The anomalous $U_A(1)$ symmetry

and a *second* order chiral phase transition

RDP & F. Rennecke, 2401.06130

F. Giacosa, G. Kovacs, P. Kovacs, RDP & F. Rennecke, 2410.?

RDP, 2410.?

or: how being wrong can feel so right…

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θ-dependence from instantons, without quarks

Topological susceptibility of QCD, no quarks:

$$\mathcal{S}_\theta = \mathcal{S} + i\theta Q \; ; \; \mathcal{S} \sim \int d^4x \, \mathrm{tr} \, G^2_{\mu\nu} \; ; \; Q \sim \int d^4x \; \mathrm{tr} \, G_{\mu\nu}\widetilde{G}_{\mu\nu} = \pm \text{ integer}$$

Q = integer → periodicity in θ:

$$\mathrm{e}^{-V E(\theta)} = \int \mathcal{D}A_\mu \, \mathrm{e}^{-\mathcal{S}_\theta} \quad ; \quad E(\theta + 2\pi) = E(\theta)$$

$Q = \pm 1 \; : \; S_{\mathrm{inst}} = \dfrac{8\pi^2}{g^2}$ ‘t Hooft: dilute instanton gas with Q = ±1, ±2...

$$E(\theta) - E(0) \sim \cos(\theta) \, \mathrm{e}^{-8\pi^2/g^2} + \# \cos(2\theta) \, \mathrm{e}^{-16\pi^2/g^2} + \ldots$$

Jing-Yuan Chen, 2406.06673:
“Instanton Density Operator in Lattice QCD from Higher Category Theory”

instanton # at *non*-zero lattice spacing! ~ generalized Villian transformation

θ-dependence from instantons, with quarks

With N_f flavors of massless quarks, *no* θ dependence:

$$\theta_{\rm eff} = \theta + \log\det m$$

Adler-Bell-Jackiw anomaly and Atiyah-Singer index theorem.

For N_f flavors, generate terms invariant under global flavor of $SU_L(N_f)$ x $SU_R(N_f)$,

but *not* the anomalous axial $U_A(1)$ symmetry.

$$\Phi = \overline{q}_L q_R$$

$$\mathcal{L}_{\rm eff} \sim \#({\rm e}^{i\theta}\,\det\,\Phi + {\rm c.c.}){\rm e}^{-8\pi^2/g^2} + \#'({\rm e}^{2i\theta}(\det\,\Phi)^2 + {\rm c.c.}){\rm e}^{-16\pi^2/g^2} + \ldots$$

Infinite series of terms which violate $U_A(1)$.
For *all* such terms, eliminate θ just by $U_A(1)$ rotation, $\Phi \rightarrow {\rm e}^{-i\theta/N_f}\,\Phi$

$\det\Phi \sim \Phi^{N_f}$ *Cubic* for 3 flavors → *1st* order chiral trans. RDP & Wilczek '84.

$(\det\,\Phi)^2$: RDP & Rennecke, 1910.14052

θ-dependence at large N_c, without quarks

Two flavors:

$$\Phi = (\sigma + i\eta)\mathbf{1} \; ; \; \mathrm{tr}\,\Phi^\dagger \Phi = \sigma^2 + \eta^2 \; ; \; -(\det\Phi + \det\Phi^\dagger) \sim -\sigma^2 + \eta^2$$

~ det Φ makes the η heavy. For $N_f = 3$, the η' is heavy.

't Hooft: QCD ~ large N_c. But instantons vanish as $N_c \to \infty$!

Keeping $\lambda = g^2 N_c$ fixed, a single instanton ~ $\exp(-8\pi^2/\lambda\; N_c)$

Witten & Veneziano '79: at large N_c, small N_f, $E(\theta)$ periodic in $2\pi/N_c$:

$$E(\theta) = N_c^2\, \widetilde{E}(\theta/N_c) = \#\, \theta^2$$

Like fractional instantons with $Q = \pm 1/N_c$. RDP & V.P. Nair, 2206.11284

Witten '79: at large N_c, small N_f, η' is light:

