NSC3 Nuclear Science Computing Center at CCNU

Microscopic Encoding of Macroscopic Universality: Scaling properties of Dirac Eigenspectra near QCD Chiral Phase Transition

How do universal behaviors at macroscale arise from quarks and gluons?

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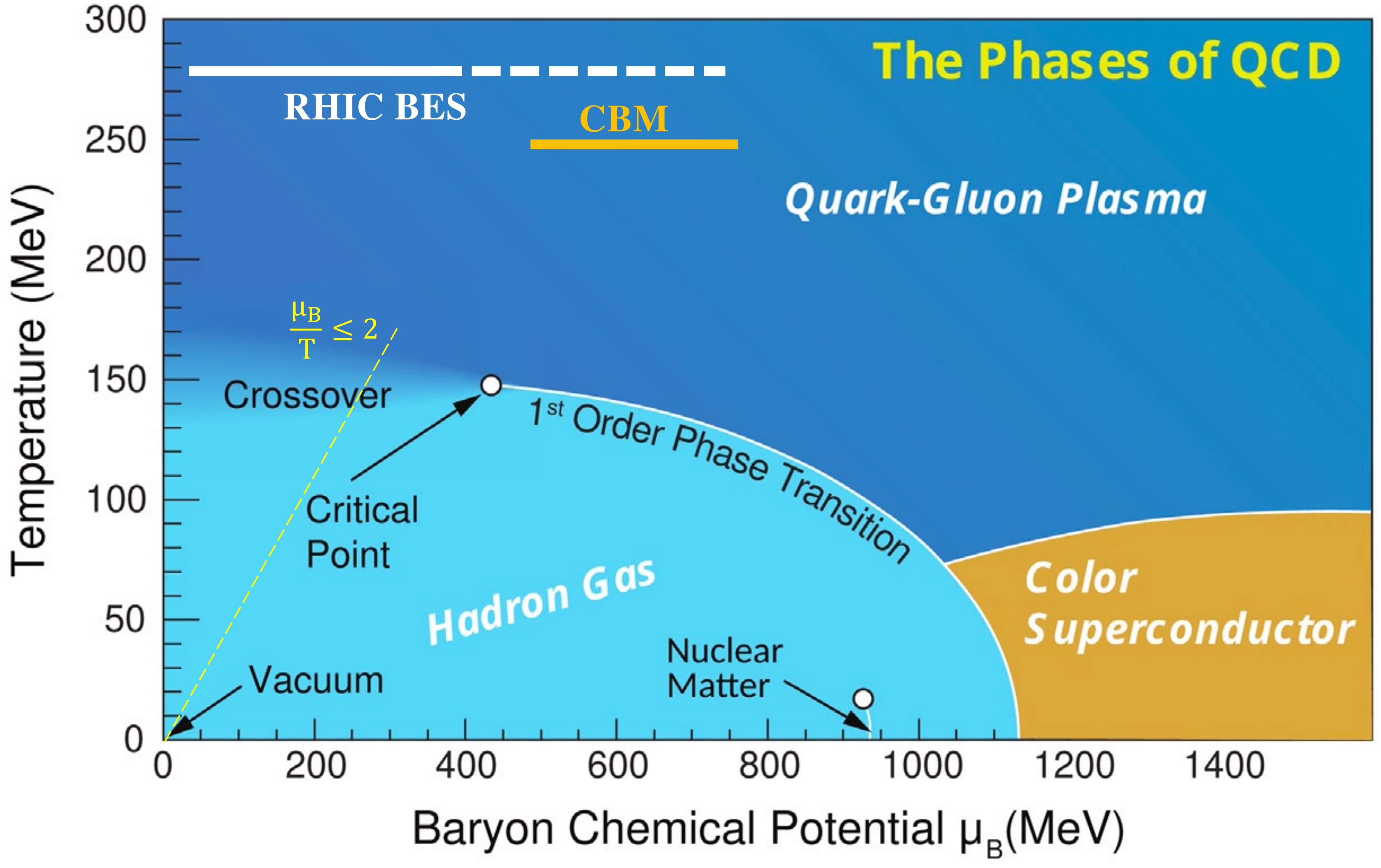
based on PRL 131 (2023), 161903, PRL 126 (2021), 082001 & work in progress,
in collaboration with
Heng-Tong Ding, Swagato Mukherjee, Peter Petreczky, Yu Zhang

New developments in studies of the QCD phase diagram,
Sep 9 - Sep 13, 2024 @ ECT* workshop

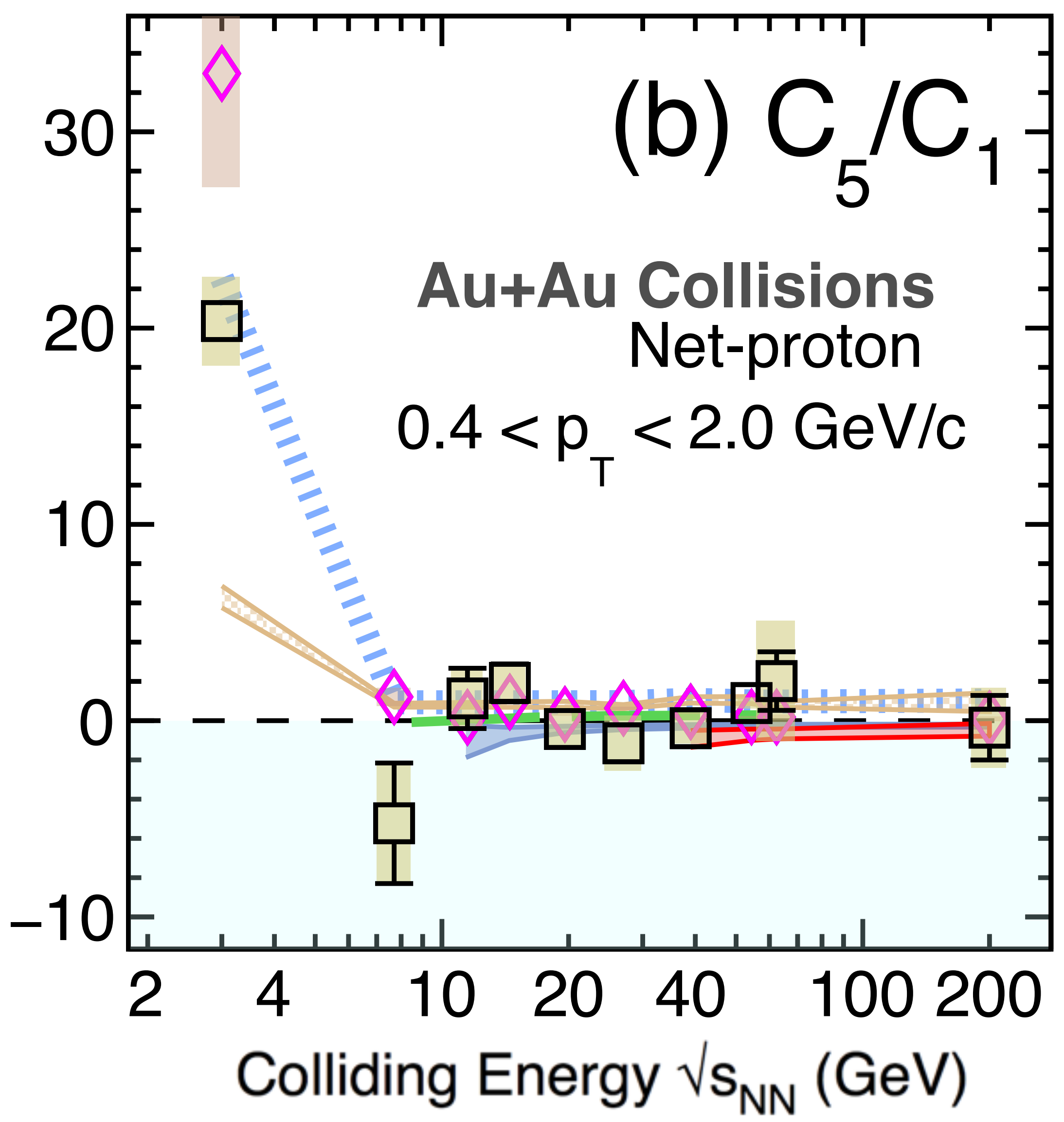
Search for Criticality in QCD

Exploration of QCD phase diagram



D. Almaalol et al., arXiv:2209.05009

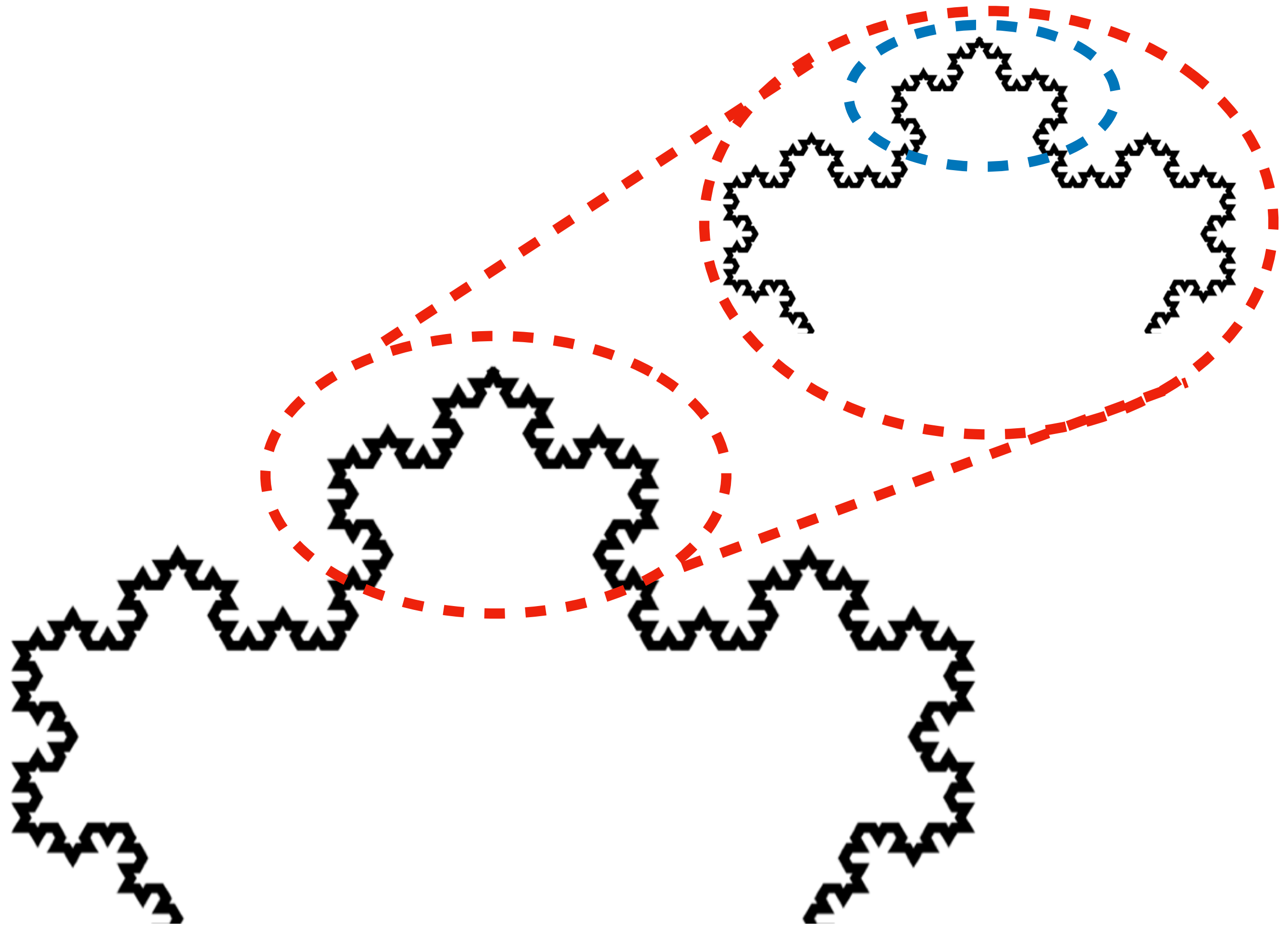
Searching for signatures of criticality in Macroscopic quantities



