Work in progress on analytic continuation from imaginary chemical potential

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In collaboration with F. Di Renzo, P. Dimopoulos (Parma)
and the Bielefeld-Parma collaboration

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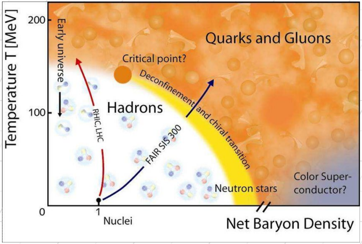
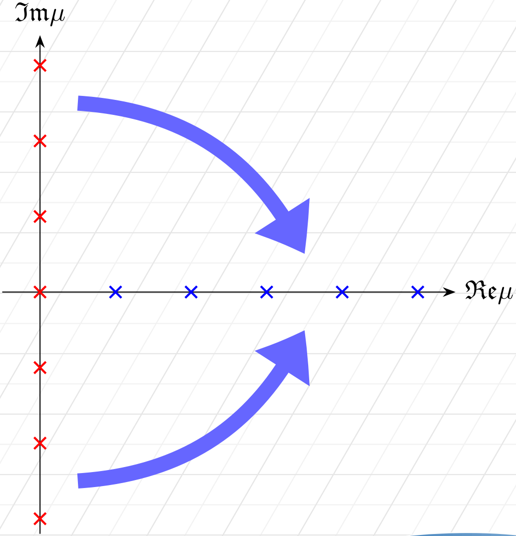
Gruppo Collegato di Parma

The Sign Problem

- The study of the phase diagram requires finite baryon number density
- Finite density lattice simulations $\Rightarrow$ chemical potential $\mu \neq 0$
- Generic $\mu \Rightarrow$ complex Dirac determinant, leads to sign problem
- For purely imaginary values of μ the Dirac determinant remains real
- Methods to extrapolate physical functions of real μ from the imaginary axis are needed



Imaginary μ



- Data from simulations at imaginary μ
- Analytic continuation to real μ
- Propagation of the statistical uncertainty?
- Radius of convergence?

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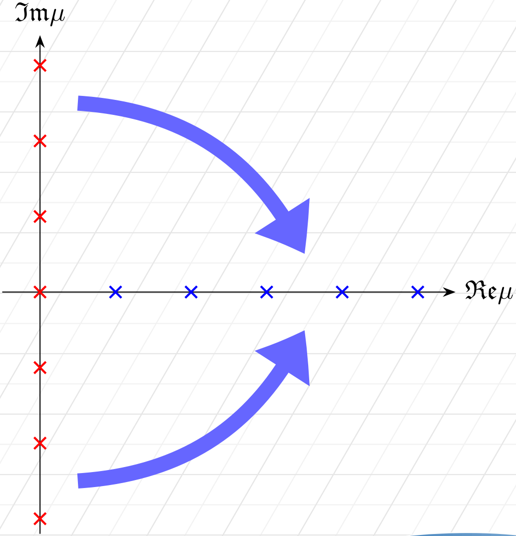

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Various different methods for analytitc continuation

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The Methods

- Taylor Expansion: $P_n(\mu) = \sum_{k=0}^{n} c_k \mu^k$
- Padé analysis[1]: $R_m^n(\mu) = \frac{\sum_{k=0}^{n} p_k \mu^k}{1 + \sum_{j=1}^{m} q_j \mu^j}$
- Cauchy's Theorem integrated numerically[2]
- Power series of μ^2, from $\mu^2 < 0$ to $\mu^2 > 0$

[1] Explained in detail by C. Schmidt on Monday
[2] Explained in detail by F. Di Renzo on Tuesday

The Methods

- Taylor Expansion: $P_n(\mu) = \sum_{k=0}^{n} c_k \mu^k$

$$\chi_n(T, V, \mu_B) = \left(\frac{\partial}{\partial \mu_B}\right)^n \frac{\ln Z(T, V, \mu_B)}{VT^3}$$

- Padé analysis[1]: $R_m^n(\mu) = \frac{\sum_{k=0}^{n} p_k \mu^k}{1 + \sum_{j=1}^{m} q_j \mu^j}$

$$\mu \equiv \mu_B / T$$

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HISQ $N_f = 2+1$, $N_\tau = 6$, $T = 157.5 MeV$

Physical pion mass

From the Bielefeld-Parma collaboration

- Power series of μ^2, from $\mu^2 < 0$ to $\mu^2 > 0$

Disclaimer: For now, only the central values will be shown (without errors).

