Evaluating light-cone distributions in Minkowski space

Giovanni Salmè
(INFN Rome)

In Collaboration with

T. Frederico[a], W. de Paula[a], J. Nogueira[ab], E. Ydrefors[a], C. Mezrag[b], E. Pace[c], M. Viviani[d]

[a]ITA, S. José dos Campos (Brazil), [b]INFN - Rome (Italy), [c]Rome Univ. Tor Vergata & INFN (Italy), [d]INFN - Pisa (Italy)

and some results....

- dFSV PRD 94, 071901(R) (2016): Two-fermion bound systems
  EPJ C 77, 764 (2017) : Light-cone singularities and structure

- FVS PRD 85, 036009 (2012): General formalism for bound and scattering states
- FVS PRD 89, 016010 (2014): Bound states and LF momentum distributions for two scalars
- FVS EPJC 75, 398 (2015): Scattering lengths for two scalars
- Gutierrez et al PLB 759, 131 (2016): Spectra of excited states and LF momentum distributions

Outline

Motivations and Tools

- M: Presently, many and collaborative efforts to address relevant dynamical quantities, like PDFs, TMDs and GPDs are carried out within LQCD, widely considered the elective tool for non perturbative studies. Of particular interest, the ongoing investigation of the X. Ji proposal (PRL 110, 262002 (2013)) and the A. Radyushkin one (PRD 96, 034025 (2017)) on Quasi-PDFs, that aim at the evaluation of PDFs from LQCD, though some subtleties are still not fully elucidated.
- M: To have workable alternatives is highly desirable, even with a lower degree of complexity than LQCD can achieve. A reference approach, quite popular and very effective in predicting the dynamical behavior inside hadrons, is the so-called Continuum QCD. Based on both the Dyson-Schwinger equation (for self-energies) and the Bethe-Salpeter one, it is able to get the ingredients for calculating dynamical observables.
- M: Our perspective evolves within a continuum approach, but directly played in Minkowski momentum-space. We aim at achieving a fully covariant and non perturbative description for bound systems, with spin dof, incorporating, step by step and in a controlled way, dynamical effects, at the level of the interaction kernel, self-energy and vertex corrections. The first milestone has been the actual solutions of the Bethe-Salpeter equation (BSE), in ladder approximation. The formal extension to DSE is under progress.

- M: Once the BSE is solved in Minkowski momentum-space, one can determine from the BS amplitude, the relevant momentum distributions. A straightforward outcome: the light-cone valence distributions can be obtained by properly projecting the BS amplitude of the bound system.

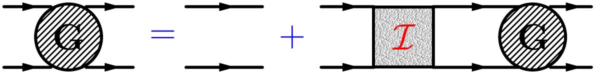
- T: Pivotal role of the Nakanishi Integral Representation (NIR) of the BS amplitude
- T: Light-front (LF) variables, $x^{\pm} = x^0 \pm x^3$ and $\mathbf{x}_\perp \equiv \{x^1, x^2\}$, very suitable for managing analytic integration and spin dof in a very effective way, in Minkowski space.
- T: Standard LAPACK routines for the numerical evaluations

The BSE in a nutshell: a simple two-body case

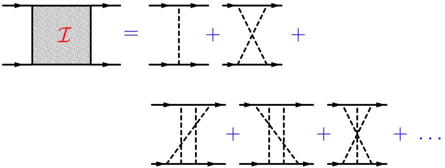
The 4-point Green's Function ($\phi_i \equiv$ scalar fields for simplicity),

$$G(x_1, x_2; y_1, y_2) = < 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 > \quad ,$$

fulfills an integral equation $G = G_0 \; + \; G_0 \, \mathcal{I} \; G$



$\mathcal{I} \; \equiv$ interaction kernel, given by the infinite sum of irreducible Feynman graphs



$\boxed{\text{All}}$ the expected (2-body reducible) contributions come from $\boxed{\text{iterations}}$!

Properly insert a complete 2-body basis in

$$G(x_1, x_2; y_1, y_2) =< 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 >$$

hence, the bound state contribution (assuming only one non degenerate bound state for the sake of simplicity) appears as a pole, in momentum space, viz

$$G(k, q; p) \Rightarrow G_B(k, q; p) \simeq \frac{i}{(2\pi)^{-4}} \, \frac{\chi(k; p_B) \; \bar{\chi}(q; p_B)}{2\omega_B(p^0 - \omega_B + i\epsilon)}$$

- $\omega_B = \sqrt{M_B^2 + |\mathbf{p}|^2}$, $p_B^\mu \equiv \{\omega_B, \mathbf{p}\}$ with M_B the mass of the bound state, $\beta \equiv$ further quantum numbers
- $\chi(k; p_B) \; \equiv$ Bethe-Salpeter Amplitude. It allows to describe the residue. Unfortunately, it has no probabilistic interpretation !