STAR, Phys. Rev. Lett. 130, 082301 (2023)

Xiaofeng Luo, talk in this workshop

Scale invariance in continuous phase transition

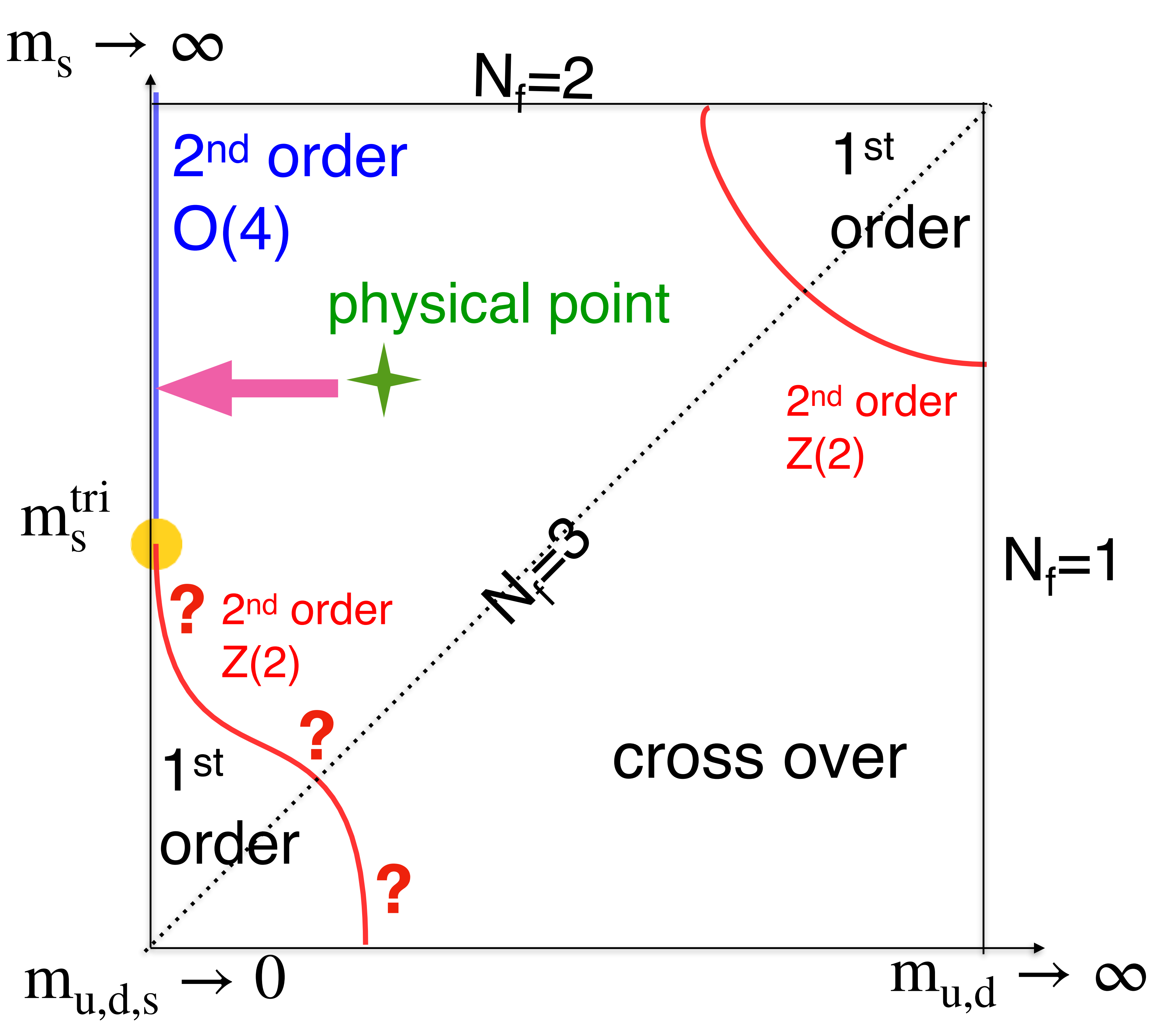


Macroscopic scaling behaviors manifested in Microscopic level

How does criticality at Macroscale arise from Microscopic *d.o.f* of QCD ?

Nature of Chiral Transition for $N_f = (2+1)$ QCD

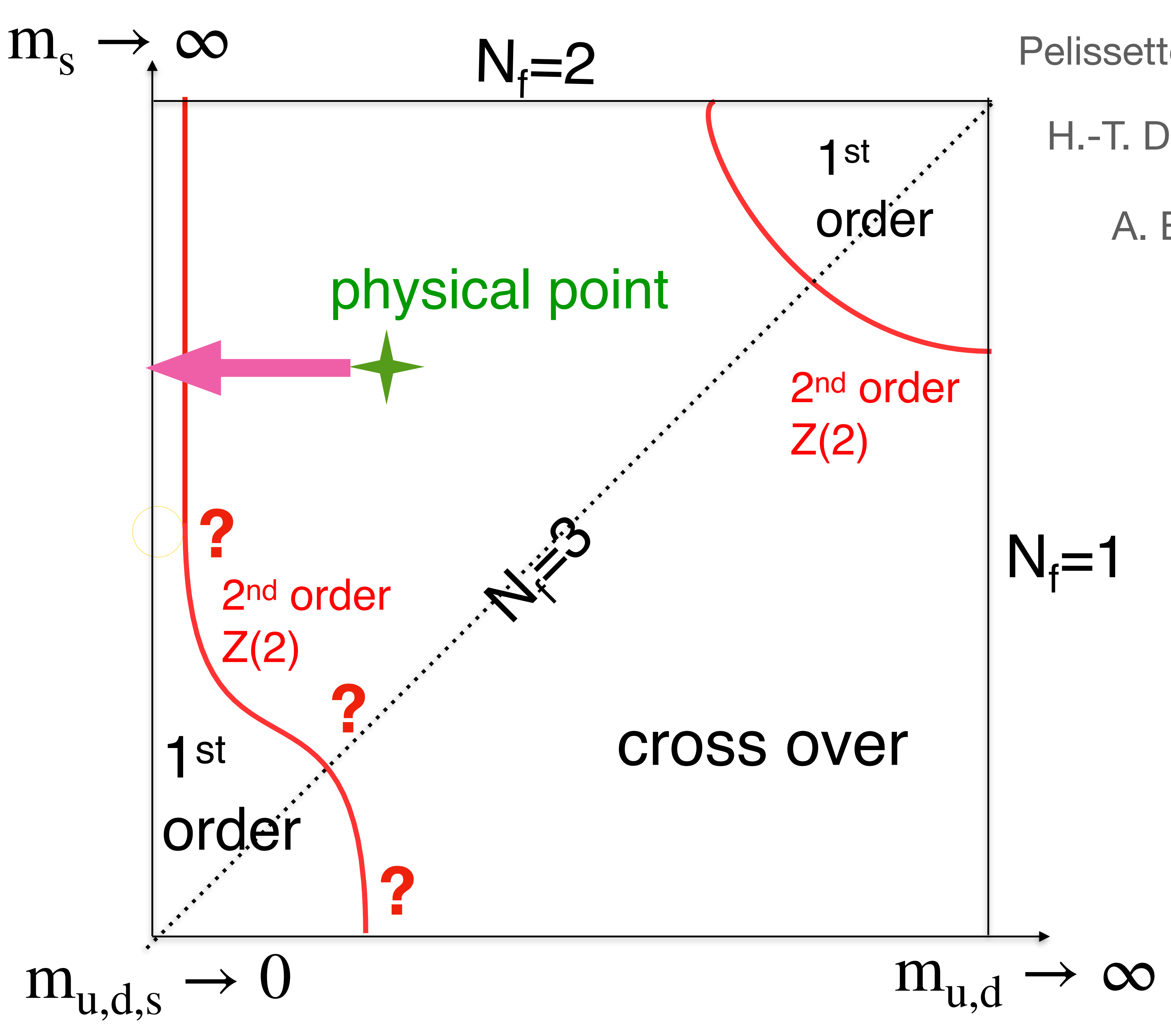
Pisarski & Wilczek, PRD 29 (1984) 338
Pelissetto & Vicari, PRD 88 (2013) 105018
H.-T. Ding et al., PRL 123 (2019) 062002
A. Bazavov et al., PLB 795 (2019) 15



$U_A(1)$ symmetry broken at $T \sim T_c$:

2nd phase transition

belonging to $O(4)$



$U_A(1)$ symmetry effectively restored at $T \sim T_c$:

1st phase transition or

2nd phase transition belonging to $U(2)_L \otimes U(2)_R / U(2)_V$

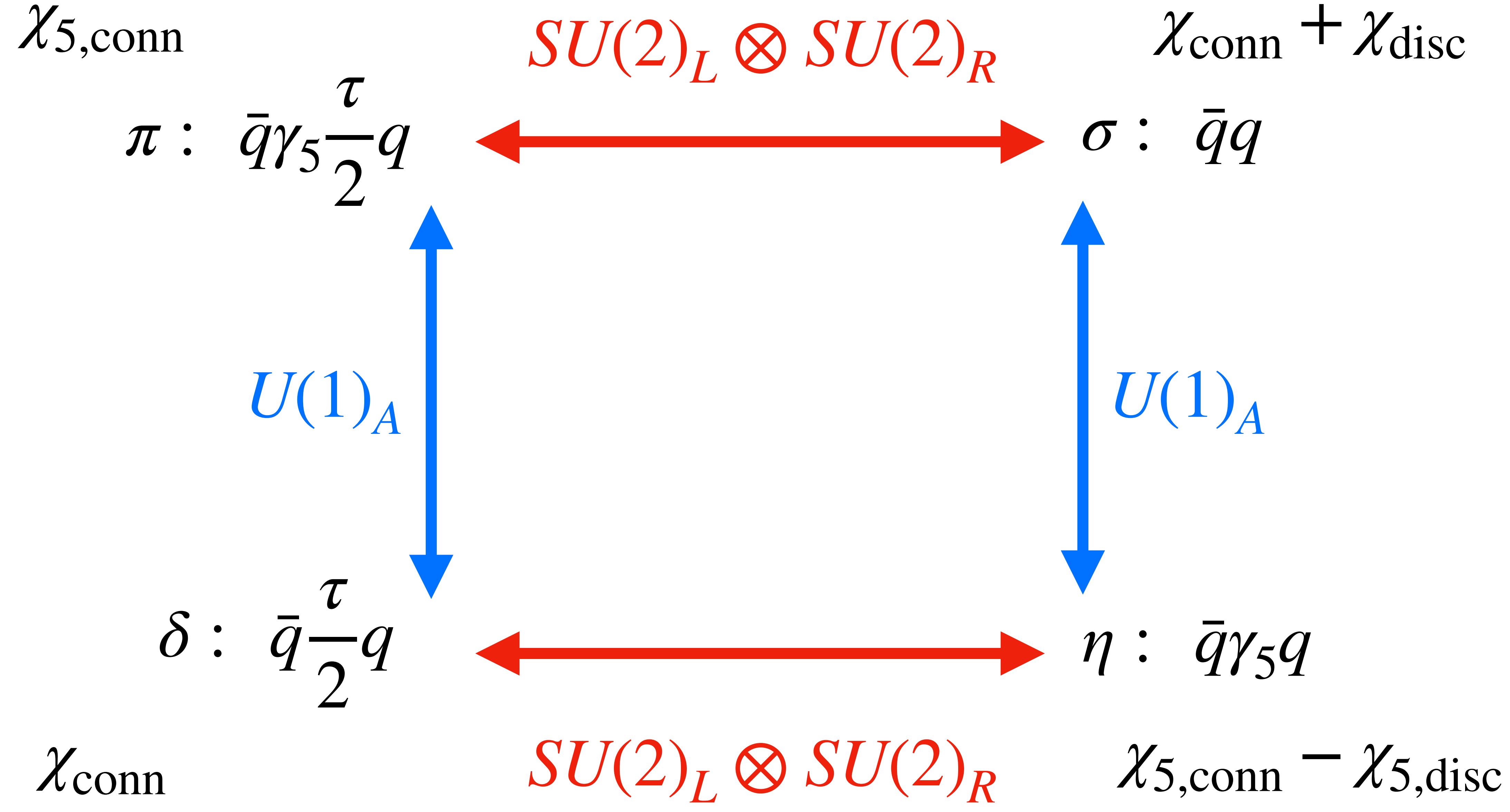
Whether / How axial anomaly manifests itself at $T \sim T_c$?

Signatures of Chiral and $U_A(1)$ Symmetry Restorations

Susceptibilities: integrated two point correlation $\chi_M = \int d^4x \left\langle J_M(x) J_M^\dagger(0) \right\rangle$ with $J_M(x) = \bar{q}(x)\Gamma_M q(x)$

A. Bazavov et al., PRD 86 (2012) 094503

N. Carabba et al., PRD 105 (2022) 5, 054034



Restoration of $SU(2)_L \otimes SU(2)_R$

$$\chi_\pi - \chi_\sigma = 0, \quad \chi_\delta - \chi_\eta = 0 \quad \Rightarrow \quad \chi_\pi - \chi_\delta = \chi_{\mathrm{disc}} = \chi_{5,\mathrm{disc}}$$

Effective restoration of $U_A(1)$ in chiral symmetric phase

$$\chi_\pi - \chi_\delta = 0, \quad \chi_\sigma - \chi_\eta = 0 \quad \Rightarrow \quad \chi_\pi - \chi_\delta = \chi_{\mathrm{disc}} = \chi_{5,\mathrm{disc}} = 0$$

Connect Macro to Micro via Dirac eigenspectra:

$$\langle\bar{\psi}\psi\rangle_l = \int_0^\infty \mathrm{d}\lambda \frac{4m_l \rho(\lambda, m_l)}{\lambda^2 + m_l^2} \qquad \chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda \frac{8m_l^2 \rho(\lambda, m_l)}{\left(\lambda^2 + m_l^2\right)^2} \qquad \chi_{\mathrm{disc}} = \int_0^\infty \mathrm{d}\lambda \frac{4m_l \cdot \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

Microscopic Origin from Dirac Eigenspectra

$$\langle\bar{\psi}\psi\rangle_l = \int_0^\infty \mathrm{d}\lambda \frac{4m_l\rho(\lambda, m_l)}{\lambda^2 + m_l^2} \qquad \chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda \frac{8m_l^2\rho(\lambda, m_l)}{\left(\lambda^2 + m_l^2\right)^2} \qquad \chi_{\mathrm{disc}} = \int_0^\infty \mathrm{d}\lambda \frac{4m_l \cdot \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

- Restoration of chiral symmetry:

$\rho(0) = 0$ from Banks-Casher relation $\lim_{m_l\to0} \langle\bar{\psi}\psi\rangle_l = \lim_{m_l\to0} 2\pi\rho\left(0,m_l\right)$ Banks and Casher, NPB 169 (1980) 103

- Effective restoration of $U_A(1)$ symmetry:

A sizable gap from zero in ρ Cohen, arXiv:nucl-th/9801061

- Possible underlying structure of $\rho(\lambda, m_l)$ responsible for chiral symmetry restoration but not $U_A(1)$:

E.g., $\rho(\lambda, m_l) = \mathrm{c}_0 + \mathrm{c}_1\lambda + \mathrm{c}_2 m_l^2\delta(\lambda) + \mathrm{c}_3 m_l + \mathrm{c}_4 m_l^2 + \cdots \Rightarrow$

$$\langle\bar{\psi}\psi\rangle = 2\mathrm{c}_0\pi - 4\mathrm{c}_1 m_l \ln(m_l) + 2\mathrm{c}_2 m_l + 2\mathrm{c}_3\pi + 2\pi\mathrm{c}_4 m_l^2$$

$$\chi_\pi - \chi_\delta = 2\mathrm{c}_0\pi/m_l + 4\mathrm{c}_1 + 4\mathrm{c}_2 + 2\mathrm{c}_3\pi + 2\mathrm{c}_4\pi m_l$$

Hard to Identify Mass Dependences in $\rho(\lambda)$

In the chiral limit: $\rho(\lambda, m_l) = \mathrm{c}_0 + \mathrm{c}_1\lambda + \mathrm{c}_2 m_l^2 \delta(\lambda) + \mathrm{c}_3 m_l + \mathrm{c}_4 m_l^2 + \cdots$

❌ Suppressed mass terms are hard to be observed;

❌ Contributions from different mass terms are hard to be distinguished

✓ Mass derivatives of $\rho(\lambda)$ are helpful to separate mass dependences from each other

$$\mathrm{e.g.,}\quad \partial\rho/\partial m_l \sim 2c_2 m_l \delta(\lambda) + c_3 + 2c_4 m_l + \cdots$$

$$\partial^2\rho/\partial m_l^2 \sim 2c_2\delta(\lambda) + 2c_4 + \cdots$$

Hard to Obtain Mass Derivatives of $\rho(\lambda)$

😖 Infeasible traditional numerical difference:

$$\frac{\partial \rho(\lambda)}{\partial m} = \lim_{\epsilon \to 0} \frac{\rho(\lambda, m+\epsilon) - \rho(\lambda, m)}{\epsilon} + \mathcal{O}\left(\epsilon^2\right)$$

❌ Discretization errors in ϵ

❌ Additional unwanted simulations at closed quark mass $(m + \epsilon)$ are required

Possible to derive analytical expressions of $\partial^n \rho(\lambda) / \partial m_l^n$?

$\partial^n \rho / \partial m_l^n$ and Correlated Dirac Eigenspectra

Eigenvalue spectrum for (2+1)-flavor QCD:

$$\rho\left(\lambda, m_l\right) = \frac{T}{VZ[U]} \int D[U] e^{-S_G[U]} \det\left[\not{D}[U] + m_s\right] \times \left(\det\left[\not{D}[U] + m_l\right]\right)^2 \rho_U(\lambda)$$

Partition function:

$$Z[U] = \int D[U] e^{-S_G[U]} \det\left[\not{D}[U] + m_s\right] \times \left(\det\left[\not{D}[U] + m_l\right]\right)^2$$

m_l dependence enters ρ:

$$\det\left[\not{D}[U] + m_l\right] = \prod_j \left(+\mathrm{i}\lambda_j + m_l\right)\left(-\mathrm{i}\lambda_j + m_l\right)$$

$$= \exp\left(\int_0^\infty \mathrm{d}\lambda \rho_U(\lambda) \mathrm{In}\left[\lambda^2 + m_l^2\right]\right)$$

Eigenvalue spectrum for a given configuration:

$$\rho_U(\lambda) = \sum_j \delta\left(\lambda - \lambda_j\right)$$

with $\not{D}[U]\, \psi_j = i\lambda_j \psi_j$

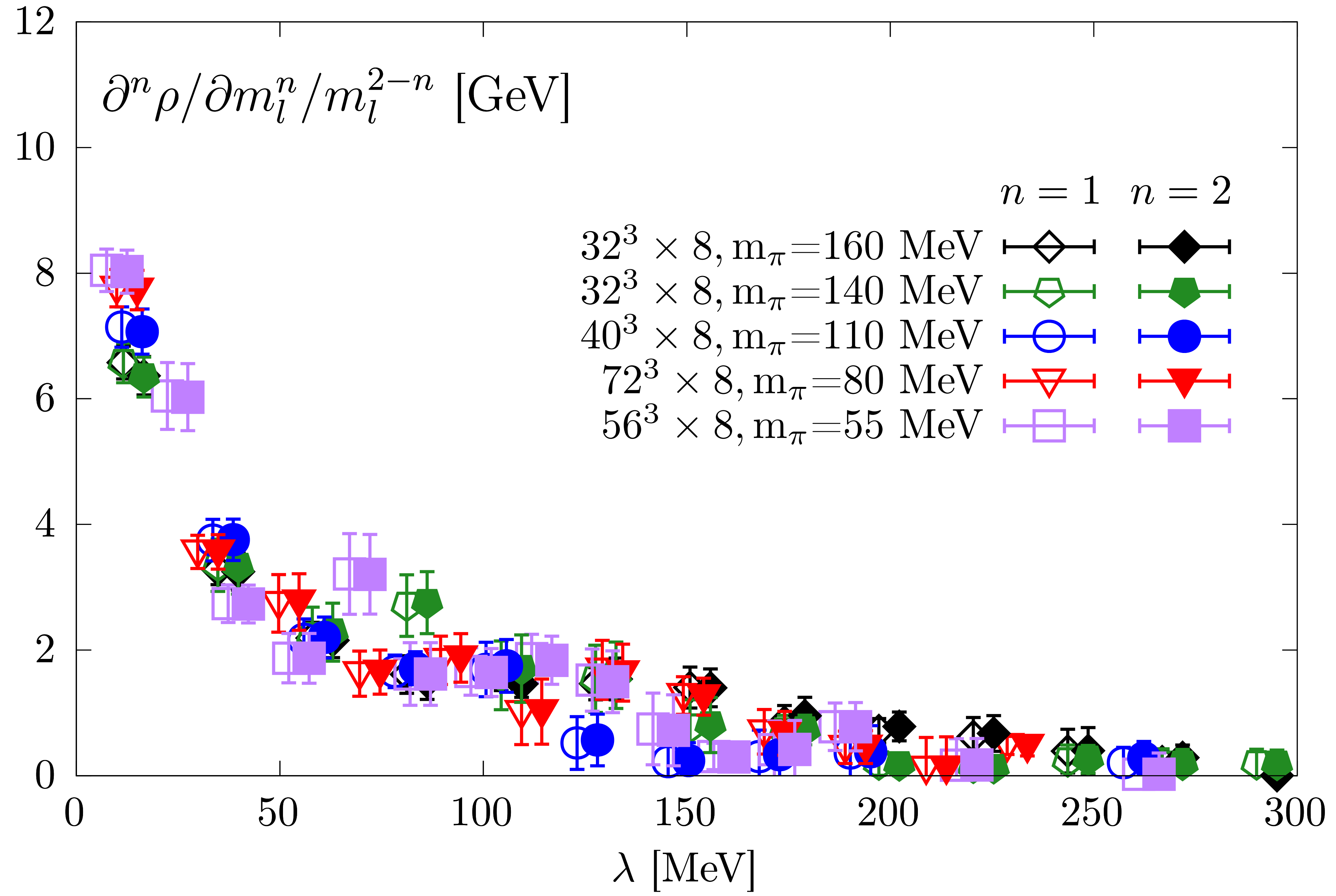
Mass derivative of $\rho(\lambda, m_l)$:

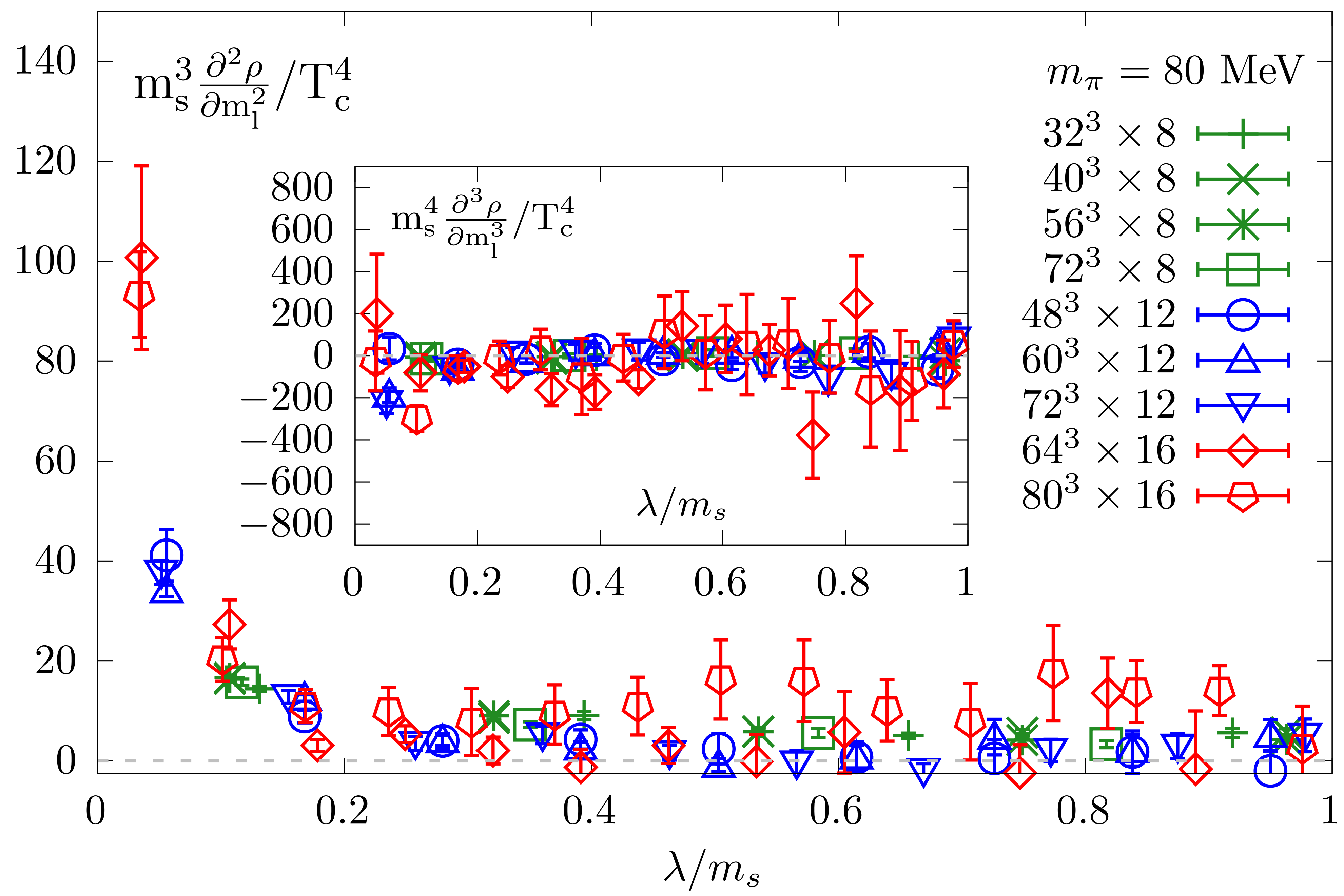
$$\frac{V}{T}\frac{\partial \rho(\lambda, m_l)}{\partial m_l} = \int_0^\infty \mathrm{d}\lambda_2 \frac{4 m_l C_2}{\lambda_2^2 + m_l^2} \quad , \quad C_2\left(\lambda, \lambda_2\right) = \left\langle \rho_U(\lambda) \rho_U\left(\lambda_2\right)\right\rangle - \left\langle \rho_U(\lambda)\right\rangle \left\langle \rho_U\left(\lambda_2\right)\right\rangle$$

H.-T. Ding et al., PRL 126 (2021), 082001

$\partial^n \rho / \partial m_l^n$ at High Temperature

In (2+1)-flavor QCD at $T \approx 205$ MeV   H.-T. Ding et al., PRL 126 (2021), 082001



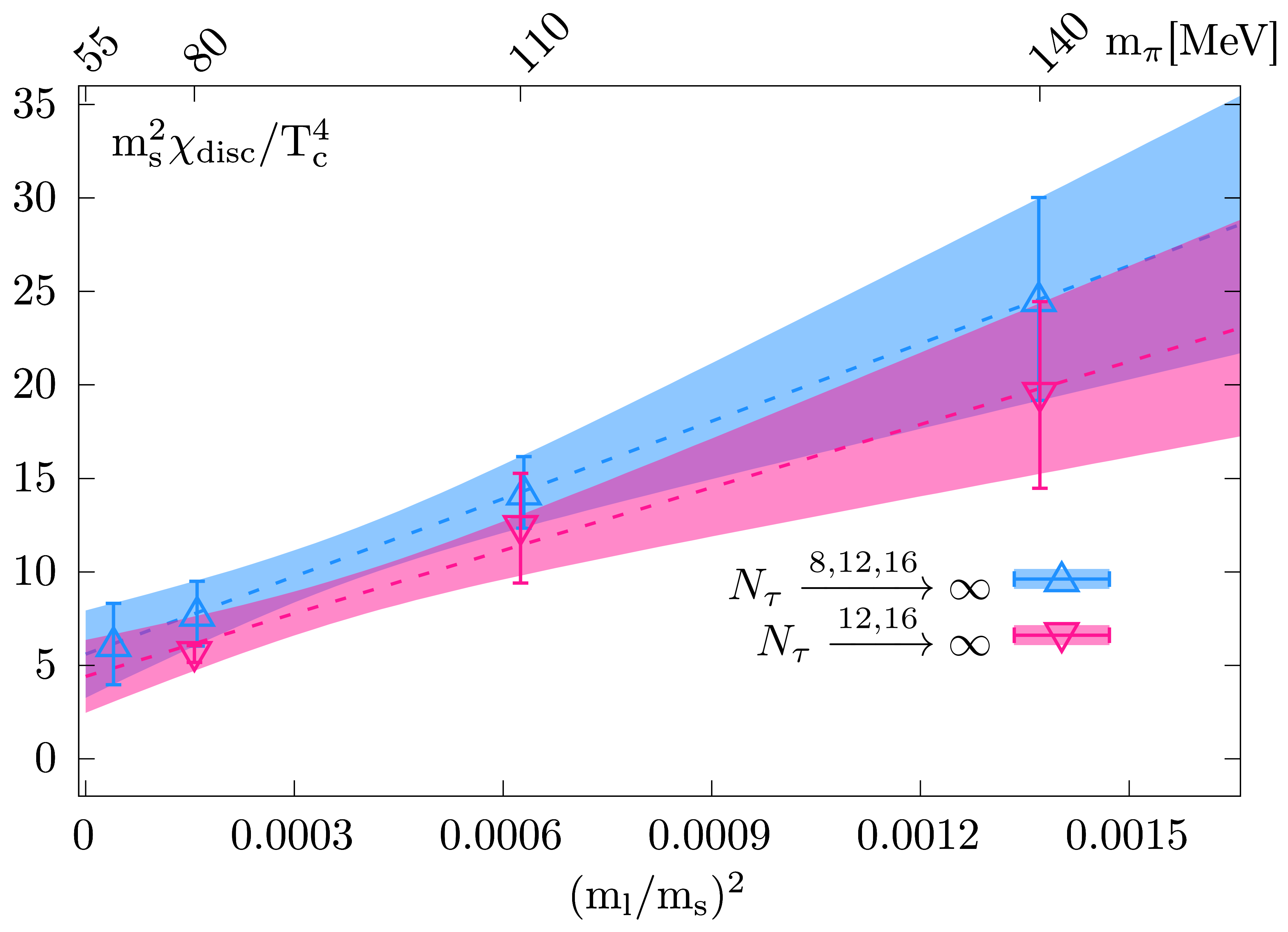


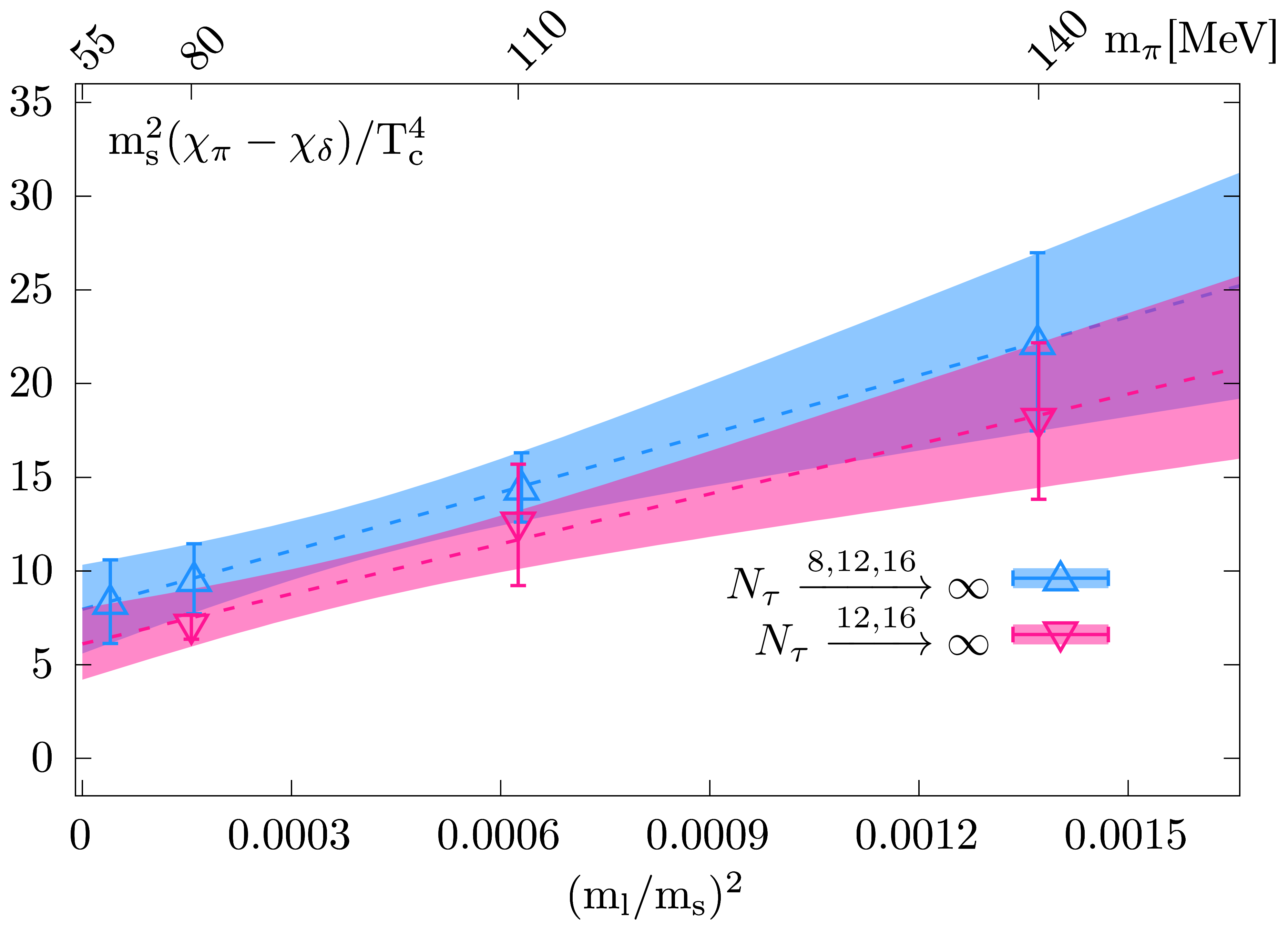
- $m_l^{-1}\partial\rho/\partial m_l \approx \partial^2\rho/\partial m_l^2$
- almost quark mass independent
- Sharper peaked structure towards continuum limit
- $\partial^3\rho/\partial m_l^3 \approx 0$

