Recent progress on the QCD phase diagram

More precision in a smaller volume

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September 12th 2024

BERGISCHE UNIVERSITÄT WUPPERTAL

WB collaboration

BMW collaboration

Motivation for a small volume

- It saves computational time:
  - fewer lattice points
  - less noise on derivatives
  - often reduces overlap problem in reweighing methods
- Investigation of transitions requires finite volume scan
- Heavy ion collisions take place in a finite volume (though it is not clear, how to translate to the lattice)
- Point of interest for DSE ([Bernhardt:2022mnx], [Bernhardt:2021iql])

1. Fluctuations in the Continuum [Borsanyi:2023wno]
   - State of the art
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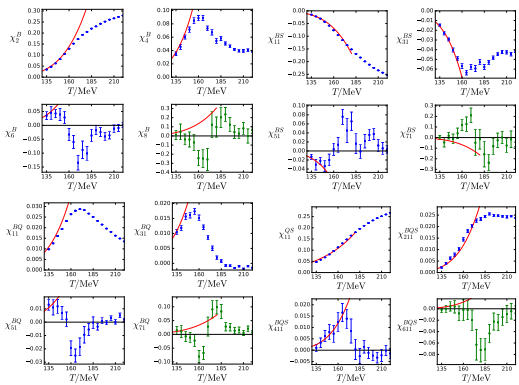
Fluctuations on the lattice

$$\chi^{B,Q,S}_{i,j,k} = \frac{\partial^{i+j+k}(p/T^4)}{(\partial\hat{\mu}_B)^i(\partial\hat{\mu}_Q)^j(\partial\hat{\mu}_S)^k}\,,\ \hat{\mu}_i = \frac{\mu}{T}$$

Today:

$$\chi^{B}_{i} = \frac{\partial^{i}(p/T^4)}{(\partial\hat{\mu}_B)^i}$$



[Borsanyi:2018grb]

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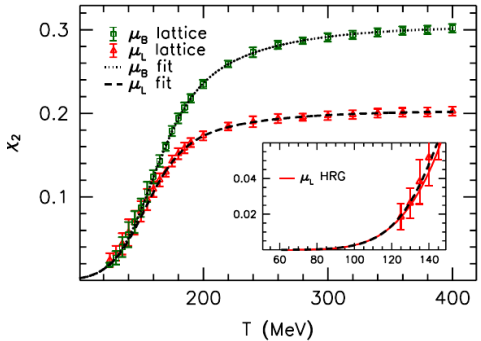
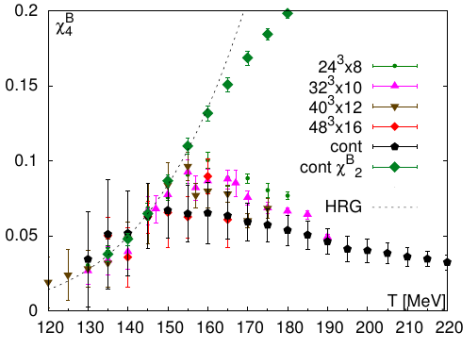
χ_2^B and χ_4^B in the continuum

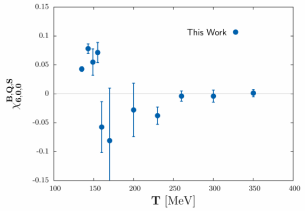


[Borsanyi:2012cr]

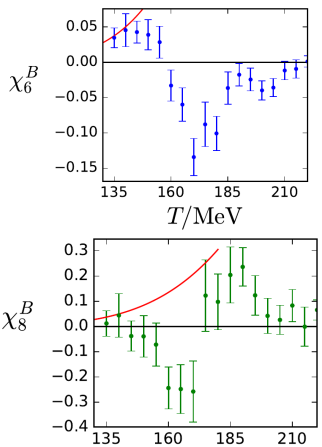


[Bellwied:2015lba]

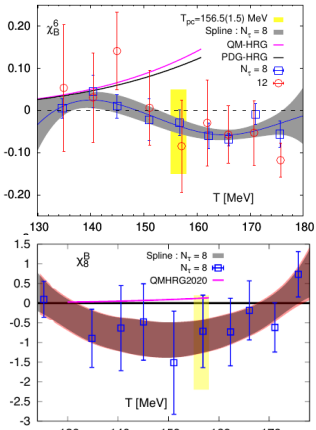
χ_6^B and χ_8^B on finite lattices



[DElia:2016jqh]
$N_t = 6$, 2stout



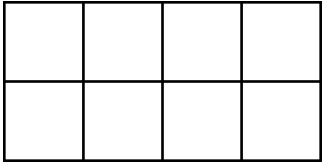
[Borsanyi:2018grb]
$N_t = 12$, 4stout



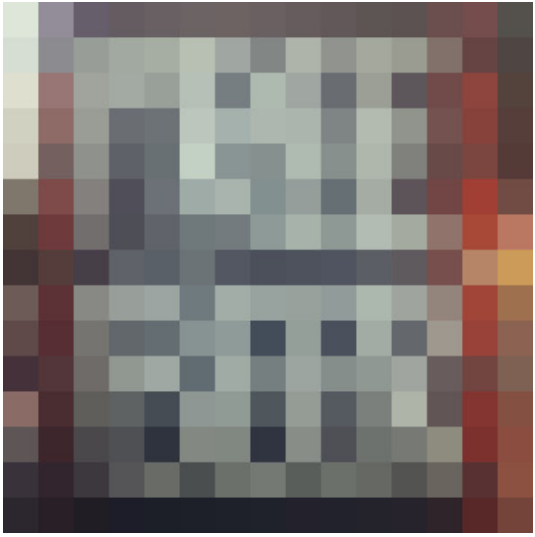
[Bazavov:2020bjn],
[Bollweg:2022rps]
$N_t = 8$, HISQ

The continuum limit

N_s



$\frac{1}{T}$



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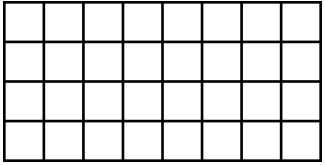

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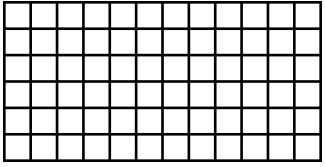

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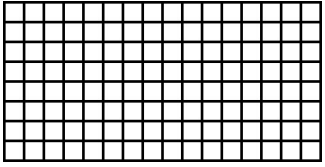

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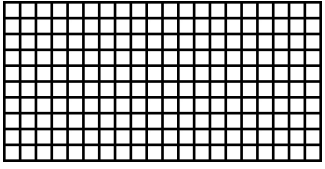