In configuration space,

$$\text{Bethe-Salpeter Amplitude} \rightarrow \langle 0| T\{\phi_1(x_1)\phi_2(x_2)\}|p_B \; \beta\rangle$$

Close to the bound-state pole, $p^0 \to \omega_B =$

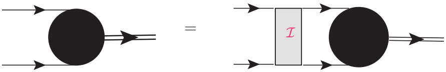
$$G \simeq G_B + \ \textit{regular terms}$$

Inserting this approximation in both sides of $G = G_0 + G_0\, \mathcal{I}\, G$ and multiplying by $(p^0 - \omega_B)$ one gets

$$G(k,q;p)\ (p^0 - \omega_B) \simeq G_B\ (p^0 - \omega_B) \simeq G_0\ \int d^4k'\ \mathcal{I}(k,k';p)\ G_B(k',q;p)\ (p^0 - \omega_B)$$

$\Rightarrow$ BS Equation

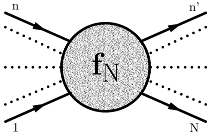
$$\chi(k;p_B,\beta) = G_0(k;p_B,\beta)\ \int d^4k'\ \mathcal{I}(k,k';p_B)\ \chi(k';p_B,\beta)$$



A non perturbative framework, like the one yielded by an integral equation, is necessary for describing a bound state (needed an infinite number of exchanges)!

Feynman parametric integrals

In the sixties, Nakanishi (PR **130**, 1230 (1963)) proposed an integral representation of N-leg transition amplitudes, based on the parametric formula for the Feynman diagrams.



Within the perturbation theory, the N-(external)-leg transition amplitude for a scalar theory (simple case) gets an infinite set of contributions, each of them with a generic form like

$$f_{\mathcal{G}}(p_1, p_2, ..., p_N) \propto \prod_{r=1}^{k} \int d^4 q_r \; \frac{1}{(\ell_1^2 - m_1^2)(\ell_2^2 - m_2^2) \; \ldots \; (\ell_n^2 - m_n^2)}$$

where one has n propagators and k loops ($\equiv$ number of integration variables).

The label $\mathcal{G} \to \{n, k\}$

N.B. the dependence upon $\{n, k\}$ is in the denominator

Nakanishi Integral Representation - I

Nakanishi proposal for a compact and elegant expression of the full N-leg amplitude $f_N(s) = \sum_{\mathcal{G}} \; f_{\mathcal{G}}(s)$:

Introducing the identity

$$1 \doteq \prod_h \int_0^1 dz_h \delta\left(z_h - \frac{\eta_h}{\beta}\right) \int_0^\infty d\gamma \; \delta\left(\gamma - \sum_l \frac{\alpha_l m_l^2}{\beta}\right)$$

with $\beta = \sum \eta_i(\vec{\alpha})$ and integrating by parts $n - 2k - 1$ times

$$f_{\mathcal{G}}(\tilde{s}) \propto \; \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1 - \sum_h z_h) \; \tilde{\phi}_{\mathcal{G}}(\vec{z}, \gamma)}{(\gamma - \sum_h z_h s_h)}$$

$\tilde{\phi}_{\mathcal{G}}(\vec{z}, \gamma) \equiv$ a proper weight function, with $\vec{z} \equiv \{z_1, z_2, \; \ldots \; , z_N\}$
$\tilde{s} \equiv \{s_1, s_2, \; \ldots \; , s_N\} \Rightarrow$ all the N *independent* scalar products, obtained from the external momenta

The dependence upon the details of the diagram, $\{n, k\}$, moves from the denominator $\rightarrow$ the numerator!!

