Spin Hydrodynamics from the Entropy Principle: flaws and remedies

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— Spin and quantum features of QCD plasma —

General Introduction

Thermal System with Vorticity

A black hole / a QGP has no hair:

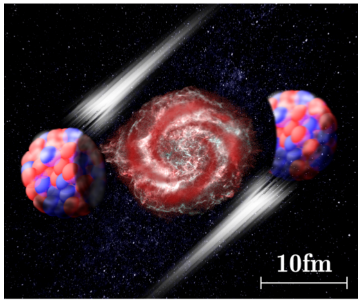
Stable black holes are characterized by

[From Forbes]

Mass ⟷ **Temperature**

Charge ⟷ **Density**

Angular Momentum

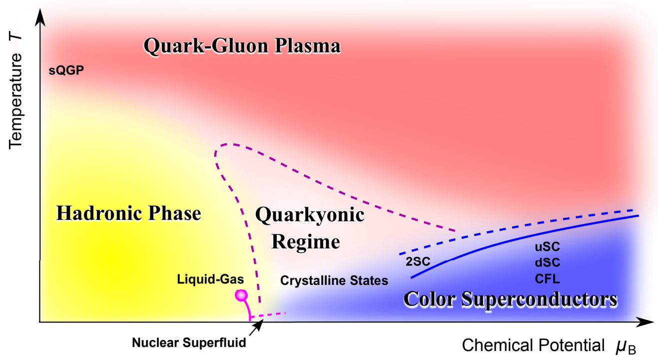


[Femto-Novae]

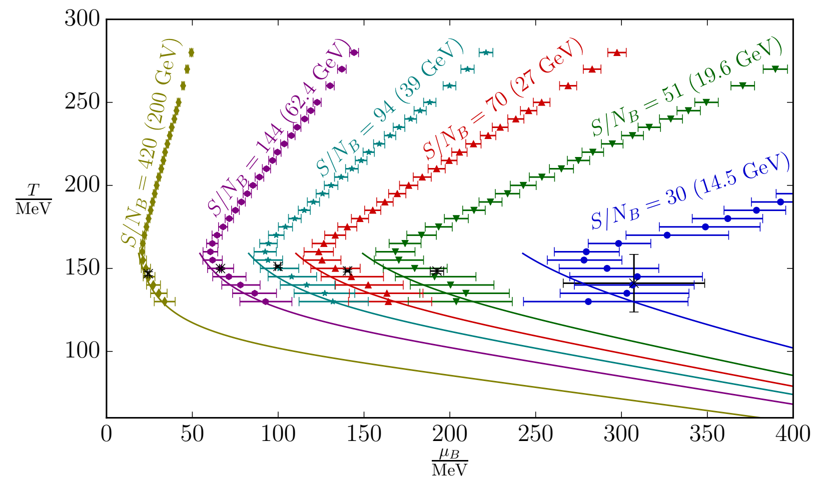
Thermal System ~~with Vorticity~~

Baryon Density

Speculated Phase Diagram



Gunther et al. (2016)



Ideal hydrodynamics: $\mathcal{S}^\mu = s u^\mu \quad j^\mu = n u^\mu$

$$\partial_\mu T^{\mu\nu} = 0 \quad \rightarrow \quad \partial_\mu \mathcal{S}^\mu = 0$$

Eliminating $(\partial \cdot u) \ \rightarrow \ n\dot{s} - s\dot{n} = 0 \ \rightarrow \ s/n =$ **(const.)**

Thermal System with Vorticity

Grand Canonical Descriptions

[EoSs]

$$de = Tds + \mu dn + \omega_{\mu\nu} dJ^{\mu\nu}$$

$$dp = sdT + nd\mu + J_{\mu\nu} d\omega^{\mu\nu}$$

$$\langle n \rangle = \frac{\partial p}{\partial \mu}$$

$$\langle J^z \rangle = \frac{\partial p}{\partial \omega^z}$$

n : baryon density

μ : baryon chemical pot.

J : angular mom.

ω : spin chemical pot.