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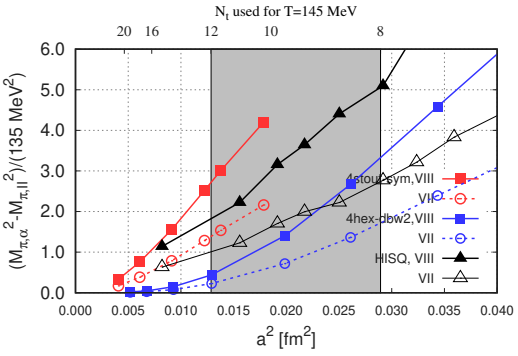
Lattice set-up



- 4Hex + dbw2 action
- lattices: $16^3 \times 8$, $20^3 \times 10$, $24^3 \times 12$
- $\mu_S = 0$
- scale setting with f_π and w_1
- Exponential definition of the chemical potential (introduced like a constant imaginary gauge field) $\rightarrow$ derivatives with respect to the chemical potential can be shown to be UV finite by virtue of a $U(1)$ symmetry [Hasenfratz:1983ba]

Systematic errors

We have to deal with systematic errors from the continuum extrapolation and the scale setting.

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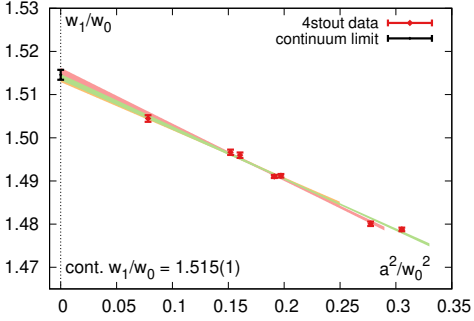
$$W(t)|_{t=w_1^2} = 0.7 \ , W(t) \equiv t\frac{d}{dt}\left\{t^2 \langle E(t)\rangle\right\}$$



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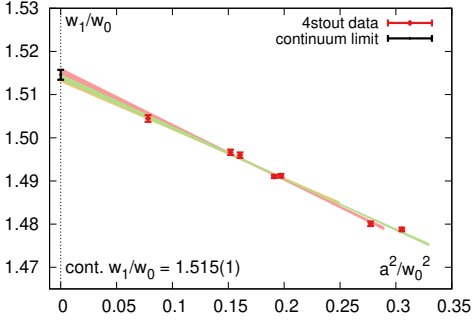

Continuum extrapolation:

$$\hat{O}(T, 1/N_\tau^2) = \sum_{i=1}^{M}\left(\alpha_i + \beta_i \frac{1}{N_\tau^2}\right) s_i(T) \ ,$$

s_i: set of basis spline function. We take three different sets of node points.

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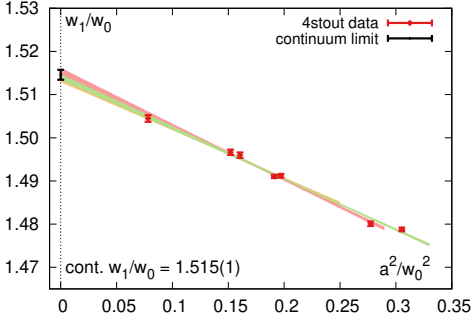

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The final results are obtained by combining the $6 = 2 \times 3$ analyses to construct a histogram.

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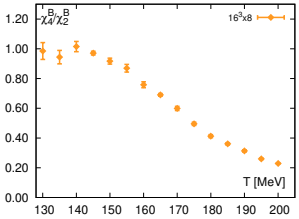
New continuum results



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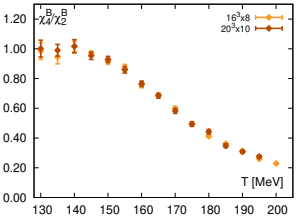

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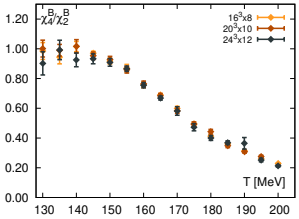

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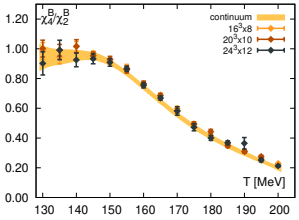

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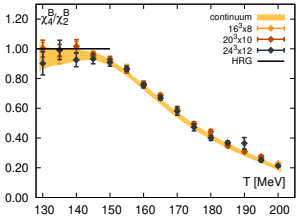

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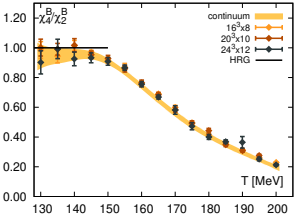

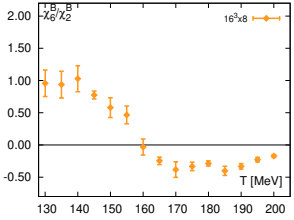

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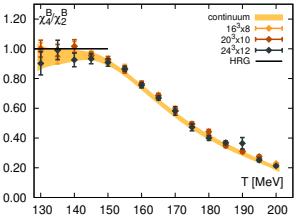

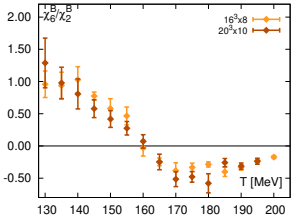

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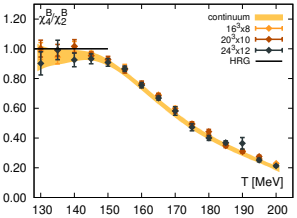

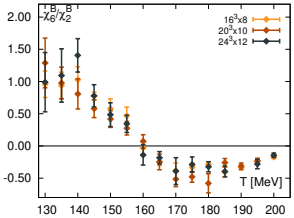

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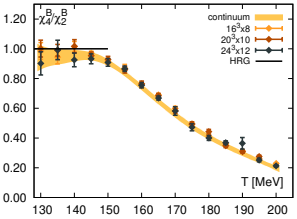

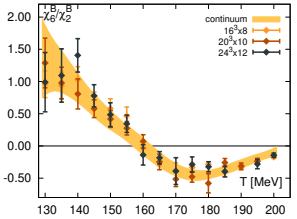

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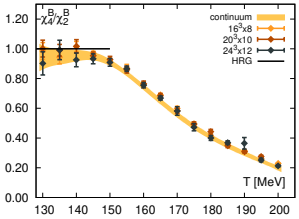