$\widetilde{f}_\pi \sim 1$ as $N_c \to \infty$

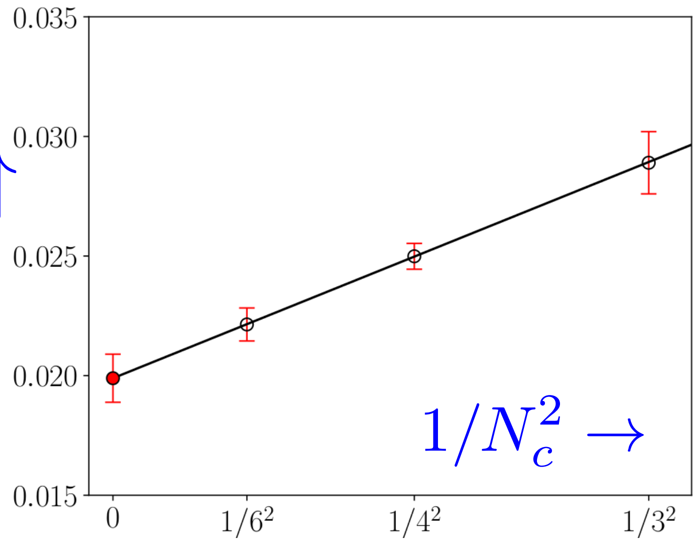
$$m_{\eta'}^2 = \frac{4}{\widetilde{f}_\pi^2} \frac{N_f}{N_c} \left(\frac{d^2 E}{d\theta^2} \right)^{\text{no qks}}_{\theta=0}$$

θ-dependence at large N_c, without quarks: lattice @ T=0

Bonanno, Bonati & D'Elia, 2012.14000 SU(N_c), N_c = 3, 4, 6, without quarks

$$E(\theta) - E(0) = \frac{\chi}{2}\,\theta^2\,(1 + b_2\,\theta^2 + \ldots)$$

$\frac{\chi}{\sigma^2}$ ↑



$$\frac{\chi}{\sigma^2} = 0.19 + \frac{.08}{N_c^2}$$

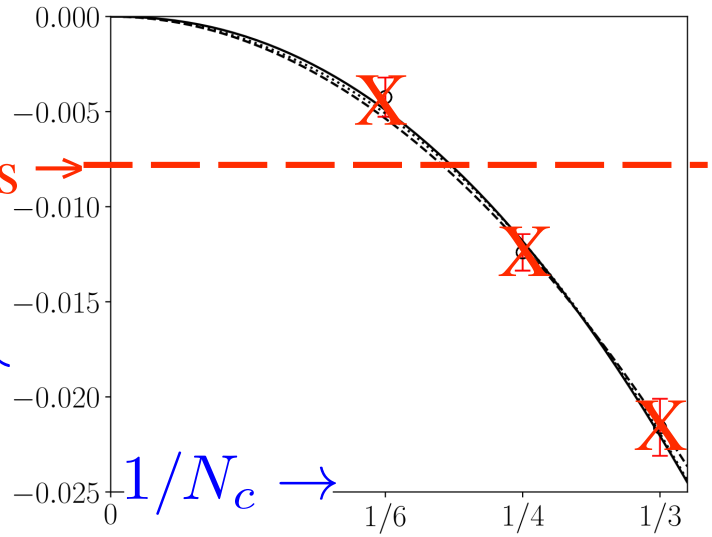
χ=topological susceptibility
σ = string tension

b₂: *if* instantons, Q = integer, $b_2 = -1/12 \sim -.08$, *independent* of N_c! Lattice finds

$$b_2 \approx \frac{-0.19}{N_c^2}$$

inst.'s →

b_2 ↑



Dilute gas of frac. inst.'s, $Q=1/N_c$: $b_2 = -.08/N_c^2$

So *dense* liquid of *fractional* inst's, $Q \sim 1/N_c$.

θ-dependence in QCD: lattice, T ≠ 0

Petreczky, Schadler & Sharma, 1606.03145:
QCD, $N_c=3$, $N_f = 2+1$.