The SAME formal expression for the denominator of ANY diagram $\mathcal{G}$ appears

NIR - II

The full N-leg transition amplitude is the sum of infinite diagrams $\mathcal{G}(n,k)$ and it can be formally written as

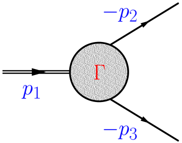
$$f_N(\tilde{s}) = \sum_{\mathcal{G}} f_{\mathcal{G}}(\tilde{s}) \propto \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1-\sum_h z_h)\ \phi_N(\vec{z},\gamma)}{(\gamma - \sum_h z_h s_h)}$$

where

$$\phi_N(\vec{z},\gamma) = \sum_{\mathcal{G}} \tilde{\phi}_{\mathcal{G}}(\vec{z},\gamma)$$

is called a Nakanishi weight function and it is REAL.

Application: 3-leg transition amplitude $\to$ vertex function for a scalar theory (N.B. for fermions $\to$ spinor indexes)



$$f_3(\tilde{s}) = \int_0^1 dz \int_0^\infty d\gamma \frac{\phi_3(z,\gamma)}{\gamma - \frac{p^2}{4} - k^2 - zk\cdot p - i\epsilon}$$

with $p = p_1 + p_2$ and $k = (p_1 - p_2)/2$

The expression holds at any order in PT !
Natural choice as a general trial function for obtaining actual solution of BSE. N.B. the variables z and γ are real and the analytic structure is made explicit.

A vertex function $f_3(\tilde{s})$ (N.B. $\tilde{s} \equiv$ all the independent scalar products involving the external momenta) with one leg on mass-shell is related to the BS amplitude Φ_{BS}. Schematically (G_1 and G_2: constituent propagators)

$$\Phi_{BS} = G_1 \otimes G_2 \otimes f_3(\tilde{s})$$

Milestones

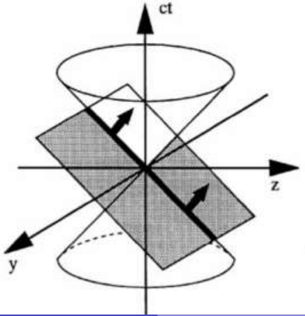
★ The BSE for the celebrated Wick-Cutkosky model (1954), i.e. two massive scalars interacting through a massless scalar can be **exactly solved** by using an integral representation like the one introduced by Nakanishi.

★ ★ The generalization to massive exchange was validated numerically by

- Kusaka et al, PRD **56** (1997) by exploiting the *uniqueness* of the NIR weight-function for a two scalar system;
- Carbonell and Karmanov [EPJA **27**, 1 (2006)] that properly integrated both sides of the BSE exploiting LF variables and without using *uniqueness*. They evaluated a fermionic system, as well [EPJA **46** 387 (2010)];
- Frederico, Viviani and G.S., that extended the NIR+LF formalism to the scattering-state BSE [FSV PRD **85**, 036009 (2012)]and successfully cross-checked the two-scalar results, *with and without uniqueness*, using integration LF variables in a different context [FSV PRD **89**, 016010 (2014)]. The scattering BSE in the zero-energy limit was also calculated [FSV EPJC 75, 398 (2015)]. A difficulty with spin dof was clarified and fixed, so that the fermionic case was fully explored [dFSV PRD 94, 071901(R) (2016); EPJ C 77, 764 (2017)].

Projecting BSE onto the LF hyper-plane $x^+ = 0$

- NIR contains the *needed freedom* for exploring non perturbative problems, once the Nakanishi weight functions are taken as unknown REAL quantities.
- Even adopting NIR, BSE still remains a highly singular integral equation in the 4D Minkowski momentum space. BUT exploiting an expression á la Nakanishi for the BS amplitude, then its analytic structure is displayed in full, allowing formal manipulations
- Noteworthy, in the LF framework one recovers a probabilistic interpretation by expanding the BS amplitude on a Fock basis, and then singling out the *valence component*. Hence the probability of finding two constituents in the fully interacting state can be evaluated.



The valence component is formally obtained by integrating on $k^- = k^0 - k^3$ the BS amplitude. This mathematical step is equivalent to restrict the *LF-time* x^+ to the null plane, i.e. putting $x^+ = 0$ in Φ_{BS}

For illustrative purpose, a fermion-scalar system, interacting through the exchange of a scalar, can be expanded on a Fock basis (see e.g. Brodsky-Pauli-Pinsky (PR 301, 299 (1998)).