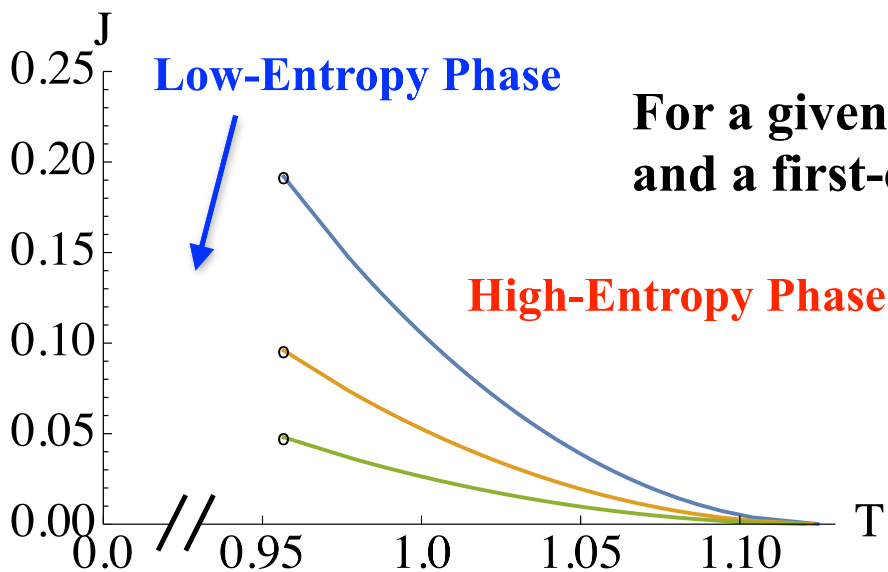
Phase Diagram
Hydro Trajectories

Counterparts?

Thermal System with Vorticity

Ahmed-Cong-Kubiznak-Mann-Visser (2023)

fixed $(J, \mathcal{V}, C)$:	$F \equiv E - TS$	**Canonical**
fixed $(\tilde{\Omega}, \mathcal{V}, C)$:	$W \equiv E - TS - \tilde{\Omega} J$	**Grand Canonical**
fixed $(J, \mathcal{V}, \mu)$:	$G \equiv E - TS - \mu C$	



For a given *J*, there are multiple solutions and a first-order phase transition occurs.

Also, the "end-point" appears that is of second-order.

Thermal System with Vorticity

Ahmed-Cong-Kubiznak-Mann-Visser (2023)

fixed $(J, \mathcal{V}, C)$:	$F \equiv E - TS$	**Canonical**
fixed $(\tilde{\Omega}, \mathcal{V}, C)$:	$W \equiv E - TS - \tilde{\Omega}J$	**Grand Canonical**
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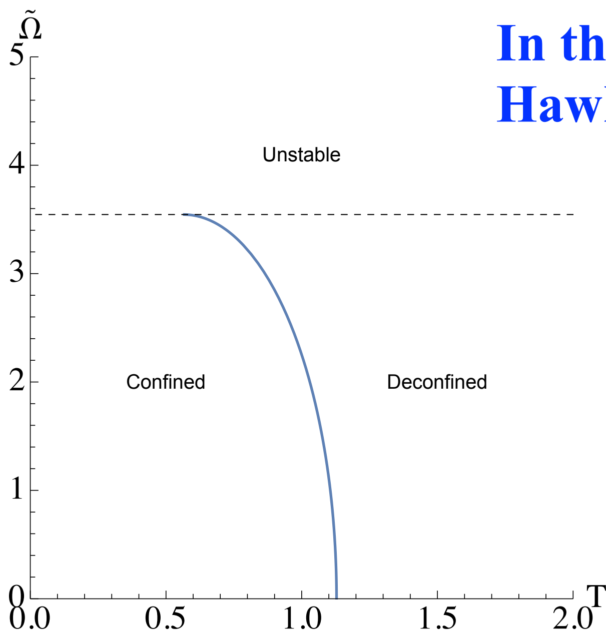
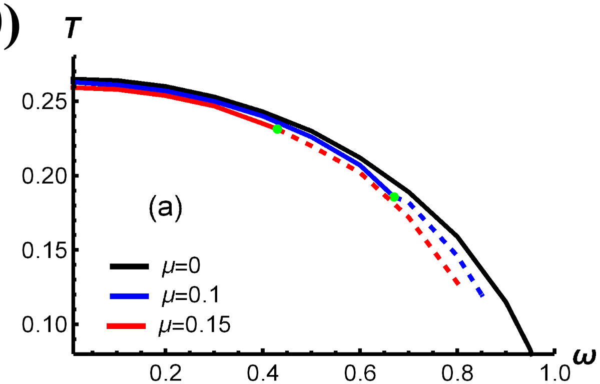