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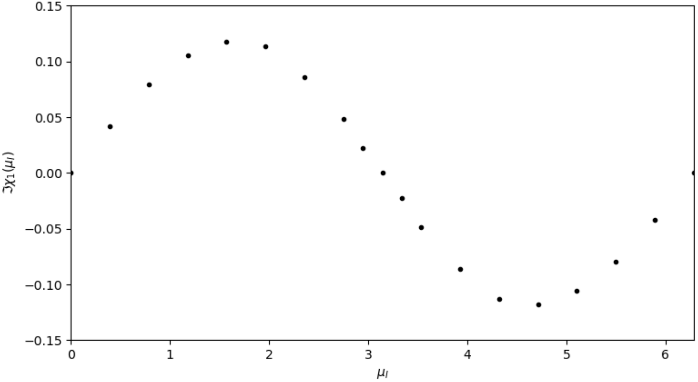
Input Data



Taylor Expansion

Dataset

$\{\chi_1(\mu_0), \ldots, \chi_1(\mu_{N-1})\}$

$$\chi_1(\mu_i) = \sum_{k=0}^{N-1} \frac{1}{k!} \chi_{1+k}(0) \mu_i^k + O(\mu^N)$$

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Taylor Expansion

Dataset

$$\{\chi_1(\mu_0), \dots, \chi_1(\mu_{N-1})\}, \quad \{\chi_2(\mu_N), \dots, \chi_2(\mu_{N+M-1})\}$$

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- Straightforward to implement

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- Radius of convergence: (way before) nearest singularity

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- Straightforward to implement
- Radius of convergence: (way before) nearest singularity
- No singularity structure (polynomial)

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Taylor Expansion - Fixed Parity

Charge conjugation symmetry $\Longrightarrow \chi_1(-\mu) = -\chi_1(\mu)$

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$$\begin{cases} \chi_1(\mu_i) = \displaystyle\sum_{k=0}^{N+M-1} \frac{1}{(2k+1)!}\chi_{2+2k}(0)\mu_i^{2k+1} + O(\mu^{2N+2M}) \\ \chi_2(\mu_j) = \displaystyle\sum_{k=1}^{N+M-1} \frac{2k+1}{(2k+1)!}\chi_{2+2k}(0)\mu_j^{2k} + O(\mu^{2N+2M-1}) \end{cases}$$

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- Higher number of *significant* derivatives (with same input data)

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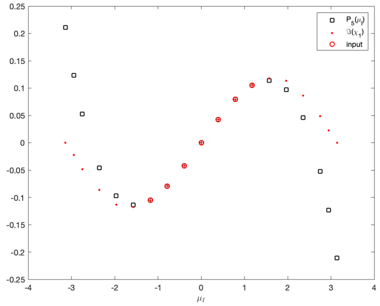
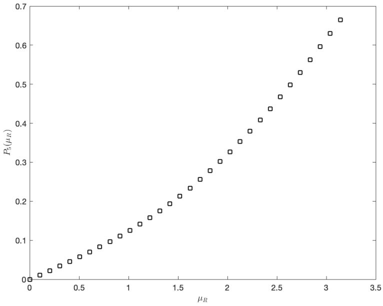
- Higher number of *significant* derivatives (with same input data)
- Worse condition number