$$|\tilde{P}; M, JJ_z; \pi; \rangle = 2(2\pi)^3 \sum_{n\geq 2} \sum_{\{\sigma_i\}_{n_F}} \int [d\xi_i] \int [d\boldsymbol{\kappa}_{i\perp}]$$

$$\times\ \psi_n^{J\pi}(\{\xi_i p^+\}_n; \{\kappa_{i\perp}\}_n; \{\sigma_i\}_{n_F}; J_z)\ |\{\tilde{q}_i\}_{n-n_F}; \{\tilde{k}_i\}_{n_F}; \{\sigma_i\}_{n_F}\rangle$$

where the integration symbols mean

$$\int [d\xi_i] \equiv \prod_{i=1}^{n} \int \frac{d\xi_i}{2\ (2\pi)\xi_i}\ \delta\left(1 - \sum_{j=1}^{n} \xi_j\right) \quad , \quad \int [d\boldsymbol{\kappa}_{i\perp}] \equiv \prod_{i=1}^{n} \int \frac{d\boldsymbol{\kappa}_{i\perp}}{(2\pi)^2}\ \delta^2\left(\sum_{j=1}^{n} \boldsymbol{\kappa}_{j\perp}\right)$$

and the generic Fock state is given by

$$|\{\tilde{q}_i\}_{n_s}; \{\tilde{k}_i\}_{n_F}; \{\sigma_i\}_{n_F}\rangle = (2\pi)^{3(n_s+n_F)/2}\ \frac{1}{\sqrt{n_s!}}\ \frac{1}{\sqrt{n_F!}}\ \sqrt{2\tilde{q}_1}\ldots\ \ldots\sqrt{2\tilde{q}_{n_s}}$$

$$\times\ \sqrt{2\tilde{k}_1}\ldots\ \ldots\sqrt{2\tilde{k}_{n_F}}\, a^\dagger(\tilde{q}_1)\ldots\ \ldots a^\dagger(\tilde{q}_{n_s})\ b^\dagger(\tilde{k}_1, \sigma_1)\ldots\ \ldots b^\dagger(\tilde{k}_{n_F}, \sigma_{n_F})\ |0\rangle$$

In the Fock expansion,
the amplitudes $\psi_n^{J\pi}(\dots \quad \dots)$ are called LF wave functions
The one with the lowest number of constituents, i.e. $n = 2$ in the present example, is the *valence wave function*.
If the interacting-system state is normalized then the LF wave functions are normalized, viz

$$2(2\pi)^3 \sum_{n\geq 2} \sum_{\{\sigma_i\}_{n_F}} \int \left[d\xi_i\right] \left[d^2\boldsymbol{\kappa}_{i\perp}\right] \left|\psi_n^{J\pi}(\{\xi_i p^+\}_n; \{\kappa_{i\perp}\}_n; \{\sigma_i\}_{n_F}; J_z)\right|^2 = 1$$

It turns out that the bridge between valence wave function and BS amplitude is

$$p^+\psi_{n=2}^{J\pi}(q_1^+/p^+; \mathbf{q}_{1\perp}; \sigma_1; J_z) = \frac{q_2^+}{2} \int \frac{dk^-}{2\pi} \bar{u}_\alpha(\tilde{q}_1, \sigma_1)\gamma^+_{\alpha\beta}\Phi_\beta(k, p; J^\pi, J_z)$$

★ the presence of $\gamma^+ = \gamma^0 + \gamma^3$ is dictated by the features of the QFT onto the light-cone (Yan et al PRD 7, 1780 (1973))
★★ the *macroscopic* structure of the BS amplitude for a fermion-scalar system is

$$\Phi(k, p) = \left[\mathbb{I}\ \phi_1(k, p) + \frac{\not{k}}{M}\ \phi_2(k, p)\right]\ U(p, s)$$

$$(\not{p} - M)\ U(p, s) = 0 \quad , \quad k = \frac{q_1 - q_2}{2} \quad , \quad p = q_1 + q_2$$

$\phi_i(k, p) \equiv$ scalar functions, that depend upon the independent scalar products,and have to be determined through the BSE.