In the grand canonical case, a first-order Hawking-Page transition is seen.

Consistent with Chen-Zhang-Li-Hou-Huang (2020)



Thermal System with Vorticity

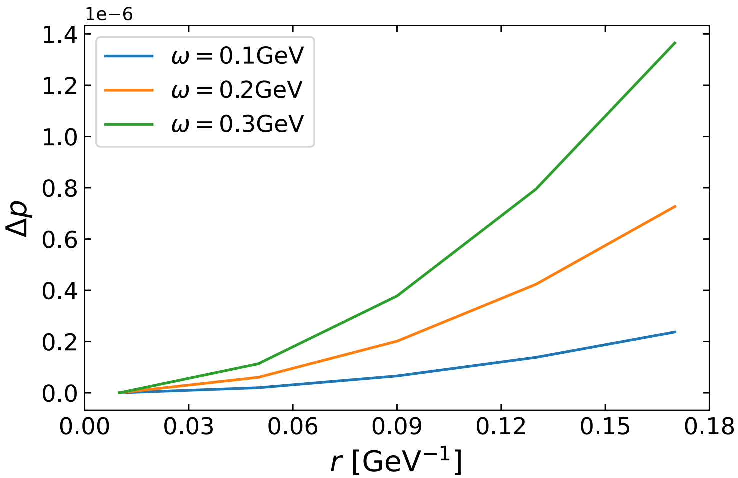
EoS calculable from the rotating HRG model

Pressure:

~~Orbital~~

$$p_i^{\pm} = \pm \frac{T}{8\pi^2} \sum_{\ell=-\infty}^{\infty} \int_{(\Lambda_\ell^{\mathrm{IR}})^2} dk_r^2 \int dk_z \sum_{\nu=\ell}^{\ell+2S_i} J_\nu^2(k_r r)$$

$$\times \log\{1 \pm \exp[-(\varepsilon_{\ell,i} - \mu_i)/T]\}$$



$$\varepsilon_{\ell,i} = \sqrt{k_r^2 + k_z^2 + m_i^2} - (\ell + S_i)\omega$$

$$\Lambda_\ell^{\mathrm{IR}} = \xi_{\ell,1}\omega$$

Fujimoto-Fukushima-Hidaka (2021)

Thermal System with Vorticity

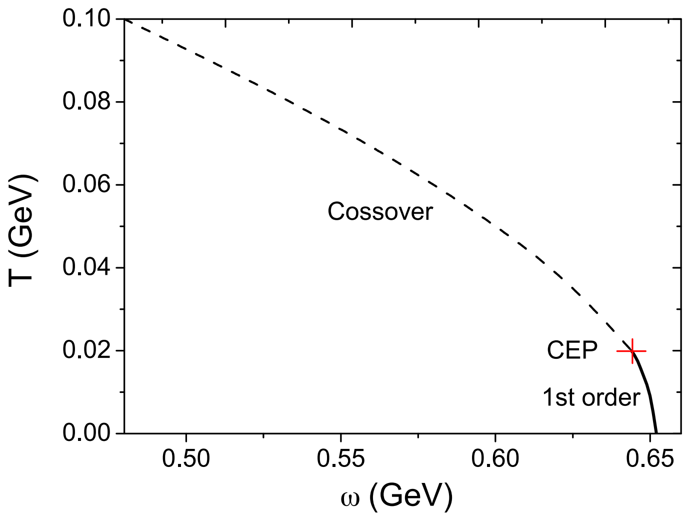
Angular Momentum

Ideal hydrodynamics ?

$$\mathcal{S}^\mu = s u^\mu \quad j^\mu = n u^\mu$$

$$J^{\lambda\mu\nu} \stackrel{?}{=} S^{\mu\nu} u^\lambda$$

Speculated Phase Diagram
Jiang-Liao (2017)



What are the hydro trajectories?
We need the Spin Hydrodynamics.
Power counting??

