On the local thermodynamic relations in relativistic spin hydrodynamics

Rajeev Singh

Stony Brook University
NISER

Ongoing work
with
Francesco Becattini

Spin and quantum features of QCD plasma

16—20 Sep 2024

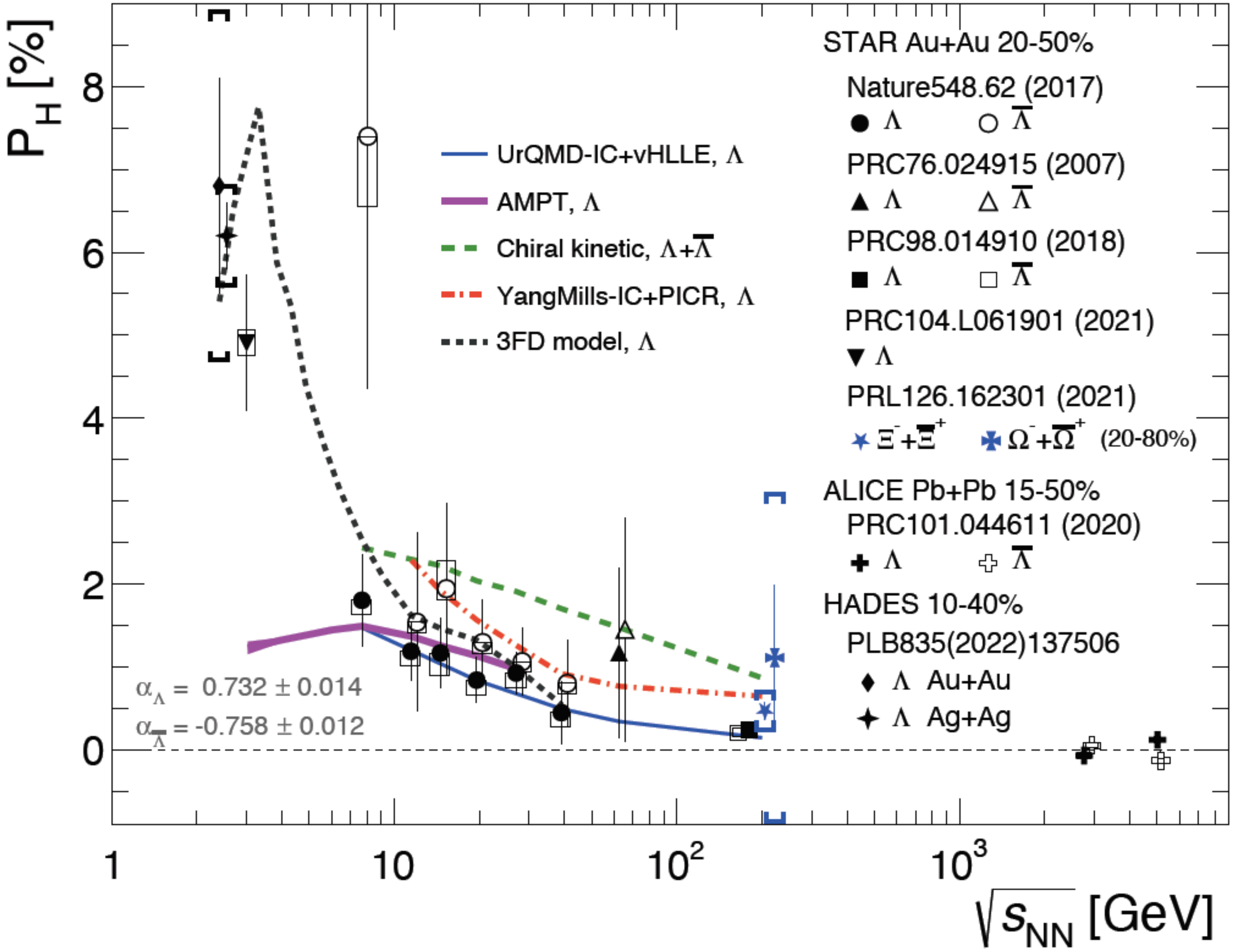
"Effetti quantistici nei fluidi relativistici"

Outline

- Motivation
- Entropy current using Quantum statistical method
- Compute thermodynamic potential current
- Local thermodynamic relations

