPT, comprises LO, NLO, and all NNLO terms accept one (T_c—calculation in progress)
- TFF predictions based on BMS and platykurtic DA are presented, which agree with all data compatible with QCD scaling behavior at large Q^2
- Auxetic branch of *BABAR* data beyond 10 GeV^2 not reproduced— waiting for Belle-II data
- LCSRs augmented with RG summation of all (logarithmic) radiative corrections yield (with endpoint-suppressed pion DAs) TFF with improved Q^2 behavior in range $[1-5]$ GeV^2
- Announced BESIII data with high statistical precision best-suited to test these new predictions
- Calibrated FAPT contains new effective analytic couplings that depend on three arguments $\mathbb{A}_\nu(m^2, s_0; x)$ and reduce to FAPT couplings away from $Q^2 = 0$ and $0 < \nu \leqslant 1$ with $m^2 = 0$ by demanding $\mathcal{A}_\nu^{(1)}[L \to -\infty] = \mathfrak{A}_\nu^{(1)}[L \to -\infty] = 0$

Appendix: Standard FAPT couplings

One-loop running couplings in QCD and FAPT in terms of $L = \ln(Q^2/\Lambda^2_{\text{QCD}})$, multiplied by β_0^ν, i.e., we shift origin of different coupling images to $a_s \to A_s = \beta_0 a_s = \beta_0 \alpha_s/(4\pi)$

$$A_s^\nu[L] = \frac{1}{L^\nu} \qquad \text{standard pQCD}$$

$$\mathcal{A}_\nu^{(1)}[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)} \qquad \text{spacelike FAPT}$$

$$\mathfrak{A}_\nu^{(1)}[L_s] = \frac{\sin\left[(\nu-1)\arccos\left(L_s/\sqrt{(L_s^2+\pi^2)}\right)\right]}{\pi\,(\nu-1)\,(L_s^2+\pi^2)^{(\nu-1)/2}} \qquad \text{timelike FAPT}$$

$\dfrac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$ is "pole remover", expressed via Lerch transcendental function $F(z,s)$ $(= Li_s(z))$

$$F(z, 1-\nu) + \exp\left(i\pi(1-\nu)\right) F\left(1/z, 1-\nu\right) = \frac{(2i\pi)^{1-\nu}}{\Gamma(1-\nu)} \zeta\left(\nu, \frac{\ln(z)}{2i\pi}\right)$$

determines analytic continuation into outer region of radius of convergence, making use of Hurwitz zeta function $\zeta(\nu, z)$