Technical Introduction

[Disclaimer]

Instead of an incomplete review, the general review is completely omitted… sorry!

Entropy Principle — review

Very powerful strategy — Good example from Son-Surowka (2009)

(The notation is slightly changed here)

Ideal Hydrodynamics $\sim \mathcal{O}(\partial^0)$

Entropy

$$s = \beta(e+p) - n\alpha \qquad (\alpha = \beta\mu)$$

Entropy Current

$$\mathcal{S}^\mu_{(0)} = su^\mu = \beta(e+p)u^\mu - n\alpha u^\mu$$
$$= \beta(u_\nu \Theta^{\mu\nu}_{(0)} + pu^\mu - \mu j^\mu_{(0)})$$ **→ Generalized later!**

$\partial_\mu \mathcal{S}^\mu_{(0)} = 0$ **is easily concluded (ideal).**

Entropy Principle — review

Anomalous Hydrodynamics ~ $\mathcal{O}(\partial)$

$$\mathcal{S}^\mu = \beta(u_\nu \Theta^{\mu\nu} + p u^\mu - \mu j^\mu) \qquad \partial_\mu j^\mu = C_{\mathrm{anom}} E \cdot B$$

$$\partial_\mu \mathcal{S}^\mu = \Theta^{\mu\nu}_{(1)} \partial_\mu(\beta u_\nu) + j^\mu_{(1)}(-\partial_\mu \alpha + \beta E_\mu) - C_{\mathrm{anom}} \alpha E \cdot B$$

How can we make this non-negative?

$$\Theta^{\mu\nu}_{(1)} = 2h^{(\mu}u^{\nu)} + \pi^{\mu\nu} \qquad u \cdot h = (u \cdot \pi)^\mu = 0$$

Heat Flow Shear Tensor

(Here, we do not care about the frame choice.)

Entropy Principle — review

$$\Theta^{\mu\nu}_{(1)} = 2h^{(\mu}u^{\nu)} + \pi^{\mu\nu}$$

$$u \cdot h = (u \cdot \pi)^\mu = 0$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

$$2h^{(\mu}u^{\nu)}\partial_\mu(\beta u_\nu) - j^\mu_{(1)}\underline{\partial_\mu \alpha} = (h^\mu - \mathscr{H} j^\mu_{(1)})(\Delta_{\mu\nu}\partial^\nu \beta + \underline{\beta \dot{u}_\mu})$$

$$\mathscr{H}^{-1} \quad \frac{n}{e+p}\Delta_{\mu\nu}\partial^\nu \alpha = \Delta_{\mu\nu}\partial^\nu \beta + \beta \dot{u}^\mu \qquad \beta u^\nu \partial_\nu u_\mu$$

$$\pi^{\mu\nu}\partial_\mu(\beta u_\nu) = \beta \pi^{\mu\nu}\left[\partial_{<\mu}u_{\nu>} + \frac{1}{3}\Delta_{\mu\nu}(\partial \cdot u)\right]$$

$$A^{<\mu\nu>} = \Delta^{(\mu\alpha}\Delta^{\beta\nu)}A_{\alpha\beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}A_{\alpha\beta}$$