Motivation

→ Due to the evidence of spin polarization, there has been huge interest in the formulation of rel. fluid dynamics with spin



Motivation

→ Due to the evidence of spin polarization, there has been huge interest in the formulation of rel. fluid dynamics with spin

→ For which, beside having energy-momentum tensor, it requires to have an addition of spin tensor, through

$$\hat{J}^{\lambda\mu\nu} = x^{\mu}\hat{T}^{\lambda\nu} - x^{\nu}\hat{T}^{\lambda\mu} + \hat{S}^{\lambda\mu\nu}$$

→ Conservation of total angular momentum gives

$$\partial_{\lambda}\hat{J}^{\lambda\mu\nu} = 0 \implies \partial_{\lambda}\hat{S}^{\lambda\mu\nu} = \hat{T}^{\nu\mu} - \hat{T}^{\mu\nu}$$

Motivation

QFT, Itzykson and Zuber (Saclay 1980)

→ But definitions of $\hat{T}^{\mu\nu}$ and $\hat{S}^{\lambda,\mu\nu}$ are not unique.

→ One obtains new pair of $\hat{T}^{\mu\nu}$ and $\hat{S}^{\lambda,\mu\nu}$ using $\hat{T}^{\mu\nu}_{\rm Can}$ and $\hat{S}^{\lambda,\mu\nu}_{\rm Can}$ through pseudo-gauge transformation.

Rept.Math.Phys. 9 (1976) 55-82,

$$\hat{T}^{\mu\nu} = \hat{T}^{\mu\nu}_{\rm Can} + \frac{1}{2}\partial_\lambda(\hat{\Pi}^{\lambda,\mu\nu} + \hat{\Pi}^{\nu,\mu\lambda} + \hat{\Pi}^{\mu,\nu\lambda})$$

$$\hat{S}^{\lambda,\mu\nu} = \hat{S}^{\lambda,\mu\nu}_{\rm Can} - \hat{\Pi}^{\lambda,\mu\nu} + \partial_\rho \hat{\Upsilon}^{\mu\nu,\lambda\rho}$$

$$\hat{\Pi}^{\lambda,\mu\nu} = -\hat{\Pi}^{\lambda,\nu\mu}$$

$$\hat{\Upsilon}^{\mu\nu,\lambda\rho} = -\hat{\Upsilon}^{\nu\mu,\lambda\rho} = -\hat{\Upsilon}^{\mu\nu,\rho\lambda}$$

$$\sigma^{\mu\nu} = (i/2)\left[\gamma^\mu, \gamma^\nu\right]$$

→ But the conserved charges do not change

S. De Groot, W. Van Leeuwen, and C. Van Weert, Relativistic Kinetic Theory. Principles and Applications. North Holland, 1, 1980

Eur.Phys.J.A 57 (2021) 5, 155

Prog.Part.Nucl.Phys. 108 (2019) 103709

Motivation

- Hence, the dynamical meaning of the spin tensor is still not clear.
- For a fluid in local thermodynamic equilibrium, the quantum state of a system is not invariant under pseudogauge transformations
- It seems that the physical measurements depend on pseudogauge

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Motivation

- Based on the requirement of the positivity of local entropy production rate, there are many formulations that derive constitutive relations of spin hydrodynamics

- However, in the traditional approach, entropy current is obtained from an educated guess of the thermodynamic relations

$$Ts + \mu n = \rho + p - \frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}$$

$$\mathrm{d}p = s\mathrm{d}T + n\mathrm{d}\mu + \frac{1}{2}S^{\mu\nu}\mathrm{d}\omega_{\mu\nu}$$

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Motivation: Two questions

→ Rather than assuming a form, can one derive the form of entropy current from first principle?

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Motivation: Two questions

→ Rather than assuming a form, can one derive the form of entropy current from first principle?

Yes, using the quantum statistical method

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Motivation: Two questions

→ Rather than assuming a form, can one derive the form of entropy current from first principle?

Yes, using the quantum statistical method

→ One may also wonder, whether the relations are complete for the case of spin hydrodynamics?

$$Ts + \mu n = \rho + p - \frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}$$

$$\mathrm{d}p = s\mathrm{d}T + n\mathrm{d}\mu + \frac{1}{2}S^{\mu\nu}\mathrm{d}\omega_{\mu\nu}$$

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Motivation: Two questions

→ Rather than assuming a form, can one derive the form of entropy current from first principle?