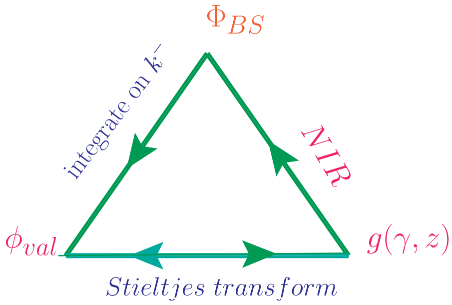
A regular integral equation equivalent to BSE

$$2-\text{boson Valence w.f.} = \psi_{n=2}(\xi, k_\perp) = \frac{p^+}{\sqrt{2}}\,\xi\,(1-\xi)\int \frac{dk^-}{2\pi}\overbrace{\Phi_b(k,p)}^{\text{BS Amplitude}} =$$

$$= \frac{1}{\sqrt{2}}\xi\,(1-\xi)\underbrace{\int_0^\infty d\gamma'\,\frac{g_b(\gamma', 1-2\xi;\kappa^2)}{[\gamma' + k_\perp^2 + \kappa^2 + (2\xi-1)^2\,\frac{M^2}{4} - i\epsilon]^2}}_{\text{NIR}}$$

with $\kappa^2 = 4m^2 - M^2$ and $M = 2m - B$.($B \equiv$ binding energy)

The step for recovering the probabilistic interpretation strongly suggests to apply the projection on both sides of BSE. This can be actually done by introducing NIR !!



N.B. The valence w.f. $\psi_{n=2}$ is a generalized Stieltjes transform (invertible) of the Nakanishi weight funct. g_b (Carbonell, Frederico, Karmanov PLB 769 (2017), 418). This observation enforces the idea that NIR can be a general trial function for solving BSE, given the very general hypotheses on the existence of a Stieltjes transform.

LF projection of the homogeneous BSE and the NIR

$$(4D) \qquad \Phi(k,p) = G_0(k,p) \int d^4k' \, \mathcal{K}_{BS}(k,k',p) \, \Phi(k',p)$$

$$\underrightarrow{NIR+LF}$$

$$\text{valence w.f.} \;\propto\; \int_0^\infty d\gamma' \frac{g_b(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1 - z^2)\kappa^2 - i\epsilon]^2} =$$

$$= \alpha \int_0^\infty d\gamma' \int_{-1}^1 dz' \; V_b^{LF}(\alpha; \gamma, z; \gamma', z') \;\; g_b(\gamma', z'; \kappa^2).$$

with $V_b^{LF}(\alpha; \gamma, z; \gamma', z')$ determined by the irreducible kernel $\mathcal{I}(k, k', p)$ (!) and α is the coupling constant ($\equiv g^2/16\pi$ for the scalar case).

In turn, by adopting an orthonormal basis (Laguerre $\times$ Gegenbauer) for expanding $g_b(\gamma, z; \kappa^2 = 4m^2 - M^2)$, the integral equation becomes a generalized eigen-equation, with eigenvalue α , and the eigenvector composed by the coefficients of the expansion.

★ If the eigen-equation admits a solution, for a given mass M of the system, then we know how to reconstruct the whole BS amplitude

The investigation of i) two scalars ii) two fermions and iii) fermion + scalar, has been carried out in ladder approximation, by assuming massive exchanges.
i) two scalars ii) two fermions and iii) fermion + scalar

$$i\mathcal{K}_S^{(Ld)}(k,k') = -ig^2 \ \frac{1}{(k-k')^2 - \mu^2 + i\epsilon},$$

i) two fermions (γ_5 vertexes)

$$i\mathcal{K}_{PS}^{(Ld)}(k,k') = ig^2 \ \frac{1}{(k-k')^2 - \mu^2 + i\epsilon},$$

ii) two fermions and iii) fermion + scalar (in Feynman gauge)

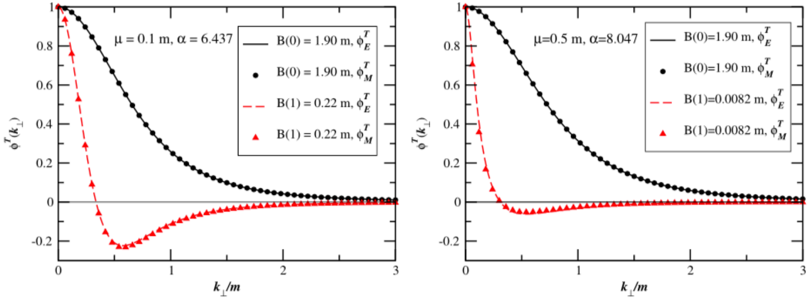
$$i\mathcal{K}_V^{(Ld)\mu\nu}(k,k') = -ig^2 \ \frac{g^{\mu\nu}}{(k-k')^2 - \mu^2 + i\epsilon}.$$

For the two-scalar system:

Ladder approx. by Carbonell and Karmanov within the explicitly-covariant LF framework (EPJA 27 (2006) 1; also the cross-ladder kernel in EPJA 27 (2006) 11), and by Frederico, Viviani & GS(PRD 89 (2014) 016010), in the non explicitly-covariant version.