Entropy Principle — review

$$\partial_\mu \mathcal{S}^\mu = \Theta^{\mu\nu}_{(1)} \partial_\mu (\beta u_\nu) + j^\mu_{(1)} (-\partial_\mu \alpha + \beta E_\mu) - C_{\text{anom}} \alpha E \cdot B$$

$$= (h^\mu - \mathscr{H} j^\mu_{(1)})(\Delta_{\mu\nu} \partial^\nu \beta + \beta \dot{u}_\mu)$$

$$+\beta \pi^{\mu\nu} \left[\partial_{<\mu} u_{\nu>} + \frac{1}{3} \Delta_{\mu\nu} (\partial \cdot u) \right] - C_{\text{anom}} E \cdot B$$

$$h^\mu - \mathscr{H} j^\mu_{(1)} = \sigma \Delta^{\mu\nu} (\partial_\nu \beta + \beta \dot{u}_\nu)$$

Sum of squared quantities ≥ 0

$$\pi^{\mu\nu} = 2\eta \partial^{<\mu} u^{\nu>} \qquad \Pi = -\zeta (\partial \cdot u)$$

Entropy Principle — review

$$\partial_\mu \mathcal{S}^\mu = \Theta^{\mu\nu}_{(1)} \partial_\mu (\beta u_\nu) + j^\mu_{(1)} (-\partial_\mu \alpha + \beta E_\mu) - C_{\mathrm{anom}} \alpha E \cdot B$$

$$= (h^\mu - \mathscr{H} j^\mu_{(1)})(\Delta_{\mu\nu} \partial^\nu \beta + \beta \dot{u}_\mu)$$

$$+ \beta \pi^{\mu\nu} \left[\partial_{<\mu} u_{\nu>} + \frac{1}{3} \Delta_{\mu\nu} (\partial \cdot u) \right] - C_{\mathrm{anom}} E \cdot B$$

The last unwanted term is "renormalized" in the currents.

$$\delta j^\mu_{(1)} = \xi_V \omega^\mu + \xi_B B^\mu \qquad \delta \mathcal{S}^\mu = -\alpha \delta j^\mu_{(1)} + D_V \omega^\mu + D_B B^\mu$$

$$\longrightarrow \quad \xi_V = C_{\mathrm{anom}} \left(\mu^2 - \frac{2}{3} \frac{\mu^3}{\mathscr{H}} \right) \qquad \xi_B = C_{\mathrm{anom}} \left(\mu - \frac{1}{2} \frac{\mu^2}{\mathscr{H}} \right)$$

Son-Surowka (2009)

Spin Hydro Introduction

Spin Hydro — review

Spin Hydrodynamics (Hattori-Hongo-Huang-Matsuo-Taya 2019)

Noether current from rotational symmetry

$$J^{\lambda\mu\nu} = \underbrace{x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\nu}}_{\text{Orbital}} + \underbrace{\Sigma^{\lambda\mu\nu}}_{\text{Spin}}$$

$$\partial_\mu T^{\mu\nu} = \partial_\lambda J^{\lambda\mu\nu} = 0 \longrightarrow \partial_\lambda \Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}$$

Anti-symmetric part of the energy-momentum tensor is the source of the spin (from the orbital part).

Spin Hydro — review

Spin Hydrodynamics (Hattori-Hongo-Huang-Matsuo-Taya 2019)

$$\mathcal{S}^\mu = \beta(u_\nu \Theta^{\mu\nu} + p u^\mu - \mu j^\mu - \underline{\omega_{\rho\sigma} S^{\rho\sigma} u^\mu})$$

Induced by the spin potential

$$\Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + \Sigma^{\lambda\mu\nu}_{(1)}$$

$$\Theta^{\mu\nu} = \Theta^{(\mu\nu)} + \underline{2q^{[\mu}u^{\nu]} + \phi^{\mu\nu}}$$

$$u \cdot q = (u \cdot \phi)^\mu = 0$$

Anti-symmetric part

$$\partial_\mu \mathcal{S}^\mu = \cdots (2q^{[\mu}u^{\nu]} + \phi^{\mu\nu})\partial_\mu(\beta u_\nu) - \partial_\mu(u^\mu S^{\rho\sigma})\beta\omega_{\rho\sigma}$$

Spin Hydro — review

Spin Hydrodynamics (Hattori-Hongo-Huang-Matsuo-Taya 2019)

$$\partial_\mu \mathcal{S}^\mu = \cdots + (2q^{[\mu}u^{\nu]} + \phi^{\mu\nu})\partial_\mu(\beta u_\nu) \underline{- \partial_\mu(u^\mu S^{\rho\sigma})}\beta\omega_{\rho\sigma}$$

