Quantum kinetic theory in curved spacetime

Annamaria Chiarini, Masoud Shokri,
& Dirk Rischke

Goethe university

Outline of the talk

Motivation and Background

Background

- Ideal relativistic hydrodynamics

$$\partial_\mu T^{\mu\nu} = 0 \qquad T^{\mu\nu} \equiv T^{\mu\nu}_{\text{id}} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} \tag{1}$$

- Including dissipative effects

$$\partial_\mu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = T^{\mu\nu}_{\text{id}} + \pi^{\mu\nu} \tag{2}$$

$\pi^{\mu\nu}$ has to be built, e.g. $1^{\text{st}}, 2^{\text{nd}}$ order gradients of $\{\varepsilon, p, u^\mu, g^{\mu\nu}\}$

- **1st order** $\Rightarrow$ Relativistic Navier-Stokes (NS) equations

$$\pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>} + \zeta\Delta^{\mu\nu}\nabla\cdot u \qquad \textbf{Causality problem} \tag{3}$$

- **2nd order** viscous stresses relax to NS on a finite relaxation time

$$\tau_\pi \pi^{\mu\nu} + \pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>} + \zeta\Delta^{\mu\nu}\nabla\cdot u \tag{4}$$

Background

BRSSS Hydrodynamics

[Baier, Romatschke, Son, Starinets, Stephanov, (2008)]

- all the possible structures for $\pi_{\mu\nu}$ up to second orders in gradients

$$\pi_{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_\pi \left[{}^{<}D\pi^{\mu\nu>} + \frac{d}{d-1}\pi^{\mu\nu}\nabla_\lambda u^\lambda \right] + \kappa \big[R^{\langle\mu\nu\rangle} - 2u_\lambda u_\rho R^{\lambda\langle\mu\nu\rangle\rho} \big] + \frac{\lambda_1}{\eta^2}\pi_\lambda^{<\mu}\pi^{\nu>\lambda} - \frac{\lambda_2}{\eta}\pi_\lambda^{<\mu}\Omega^{\nu>\lambda} + \lambda_3 \Omega_\lambda^{<\mu}\Omega^{\nu>\lambda}$$

- terms build up with symmetry considerations

Background

BRSSS Hydrodynamics

[Baier, Romatschke, Son, Starinets, Stephanov, (2008)]

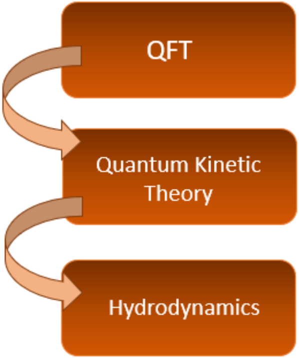
- all the possible constituitive relations for $\pi_{\mu\nu}$ up to second orders in gradients

$$\pi_{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_\pi \left[{}^{<}D\pi^{\mu\nu>} + \frac{d}{d-1}\pi^{\mu\nu}\nabla_\lambda u^\lambda \right] + \kappa \Big[R^{\langle\mu\nu\rangle} - 2u_\lambda u_\rho R^{\lambda\langle\mu\nu\rangle\rho} \Big] + \frac{\lambda_1}{\eta^2}\pi_\lambda^{<\mu}\pi^{\nu>\lambda} - \frac{\lambda_2}{\eta}\pi_\lambda^{<\mu}\Omega^{\nu>\lambda} + \lambda_3\Omega_\lambda^{<\mu}\Omega^{\nu>\lambda}$$

* gravity terms appear in second order and are non trivial in equilibrium
* they might not satisfy the equation of motion

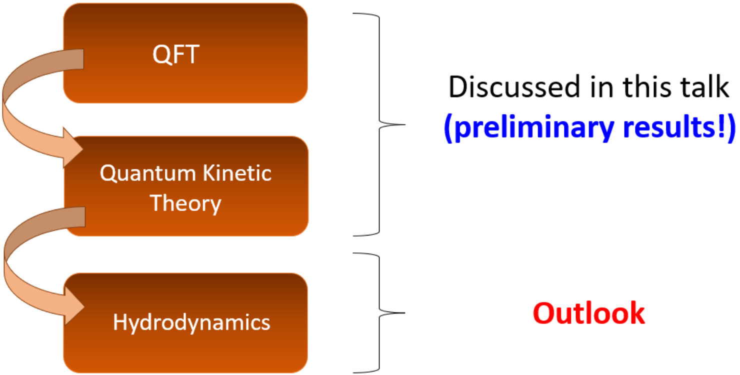
How can we find them in a rigorous way?