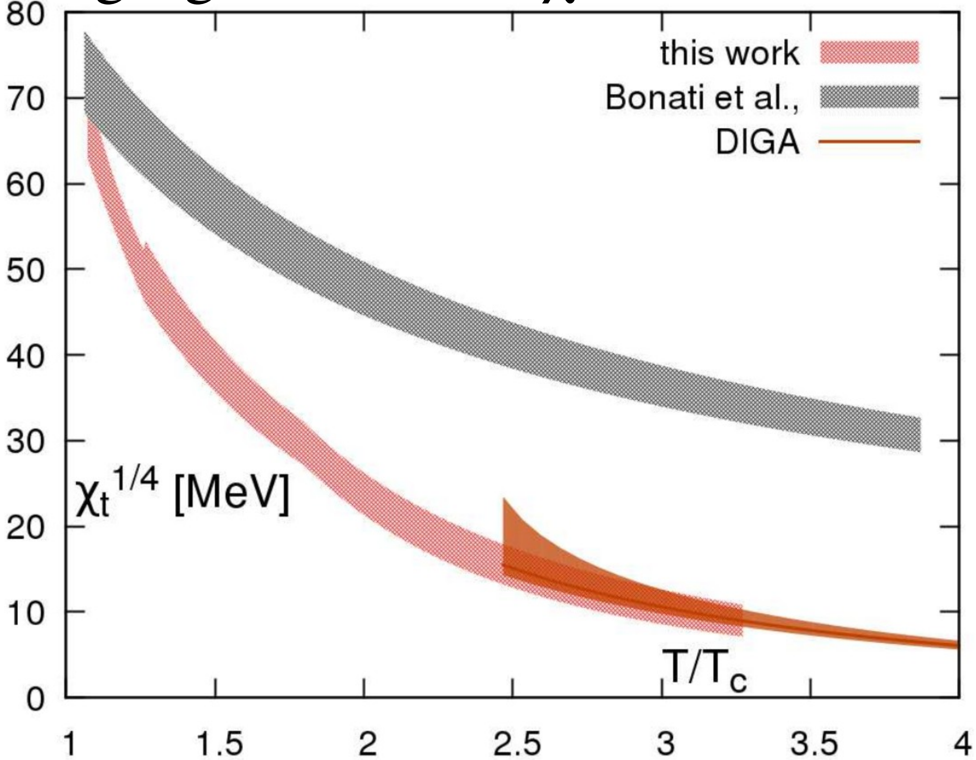
$$E(\theta) - E(0) = \frac{\chi}{2}\,\theta^2 + \ldots$$

Three regimes: $T < T_{ch} \sim 155$ MeV: $\chi(T) \approx$ const. "Veneziano-Witten" (VW)

T: $T_{ch} \to T_{deconf} \sim 300$ MeV: $\chi(T)$ *not* DIG, falls slower with T

$T > T_{deconf}$: Dilute Instanton Gas (DIG)

Pure gauge: $T<T_{deconf}$, $\chi \approx$ const. , VW. $T>T_{deconf}$, DIG



DIG: (leading) $1/T^c$ given by classical S!
Corrections "a" ~ 1/10 lattice;
sensitive to 2-loop corrections?

$$\chi_{\rm DIG} \sim T^4 {\rm e}^{-8\pi^2/g^2}$$

$$g^2(T) \sim \frac{8\pi^2}{c\log T} \to \chi_{\rm DIG} \sim \frac{a}{T^{c-4}}$$

Veneziano-Witten: large $N_c \gg N_f$, T=0

Under global axial $U_A(1)$ transformation $q_{L,R} \to \mathrm{e}^{\mp i\phi_A/2}$, $\Phi = \overline{q}_L q_R \to \mathrm{e}^{i\phi_A}$

With massless quarks, can *always* eliminate θ by $\Phi \to \mathrm{e}^{-i\theta/N_f}\,\Phi$

At large $N_c \gg N_f$, *uniquely* fixes effective Lag.

$$\mathcal{L}_\theta(\Phi, Q) = \mathcal{L}_0(\Phi) + i(\theta - i/2\,\mathrm{tr}\log(\Phi/\Phi^\dagger))Q \quad , \quad Q \sim \int d^4x\,\mathrm{tr}\,G_{\mu\nu}\widetilde{G}^{\mu\nu}$$

Introduce Abelian "ghost", $Q \sim \epsilon_{\mu\nu\alpha\beta}F^{\mu\nu\alpha\beta}$ and a term ~ Q^2. Int.'g out Q,

$$\mathcal{L}_\theta(\Phi, Q) = \mathcal{L}_0(\Phi) + \widetilde{f}_\pi^{\,2}\, m_{\eta'}^2 (\theta - i/2\,\mathrm{tr}\log(\Phi/\Phi^\dagger))^2 + \ldots$$

First term in a low energy expansion.
Veneziano, diVecchia, Luscher, Witten...'79-'82

$$m_{\eta'}^2 = \frac{4}{\widetilde{f}_\pi^2}\frac{N_f}{N_c}\left(\frac{d^2E}{d\theta^2}\right)^{\mathrm{no\ qks}}_{\theta=0}$$

Corrections power series in ~ Q^2 and $(\theta - i/2\,\mathrm{tr}\log(\Phi/\Phi^\dagger))^2$

Veneziano-Witten: $N_f \sim N_c$, first try

At large $N_c \gg N_f$ the η' is light, $m^2_{\eta'} \sim N_f/N_c$.

When $N_c \sim N_f$ the η' is heavy, $m^2_{\eta'} \sim N_f/N_c \sim 1$. Includes QCD.