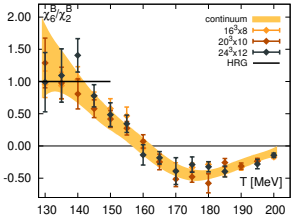

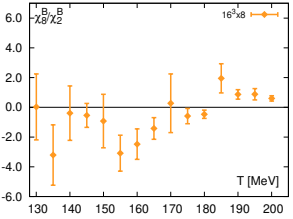

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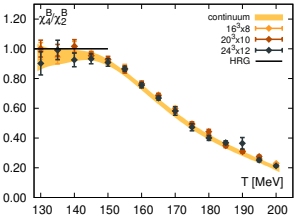

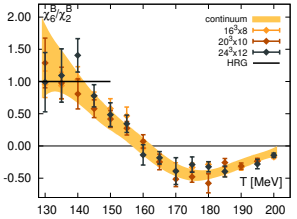

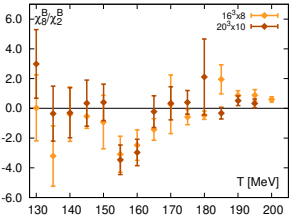

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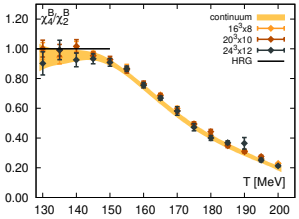

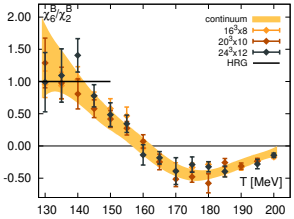

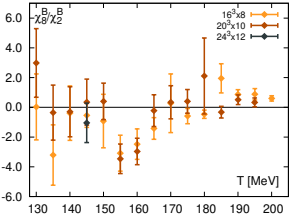

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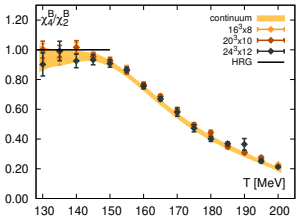

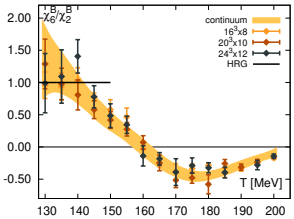

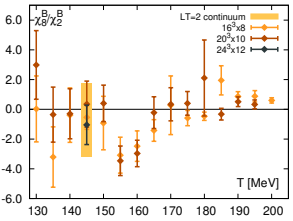

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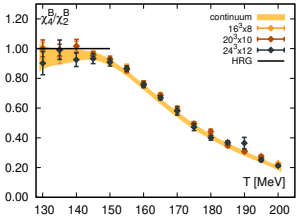
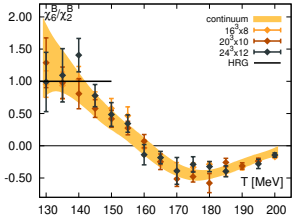
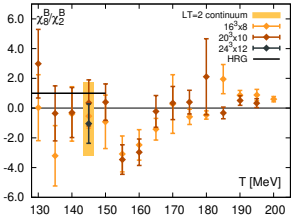
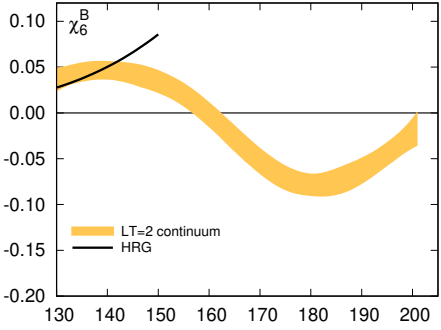






Comparision with different actions



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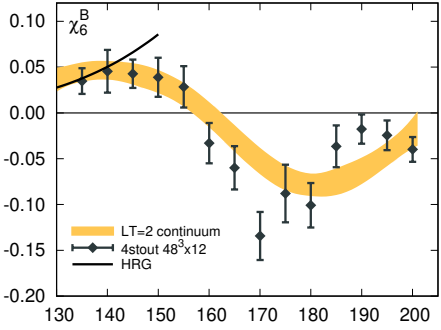

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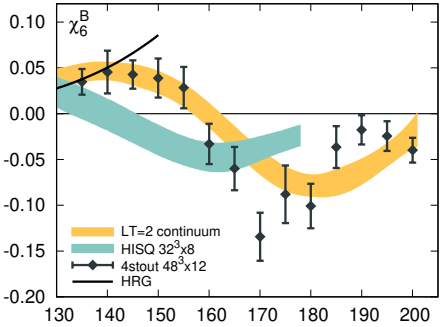

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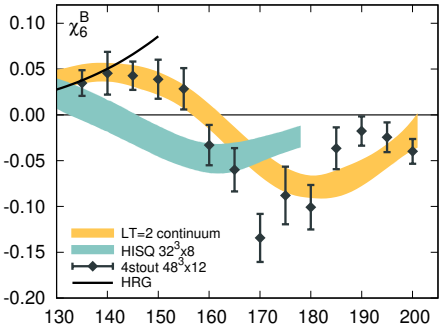

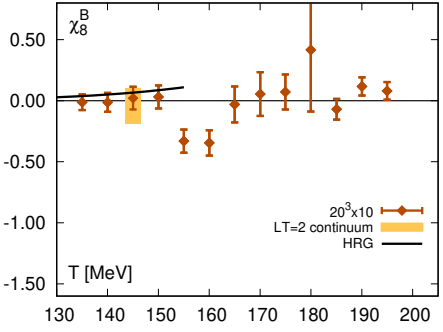

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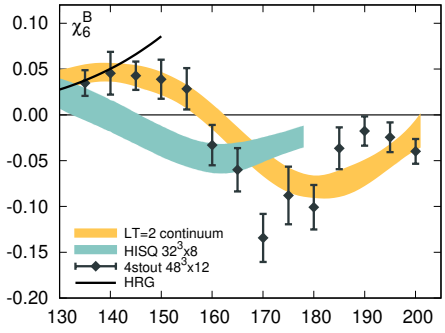