How can we find them in a rigorous way?
[cfr D. Wagner's talk on 20/09 h 9.30]



How can we find them in a rigorous way?

[cfr D. Wagner's talk on 20/09 h 9.30]

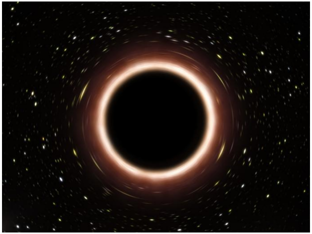


Motivation

- **relativistic hydrodynamics** description of fluid (with **spin**) able to explain spin-dependent observables
  - * **kinetic theory** as the underlying microscopic theory
  - * introduction of spin and quantum effects by yielding a quantum transport theory from **quantum field theory**

- achievement of **covariant formalism** to investigate transport under the effect of **gravity**

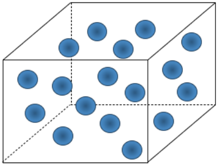
⇒ Possible **Astrophysical** and **cosmological** application
  - * neutron stars
  - * black holes



Quantum kinetic theory

Kinetic Theory

Goal of kinetic theory: replacing the field equations by a transport equation for a distribution function



Boltzmann equation

$$p_\mu \partial^\mu f = \mathcal{C}[f] \tag{5}$$

f connects the microscopic dynamics to macroscopic quantities

$$N^\mu = \int \frac{d^3p}{p^0} p^\mu f \tag{6}$$

$$T^{\mu\nu} = \int \frac{d^3p}{p^0} p^\mu p^\nu f \tag{7}$$

Quantum kinetic theory

- Given the Wigner operator as defined in terms of quantum fields Φ

$$\hat{W}(x,p) \equiv \kappa \int d^4v e^{-ip\cdot v/\hbar}\Phi(x-v/2)\otimes\Phi^\dagger(x+v/2)\ , \qquad (8)$$

  the **Wigner function**, as the closest quantum analogue of the classical distribution function is obtained by taking $\langle : \hat{W} : \rangle$

- Using the equation of motion for the fields, one can find the so-called **transport equation** for the Wigner operator.

QKT in flat spacetime (brief review)

- The Klein-Gordon Lagrangian for interacting complex scalar fields $\phi(x)$ reads

$$\mathcal{L} = -\hbar^2 \left[\eta^{\mu\nu} \partial_\mu \phi^\star(x) \partial_\nu \phi(x) + \frac{m^2}{\hbar^2} \phi^\star(x)\phi(x) \right] + \mathcal{L}_{\text{int}} \ ,$$

- Varying the action $S = \int d^4x \mathcal{L}$ with respect to ϕ and $\phi^\star$ then gives rise to the equation of motion,

$$\left(\Box - \frac{m^2}{\hbar^2} \right) \phi(x) = -\frac{1}{\hbar} \frac{\partial \mathcal{L}_{int}}{\partial \phi^\star} \equiv \hat{\rho} \ .$$

- derivative operator on the Wigner function

$$\partial^\mu \hat{W} = \frac{2i}{\hbar} p^\mu \hat{W} + 2\kappa \int d^4v \, e^{-ip\cdot v/\hbar} \phi^\star(x + v/2) \partial^\mu \phi(x - v/2) \ .$$

QKT in flat spacetime (brief review)

- defined the so-called Bopp operator as

$$D^\mu := \frac{i\hbar}{2}\partial^\mu + p^\mu \ .$$

- in terms of which deriving the dynamics for the Wigner function

$$(D^2 + m^2)\hat{W}(x,p) \equiv \left(p^2 + m^2 + i\hbar p_\mu \partial^\mu - \frac{\hbar^2}{4}\Box\right)\hat{W}(x,p)$$
$$= -\kappa \int \mathrm{d}^4 v \ e^{-ip\cdot v/\hbar}\phi^\star(x+v/2)\hat{\rho}(x-v/2) \ ,$$