Yes, using the quantum statistical method

→ One may also wonder, whether the relations are complete for the case of spin hydrodynamics?

It seems no!

$$Ts + \mu n = \rho + p - \frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}$$

$$\mathrm{d}p = s\mathrm{d}T + n\mathrm{d}\mu + \frac{1}{2}S^{\mu\nu}\mathrm{d}\omega_{\mu\nu}$$

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Entropy and thermodynamic potential current

→ In the quantum statistical description of a rel. fluid, $\rho_{\rm LE}$ is obtained by maximizing entropy $S = -\,{\rm Tr}\,(\hat\rho \log \hat\rho)$

$$\hat\rho_{\rm LE} = \frac{1}{Z_{\rm LE}}\exp(-\hat\Upsilon)$$
$$= \frac{1}{Z_{\rm LE}}\exp\left(-\int_\Sigma {\rm d}\Sigma_\mu \left(\hat T^{\mu\nu}\beta_\nu - \zeta \hat j^\mu - \frac{1}{2}\Omega_{\lambda\nu}\hat S^{\mu\lambda\nu}\right)\right)$$

with constraints

$$n_\mu T^{\mu\nu} = n_\mu T^{\mu\nu}_{\rm LE}\,, \quad n_\mu j^\mu = n_\mu j^\mu_{\rm LE}\,, \quad n_\mu S^{\mu\lambda\nu} = n_\mu S^{\mu\lambda\nu}_{\rm LE}$$

$$\beta_\nu = u_\nu/T$$
$$\zeta = \mu/T$$
$$\Omega_{\mu\nu} = \omega_{\mu\nu}/T$$
$${\rm d}\Sigma_\mu = {\rm d}\Sigma\, n_\mu$$

→ where LE values are defined as: $X_{\rm LE} \equiv {\rm Tr}\left(\hat\rho_{\rm LE}\hat X\right) - \langle 0|\hat X|0\rangle$

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Entropy and thermodynamic potential current

→ In global equilibrium, these Lagrange multipliers become

$$\beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu \quad \text{with} \quad b, \varpi, \zeta \quad \text{being constants}$$

$$\Omega = \varpi$$

$$\beta_\nu = u_\nu / T$$

$$\zeta = \mu / T$$

$$\Omega_{\mu\nu} = \omega_{\mu\nu} / T$$

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Entropy and thermodynamic potential current

→ If, $\hat{\Upsilon} = \int_\Sigma \mathrm{d}\Sigma_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu - \frac{1}{2}\Omega_{\lambda\nu}\hat{S}^{\mu\lambda\nu} \right)$ is bounded from below and lowest lying eigenstate $|0\rangle$ is non-degenerate

→ then, $\log Z_{\rm LE}$ can be proved to be extensive, i.e.

$$\log Z_{\rm LE} = \int_\Sigma \mathrm{d}\Sigma_\mu \, \phi^\mu - \langle 0 | \hat{\Upsilon} | 0 \rangle$$
$$= \int_\Sigma \mathrm{d}\Sigma_\mu \left[\phi^\mu - \langle 0 | \left(\hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu - \frac{1}{2}\Omega_{\lambda\nu}\hat{S}^{\mu\lambda\nu} \right) | 0 \rangle \right]$$

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

F. Becattini and D. Rindori, PRD 99, 125011 (2019)

Entropy and thermodynamic potential current

$$\phi^\mu = \int_1^\infty \mathrm{d}\lambda \left(T^{\mu\nu}_{\mathrm{LE}}(\lambda)\beta_\nu - \zeta j^\mu_{\mathrm{LE}}(\lambda) - \frac{1}{2}\Omega_{\lambda\nu} S^{\mu\lambda\nu}_{\mathrm{LE}}(\lambda) \right)$$

$$\phi^\mu(x) = \int_0^{T(x)} \frac{\mathrm{d}T'}{T'^2} \left(T^{\mu\nu}_{\mathrm{LE}}(x)[T',\mu,\omega]u_\nu(x) - \mu(x) j^\mu_{\mathrm{LE}}(x)[T',\mu,\omega] - \frac{1}{2}\omega_{\lambda\nu}(x) S^{\mu\lambda\nu}_{\mathrm{LE}}(x)[T',\mu,\omega] \right),$$