Taylor Plots





Padé Analysis

- Rational function
- Interpolates singularities
- Can have only poles
- Branch cuts are represented as a series of poles and zeros

$$R_m^n(\mu) = \frac{\sum_{k=0}^{n} p_k \mu^k}{1 + \sum_{j=1}^{m} q_j \mu^j}$$

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Single-Point Padé

Dataset

$$\chi_1(0), \chi_2(0), \ldots, \chi_N(0)$$

$$R^n_m(\mu) = \frac{\sum_{k=0}^{n} p_k \mu^k}{1 + \sum_{j=1}^{m} q_j \mu^j} \qquad \text{with} \qquad n + m = N$$

Conditions: $R(0) = \chi_1(0)$, $R'(0) = \chi_2(0)$, $\ldots$, $R^{(N-1)}(0) = \chi_N(0)$

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Problem: Noisy high derivatives $\Longrightarrow$ Few parameters

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Multi-Point Padé

Dataset

$$\chi_1(\mu_1), \chi_1(\mu_2), \ldots, \chi_1(\mu_N), \quad \chi_2(\mu_{N+1}), \chi_2(\mu_{N+2}), \ldots, \chi_2(\mu_{N+M})$$

$$R^n_m(\mu) = \frac{\sum_{k=0}^{n} p_k \mu^k}{1 + \sum_{j=1}^{m} q_j \mu^j} \qquad \text{with} \qquad n + m = N + M$$

Conditions:

$$R(\mu_1) = \chi_1(\mu_1),\ R(\mu_2) = \chi_1(\mu_2),\ \ldots,\ R(\mu_N) = \chi_1(\mu_N)$$

$$R'(\mu_{N+1}) = \chi_2(\mu_{N+1}),\ \ldots,\ R'(\mu_{N+M}) = \chi_2(\mu_{N+M})$$

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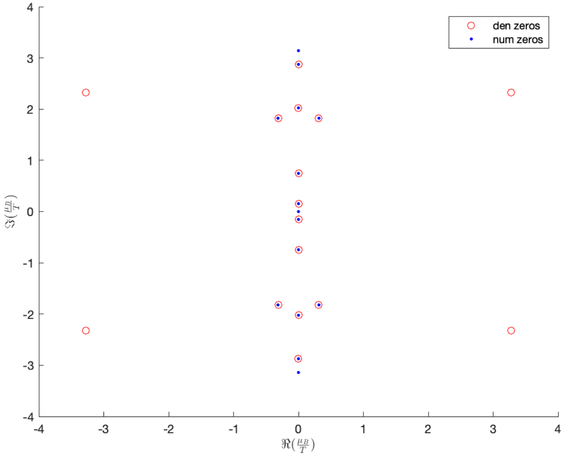
Problem: Convergence not rigorously defined

"Simplified" Multi-Point Padé



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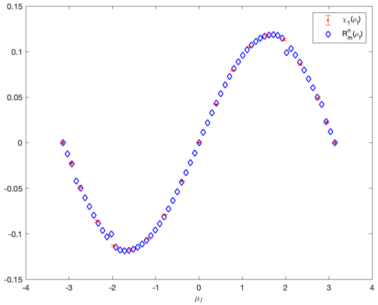
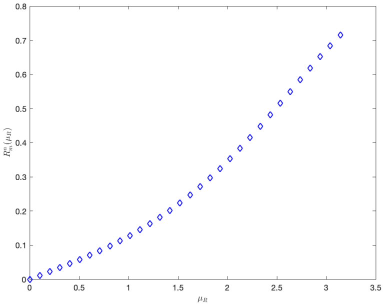
Padé Plots