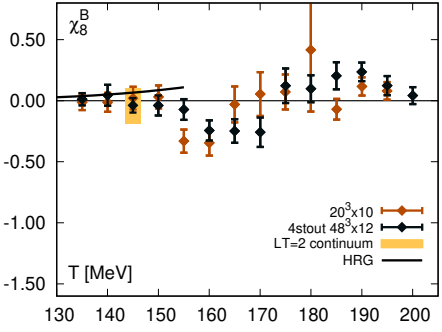

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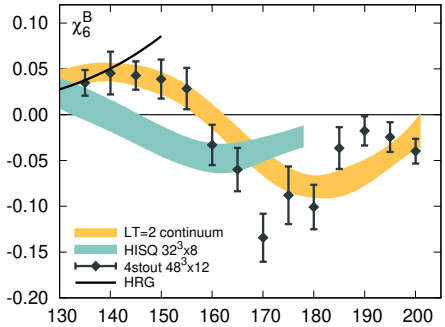
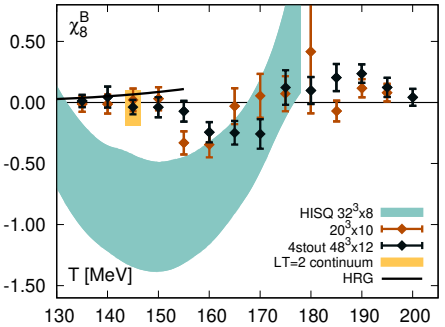
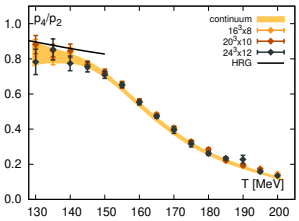
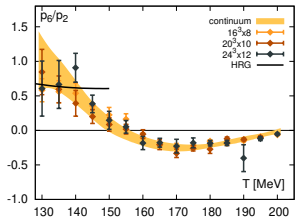
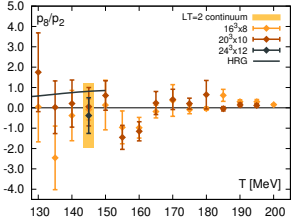




Strangness neutrality: $\langle n_S \rangle = 0$ – Continuum results







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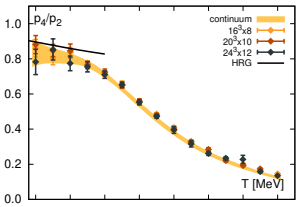
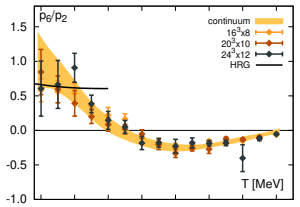
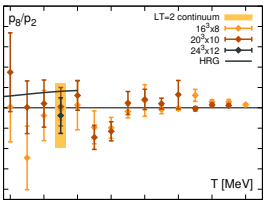
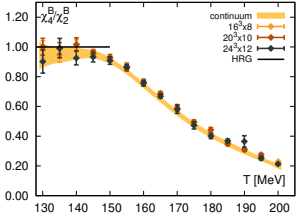
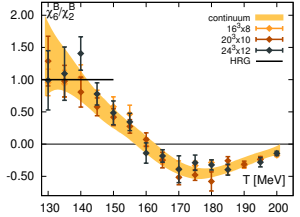
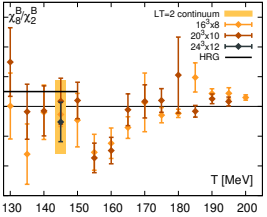




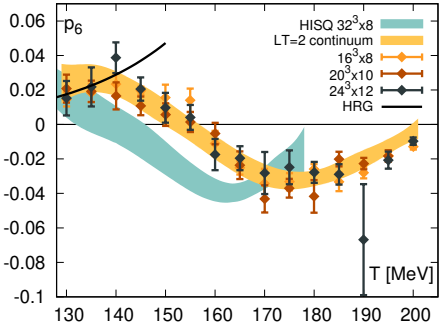


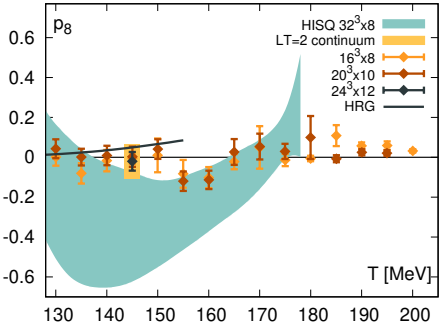




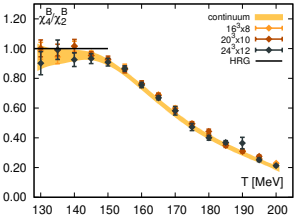


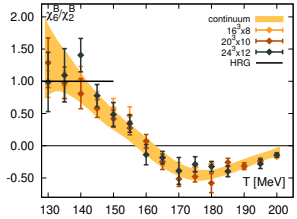
Strangness neutrality: $\langle n_S \rangle = 0$ –Comparision with different actions

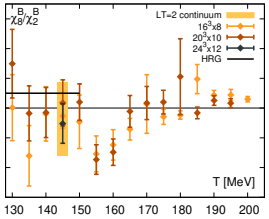




Conclusion







- First continuum extrapolated results for high order baryon number fluctuations
- A 4Hex + dbw2 action allowed for a continuum limit from $N_t = 8, 10, 12$
- With $LT = 2$ the volume effects are under control in the low temperature region.

1. Fluctuations in the Continuum [Borsanyi:2023wno]
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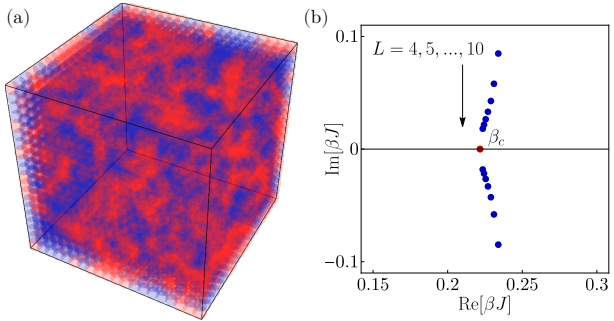

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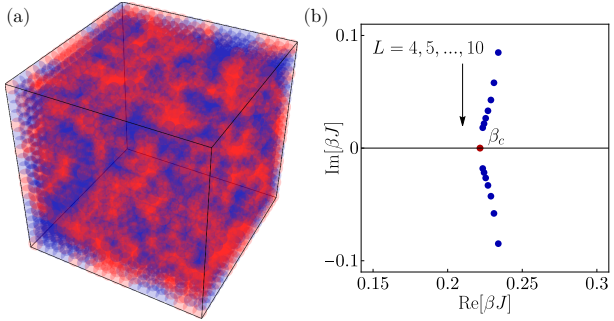

- a small excerpt of other work by others:
  - Parma Bielefeld [2405.10196]
  - Simran Singh PhD Thesis
  - Gökçe Başar [2312.06952]
  - Giordano, Pásztor [1904.01974]
  - Mukherjee, Skokov [1909.04639]
  - Wakayama et al. [1802.02014]