Assume: for massless quarks, θ-independence implies for $N_c \sim N_f$, L_{eff} is constructed from powers of:

$$(\theta - i/2\,\mathrm{tr}\log(\Phi/\Phi^\dagger))^2$$

Need the log to ensure an axial rotation can eliminate θ.

First such term is a mass for the η':

$$\mathcal{L}_\theta(\Phi, Q) = \mathcal{L}_0(\Phi) + M^4(\theta - i/2\,\mathrm{tr}\log(\Phi/\Phi^\dagger))^2 + \ldots$$

What is M^4? For $Nc \rightarrow \infty \gg N_f$, $M^4 \sim d^2 E(\theta)/d\theta^2$.

But in general, E(θ) is independent of θ with massless quarks, $d^2 E(\theta)/d\theta^2 = 0$!

Can $M^4 \sim$ constant? But then $(\theta - i/2\,\mathrm{tr}\log(\Phi/\Phi^\dagger))^2$ is singular as $\Phi \rightarrow 0$!

Veneziano-Witten: $N_f \sim N_c$, improved

Second guess:

$$\mathcal{L}_\theta(\Phi, Q) = \mathcal{L}_0(\Phi) + M^2\, \mathrm{tr}(\Phi^\dagger \Phi)\, (\theta - i/2\, \mathrm{tr} \log(\Phi/\Phi^\dagger))^2 + \ldots$$

Regular as $\Phi \rightarrow 0$. Usual chiral Lagrangian:

$$\mathcal{L}_0(\Phi) = \mathrm{tr}|\partial_\mu \Phi|^2 + m^2 \mathrm{tr}\Phi^\dagger\Phi + \lambda_1(\mathrm{tr}\Phi^\dagger\Phi)^2 + \lambda_2 \mathrm{tr}(\Phi^\dagger\Phi)^2$$

Can also add (θ=0):

$$(\xi_1(\mathrm{tr}(\Phi^\dagger\Phi))^2 + \xi_2 \mathrm{tr}(\Phi^\dagger\Phi)^2)|\mathrm{tr} \log(\Phi/\Phi^\dagger)|^2 + \ldots$$

$$(\kappa_1(\mathrm{tr}(\Phi^\dagger\Phi))^2 + \kappa_2 \mathrm{tr}(\Phi^\dagger\Phi)^2)(|\mathrm{tr} \log(\Phi/\Phi^\dagger)|^2)^2 + \ldots$$

2nd order chiral transition possible for *all* N_f = 1, 2, 3, 4..

Only invariant under $SU_L(N_f)$ x $SU_R(N_f)$, *not* $U_A(1)$.

New universality class with *non*-polynomial interactions?

Going up in T, without quarks

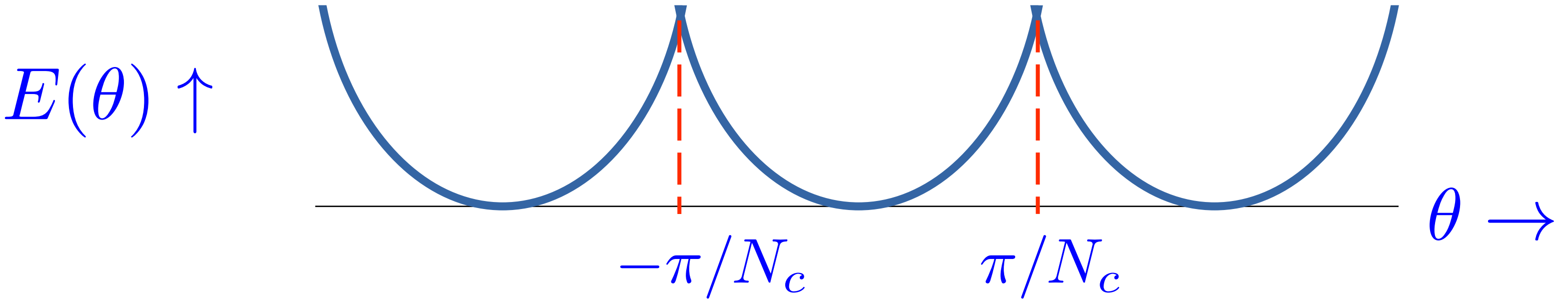
θ-dependence for SU(N_c) without quarks.

T=0: E(θ) not just simple quadratic form:

$$E(\theta) - E(0) = \frac{\chi}{2}\,\theta^2 \;,\; \theta < \frac{\pi}{N_c}$$

cusps @ integer * π/N_c.

Periodic in θ → θ + $2\pi/N_c$.