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Entropy and thermodynamic potential current

→ Once, ϕ_μ is known

$$S = -\operatorname{Tr}(\widehat{\rho}_{\rm LE} \log \widehat{\rho}_{\rm LE}) = \log Z_{\rm LE} + \int_\Sigma {\rm d}\Sigma_\mu \left(\operatorname{Tr}(\widehat{\rho}_{\rm LE} \widehat{T}^{\mu\nu})\beta_\nu - \zeta \operatorname{Tr}(\widehat{\rho}_{\rm LE} \widehat{j}^\mu) - \frac{1}{2}\Omega_{\lambda\nu} \operatorname{Tr}(\widehat{\rho}_{\rm LE} \widehat{S}^{\mu\lambda\nu}) \right)$$

$$= \int_\Sigma {\rm d}\Sigma_\mu \left(\phi^\mu + T^{\mu\nu}_{\rm LE}\beta_\nu - \zeta j^\mu_{\rm LE} - \frac{1}{2}\Omega_{\lambda\nu} S^{\mu\lambda\nu}_{\rm LE} \right),$$

→ entropy current is: $s^\mu = \phi^\mu + T^{\mu\nu}_{\rm LE}\beta_\nu - \zeta j^\mu_{\rm LE} - \frac{1}{2}\Omega_{\lambda\nu} S^{\mu\lambda\nu}_{\rm LE}$

$$\beta_\nu = u_\nu / T$$

$$\zeta = \mu / T$$

$$\Omega_{\mu\nu} = \omega_{\mu\nu} / T$$

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Entropy and thermodynamic potential current

- However, in global equilibrium, s_μ is defined more generally

$$s^\mu = \phi^\mu + T^{\mu\nu}\beta_\nu - \zeta j^\mu - \frac{1}{2}\Omega_{\lambda\nu}S^{\mu\lambda\nu}$$

- where

$$\phi^\mu = \int_0^T \frac{\mathrm{d}T'}{T'^2}\left(T^{\mu\nu}[T']u_\nu - \mu j^\mu[T'] - \frac{1}{2}\omega_{\lambda\nu}S^{\mu\lambda\nu}[T']\right)$$

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Local thermodynamic relations

→ Contracting s^μ with u_μ gives local thermodynamic relations

$$s \equiv s^\mu u_\mu = \phi \cdot u + \frac{1}{T}\rho - \zeta n - \frac{1}{2}\Omega_{\lambda\nu} u_\mu S^{\mu\lambda\nu} \equiv \phi \cdot u + \frac{1}{T}\rho - \frac{\mu}{T} n - \frac{1}{2}\Omega_{\lambda\nu} S^{\lambda\nu}$$

where $\rho = u_\mu u_\nu T^{\mu\nu}$ and $n = u_\mu j^\mu$

→ Defining: $p \equiv T\phi \cdot u$ <— valid in global equilibrium

we get $$Ts + \mu n = \rho + p - \frac{1}{2}\omega_{\mu\nu} S^{\mu\nu}$$

$\beta_\nu = u_\nu / T$

$\zeta = \mu / T$

$\Omega_{\mu\nu} = \omega_{\mu\nu} / T$

Francesco Becattini, Asaad Daher, Xin-Li Sheng, PLB 850 (2024) 138533

Local thermodynamic relations

→ Using ϕ^μ, one obtains p

$$p = T\phi \cdot u = T \int_0^T \frac{\mathrm{d}T'}{T'^2} \left(u_\mu T^{\mu\nu}[T'] u_\nu - \mu u_\mu j^\mu[T'] - \frac{1}{2}\omega_{\lambda\nu} u_\mu S^{\mu\lambda\nu}[T'] \right)$$

$$= T \int_0^T \frac{\mathrm{d}T'}{T'^2} \left(\rho[T'] - \mu n[T'] - \frac{1}{2}\omega_{\lambda\nu} S^{\lambda\nu}[T'] \right),$$

whence the following relation can be readily obtained

$$\left.\frac{\partial p}{\partial T}\right|_{\mu,\omega} = s$$

→ But do these relations hold in general?

$$\left.\frac{\partial p}{\partial \mu}\right|_{T,\omega} = n \qquad \left.\frac{\partial p}{\partial \omega_{\lambda\nu}}\right|_{T,\mu} = S^{\lambda\nu}$$