What we work with

- To access zeros of Z, we can look at $\log(Z) = p$
- Written as a Tayor series

$$\Delta p = \frac{p(T,\mu_B) - p(T,0)}{T^4} = \sum_{n=0} \frac{\chi_{2n}(T)}{(2n)!} \left(\frac{\mu_B}{T}\right)^{2n} \qquad \chi_n = \frac{\partial^n (p/T^4)}{(\partial \mu/T)^2}$$

- χ_{2n} is given by simulations at $\mu_B = 0$
- These coefficients can be used in conjunction with a scaling relation for extrapolation
- In addition we can model:

$$\chi_1(T,\mu_B) = \sum_{n=1} \frac{\chi_{2n}(T)}{(2n-1)!} \left(\frac{\mu_B}{T}\right)^{2n+1} \qquad \chi_2(T,\mu_B) = \sum_{n=0} \frac{\chi_{2n+2}(T)}{(2n)!} \left(\frac{\mu_B}{T}\right)^{2n}$$

$\mathrm{Pade}[m,n] : \frac{P_m(x)}{1+Q_n(x)} = \frac{\sum_{i=0}^{m} a_i x^i}{1+\sum_{i=1}^{n} b_i x^i}$

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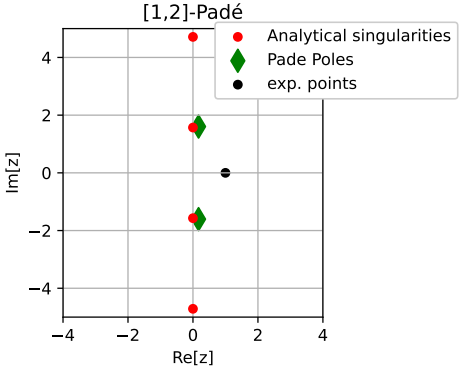

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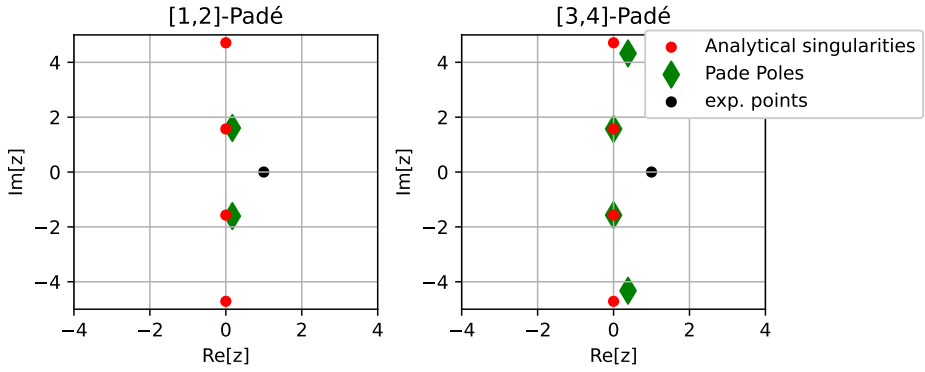
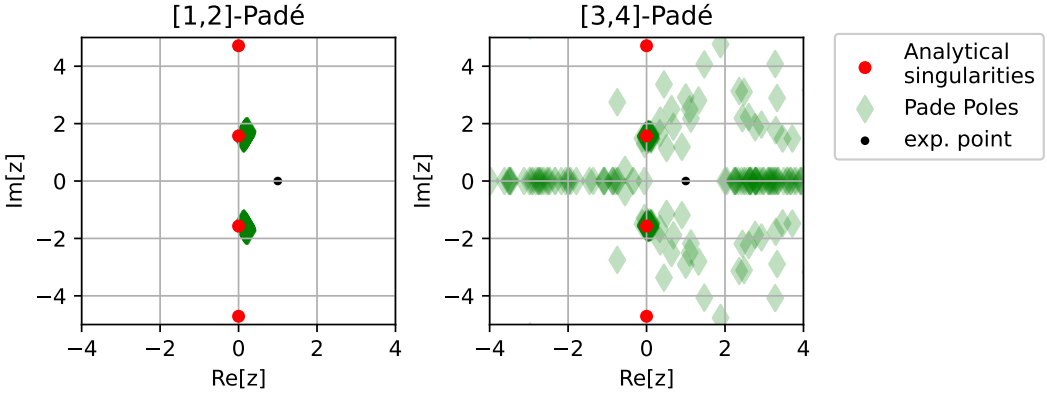


Padé with Noise

- Adding 3% of noise to each of the derivatives of $1/\cosh(z)$:



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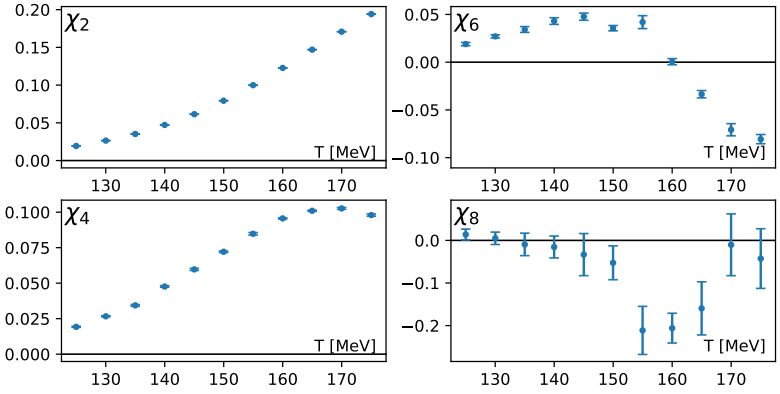
Lattice Setup

- Volume : $16^3 \times 8$
- $\mathcal{O}(5 \cdot 10^5)$ configurations per T
- 4 HEX smearing
- Simulated at physical quark mass



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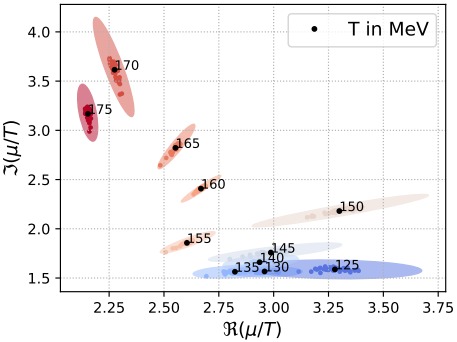

modeling χ_2

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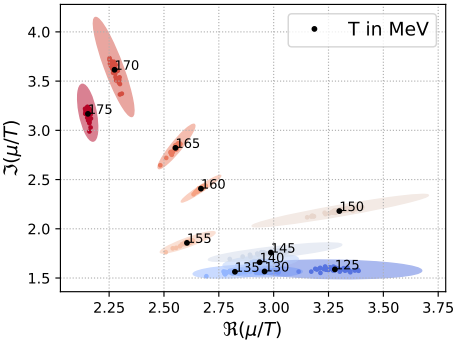