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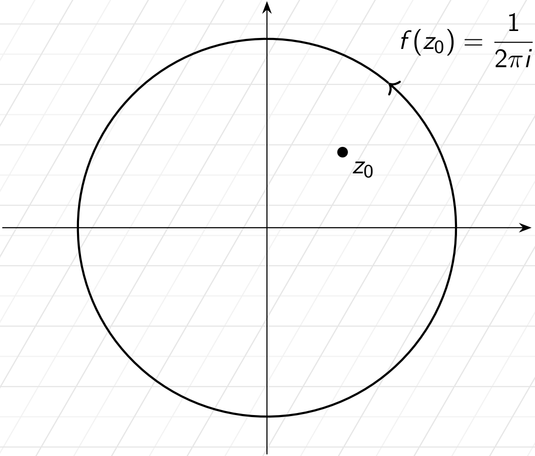
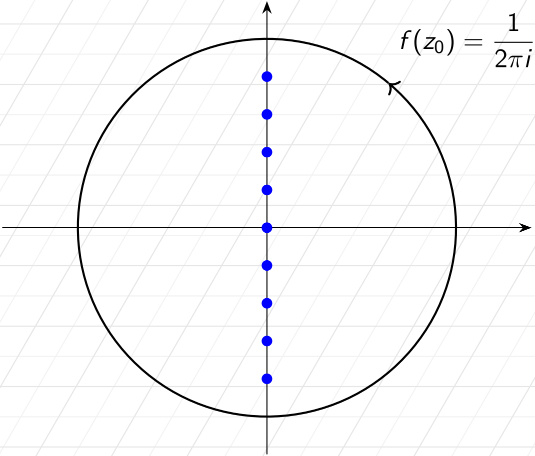
Inverse Problem



$$f(z_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z - z_0} dz$$

Inverse Problem



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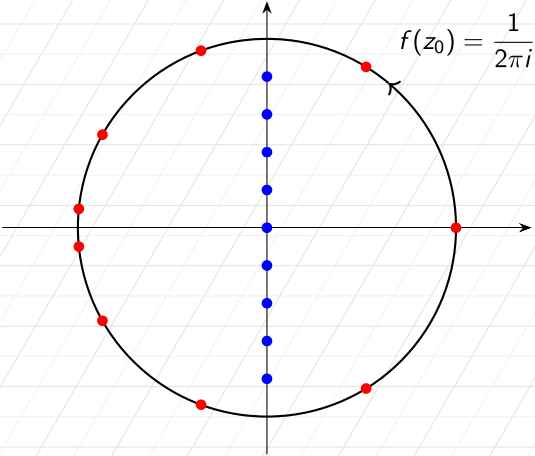

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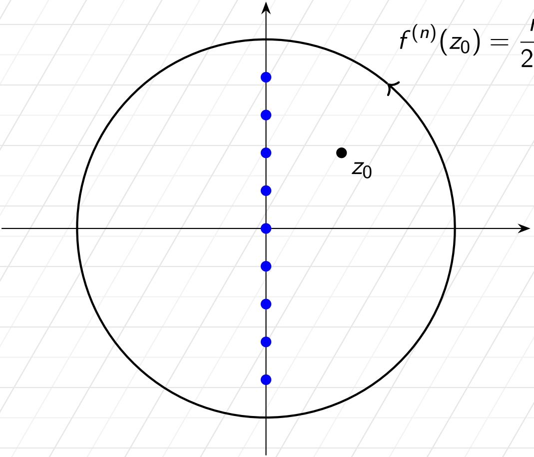
Gauss-Legendre

$$\chi_1(\mu_j) \simeq \frac{1}{2\pi}\sum_{k=1}^{n} w_k \frac{Re^{i\theta_k}}{Re^{i\theta_k} - \mu_j}\hat{\chi}_{1\,k}$$

$$j = 1, \ldots, n$$

Inverse Problem With Derivatives



$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

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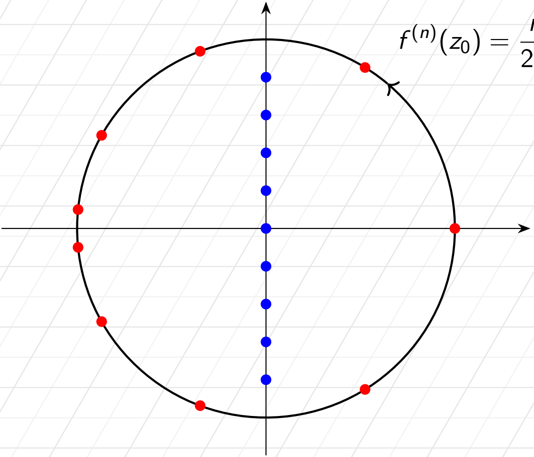