Very good agreement for both eigenvalues (the coupling constants at given binding energies) and LF distributions, namely $|\psi_{val}(x, |\mathbf{k}_\perp|)|^2$ (with $x =$ the Bjorken variable).

Also: (i) Scattering lengths in FVS EPJC 75 (2015) 398, (ii) spectra of excited states and LF momentum distributions in Gutierrez et al PLB 759 (2016) 131.



Transverse amplitudes, in Minkowski space and in the Wick-rotated one, obtained from the BS amplitude by integration on the remaining variables ($\{k^+, k^-\}$ or $\{ik^0, k^3\}$). Ground and first excited states.

Spin dof and BSE

Adding spin dof is a challenge, both on formal and numerical sides.
While projecting onto the null plane, one faces with integrals that could become singular for some values of an external variable.

Fortunately, the prototype of such singular integrals was studied by Yan (PRD 7 (1973) 1780) in the context of the field theory in the Infinite Momentum frame viz

$$\mathcal{I}(\beta, y) = \int_{-\infty}^{\infty} \frac{dx}{\left[\beta x - y \mp i\epsilon\right]^2} = \pm \frac{2\pi i \ \delta(\beta)}{\left[-y \mp i\epsilon\right]}$$

★ In the fermionic BSE case, one can rigorously evaluate the singular integrals by applying the Yan result and some simple extensions, leading to derivative of the delta-functions. This is not an issue since we use an orthonormal basis (infinitely derivable...) for expanding the Nakanishi weight functions.
★★ In the fermion-scalar case, the singular behavior is avoided, due to the presence of bigger power in the denominator, carried by the scalar propagator.

BSE for fermions

$$\Phi_{BS}(k,p) = S(p/2+k) \int d^4k' \; F^2(k-k') i\mathcal{K}(k,k')\Gamma_1 \; \Phi_{BS}(k',p) \; \bar{\Gamma}_2 \; S(k-p/2)$$

$$S(q) = i\frac{\not{q}+m}{q^2-m^2+i\epsilon} \quad , \qquad F(k-k') = \frac{(\mu^2-\Lambda^2)}{[(k-k')^2-\Lambda^2+i\epsilon]}$$

$$\Gamma_1 = \Gamma_2 = 1 \; (scalar), \; \gamma_5 \; (pseudo), \; \gamma^\mu \; (vector)$$

For a 0^+ state, one can decompose Φ_{BS} $\Rightarrow$

$$\Phi_{BS}(k,p) = S_1 \; \phi_1(k,p) + S_2 \; \phi_2(k,p) + S_3 \; \phi_3(k,p) + S_4 \; \phi_4(k,p)$$

$\phi_i \equiv$ unknown scalar functions, with well-defined symmetry under the exchange $1 \to 2$, from the symmetry of both $\Phi(k,p)$ and S_i.

NIR applied to ϕ_i !!

$Tr\{S_i \; S_j\} = \mathcal{N}_i \; \delta_{ij}$ with

$$S_1 = \gamma_5 \quad , \quad S_2 = \frac{\not{p}}{M}\;\gamma_5 \quad , \quad S_3 = \frac{k\cdot p}{M^3}\not{p}\;\gamma_5 - \frac{1}{M}\not{k}\gamma_5 \quad , S_4 = \frac{i}{M^2}\sigma^{\mu\nu}p_\mu k_\nu \; \gamma_5$$

LF projection $\Rightarrow$ integral-equation system

★ For each ϕ_i, use NIR and apply LF projection

$$\psi_i(\gamma, z) = \int \frac{dk^-}{2\pi}\ \phi_i(k, p) = -\frac{i}{M}\int_0^\infty d\gamma' \frac{g_i(\gamma', z; \kappa^2)}{[\gamma + \gamma' + m^2 z^2 + (1 - z^2)\kappa^2 - i\epsilon]^2}$$

- $\gamma \equiv |\mathbf{k}_\perp|^2 \in [0, \infty]$
- $z \equiv\in [-1, 1]$
- $\kappa^2 = 4m^2 - M^2$ with $M = 2m - B$.($B \equiv$ binding energy).