- $\rho(\lambda \to 0,\, m_l \to 0) \propto m_l^2\delta(\lambda)$ gives rise to $U_A(1)$ anomaly, indicating $U_A(1)$ still broken around T_c
- m^2 behavior in $\rho(\lambda)$ at high temperature is consistent with dilute instanton gas picture

Continuum and Chiral Extrapolations of $U_A(1)$ Measures

In (2+1)-flavor QCD at $T \approx 205$ MeV H.-T. Ding et al., PRL 126 (2021), 082001





Axial anomaly remains manifested in the $U_A(1)$ measures at $2 \sim 3\sigma$ level

$\Rightarrow$ Chiral phase transition for $N_f = (2+1)$ should be of 2nd order belonging to $O(4)$ universality class

How does Macroscopic criticality in 2nd $O(4)$ chiral phase transition arise at Microscopic level?

Universal Scaling in QCD Chiral Transition

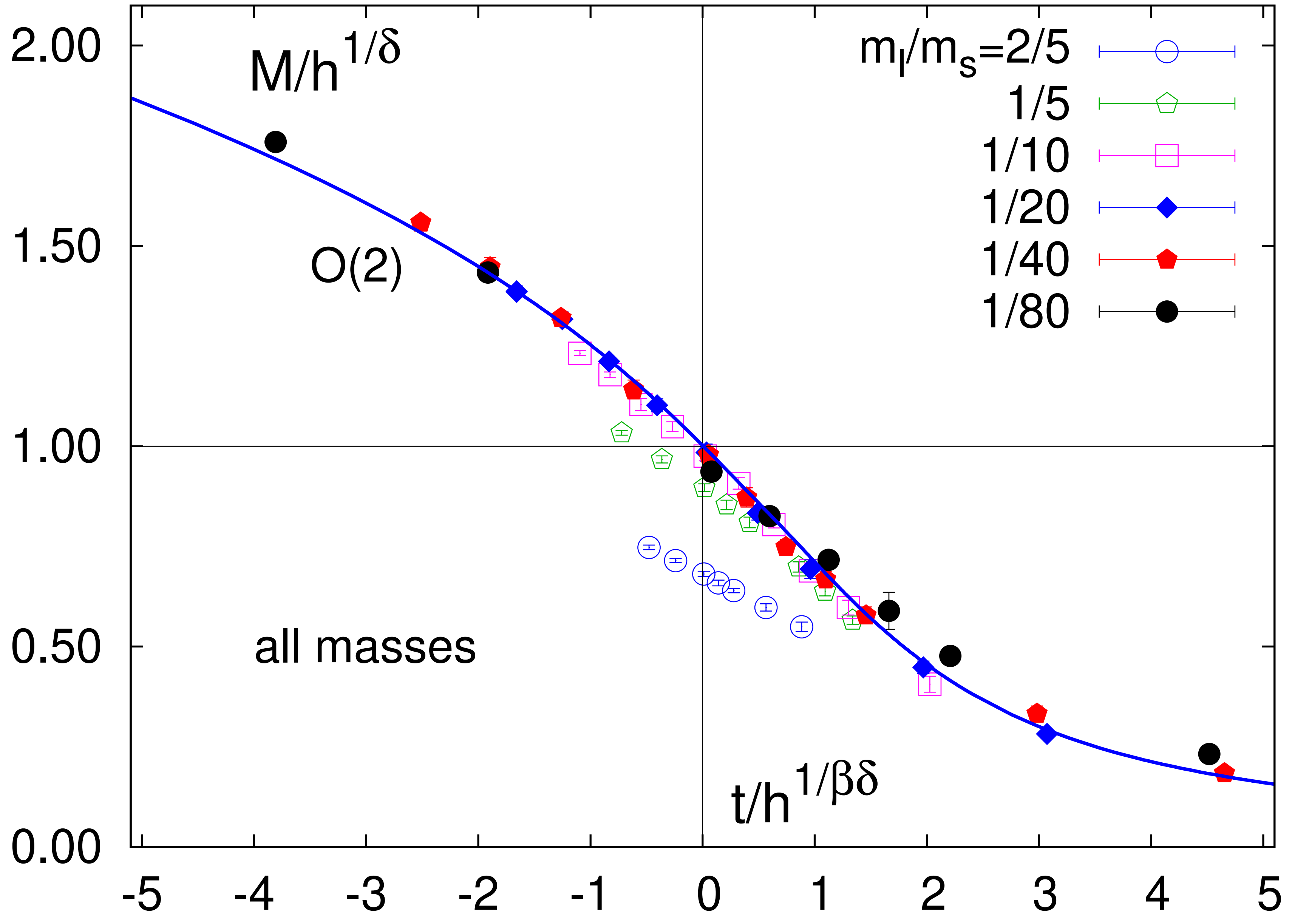
Universal O(2) scaling behaviors in staggered discretization scheme

Free energy in continuous phase transition:

$$F(m,T) = F_{\text{singular}}(z) + F_{\text{regular}} \qquad z = t/h^{1/\beta\delta} \quad t = \frac{T-T_c}{t_0 T_c} \quad h = \frac{m_l}{h_0 m_s}$$

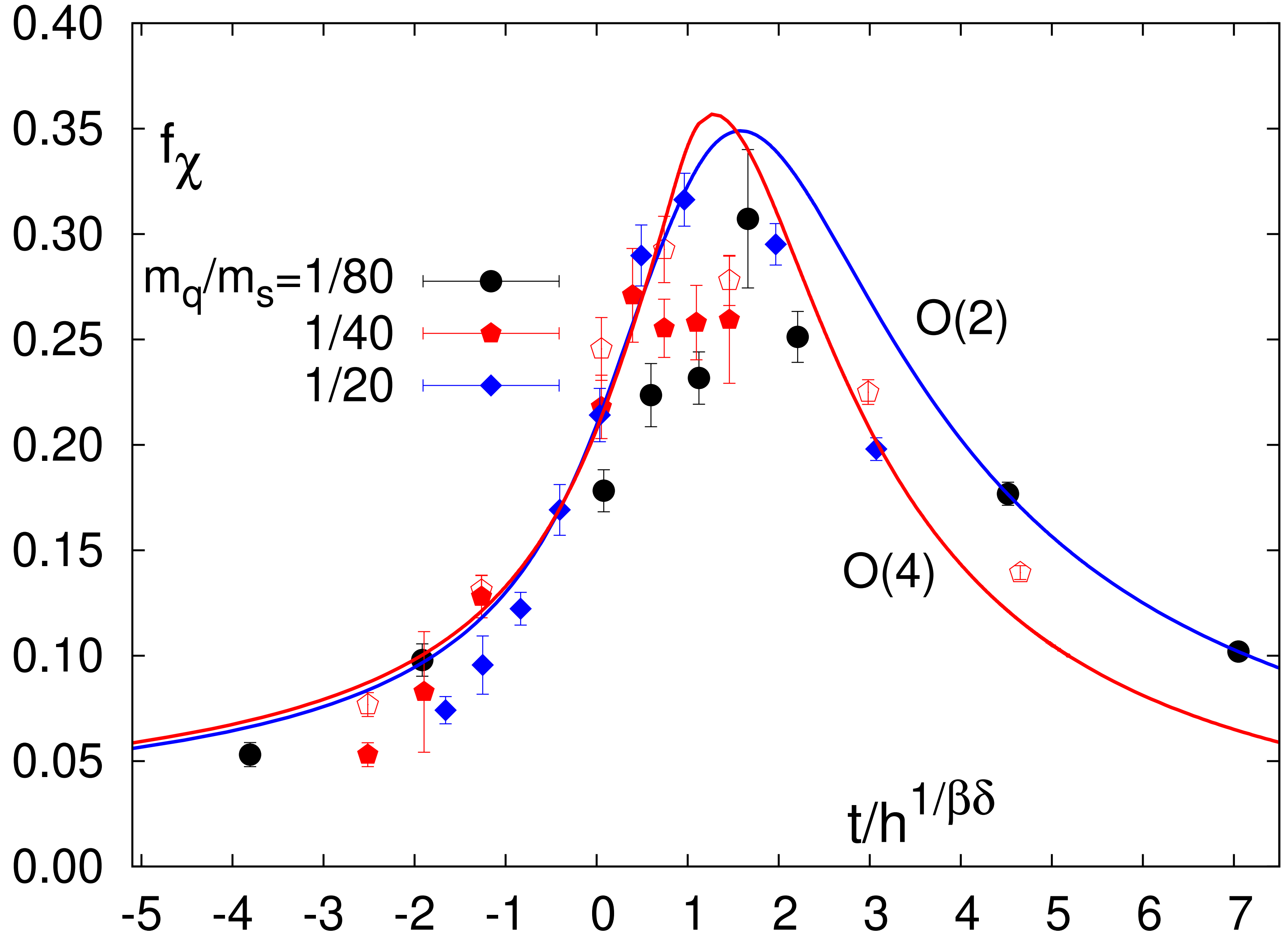
Order parameter :

$$M(t,h) = \partial F/\partial H = h^{1/\delta} f_1(z) + f_{\text{reg}}(T,H)$$



Order parameter susceptibility :

$$\chi_M(t,h) = \partial^2 F/\partial H^2 = h_0^{-1} h^{1/\delta - 1} f_2(z) + f'_{\text{reg}}$$