$2\Theta^{[\rho\sigma]}$

$$= \cdots + q^\mu(\partial_\mu\beta - \beta\dot{u}_\mu + 4\beta\omega_{\mu\nu}u^\nu)$$
$$+\beta\phi^{\mu\nu}(2\omega_{\mu\nu} + \partial_\mu u_\nu)$$

$$q^\mu = \lambda\Delta^{\mu\nu}(\partial_\nu\beta - \beta\dot{u}_\nu + 4\beta\omega_{\nu\rho}u^\rho)$$

$$\phi^{\mu\nu} = \gamma\Delta^{\mu\rho}\Delta^{\nu\sigma}(2\omega_{\rho\sigma} + \partial_{[\rho}u_{\sigma]})$$

Same conclusion in the Belinfante symmetric form Fukushima-Pu (2020)

Spin Hydro Revisited

Spin Hydro — puzzle

$$\partial_\lambda \Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}$$

$$\Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + \Sigma^{\lambda\mu\nu}_{(1)}$$

$$\Theta^{\mu\nu} = \Theta^{(\mu\nu)} + 2q^{[\mu}u^{\nu]} + \phi^{\mu\nu}$$

$$q^\mu = -\frac{1}{2} u_\nu \partial_\lambda \Sigma^{\lambda\mu\nu}$$

This current is already solved in terms of spin!?

In the whole formulation, the d.o.f. are redundant.

Hongo-Huang-Kaminski-Stephanov-Yee (2021)

Spin Hydro — puzzle

Should we really keep 6 generators? — No!

3 rotations + 3 boosts

Hermitian **non-Hermitian**

Lorentz group has no hermitian representation with finite dimensions.

Fully anti-symmetrized $\Sigma^{\lambda\mu\nu}$ should correspond to the physical observables, i.e., $\Sigma^{00i} = 0$

$$\Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + u^\mu S^{\nu\lambda} + u^\nu S^{\lambda\mu} + \Sigma^{\lambda\mu\nu}_{(1)}$$

Spin Hydro — challenge

Shuo Fang-KF-SP-Dong-Lin Wang (soon!)

Can we really construct the hydro? — Yes!

(Discussions did not converge yet.)

Differences:

q^μ **is already solved as**

$$q^\mu = -\frac{1}{2} u_\nu \partial_\lambda \Sigma^{\lambda\mu\nu} = \frac{1}{2}(S^{\mu\nu}\dot{u}_\nu + \Delta^\mu_\nu \partial_\lambda S^{\nu\lambda})$$

Some extra terms from

$$-\partial_\mu(u^\mu S^{\rho\sigma})\beta\omega_{\rho\sigma} \neq 2\Theta^{[\rho\sigma]}\beta\omega_{\rho\sigma}$$

Simplification

$$(u \cdot S)^\mu = 0$$
$$(u \cdot \omega)^\mu = 0$$

Spin Hydro — challenge

Shuo Fang-KF-SP-Dong-Lin Wang (soon!)

This is easily "renormalized".

$$\partial_\mu \mathcal{S}^\mu$$
$$= (h^\mu - \mathcal{H}\nu^\mu)(\partial_\mu \beta + \beta \dot{u}_\mu) + \beta \pi^{\mu\nu} \partial_\mu u_\nu$$
$$+ \boxed{q^\mu (\partial_\mu \beta - \beta \dot{u}_\mu)} + \boxed{2\beta \omega_{\mu\nu} S^{\lambda\mu} \partial_\lambda u^\nu}$$
$$+ \underline{\beta \phi^{\mu\nu} (2\omega_{\mu\nu} + \partial_\mu u_\nu)} + \mathcal{O}(\partial^3)$$

Unchanged

Spin Hydro — challenge

Shuo Fang-KF-SP-Dong-Lin Wang (soon!)

$$\beta S^{\lambda\rho}\partial_\lambda u^\sigma \Delta^\mu_\rho \Delta^\nu_\sigma (2\omega_{\mu\nu} + \underline{\partial_\mu u_\nu})$$

Vanishing

Symmetric Part

Anti-symmetric Part

$$\pi^{\rho\sigma} + S^{\lambda(\rho}\partial_\lambda u^{\sigma)} \qquad \phi^{\rho\sigma} + S^{\lambda[\rho}\partial_\lambda u^{\sigma]}$$

Maybe…, these do not appear,
if the entropy current is chosen better.