★ ★ Hence, BSE formally reduces to a system of 4 coupled integral equations for a 0^+ state

$$\psi_i(\gamma, z) = g^2 \sum_j \int_{-1}^{1} dz' \int_0^\infty d\gamma'\ g_j(\gamma', z'; \kappa^2)\ \mathcal{L}_{ij}(\gamma, z, \gamma', z'; p)$$

Again, if the coupled system admits solutions, then we know how to reconstruct the BS amplitude !.

For the fermionic case, besides the scalar exchange other two exchanges have been investigated (still in ladder approx.):

a massive pseudoscalar and massless/massive vectors

Numerical comparison: Scalar coupling

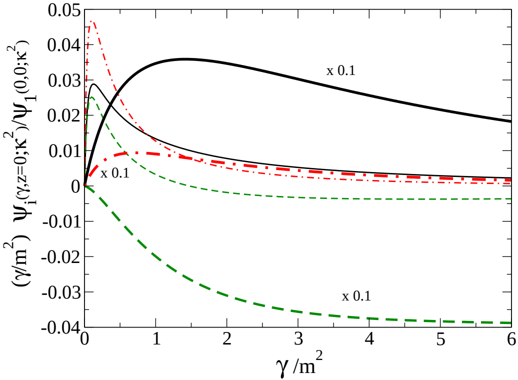
	$\mu/m = 0.15$			$\mu/m = 0.50$		
B/m	g^2_{dFSV}(full)	g^2_{CK}		g^2_{dFSV}(full)	g^2_{CK}	g^2_E
0.01	7.844	7.813		25.327	25.23	-
0.02	10.040	10.05		29.487	29.49	-
0.04	13.675	13.69		36.183	36.19	36.19
0.05	15.336	15.35		39.178	39.19	39.18
0.10	23.122	23.12		52.817	52.82	-
0.20	38.324	38.32		78.259	78.25	-
0.40	71.060	71.07		130.177	130.7	130.3
0.50	88.964	86.95		157.419	157.4	157.5
1.00	187.855	-		295.61	-	-
1.40	254.483	-		379.48	-	-
1.80	288.31	-		421.05	-	-

First column: binding energy per unit mass.
Red digits: coupling constant g^2 ,for two values of the exchanged-boson mass $\mu/m = 0.15$ and 0.50, and using exact formula, á la Yan, for the fermionic singularities.
Black digits: results from Carbonell & Karmanov [EPJA **46**, (2010) 387)].
Blue digits: results in Euclidean space from Dorkin et al FBS. **42** (2008) 1.

Vector coupling and high-momentum tails: $\gamma \equiv |\mathbf{k}_\perp|^2$

The LF amplitudes ψ_i, components of the valence momentum distributions have the correct tail $\big(!\big)$, for the massless-vector coupling.

Power one is expected for the pion valence amplitude from dimensional arguments by X. Ji et al, PRL 90 (2003) 241601 (cf also Brodsky & Farrar (PRL 31 (1973) 1153) for the counting rules of exclusive amplitudes)



$\psi_i \times \gamma/m^2$ at fixed $z = 0$ ($\xi = 1/2$), for the massless-vector coupling.

$B/m = 0.1$ (thin lines) and 1.0 (thick lines).

—— : $(\gamma/m^2)\ \psi_1$.
— —: $(\gamma/m^2)\ \psi_2$.
— • : $(\gamma/m^2)\ \psi_4$.
$\psi_3 = 0$ for $z = 0$ (odd function)

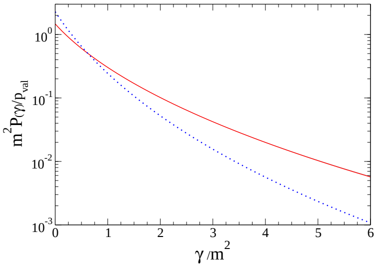
Valence LF distributions: a mock pion

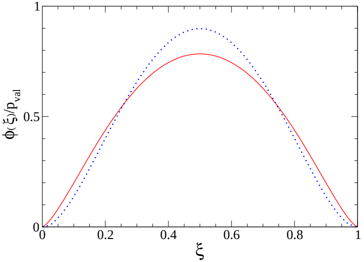
A fermion-antifermion 0^- system, bound through a massive-vector exchange, is a by product of the fermion-fermion calculations.

For illustrative purpose (no fine tuning...), in the Feynman gauge, by using the ladder approximation and two relevant parameters (quark mass and gluon mass) inspired by LQCD, we have evaluated :

the two *valence* momentum distributions depending upon i) the transverse-momentum $\gamma = [k_T]^2$ and ii) the Bjorken variable ξ.