Local thermodynamic relations

We shall see that for a system of massless free fermions with rotation and acceleration at global equilibrium

$$\left.\frac{\partial p}{\partial \mu}\right|_{T,\omega} = n \qquad\qquad \left.\frac{\partial p}{\partial \omega_{\lambda\nu}}\right|_{T,\mu} \neq S^{\lambda\nu}$$

Local thermodynamic relations

$$\alpha^\mu = \frac{\omega^{\mu\nu}}{T}u_\nu, \quad w^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\frac{\omega_{\nu\rho}}{T}u_\sigma$$

We have pressure defined as:

$$p = T\phi^\mu u_\mu = T\int_0^T \frac{\mathrm{d}T'}{T'^2}\left(T^{\mu\nu}_{\mathrm{Can}}[T']u_\nu u_\mu - \mu u_\mu j^\mu[T'] - \frac{1}{2}u_\mu\omega_{\lambda\nu}S^{\mu\lambda\nu}_{\mathrm{Can}}[T']\right)$$

$$w^2 = \omega^2/T^2, \quad \alpha^2 = A^2/T^2$$

where:

$$u_\mu u_\nu T^{\mu\nu}_{\mathrm{Can}} = \frac{7\pi^4 + 30\pi^2\zeta^2 + 15\zeta^4}{60\pi^2\beta^4} - \frac{\alpha^2(\pi^2+3\zeta^2)}{24\pi^2\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4} - \frac{w^2(\pi^2+3\zeta^2)}{8\pi^2\beta^4} + \frac{23\alpha^2 w^2}{1440\pi^2\beta^4} + \frac{w^4}{64\pi^2\beta^4} + \frac{11(\alpha\cdot w)^2}{720\pi^2\beta^4}$$

$$\mu u_\mu j^\mu = \frac{\mu\zeta}{\sqrt{\beta^2}}\left(\frac{\pi^2+\zeta^2}{3\pi^2\beta^2} - \frac{\alpha^2}{4\pi^2\beta^2} + \frac{w^2}{4\pi^2\beta^2}\right)$$

$$u_\mu\omega_{\lambda\nu}S^{\mu\lambda\nu}_{\mathrm{Can}} = -\frac{1}{2}\epsilon^{\mu\lambda\nu\rho}u_\mu\omega_{\lambda\nu}w_\rho\frac{1}{\beta^2\sqrt{\beta^2}}\left(\frac{1}{6} + \frac{\zeta^2}{2\pi^2} - \frac{w^2}{24\pi^2} - \frac{\alpha^2}{8\pi^2}\right)$$

Andrea Palermo PHD thesis

Andrea Palermo, Matteo Buzzegoli, Francesco Becattini, JHEP 10 (2021) 077

Local thermodynamic relations

After removing vacuum terms

$$p = \frac{7\pi^2 T^4}{180} + \frac{1}{6}\mu^2 T^2 - \frac{A^2}{24}T^2 - \frac{\omega^2}{24}T^2$$

$$\left.\frac{\partial p}{\partial \mu}\right|_{T,\omega} = \frac{1}{3}\mu T^2 = n$$

Local thermodynamic relations

$$p = \frac{7\pi^2 T^4}{180} + \frac{1}{6}\mu^2 T^2 - \frac{A^2}{24}T^2 - \frac{\omega^2}{24}T^2$$

$$A^2 = \omega^{\alpha\beta}\omega_{\alpha\gamma}u_\beta u^\gamma\,, \qquad \omega^2 = \frac{1}{4}\epsilon^{\mu\delta\rho\sigma}\epsilon_{\mu\alpha\beta\gamma}\omega_{\delta\rho}\omega^{\alpha\beta}u_\sigma u^\gamma$$