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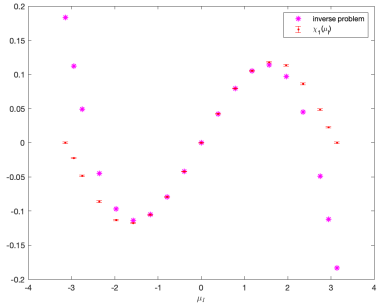
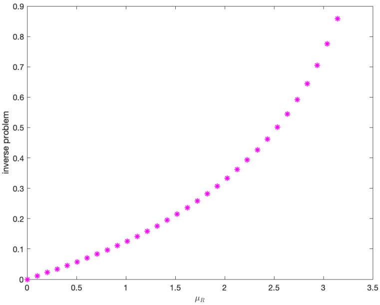
Inverse Problem Plots





Analytic continuation from $\mu^2 < 0$

$$\mu = \{+0.3928i, +0.7853i, +1.178i, +1.571i, \dots\}$$

$$\chi_1(\mu) = \sum_{k=0}^{\infty} \frac{\chi_{2k+1}(0)}{(2k+1)!}\mu^{2k+1}$$

Analytic continuation from $\mu^2 < 0$

$$\mu = \{+0.3928i, +0.7853i, +1.178i, +1.571i, \dots\}$$
$$\Downarrow$$
$$\mu^2 = \{-0.1543, -0.6167, -1.388, -2.468, \dots\}$$

$$\chi_1(\mu) = \sum_{k=0}^{\infty} \frac{\chi_{2k+1}(0)}{(2k+1)!}\mu^{2k+1}$$

$$\tilde{\chi}_1(\mu^2) = \chi_1(\mu)/\mu$$

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- Polynomial fit in μ^2

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- Polynomial fit in μ^2
- Recover the original function by multiplying by μ

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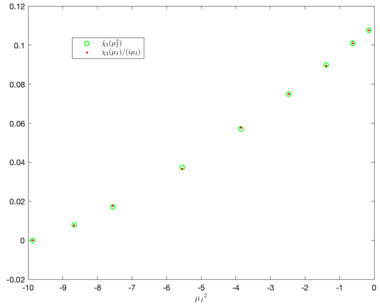
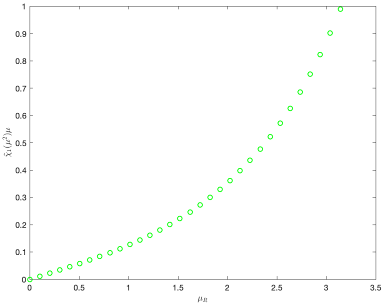
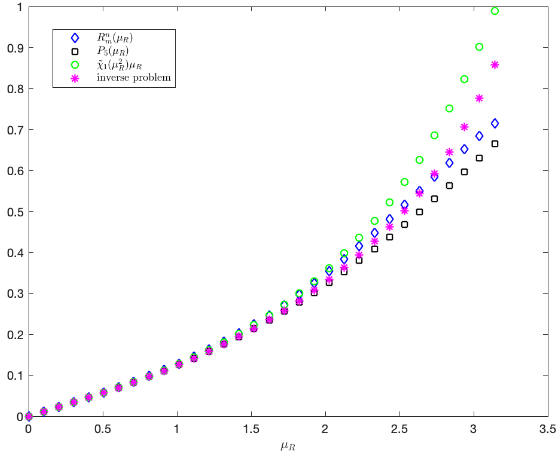
μ^2 Fit Plots