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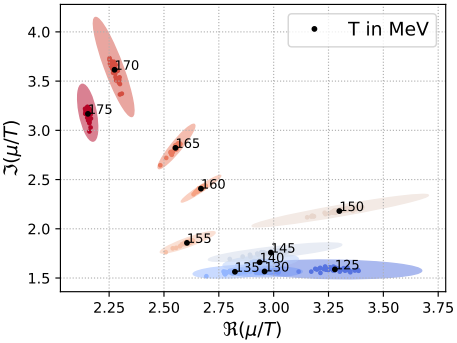
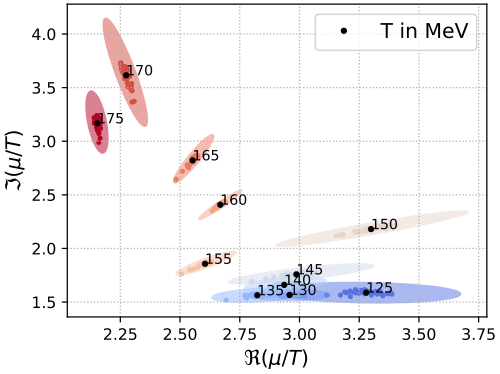
- Extrapolate T_c, via eg. $\mathrm{Im}[\mu_B] = \kappa\,(\Delta T)^{\beta\delta}$
- Estimate systematic effects
  - Use Δp or χ_1 or χ_2
  - Vary fit range in temperature
  - Use different scaling ansatz



modeling χ_2

Varying the approximated function



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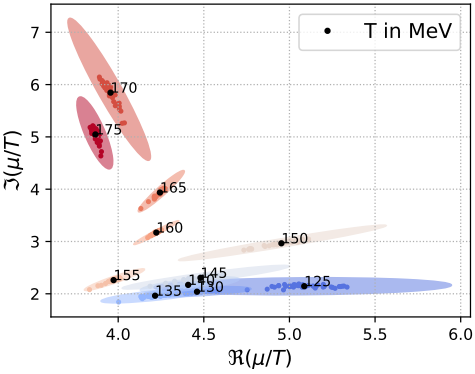
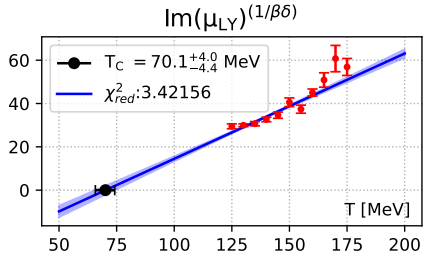
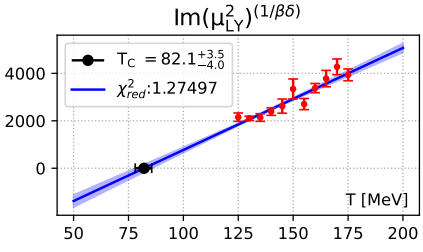
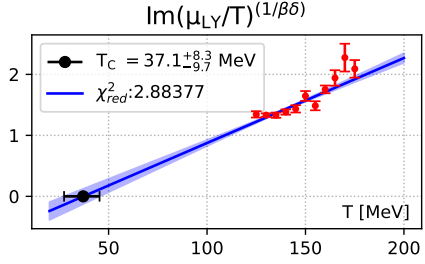


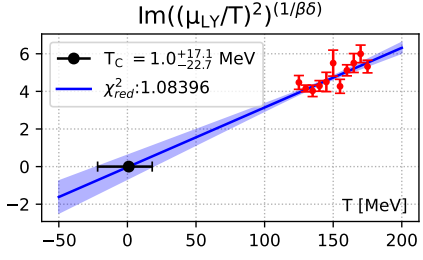
modeling $p - p_0$

Varying the scaling variable for χ_2 : $\kappa\Delta T = \mathrm{Im}(x)^{1/\beta\delta}$

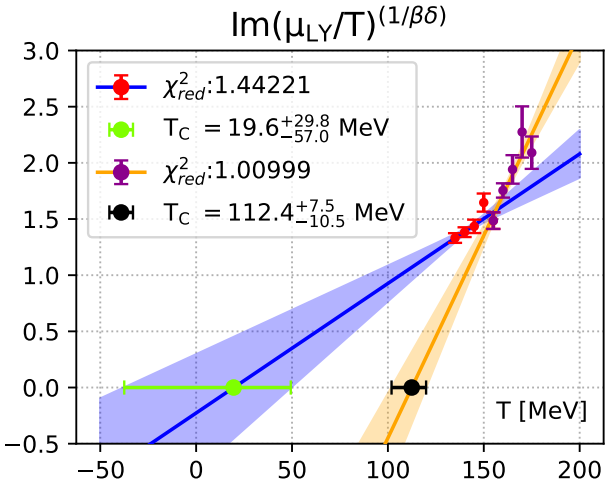






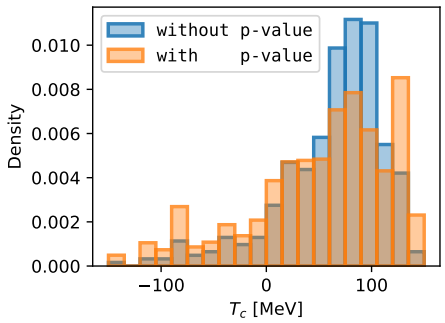


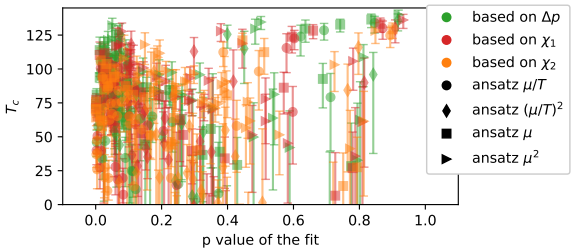
Varying the fit range for χ_2



Combination

Based on $3\,[\Delta p, \chi_1, \chi_2] \times 4\left[\mu, \mu^2, \frac{\mu}{T}, \frac{\mu^2}{T^2}\right] \times 36$ Temp. ranges $= 432$ Fits





Conclusion

- We used a high statistics campaign to look for the Lang Yee Zeros
- We can estimate T_c with a reasonable statistical error but a high systematic error
- No systematic control of the Padé order
- Approximation from a great distance requires strong assumptions

- Reliable prediction of the CEP with LYZ from the lattice data requires great care and consideration