  * **mass-shell condition** for the Wigner function

$$\left(p^2 + m^2 - \frac{\hbar^2}{4}\Box\right)W(x,p) = \hbar\mathcal{A}(x,p) \ ,$$

  * **transport equation** for W,

$$p^\mu \partial_\mu W(x,p) = \mathcal{C}(x,p) \ .$$

Curved background

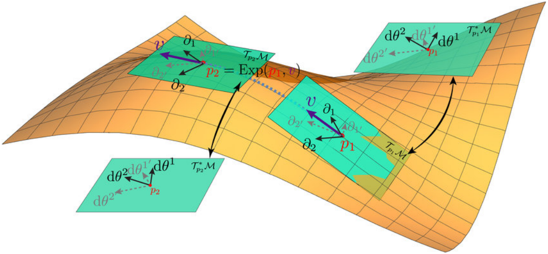
Generalization to a curved geometry

Spacetime as a curved manifold $\mathcal{M}$. Given x^μ as label of the particle position in $\mathcal{M}$, we can define

- v^μ as the particle position in the tangent space $\mathbb{T}_x\mathcal{M}$
- p_μ as the particle position in the cotangent space $\mathbb{T}^\star_x\mathcal{M}$

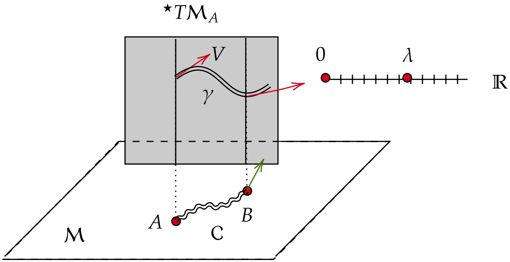
⇒ **vectors** live in tangent space

⇒ **covectors** live in cotangent space



Tangent and cotangent spaces are local and defined for every point in $\mathcal{M}$.

- The ensemble of all tangent spaces $\mathbb{T}_x\mathcal{M}$ is the tangent bundle $\mathbb{T}\mathcal{M}$
- The ensemble of all cotangent spaces $\mathbb{T}^\star_x\mathcal{M}$ is the cotangent bundle $\mathbb{T}^\star\mathcal{M}$
  - $\Rightarrow$ **Phase space** as the cotangent bundle, including (x, p)



Mathematical tools

$\longrightarrow$ We need **covariance** and **gauge invariance** of the derivative operator

- Lifting of the covariant derivative ∇_μ to the *tangent bundle*

$$\mathcal{D}_\mu = \nabla_\mu - \Gamma^\lambda_{\mu\nu} v^\nu \frac{\partial}{\partial v^\lambda} \tag{9}$$

- Lifting of the covariant derivative ∇_μ to the *cotangent bundle*

$$\tilde{\mathcal{D}}_\mu = \nabla_\mu + \Gamma^\lambda_{\mu\nu} p_\lambda \frac{\partial}{\partial p_\nu} \tag{10}$$

- The collisionless Boltzmann equation for classical distribution function f in curved spacetime is

$$p_\alpha \tilde{\mathcal{D}}^\alpha f = 0$$

Wigner formalism in curved spacetime

- Fields are extended to the tangent bundle as [Fornarev (1993)]

$$\Phi(x,v) \equiv \left(1 + v^\alpha \nabla_\alpha + \frac{1}{2} v^\alpha v^\beta \nabla_\alpha \nabla_\beta + \cdots \right) \Phi(x) \,, \qquad (11)$$

- Since v is parallel transported, fields are horizontally lifted to $\mathbb{T}\mathcal{M}$

$$\Phi(x+v) \;\longrightarrow\; \Phi(x,v) \equiv e^{v\cdot\mathcal{D}} \phi(x) \qquad (12)$$