Quark mass: $m_q = 187$ MeV





Solid line: $m_g = 280$ MeV, $P_{val} = 0.78$, $f_\pi = 99$ MeV ($f_\pi^{exp} = 92.2$ MeV) Dotted line: $m_g = $ 28 MeV, $P_{val} = 0.64$, $f_\pi = 77$ MeV. Vertex form-factor parameter, $\Lambda/m_q = 2$

Valence LF distributions (preliminary): fermion+scalar

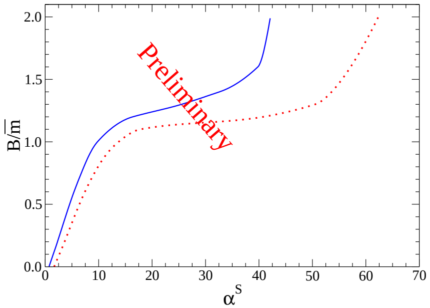
The BS amplitude for the fermion-scalar system contains two unknown scalar functions ϕ_i

$$\Phi_{BS}(k,p) = \Big[O_1(k)\ \phi_1(k,p) + O_2(k)\ \phi_2(k,p)\Big]\ U(p,s)$$

with

$$O_1(k) = \mathbb{I}\ , \qquad O_2(k) = \frac{\not{k}}{M}\ , \qquad (\not{p} - M)\ U(p,s) = 0\ .$$

Preliminary results for the scalar exchange

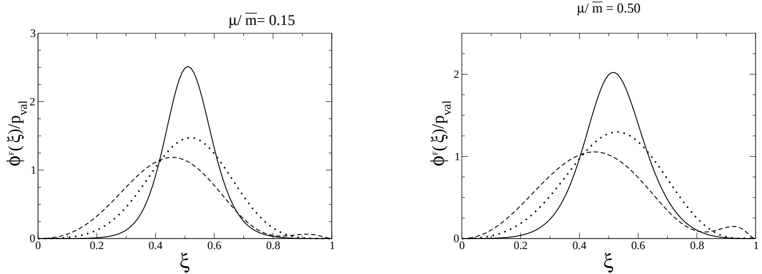


Blue solid line: light exchanged mass, $\mu/\bar{m} = 0.15$.

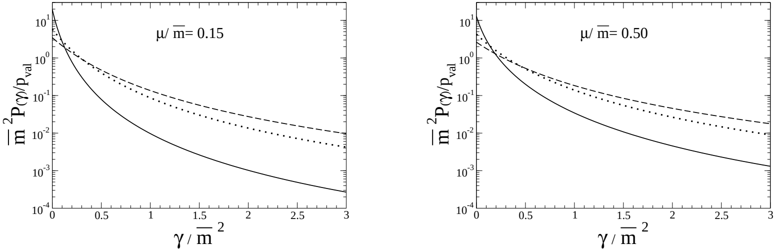
Red dotted line: heavy exchanged mass, $\mu/\bar{m} = 0.5$.

★. The attraction is softened for increasing binding energy, due to the competition between the large and small components in $\bar{u}\ u$, at the fermionic vertex. Recall that the small components are driven by the kinetic energy.

Fermion-scalar system interacting through a massive scalar exchange



Longitudinal light-cone distribution for a fermion in the valence component. Solid line : $B/\bar{m} = 0.1$. Dotted line: $B/\bar{m} = 0.5$. Dotted line: $B/\bar{m} = 1.0$



Transverse light-cone distribution for a fermion in the valence component.

Conclusions & Perspectives

- A systematization of the technique for solving BSE with and without spin dof has been reached, and the cross-check among results obtained by different groups, for different interacting systems (with kernels in ladder and cross-ladder contributions) has produced a clear numerical evidence of the validity of NIR for obtaining actual solutions. $\Rightarrow$ more refined phenomenological models...
- A general comment to be reminded: the LF framework has well-known advantages in performing analytical integrations, and in the investigation of the fermionic case its effectiveness has been shown in its full glory.
- The achieved numerical validation of NIR strongly encourages to proceed by including needed improvements, i.e. self-energies and vertex corrections evaluated within the same framework (work in progress on the gap Equation by C. Mezrag)
- An interesting possibility for the tomography of the nucleon: Fragmentation functions?