S. Ejiri et al., PRD 80, 094505 (2009)

Connect Macro to Micro: Banks-Casher relation and its limitation

$$\langle\bar\psi\psi(m)\rangle = \frac{T}{V}\langle 2\ \mathrm{Tr}(\not{D}[U]+m)^{-1}\rangle$$

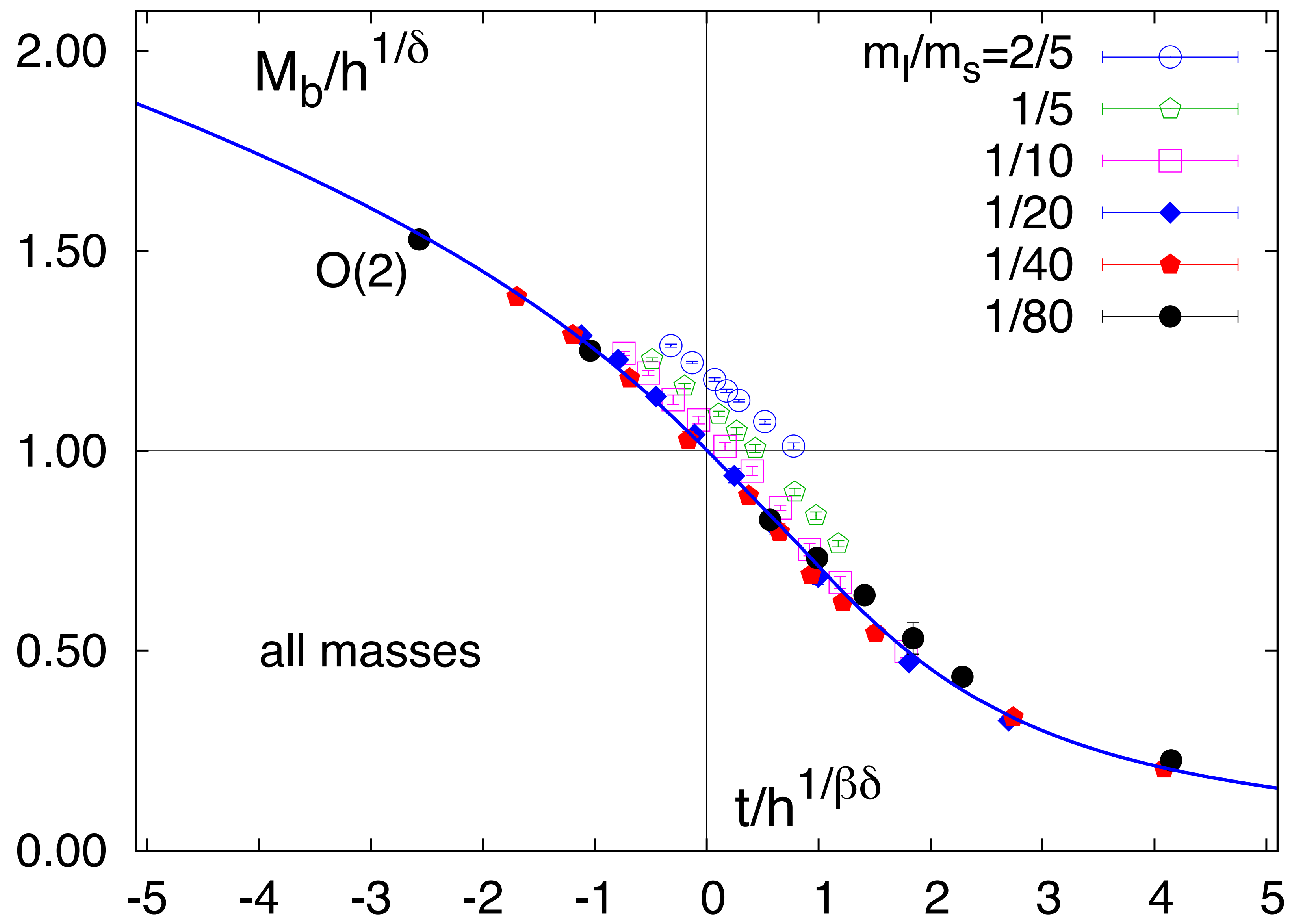
$$\rho(\lambda,m) = \frac{T}{V}\langle\rho_U(\lambda)\rangle \equiv \frac{T}{V}\langle\sum_j \delta(\lambda-\lambda_j)\rangle \quad \text{with} \quad \not{D}\psi_j = i\lambda_j\psi_j$$

$$\langle\bar\psi\psi\rangle = \int_0^\infty \frac{4m\,\rho(\lambda,m)}{\lambda^2+m^2}d\lambda \xrightarrow{m\to 0} \pi\rho(\lambda=0)$$

Banks & Casher, NPB 169(1980) 103

❌ Complicated to deal with regular parts in $\langle\bar\psi\psi\rangle$

$$\langle\bar\psi\psi\rangle = h^{1/\delta}f_1(z)+c_1H^1+c_2H^2+\cdots$$



✓ Singularity is more pronounced in higher order derivatives

❌ Additional connected part involved

$$\chi_M = \partial\langle\bar\psi\psi\rangle/\partial H = h_0^{-1}h^{1/\delta-1}f_2(z)+c_1+2c_2H+\cdots$$
$$\sim \langle[\bar\psi\psi(m)-\langle\bar\psi\psi(m)\rangle]^2\rangle + \#\langle\mathrm{Tr}\,M_l^{-2}\rangle$$

$$\partial^2\langle\bar\psi\psi\rangle/\partial H^2 = h_0^{-2}h^{1/\delta-2}f_3(z)+2c_2+\cdots$$
$$\sim \langle[\bar\psi\psi(m)-\langle\bar\psi\psi(m)\rangle]^3\rangle + \#\langle\mathrm{Tr}\,M_l^{-3}\rangle+\cdots$$

⋮

How to define a Macroscopic observable with:
suppressed regular contribution & _connected pieces excluded_,
and then find its Microscopic counterpart?

Cumulants of order parameter

n-th order cumulant of $\bar{\psi}\psi$: $\mathbb{K}_n[\bar{\psi}\psi] = \frac{T}{V}(-1)^n \frac{\partial^n \mathbb{G}(m;\epsilon)}{\partial m^n}\Big|_{\epsilon=m}$

Generating functional :

$$\mathbb{G}(m;\epsilon) = \ln \left\langle \exp\left\{-m\,\bar{\psi}\psi(\epsilon)\right\}\right\rangle_0$$

$\langle\cdots\rangle_0$: average over QCD partition function in the chiral limit

Probe operator with valance quark mass ϵ :
$\bar{\psi}\psi(\epsilon) \equiv 2\ \mathrm{Tr}(\not{D}[U]+\epsilon)^{-1}$

$$\partial\langle\bar{\psi}\psi\rangle/\partial m = \frac{T}{V}\langle[\bar{\psi}\psi(m)-\langle\bar{\psi}\psi(m)\rangle]^2\rangle - \frac{2T}{V}\langle\mathrm{Tr}\,M_l^{-2}\rangle$$

$\mathbb{K}_2[\bar{\psi}\psi]$

$$\partial^2\langle\bar{\psi}\psi\rangle/\partial m^2 \sim \frac{T}{V}\langle[\bar{\psi}\psi(m)-\langle\bar{\psi}\psi(m)\rangle]^3\rangle + \#\langle\mathrm{Tr}\,M_l^{-3}\rangle$$

$\mathbb{K}_3[\bar{\psi}\psi]$

✓ *More pronounced singular part contribution*

✓ *Only disconnected part needed*

H.-T. Ding, **W.-P. Huang,** S. Mukherjee, P. Petreczky, PRL 131 (2023), 161903

Connect Condensate Cumulant to Quark Energy Correlation

n-th order cumulant of $\bar{\psi}\psi$ **:** $\mathbb{K}_n[\bar{\psi}\psi] = \frac{T}{V}(-1)^n \frac{\partial^n \mathbb{G}(m;\epsilon)}{\partial m^n}\Big|_{\epsilon=m}$

$$\mathbb{G}(m;\epsilon) = \ln\left\langle \exp\left\{-m\ \bar{\psi}\psi(\epsilon)\right\}\right\rangle_0 = \ln\left\langle \exp\left\{-m\int_0^\infty P_U(\lambda;\epsilon)\ d\lambda\right\}\right\rangle_0$$

Define $P_U(\lambda;\epsilon) \equiv \frac{4\epsilon\rho_U(\lambda)}{\lambda^2+\epsilon^2}$

$$\mathbb{K}_n[\bar{\psi}\psi(m)] = \int_0^\infty K_1[P_U(\lambda_1;m), P_U(\lambda_2;m), \cdots, P_U(\lambda_n;m)]\prod_{i=1}^{n} d\lambda_i$$

$K_1[X_1, X_2, \cdots, X_n]$ denotes 1st order joint cumulant of n-variables

H.-T. Ding, **W.-P. Huang,** S. Mukherjee, P. Petreczky, PRL 131 (2023), 161903

Connect Condensate Cumulant to Quark Energy Correlation

!!

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1;m), P_U(\lambda_2;m), \cdots, P_U(\lambda_n;m)] \prod_{i=1}^{n} d\lambda_i \equiv \int_0^\infty P_n(\lambda)\ d\lambda$$

n-th order cumulant of the chiral order parameter

n-point correlation of the quark energy spectra

H.-T. Ding, **W.-P. Huang,** S. Mukherjee, P. Petreczky, PRL 131 (2023), 161903

$$\partial\langle\bar{\psi}\psi\rangle/\partial m = \frac{T}{V}\langle[\bar{\psi}\psi(m) - \langle\bar{\psi}\psi(m)\rangle]^2\rangle - \frac{2T}{V}\langle \mathrm{Tr}\, M_l^{-2}\rangle \sim m_l^{1/\delta-1} f_2(z)$$

$\mathbb{K}_2[\bar{\psi}\psi]$

$$\partial^2\langle\bar{\psi}\psi\rangle/\partial m^2 \sim \frac{T}{V}\langle[\bar{\psi}\psi(m) - \langle\bar{\psi}\psi(m)\rangle]^3\rangle + \#\text{other} \sim m_l^{1/\delta-2} f_3(z)$$

$\mathbb{K}_3[\bar{\psi}\psi]$

⋮

$$\partial^n\langle\bar{\psi}\psi\rangle/\partial m^n \sim \mathbb{K}_n[\bar{\psi}\psi] + \#\text{other} \sim m_l^{1/\delta-n+1} f_n(z)$$