1. Fluctuations in the Continuum [Borsanyi:2023wno]
   - State of the art
   - Our set-up and analysis
   - Results
2. Criticallity from fluctuations
   - Lee-Yang zeros and pad[Pleaseinsertintopreamble] approximations
   - Results
3. Observerbles taylored to small voulme [Borsanyi:2024dro]

Properties of QCD crossover

Chiral

- $SU(2) \times SU(2)$ symmetry in the limit $m_q \to 0$
- order parameter: chiral condensate $\langle \bar{\psi}\psi \rangle$
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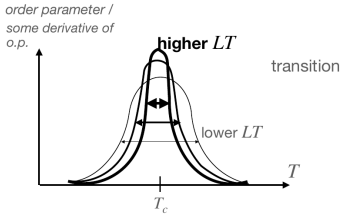
We aim to investigate the volume dependence of T_c and the strength of the transition for $\mu_B > 0$ for different definitions

- $m_q \neq 0, \infty \to$ both only approximate order parameters
- Lattice $\to$ true transitions do not happen in a finite volume
- how do chiral and deconfinement observables behave in the limit $LT \to \infty$

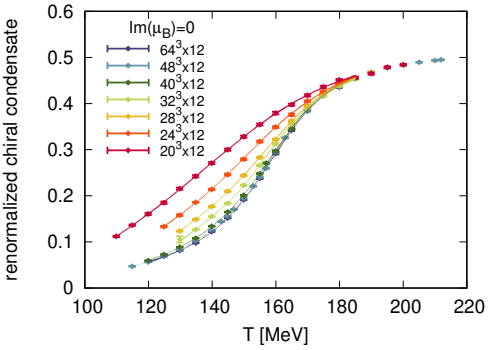


Lattice set-up

- tree-lever Symanzik improved gauge action
- $N_f = 2 + 1 + 1$ staggered fermions with 4 stout smearing
- $N_t = 12,\ N_s = 20, 24, 28, 32, 40, 48, 64$
- For $N_s = 32, 40, 48$ simulations also at $\frac{\mu_B^I \pi}{8T} = 3, 4, 5, 6, 6.5, 7$
- strangness neutrality setting $\langle n_S \rangle = 0$

Chiral observables

$$\langle \bar{\psi}\psi \rangle = \frac{T}{V}\frac{\partial \log Z}{\partial m_{ud}}$$



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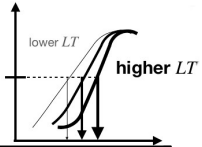

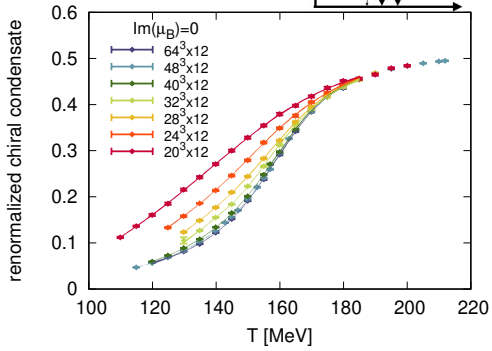

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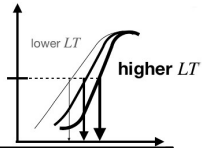
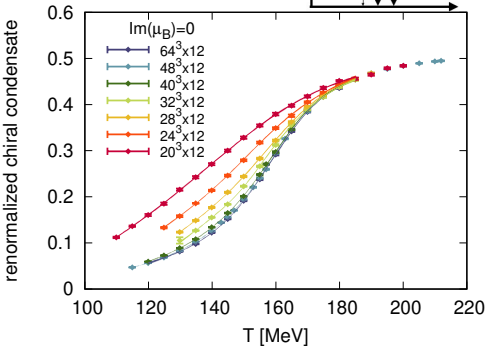
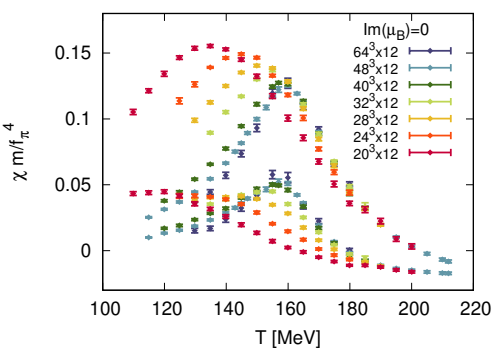




$$\chi = \frac{T}{V}\frac{\partial^2 \log Z}{\partial^2 m_{ud}},\ \chi_{\rm disc} = \frac{T}{V}\frac{\partial^2 \log Z}{\partial m_u \partial m_d}\bigg|_{m_u=m_d}$$



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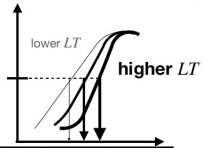

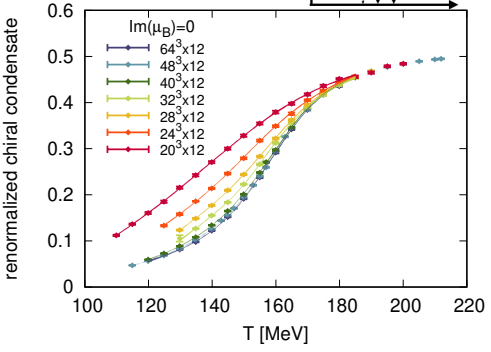

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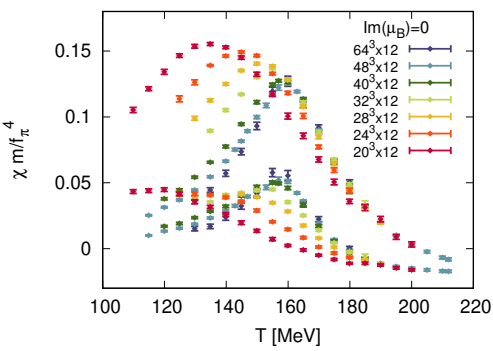
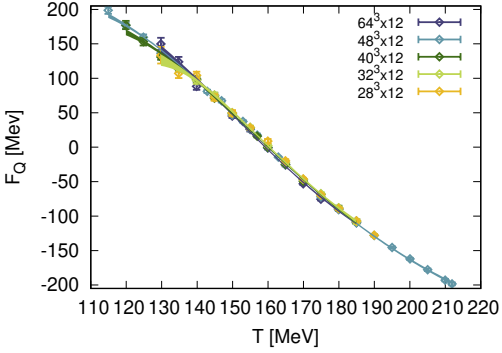


T_c increases with the volume

Deconfinement observables

$$F_Q = -T\log\left(\tfrac{1}{V}|\langle\textstyle\sum_{\vec{x}} P(\vec{x})\rangle_T|\right)$$