- The Wigner function can then be rewritten as

$$W(x,p) = \kappa \int_{\mathbb{T}_x \mathrm{M}} d^4v \sqrt{-g(x)} e^{-ip\cdot v/\hbar} \langle \Phi^\star \left(x, \tfrac{v}{2}\right) \Phi\left(x, -\tfrac{v}{2}\right) \rangle, \qquad (13)$$

which correctly expresses the macroscopic current as

$$J_\mu = \int_{\mathbb{T}_x \mathrm{M}^\star} \frac{d^4p}{(2\pi\hbar)^4 \sqrt{-g(x)}} p_\mu W(x,p). \qquad (14)$$

QFT in curved spacetime

The action for fields coupled with gravity is

$$S = S_{\text{HE}} + S_{\text{B}} + S_{\text{m}}$$

- S_{HE} is the **Hilbert-Einstein action**

$$S_{\text{HE}} = \frac{1}{16\pi G} \int_V d^4x \sqrt{-g} R$$

- S_{m} is the **matter action**

$$S_{\text{m}} = \int_V d^4x \sqrt{-g} \left[-\hbar g^{\mu\nu} \nabla_\mu \phi^\star \nabla_\nu \phi - \frac{m^2}{\hbar} \phi^\star \phi - \underbrace{\hbar \xi R \phi^\star \phi}_{\text{non-minimal}} + \mathcal{L}_{\text{int}} \right]$$

- S_{B} is the **modified Gibbons-Hawking term**

QFT in curved spacetime

The action for fields coupled with gravity is

$$S = S_{\mathrm{HE}} + \overbrace{S_{\mathrm{B}}}^{\text{annihilates the surface term}} + S_{\mathrm{m}}$$

- S_{HE} is the **Hilbert-Einstein action**
- S_{m} is the **matter action**
- S_{B} is the **modified Gibbons-Hawking term**

$$S_{\mathrm{B}} = \frac{1}{8\pi G} \oint_{\partial V} \mathrm{d}\Sigma \, f(\phi) K \quad \text{where} \quad f(\phi) = \mathbf{1} - 16\pi G\hbar\xi\phi^{\star}\phi$$

- K is the trace of the extrinsic curvature on ∂V.

- The action principle $\delta S = 0 \implies$

$$f(\phi)G_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) f(\phi) - 8\pi G T^{\phi}_{\mu\nu} = 0$$

- with respect to metric $\implies$ Einstein's equation,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- Energy-momentum tensor:

$$T_{\mu\nu} \equiv \frac{1}{f(\phi)}\left[T^{\phi}_{\mu\nu} - \frac{1}{8\pi G}\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) f(\phi)\right]$$

- with respect to $\phi, \phi^\star \implies$ field's equation,

$$\left(\Box - \frac{m^2}{\hbar^2} - \xi R\right)\phi(x) = \hat{\rho}(x)\ , \quad \text{where} \quad \hat{\rho} = -\frac{1}{\hbar}\frac{\partial \mathcal{L}_{\text{int}}}{\partial \phi^\star}$$

Applications and results

Dynamics of Wigner function for scalars

The **exact general relativistic quantum transport equation** is

$$\left[\left(\frac{i\hbar}{2}\mathcal{D}_\mu + p_\mu\right)^2 - m^2 - \xi R\right] W = 2\hbar^2\kappa \int_v \Big[\phi_+^\star \mathcal{Z}\phi_- + \phi_- \mathcal{U}\phi_+^\star$$

$$= +4g^{\mu\nu}\big(\phi_+^\star \partial_\mu^v \mathcal{G}_\nu \phi_- - \phi_- \partial_\nu^v \mathcal{G}_\mu \phi_+^\star + \mathcal{G}_\mu \phi_+^\star \mathcal{G}_\nu \phi_-\big)\Big]$$

$$\mathcal{G}_\mu \equiv -\frac{iv^\nu}{2\hbar}\sum_{n=0}^{\infty}\frac{\mathcal{C}[v\cdot\mathcal{D}]^n}{(n+2)!}G_{\mu\nu}, \qquad G_{\mu\nu} = -i\hbar R^\sigma_{\rho\mu\nu}v^\rho_\sigma \qquad \mathcal{C}[\mathcal{X}](\mathcal{Y}) \equiv [\mathcal{X},\mathcal{Y}]$$