$$\begin{aligned}
\left.\frac{\partial p}{\partial \omega_{\lambda\nu}}\right|_{T,\mu} &= \frac{\partial p}{\partial A^2}\frac{A^2}{\partial \omega_{\lambda\nu}} + \frac{\partial p}{\partial \omega^2}\frac{\omega^2}{\partial \omega_{\lambda\nu}}\,,\\
&= -\frac{T^2}{12}\left(A\frac{\partial A}{\partial \omega_{\lambda\nu}} + \omega\frac{\partial \omega}{\partial \omega_{\lambda\nu}}\right)\,,\\
&= -\frac{T^2}{12}\left(A^\rho u^\sigma \delta^{\lambda\nu}_{\rho\sigma} + \omega^\rho\left(-\frac{1}{2}\epsilon^{\rho\alpha\beta\gamma}u^\gamma\right)\delta^{\lambda\nu}_{\alpha\beta}\right)\,,\\
&= -\frac{T^2}{12}\left(A^\lambda u^\nu - A^\nu u^\lambda - \frac{1}{2}\omega^\rho u^\gamma(\epsilon^{\rho\lambda\nu\gamma} - \epsilon^{\rho\nu\lambda\gamma})\right)
\end{aligned}$$

$$\boxed{\left.\frac{\partial p}{\partial \omega_{\lambda\nu}}\right|_{T,\mu} = \frac{T^2}{12}\left(A^\nu u^\lambda - A^\lambda u^\nu + \epsilon^{\lambda\nu\rho\gamma}\omega_\rho u_\gamma\right) = \frac{T^2}{12}\left(A^\nu u^\lambda - A^\lambda u^\nu\right) + S^{\lambda\nu}}$$

Summary

→ It seems that thermodynamic relations are not fully appropriate in deriving divergence of entropy current

$$Ts + \mu n = \rho + p - \frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}$$

$$\mathrm{d}p = s\mathrm{d}T + n\mathrm{d}\mu + \frac{1}{2}S^{\mu\nu}\mathrm{d}\omega_{\mu\nu}$$

→ Employed a quantum-statistical approach to derive the entropy current without assuming the traditional local thermodynamic relations

→ Differential relations, $\left.\frac{\partial p}{\partial \mu}\right|_{T,\omega} = n$ and $\left.\frac{\partial p}{\partial \omega_{\lambda\nu}}\right|_{T,\mu} = S^{\lambda\nu}$ may not be true in general

→ For a system of massless free fermions with rotation and acceleration at global equilibrium

$$\left.\frac{\partial p}{\partial \mu}\right|_{T,\omega} = n \qquad \left.\frac{\partial p}{\partial \omega_{\lambda\nu}}\right|_{T,\mu} = \frac{T^2}{12}\left(A^\nu u^\lambda - A^\lambda u^\nu\right) + S^{\lambda\nu}$$

Thank you for listening!

Back Up Slides

$$\hat{\rho}_{\mathrm{LE}}(\lambda) = \frac{1}{Z_{\mathrm{LE}}(\lambda)} \exp\left[-\lambda \int_\Sigma \mathrm{d}\Sigma_\mu (\hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu)\right]$$

$$Z_{\mathrm{LE}}(\lambda) = \mathrm{tr}\left(\exp\left[-\lambda \int_\Sigma \mathrm{d}\Sigma_\mu (\hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu)\right]\right)$$

by taking the derivative of the trace we obtain

$$\frac{\partial \log Z_{\mathrm{LE}}(\lambda)}{\partial \lambda} = -\int_\Sigma \mathrm{d}\Sigma_\mu (\langle \hat{T}^{\mu\nu} \rangle_{\mathrm{LE}}(\lambda)\beta_\nu - \zeta \langle \hat{j}^\mu \rangle_{\mathrm{LE}}(\lambda))$$

and, by integrating both sides,

$$\log Z_{\mathrm{LE}} - \log Z_{\mathrm{LE}}(\lambda_0) = -\int_{\lambda_0}^{1} \mathrm{d}\lambda \int_\Sigma \mathrm{d}\Sigma_\mu (\langle \hat{T}^{\mu\nu} \rangle_{\mathrm{LE}}(\lambda)\beta_\nu - \zeta \langle \hat{j}^\mu \rangle_{\mathrm{LE}}(\lambda))$$

F. Becattini and D. Rindori, PRD 99, 125011 (2019)

$$\log Z_{\rm LE} - \log Z_{\rm LE}(\lambda_0)$$
$$= -\int_\Sigma {\rm d}\Sigma_\mu \int_{\lambda_0}^1 {\rm d}\lambda(\langle \hat{T}^{\mu\nu}\rangle_{\rm LE}(\lambda)\beta_\nu - \zeta\langle \hat{j}^\mu\rangle_{\rm LE}(\lambda))$$