Spin Hydro — challenge

Shuo Fang-KF-SP-Dong-Lin Wang (soon!)

$$\frac{1}{2}(S^{\mu\nu}\dot{u}_\nu + \Delta^\mu_\nu \partial_\lambda S^{\nu\lambda})(\partial_\mu \beta - \beta \dot{u}_\mu)$$

$$= -\frac{1}{2}(S^{\mu\nu}\dot{u}_\nu + \Delta^\mu_\nu \partial_\lambda S^{\nu\lambda})(\partial_\mu \beta + \beta \dot{u}_\mu)$$

$$+ (S^{\mu\nu}\dot{u}_\nu + \Delta^\mu_\nu \partial_\lambda S^{\nu\lambda})\partial_\mu \beta$$

$$= -\frac{1}{2}(\Delta^\mu_\nu \partial_\lambda S^{\nu\lambda} + \beta^{-1} S^{\mu\nu}\partial_\nu \beta)(\partial_\mu \beta + \beta \dot{u}_\mu)$$

$$+ \Delta^\mu_\nu \partial_\lambda S^{\nu\lambda} \partial_\mu \beta$$

Renormalized in the heat current.

Spin Hydro — challenge

Shuo Fang-KF-SP-Dong-Lin Wang (soon!)

$$\Delta^\mu_\nu \partial_\lambda S^{\nu\lambda}\partial_\mu\beta$$

$$= \partial_\lambda(\Delta^\mu_\nu S^{\nu\lambda}\partial_\mu\beta) - \dot\beta S^{\mu\nu}\partial_\mu u_\nu$$

Renormalized in the entropy current.

$$-\dot\beta S^{\mu\nu}(2\omega_{\mu\nu} + \partial_\mu u_\nu) + 2\dot\beta S^{\mu\nu}\omega_{\mu\nu}$$

Renormalized in $\phi^{\mu\nu}$ **Renormalized in Π**

$$\dot\beta = \left(\frac{\partial p}{\partial e}\right)_{n,S^{\mu\nu}}\beta(\partial\cdot u)$$

Spin Hydro — results

Shuo Fang-KF-SP-Dong-Lin Wang (soon!)

$$\delta h^\mu = -\frac{1}{2}\Delta^\mu_\nu \partial_\lambda S^{\nu\lambda} - \frac{1}{2}\beta^{-1} S^{\mu\nu}\partial_\nu \beta$$

$$\delta\pi^{\mu\nu} = S^{\lambda<\rho}\partial_\lambda u^{\sigma>}$$

$$\delta\Pi = \frac{1}{3}S^{\mu\nu}\partial_\mu u_\nu - 2\left(\frac{\partial p}{\partial e}\right)_{n,S^{\mu\nu}} S^{\mu\nu}\omega_{\mu\nu}$$

$$\phi^{\mu\nu} = \gamma \Delta^{\mu\rho}\Delta^{\nu\sigma}(2\beta\omega_{\rho\sigma} + \partial_{[\rho}u_{\sigma]}) + \beta^{-1}\dot{\beta}S^{\mu\nu}$$

We are still checking the calculations…
Coefficients may be wrong, but the strategy is correct.

Spin Hydro — puzzle again

Shuo Fang-KF-SP-Dong-Lin Wang (soon!)

Tensor decomposition and renormalization seem to be not unique…???

The present demonstration is just one simple example… we found a family of non-negative decomposition of tensors.

Any principle still overlooked???