$$\mathcal{Z}(x,v) = 2\sum_{n=1}^{\infty}\frac{\mathcal{C}[v\cdot\mathcal{D}]^n}{(n+2)!}\left(\mathcal{D}^2 + R^\nu_\mu v^\mu \partial^v_\nu + g^{\mu\nu}v^\lambda v^\rho R^\sigma_{\lambda\rho(\mu}\partial^v_{\nu)}\partial^v_\sigma\right).$$

$$\mathcal{U}(x,v) = v^\lambda v^\rho \sum_{n=0}^{\infty}\frac{\mathcal{C}[v\cdot\mathcal{D}]^n}{(n+1)!}\left[\nabla_\rho R^\sigma_\lambda - \nabla^\sigma R_{\lambda\rho} + 2g^{\mu\nu}R^\sigma_{\lambda\rho(\mu}\mathcal{D}_{\nu)}\right]\partial^v_\sigma$$

Dynamics of Wigner function for scalars

The **exact general relativistic quantum transport equation** is

$$\left[\left(\frac{i\hbar}{2}\tilde{\mathcal{D}}_\mu + p_\mu\right)^2 - m^2 - \xi R\right] W = 2\hbar^2\kappa \int_v \Big[\phi_+^\star \mathcal{Z}\phi_- + \phi_- \mathcal{U}\phi_+^\star$$
$$= +4g^{\mu\nu}\big(\phi_+^\star \partial_\mu^v \mathcal{G}_\nu \phi_- - \phi_- \partial_\nu^v \mathcal{G}_\mu \phi_+^\star + \mathcal{G}_\mu \phi_+^\star \mathcal{G}_\nu \phi_-\big)\Big]$$

where we can recognize

$$\mathfrak{D} := \left(\frac{i\hbar}{2}\tilde{\mathcal{D}}_\mu + p_\mu\right)$$ as the **generalized Bopp operator**

$$\partial_\mu \longrightarrow \tilde{\mathcal{D}}_\mu$$

What's next?

- $\hbar$ expansion of the Wigner function
- plug the expression in the transport equation up to 2^{nd} order in $\hbar$
- split into the **real** and **imaginary** part for
  - quantum corrected **mass-shell condition**
  - quantum corrected **Boltzmann equation**
- obtain the equilibrium solution up to $\mathcal{O}(\hbar^2)$
- find the high-order terms in $T^{\mu\nu}$ and J^{μ}

Dynamics of Wigner function for fermions

The **exact** **general relativistic quantum transport equation** is given by

[Liu, Mameda, Huang (2021)]

$$\left[\gamma^\mu\left(\frac{i\hbar}{2}\tilde{\mathcal{D}}_\mu + p_\mu\right)\right] W =$$

$$= i\hbar\gamma^\mu \int_v \mathrm{Tr}\langle\left(\mathcal{G}_\mu - \mathcal{H}_\mu\right)\Psi_-\bar{\Psi}_+ - \Psi_-\bar{\Psi}_+\overleftarrow{\mathcal{G}_\mu}\rangle\,.$$

where we have used

$$\mathcal{G}_\mu \equiv -\frac{iv^\nu}{2\hbar}\sum_{n=0}^{\infty}\frac{\mathcal{C}[v\cdot\mathcal{D}]^n}{(n+2)!}G_{\mu\nu}, \qquad \mathcal{H}_\mu \equiv -\frac{iv^\nu}{\hbar}\sum_{n=0}^{\infty}\frac{\mathcal{C}[v\cdot\mathcal{D}]^n}{(n+1)!}G_{\mu\nu}\,.$$

* $G_{\mu\nu} = F_{\mu\nu} + \hbar R_{\mu\nu\alpha\beta}\sigma^{\alpha\beta} - i\hbar R^\sigma_{\rho\mu\nu}v^\rho_\sigma$
* $\mathcal{C}[\mathcal{X}](\mathcal{Y}) \equiv [\mathcal{X},\mathcal{Y}]$

$\hbar$ expansion

- Power counting scheme

$$\begin{cases} v^\mu \sim i\hbar\partial_p^\mu \sim \mathcal{O}(\hbar) \\ p_\mu \sim \mathcal{O}(1) \end{cases}$$

- Clifford algebra decomposition of the Wigner function

$$W = \frac{1}{4}\left[\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right] \tag{15}$$

where the elements are not independent (2/4 independent components of $\mathcal{V}_\mu, \mathcal{A}_\mu$ for massless fermions, 4 for massive ones).