Thus, if there exists a particular λ_0 such that $\log Z_{\rm LE}(\lambda_0) = 0$, it is proved that $\log Z_{\rm LE}$ is extensive and, at the same time, we have a method to determine the thermodynamic potential current:

$$\log Z_{\rm LE} = \int_\Sigma {\rm d}\Sigma_\mu \phi^\mu,$$
$$\phi^\mu = -\int_{\lambda_0}^1 {\rm d}\lambda(\langle \hat{T}^{\mu\nu}\rangle_{\rm LE}(\lambda)\beta_\nu - \zeta\langle \hat{j}^\mu\rangle_{\rm LE}(\lambda))$$

F. Becattini and D. Rindori, PRD 99, 125011 (2019)

$$\hat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left[-\int_\Sigma {\rm d}\Sigma_\mu (\hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu)\right],$$

where

$$\hat{\Upsilon} \equiv \int_\Sigma {\rm d}\Sigma_\mu (\hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu)$$

$$Z_{\rm LE} = {\rm tr}\left(\exp\left[-\int_\Sigma {\rm d}\Sigma_\mu (\hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu)\right]\right)$$

operator $\hat{\Upsilon}$ is bounded from below, i.e., there exists a minimum eigenvalue Υ_0 with a corresponding eigenvector $|0\rangle$, which is supposedly nondegenerate. In this case, by ordering the eigenvalues $\Upsilon_0 < \Upsilon_1 < \Upsilon_2 \ldots$, and if the lowest eigenvector is nondegenerate, the trace can be written as

$$Z_{\rm LE}(\lambda) = {\rm tr}({\rm e}^{-\lambda\hat{\Upsilon}}) = {\rm e}^{-\lambda\Upsilon_0}(1 - {\rm e}^{-\lambda(\Upsilon_1-\Upsilon_0)} - {\rm e}^{-\lambda(\Upsilon_2-\Upsilon_0)} - \cdots),$$

so, if $\Upsilon_0 = 0$ and we let $\lambda \to +\infty$, we obtain the sought solution, that is,

$$\lim_{\lambda\to+\infty} Z_{\rm LE}(\lambda) = 1 \Rightarrow \lim_{\lambda\to+\infty} \log Z_{\rm LE}(\lambda) = 0.$$

F. Becattini and D. Rindori, PRD 99, 125011 (2019)

$$\hat{\Upsilon} \mapsto \hat{\Upsilon} - \Upsilon_0 = \hat{\Upsilon} - \langle 0|\hat{\Upsilon}|0\rangle = \int_\Sigma \mathrm{d}\Sigma_\mu [(\hat{T}^{\mu\nu} - \langle 0|\hat{T}^{\mu\nu}|0\rangle)\beta_\nu - \zeta(\hat{j}^\mu - \langle 0|\hat{j}^\mu|0\rangle)]$$

$$Z'_{\mathrm{LE}}(\lambda) = \mathrm{tr}\left(\exp\left\{-\lambda \int_\Sigma \mathrm{d}\Sigma_\mu [(\hat{T}^{\mu\nu} - \langle 0|\hat{T}^{\mu\nu}|0\rangle)\beta_\nu - \zeta(\hat{j}^\mu - \langle 0|\hat{j}^\mu|0\rangle)]\right\}\right).$$

The new partition function is such that $Z'_{\mathrm{LE}}(\infty) = 1$, and the thermodynamic potential current is thus given by

$$\phi^\mu = \int_1^{+\infty} \mathrm{d}\lambda [(\langle \hat{T}^{\mu\nu}\rangle_{\mathrm{LE}}(\lambda) - \langle 0|\hat{T}^{\mu\nu}|0\rangle)\beta_\nu - \zeta(\langle \hat{j}^\mu\rangle_{\mathrm{LE}}(\lambda) - \langle 0|\hat{j}^\mu|0\rangle)].$$

Consequently, the entropy current will be

$$s^\mu = \phi^\mu + (\langle \hat{T}^{\mu\nu}\rangle_{\mathrm{LE}} - \langle 0|\hat{T}^{\mu\nu}|0\rangle)\beta_\nu - \zeta(\langle \hat{j}^\mu\rangle_{\mathrm{LE}} - \langle 0|\hat{j}^\mu|0\rangle)$$

F. Becattini and D. Rindori, PRD 99, 125011 (2019)