Summary

- **Spin hydrodynamics in the canonical (and Belinfante) form have some redundancy.**
  - Spin has not 6 but 3 charge observables.
  - Symmetrized form should be physical.
- **Extra terms can be renormalized so that the second law of thermodyanmics is satisfied.**
  - Overconstrained problem is resolved.
- **Tensor decomposition seems to be non-unique.**
  - A family of decompositions with unfixed parameters was found to satisfy the entropy principle… more physical conditions?

Backup — Pseudogauge

Pseudo-Gauge Sym. in Energy-Momentum Tensor

Fukushima-Pu (2020)

$$\mathcal{T}^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \quad \text{(conserved current redefined)}$$

$$K^{\lambda\mu\nu} = \frac{1}{2}\left(\Sigma^{\lambda\mu\nu} - \Sigma^{\mu\lambda\nu} + \Sigma^{\nu\mu\lambda}\right)$$

$$\mathcal{T}^{\mu\nu} = \Theta^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(u^\lambda S^{\mu\nu} - u^\mu S^{\lambda\nu} + u^\nu S^{\mu\lambda}\right)$$

$$= \Theta^{\mu\nu}_{(\mathrm{s})} + \frac{1}{2}\left[\partial_\lambda\left(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda}\right)\right]$$

Backup — Pseudogauge

Pseudo-Gauge Sym. in Energy-Momentum Tensor

Fukushima-Pu (2020)

$$\mathcal{T}^{\mu\nu} = \Theta^{\mu\nu} + \frac{1}{2}\partial_\lambda(u^\lambda S^{\mu\nu} - u^\mu S^{\lambda\nu} + u^\nu S^{\mu\lambda})$$
$$= \Theta^{\mu\nu}_{(\mathrm{s})} + \frac{1}{2}\left[\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\right]$$

Spin induced terms are "renormalized" as

$$2h^{(\mu}u^{\nu)} + \pi^{\mu\nu} + \frac{1}{2}\left[\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\right]$$
$$= \delta e u^\mu u^\nu + 2\left(h^{(\mu} + \delta h^{(\mu}\right)u^{\nu)} + \pi^{\mu\nu} + \delta\pi^{\mu\nu}$$

Backup — Pseudogauge

Pseudo-Gauge Sym. in Energy-Momentum Tensor

Fukushima-Pu (2020)

$$\delta e = u_\rho \partial_\sigma S^{\rho\sigma}$$

$$\delta h^\mu = \frac{1}{2}\left[\Delta^\mu_\sigma \partial_\lambda S^{\sigma\lambda} + u_\rho S^{\rho\lambda}\partial_\lambda u^\mu\right]$$

$$\delta\pi^{\mu\nu} = \partial_\lambda(u^{<\mu}S^{\nu>\lambda}) + \delta\Pi\Delta^{\mu\nu}$$

$$\delta\Pi = \frac{1}{3}\partial_\lambda(u^\sigma S^{\rho\lambda})\Delta_{\rho\sigma}$$

An electric current $\boldsymbol{j} \propto \nabla \times \boldsymbol{S}$ is implied… **Spin Vorticity Effect**

Backup — Pseudogauge

Pseudo-Gauge Sym. in Energy-Momentum Tensor

Fukushima-Pu (2020)

$$\partial_\mu \mathcal{S}^\mu = \cdots + \frac{1}{2}\big[\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\big]\partial_\mu \frac{u_\nu}{T} - \frac{\omega_{\rho\sigma}}{T}\partial_\lambda(u^\lambda S^{\rho\sigma})$$

$$= \frac{1}{2}\partial_\mu\left[\partial_\lambda(u^\lambda S^{\mu\nu} + u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\frac{u_\nu}{T}\right]$$

Total derivative

$$\mathcal{S}^\mu \;\rightarrow\; \mathcal{S}'^\mu$$

$$-\frac{1}{2}\big[\partial_\lambda(u^\lambda S^{\mu\nu})\big]\partial_\mu \frac{u_\nu}{T} - \frac{\omega_{\rho\sigma}}{T}\partial_\lambda(u^\lambda S^{\rho\sigma})$$

Canonical results

Absorbed in the entropy, then it is just canonical!