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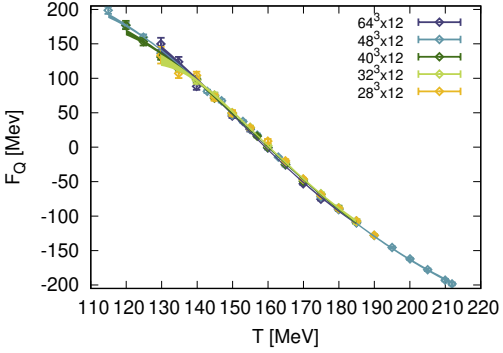
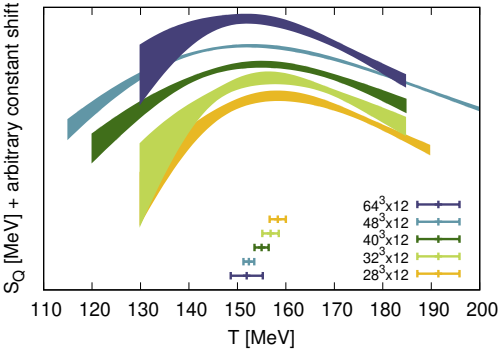


$$S_Q = -\frac{\partial F_Q}{\partial T}$$



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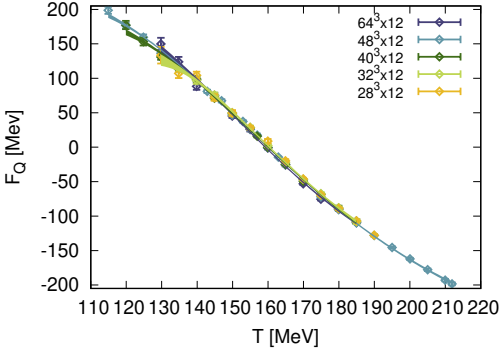

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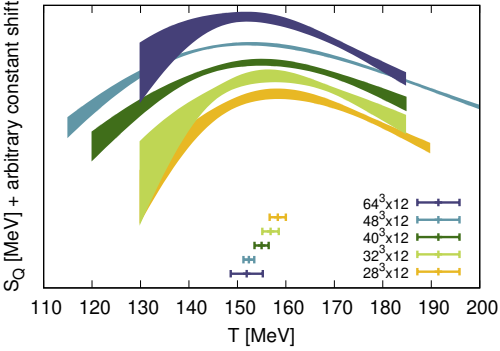
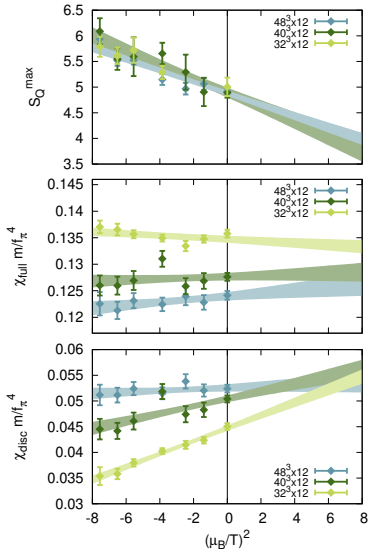


Here volume effects are milder then for the chiral observables.

$\mu_B > 0$: Height of peaks of χ, χ_{disc}, S_Q

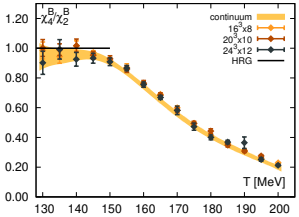


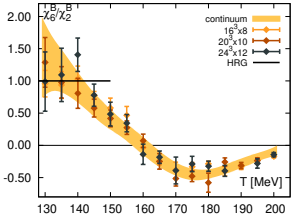
- A rise of the peak hight, could indicate a strengthening of the transition
- This only happens for χ_{disc}, but seems to decrease with volume
- S_Q only has very mild volume effects

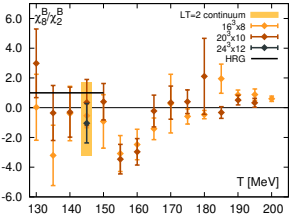
Conclusion

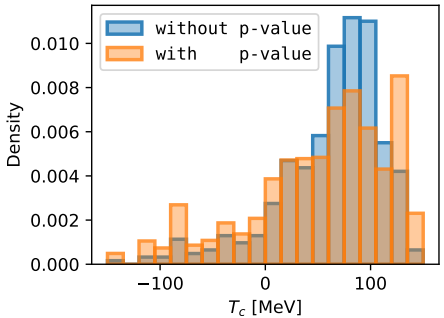
- In the thermodynamic limit $T_c^{(S_Q)} < T_c^{(\chi_{\text{disc}})} < T_c^{(\chi)}$
- The volume dependence of T_c depends on the definition
- The deconfinement observables we investigated have a milder volume dependence than the chiral ones.
- Only the peak of χ_{disc} increases with μ_B

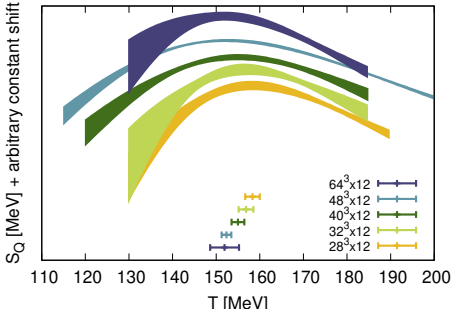
Summary


χ_4^B/χ_2^B
continuum
16^3x8
20^3x10
24^3x12
HRG
T [MeV]


χ_6^B/χ_2^B
continuum
16^3x8
20^3x10
24^3x12
HRG
T [MeV]


χ_8^B/χ_2^B
LT=2 continuum
16^3x8
20^3x10
24^3x12
HRG
T [MeV]


without p-value
with p-value
Density
T_c [MeV]


S_Q [MeV] + arbitrary constant shift
64^3x12
48^3x12
40^3x12
32^3x12
28^3x12
T [MeV]

T [MeV]	$16^3 \times 8$	$20^3 \times 10$	$24^3 \times 12$
130	31741	71090	68689
135	33528	106403	66960
140	34977	69690	75229
145	336975	188571	111435
150	65374	108481	81590
155	34057	96985	89559
160	37145	68619	94053
165	156044	67668	98744
170	34397	42314	11831
175	34180	36522	12089
180	30594	25229	12727
185	30951	18396	13066
190	30293	18267	7141
195	31276	15008	7199
200	31919	13346	7390

Table: Number of configurations analyzed on our three lattice geometries.