Conjecture: $\mathbb{K}_n[\bar{\psi}\psi] \sim m_l^{1/\delta-n+1} f_n(z)$

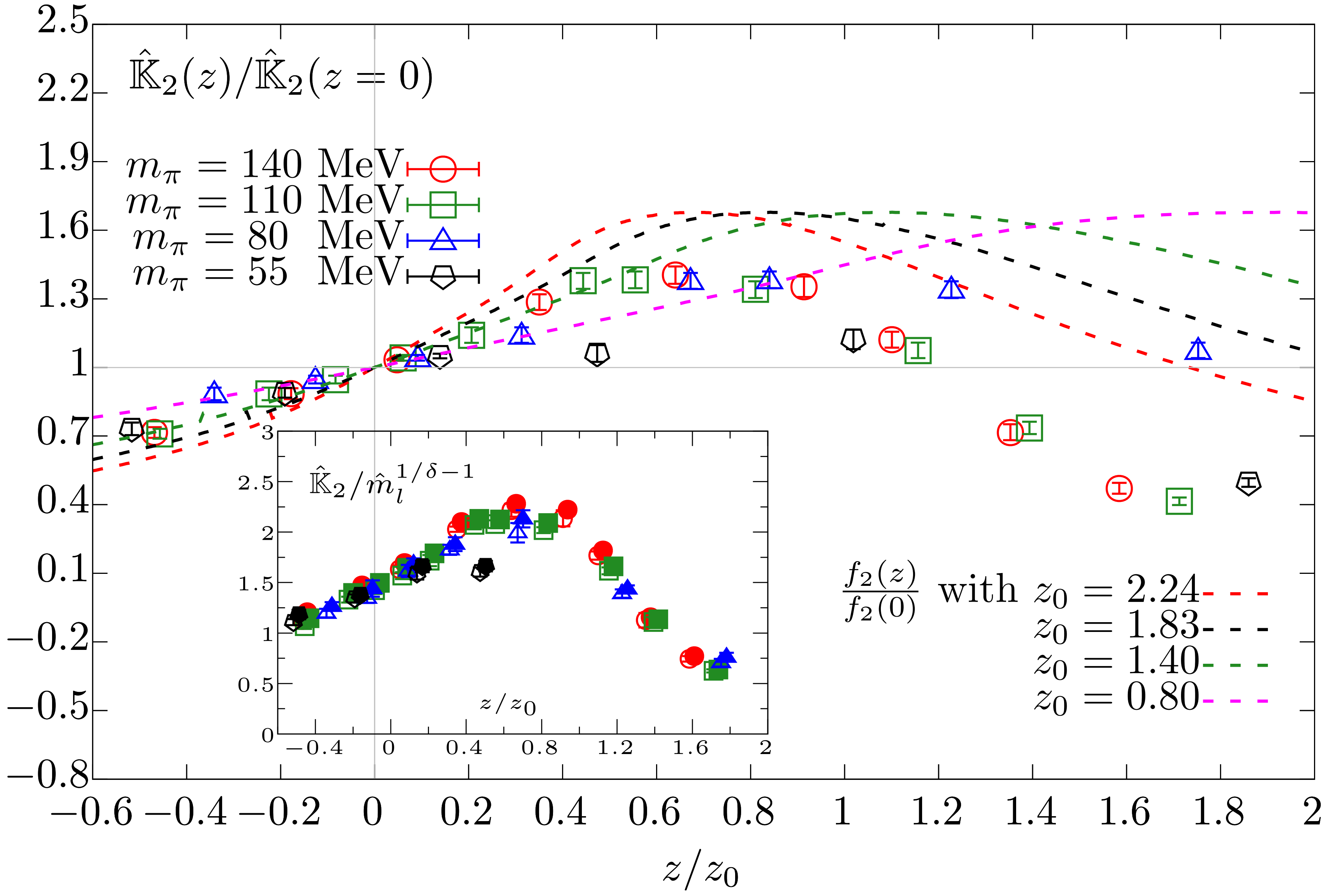
How does Macroscopic criticality in $\mathbb{K}_n[\bar{\psi}\psi]$ and then arise from Microscopic P_n ?

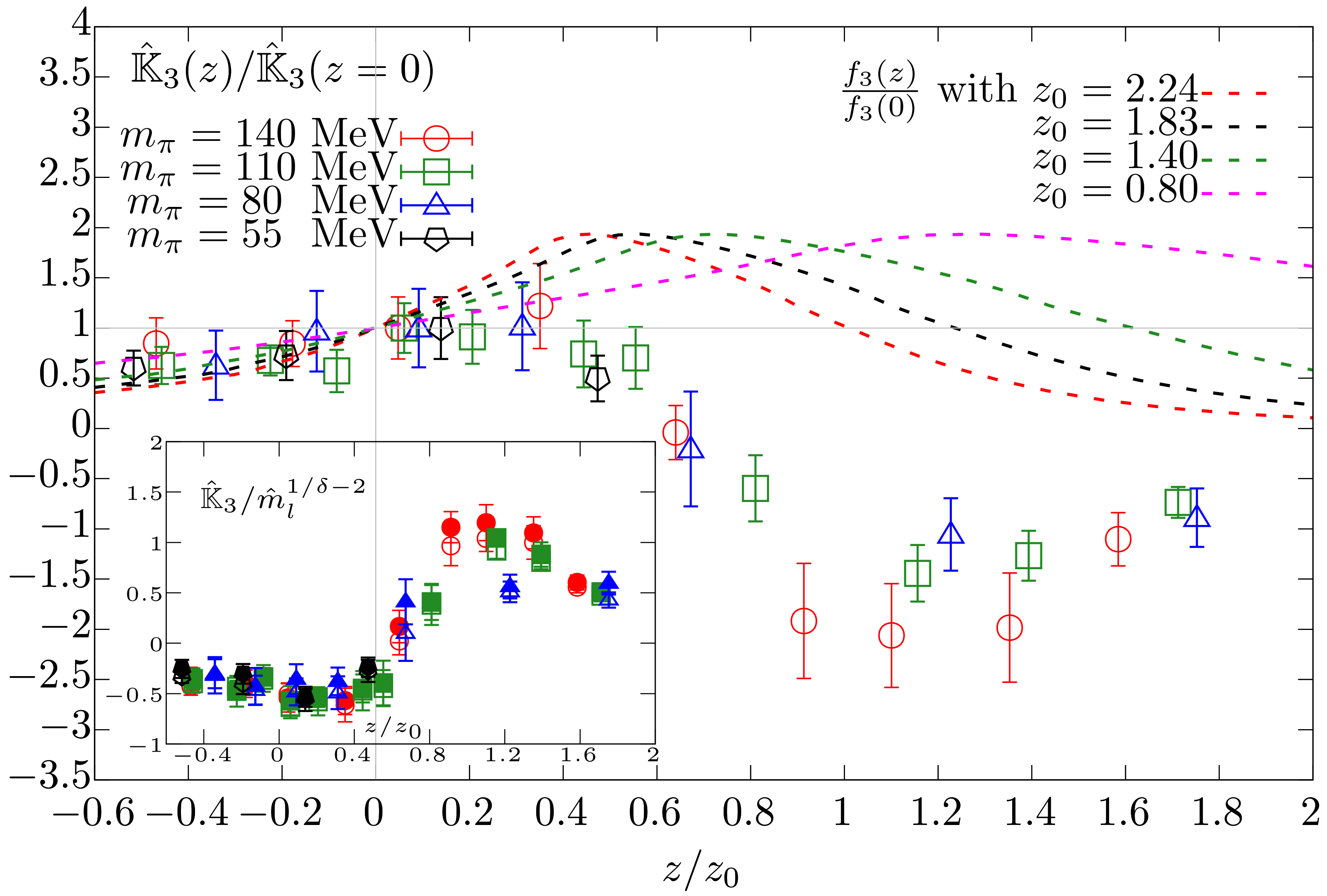
Criticality in Condensate Cumulants

Conjecture: $\mathbb{K}_n[\bar{\psi}\psi] \sim m_l^{1/\delta-n+1} f_n(z)$

$$\mathbb{K}_2[\bar{\psi}\psi] = \frac{T}{V}\langle[\bar{\psi}\psi(m) - \langle\bar{\psi}\psi(m)\rangle]^2\rangle = \int_0^\infty P_2(\lambda)\, d\lambda$$

$$\mathbb{K}_3[\bar{\psi}\psi] = \frac{T}{V}\langle[\bar{\psi}\psi(m) - \langle\bar{\psi}\psi(m)\rangle]^3\rangle = \int_0^\infty P_3(\lambda)\, d\lambda$$





Using $O(2)$ scaling with $\beta = 0.349$, $\delta = 4.78$, $z_0 = 1.83(9)$, $T_c(N_\tau = 8) = 144.2(6)$ MeV

Scaling parameters directly adopted from: S. Ejiri et al., Phys. Rev. D 80, 094505 (2009); D. A. Clarke et al., Phys. Rev. D 103, L011501 (2021)

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda)\, d\lambda \xrightarrow{\text{Around } T_c} \sim m_l^{1/\delta-n+1} f_n(z)$$

How does the criticality of $\mathbb{K}_n[\bar{\psi}\psi]$ arise from Microscopic $P_n(\lambda)$?

Microscopic Encoding of Macroscopic Criticality

Hints from the chiral limit :

$$P_U(\lambda; m) \equiv \frac{4m\rho_U(\lambda)}{\lambda^2 + m^2}$$

$$P_U(\lambda; m \to 0) = 2\pi\rho_U(\lambda)\delta(\lambda)$$

Generalized Banks-Casher relation :

$$\lim_{m\to 0} P_n(\lambda) = (2\pi)^n K_1[\rho_U(\lambda), \underbrace{\rho_U(0), \cdots, \rho_U(0)}_{(n-1)\text{ terms}}]\delta(\lambda)$$

$$\Longrightarrow \lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)]$$

$n = 1$ back to Banks-Casher relation !

Criticality in $\lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi]$ must arise from universal behaviors of ***λ-independent*** $\mathbb{K}_n[\rho_U(0)]$

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda)\, d\lambda \xrightarrow{\text{Around } T_c} \sim m_l^{1/\delta - n + 1} f_n(z)$$