Results Comparison



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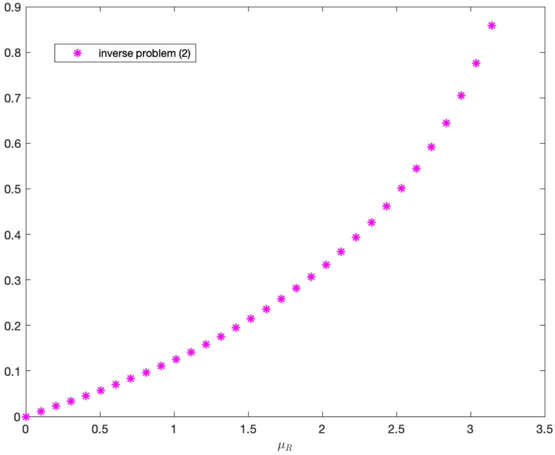
What About the Errors?



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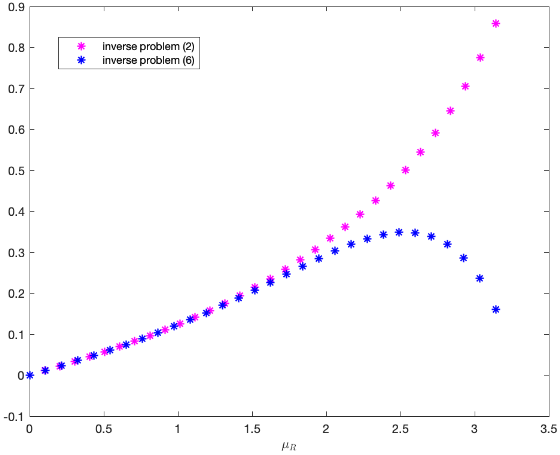

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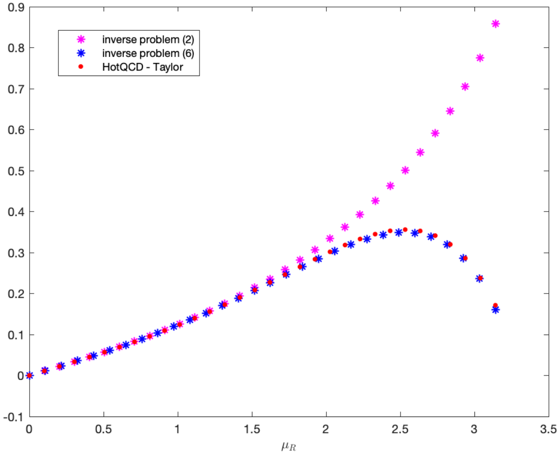

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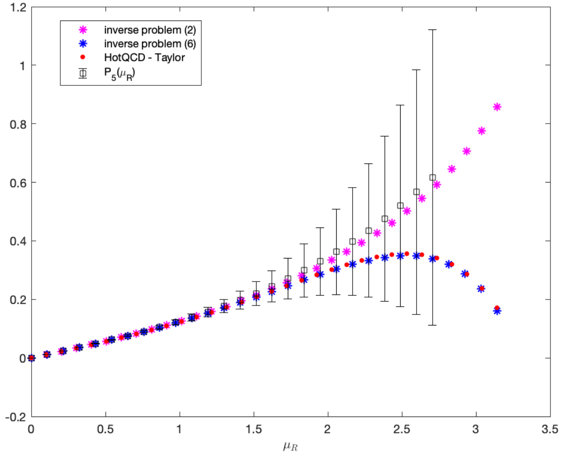

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In Conclusion...

- Various methods, with very different ways of operation, have been used for analytical continuation from imaginary μ_B/T to real μ_B/T
- There is a common region ($\mu_B/T \lesssim 1.5$) where every method agree with each other, with *small* error bars
- Outside this region ($\mu_B/T \gtrsim 1.5$), they significantly deviate from each other, but stay within the *much larger* error bars
- Beyond a given treshold, each method has a high systematic sensitivity with respect to the choice of the input data
- A thorough error analysis (both statistical and systematic) is being performed