T > T_{deconf}: dilute instanton gas

periodic in θ → θ + 2π

$$E(\theta) - E(0) \sim \cos(\theta)\, \mathrm{e}^{-8\pi^2/\lambda\, N_c} + \ldots$$

CP^N in 1+1 dim.'s: VW→ DIG smoothly

Like QCD, CP^N has one dimensionless coupling, asymptotically free, θ-vacua

z^i = complex scalar, i=1...N. Soluble as N → ∞. mass m dynamically generated.

Dynamically generated Abelian gauge field, A_μ , has θ-parameter.

$$\mathcal{S}_\theta = \frac{1}{g^2}\int d^2x \, |D_\mu z^i|^2 + i\theta\, Q \quad ; \quad \sum_i |z^i|^2 = 1 \quad ; \quad Q \sim \int d^2x \, \epsilon_{\mu\nu} F^{\mu\nu}$$

Affleck, '79 & '80: T=0: $E(\theta) - E(0) = \chi\theta^2/2 \, , \; \chi = 6\pi m^2/N$

Smooth evolution from VW to DIG! T≠ 0: $\sim - \cos(\theta) \mathrm{e}^{-N f(T)}$

T(VW→ DIG) ~ m/log(N): *no* phase transition. $f(T \to 0) \sim \mathrm{e}^{-m/T}$

Davis & Mathison '85,'86,'89, CP^N + fermions @ N → ∞

η' massive at T = 0, *massless* at T = T_{chiral}.

$U_A(1)$ *restored* at T_{chiral}! But no 2nd order trans. in 1+1 dim's.; special to N → ∞

Veneziano-Witten to Dilute Instanton Gas: chiral limit

Three regimes: $T < T_{ch} \sim 155$ MeV: $\chi(T) \approx$ constant, Veneziano-Witten

T: $T_{ch} \to T_{deconf} \sim 300$ MeV: $\chi(T)$ *not* DIG, falls slower with T

$T > T_{deconf}$: Dilute Instanton Gas (DIG)

Effective Lagrangian in the chiral limit:

$T \ll T_{ch}$: Veneziano-Witten

$$\mathcal{L}(\Phi) = \mathcal{L}_0(\Phi) + m^2_{\eta'}\,\eta'^{\,2} + \lambda_{\eta'}\,\eta'^{\,4} + \ldots\ ,\ m^2_{\eta'} \sim \Lambda^2_{\rm QCD}$$

$T > T_{deconf}$: DIG; RDP & Wilczek

$$\mathcal{L}(\Phi) = \mathcal{L}_0(\Phi) + \#\,\mathrm{e}^{-8\pi^2/g^2(T)}\,(\det(\Phi) + \mathrm{c.c.})\ + \ldots$$

$T \sim T_{chiral}$: *non*-polynomial effective Lagrangian?

$$\mathcal{L}(\Phi) = \ldots + \xi_1(\mathrm{tr}(\Phi^\dagger\Phi))^2\eta'^{\gamma_1} + \xi_2\mathrm{tr}(\Phi^\dagger\Phi)^2\eta'^{\gamma_2} + \ldots\ ;$$

$$\eta' = |\mathrm{tr}\log(\Phi/\Phi^\dagger)|^2$$

Veneziano-Witten to Dilute Instanton Gas: QCD

The "middle" regime, T: $T_{ch} \sim 155 \rightarrow T_{deconf} \sim 300$ MeV.

A middling, boring crossover? *No: VW → DIG!* *Very* interesting!

Unrealistically: measure $m^2_{\eta'}(T)$ as $T \rightarrow T_{chiral}$, generally $V(\eta')$

Generally: different m_{quark}? *Or:* $\rho(\lambda, m)$? (Horvath, Kovacs, Huang, Kotov…)

Non-polynomial terms in $L(\Phi)$ → novel terms in $\rho(\lambda, m) \sim (\log \lambda)^{\gamma 1} \lambda^{\gamma 2}$?

Vacuum: $m^2_{\eta'} > 0 \rightarrow \langle\sigma\rangle \neq 0, \quad \langle\eta'\rangle = 0$, Surely true for *all* $T \neq 0$ *iff* $\mu = 0$.

Vafa-Witten '83,'84: in vacuum ($\& T \neq 0$), QCD does *not* spontaneously break CP

Vafa-Witten theorem *fails* at $\mu \neq 0$: quark determinant complex

Perhaps: at low T, large $\mu \neq 0$, $m^2_{\eta'} < 0$: condense $\langle\eta'\rangle \neq 0$, $\langle\sigma\rangle = 0$.

-> spontaneous CP violation in some (*limited*) region of μ & T

Lee & Wick, '74; Kharzeev, RDP, Tytgat '98,'99