$\hbar$ expansion

- One can then express respectively the vectorial and axial currents

$$J^\mu = \int_p \mathrm{Tr}\left[\gamma^\mu W\right] = \int_p \mathcal{V}_\mu, \qquad J_A^\mu = \int_p \mathrm{Tr}\left[\gamma^\mu \gamma^5 W\right] = \int_p \mathcal{A}_\mu$$

- as well as energy-momentum tensor and spin tensor **[Liu, Mameda, Huang (2021)]**

$$T^{\mu\nu} = \int_p \mathrm{Tr}\left[\frac{i\hbar}{2}\gamma^\mu \tilde{\mathcal{D}}^\nu \gamma^\mu W\right] = \int_p \mathcal{V}_\mu p^\nu,$$

$$S^{\lambda\mu\nu} = \int_p \mathrm{Tr}\left[\frac{\hbar}{4}\{\gamma^\lambda, \sigma^{\mu\nu} W\right] = -\frac{\hbar}{2}\int_p \epsilon^{\lambda\mu\nu\rho}\mathcal{A}_\rho$$

where $\int_p \equiv \int_{\mathbb{T}_x\mathcal{M}} \frac{d^4p}{(2\pi\hbar)^4\sqrt{-g}}$

Conclusions

To conclude

Summary and **Outlook**

- Promoted the Wigner formalism to a covariant fashion
  - $\Rightarrow$ Exact transport equation for scalars
  - $\Rightarrow$ Corrections due to gravitational effect up to $\hbar^2$

- Hydrodynamic description
  - $\Rightarrow$ Derivation of corrected transport coefficients
  - $\Rightarrow$ Including spin and deriving a complete theory for **Spin hydrodynamics in curved spacetime**

Thank you and Stay tuned!

Appendix

References

[1] O.A. Fonarev, *Wigner function and quantum kinetic theory in curved space-time and external fields,* Journal of Mathematical Physics 35 (1994) 2105

[2] E. Calzetta, S. Habib, B.L. Hu, *Quantum Kinetic Field Theory in Curved Space-time: Covariant Wigner Function and Liouville-Vlasov equation,* Phys. Rev. D 37 (1988) 2901

[3] Y.-C. Liu, L.-L. Gao, K. Mameda and X.-G. Huang, Chiral kinetic theory in curved spacetime, Phys. Rev. D 99 (2019) 085014 [1812.10127]

[4] Y.-C. Liu, K. Mameda and X.-G. Huang, *Covariant Spin Kinetic Theory I: Collisionless Limit,* Chin. Phys. C 44 (2020) 094101 [2002.03753].

Useful identities

$$\partial^v_\mu \phi_\pm = \pm\frac{1}{2}\mathcal{D}_\mu \phi_\pm \pm \mathcal{G}_\mu \phi_\pm \tag{16}$$

$$\mathcal{D}_\mu \phi_\pm = e^{\pm v/2\cdot\mathcal{D}}\,\nabla_\mu \phi(x) \pm \frac{1}{2}\mathcal{H}_\mu \phi_\pm \tag{17}$$

$$\partial^2_v \phi_\pm = -\frac{1}{4}\mathcal{D}^2\phi_\pm \pm \partial^v\cdot\mathcal{D}\phi_\pm + \mathcal{Z}\phi_\pm \tag{18}$$

$$\mathcal{D}^2\phi_\pm = e^{\pm v/2\cdot\mathcal{D}}\,\Box\phi(x) + \mathcal{U}\phi_\pm \tag{19}$$

Energy-momentum tensor

The non minimal coupling EMT obtained by variating $\mathcal{L}_{\mathrm{m}}$

$$T^{\phi}_{\mu\nu} \equiv -2\frac{\partial \mathcal{L}_{\phi}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\phi} \ . \tag{20}$$