Conjecture: $P_n(\lambda) = m^{1/\delta - n + 1} f_n(z) g_n(\lambda)$

Scaling arise from $P_n(\lambda)$ at ***deep infrared*** λ region

Include ***all*** system-specific λ-dependence

H.-T. Ding, **W.-P. Huang,** S. Mukherjee, P. Petreczky, PRL 131 (2023), 161903

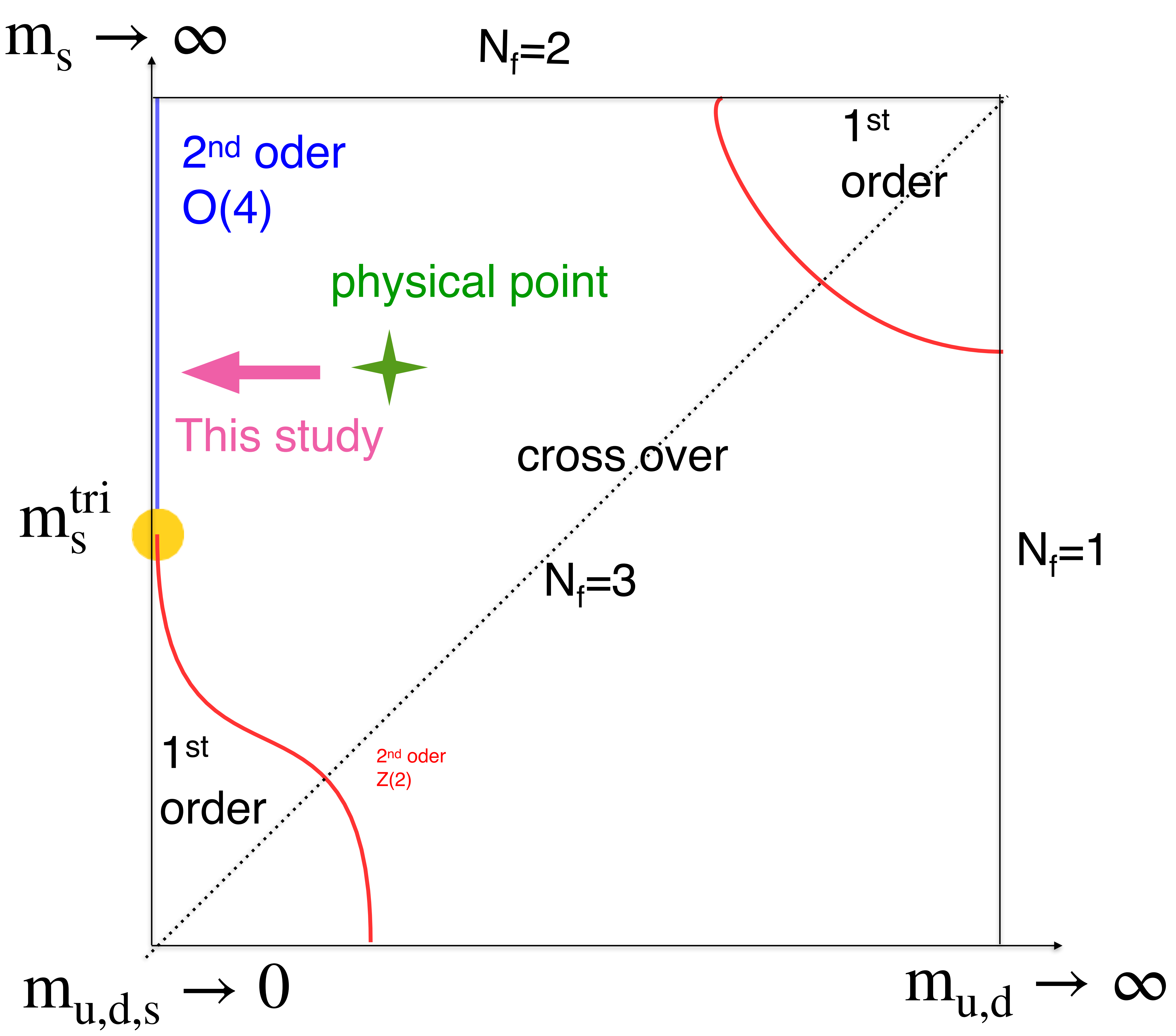
Lattice Setup

- Actions: Highly improved staggered quarks and tree-level Symanzik gauge action
- Lattice size: $N_\tau = 8, \quad N_\sigma = 32,\ 40,\ 56$
- Quark mass: $m_s^{\rm phy}/m_l = 27,\ 40,\ 80,\ 160$

  $(m_\pi \approx 140,\ 110,\ 80,\ 55\ \mathrm{MeV})$
- Temperatures: $T \in (135{,}176)\ \mathrm{MeV}$
- $\rho_U(\lambda)$ computed via Chebyshev filtering technique

  H.-T. Ding et al., PRL. 126 (2021), 082001
- HotQCD configurations; measurements carried out on NSC^3 at CCNU, Wuhan Supercomputing Center & BNL

Nuclear Science Computing Center at CCNU



$P_n(\lambda)$ around T_c

$\hat{P}_1(\hat{\lambda}) = m_s^2(m_l/m_s)P_1(\lambda)/T_c^4$ $\hat{P}_2(\hat{\lambda}) = m_s^3(m_l/m_s)P_2(\lambda)/T_c^4$ $\hat{P}_3(\hat{\lambda}) = m_s^4(m_l/m_s)P_3(\lambda)/T_c^4$

$\hat{P}_1(\hat{\lambda}), \hat{P}_2(\hat{\lambda})$ and $\hat{P}_3(\hat{\lambda})$

— Rapidly approach to zero

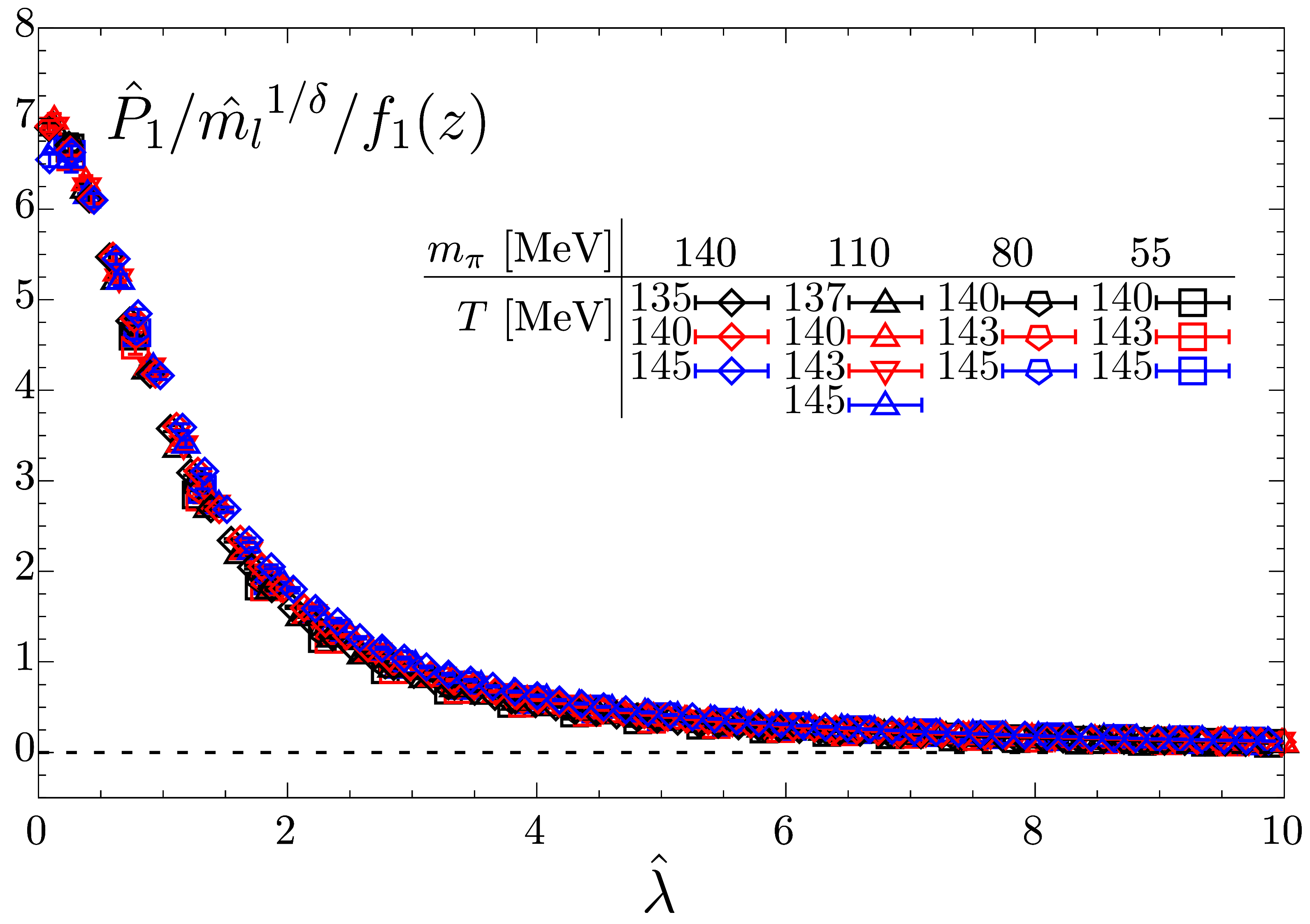
— Significant dependence on quark mass and temperature

Conjecture: $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta-n+1} f_n(z) g_n(\hat{\lambda})$

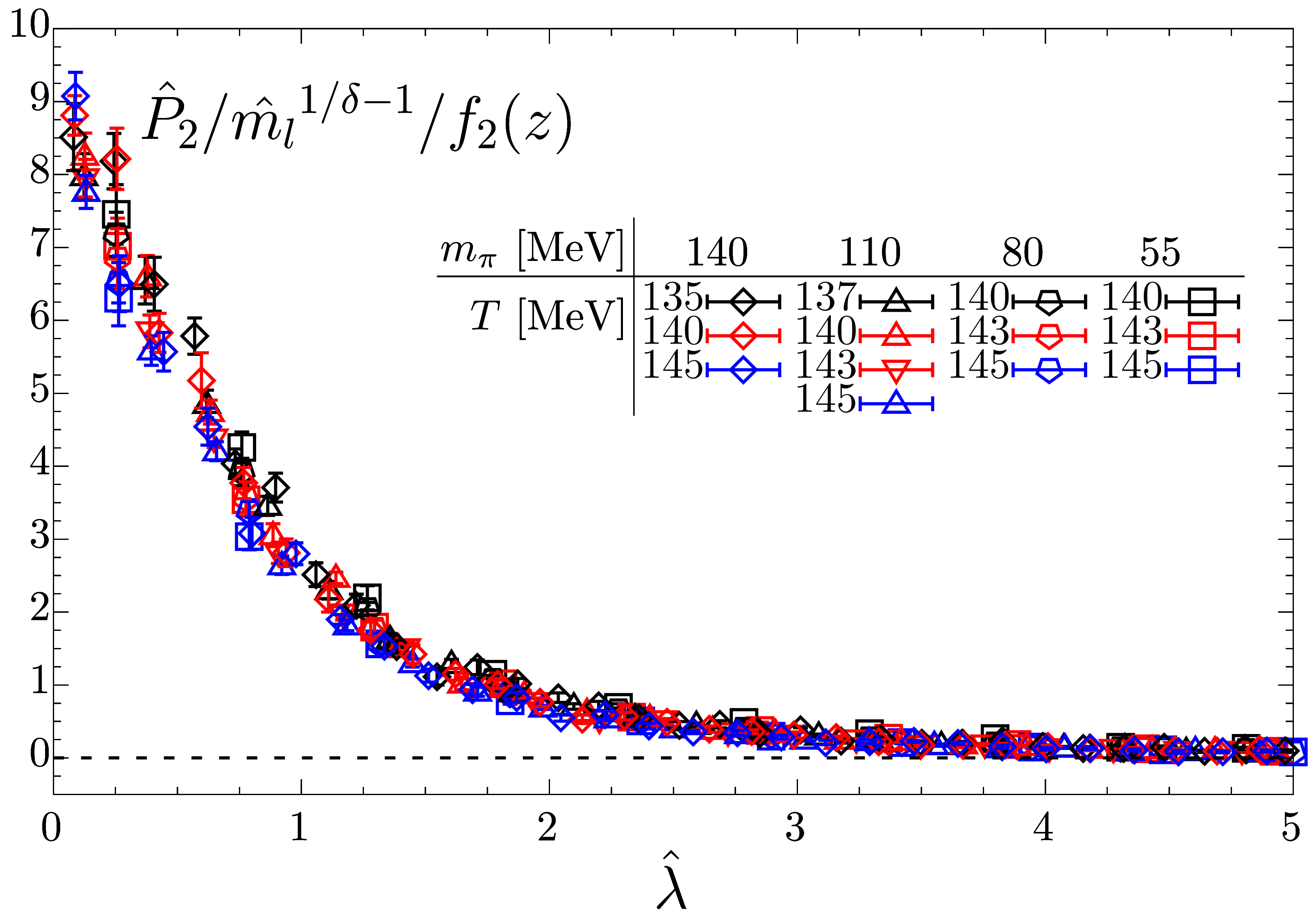
Rescaled $P_n(\lambda)$ around T_c

Conjecture: $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta - n + 1} f_n(z) g_n(\hat{\lambda})$

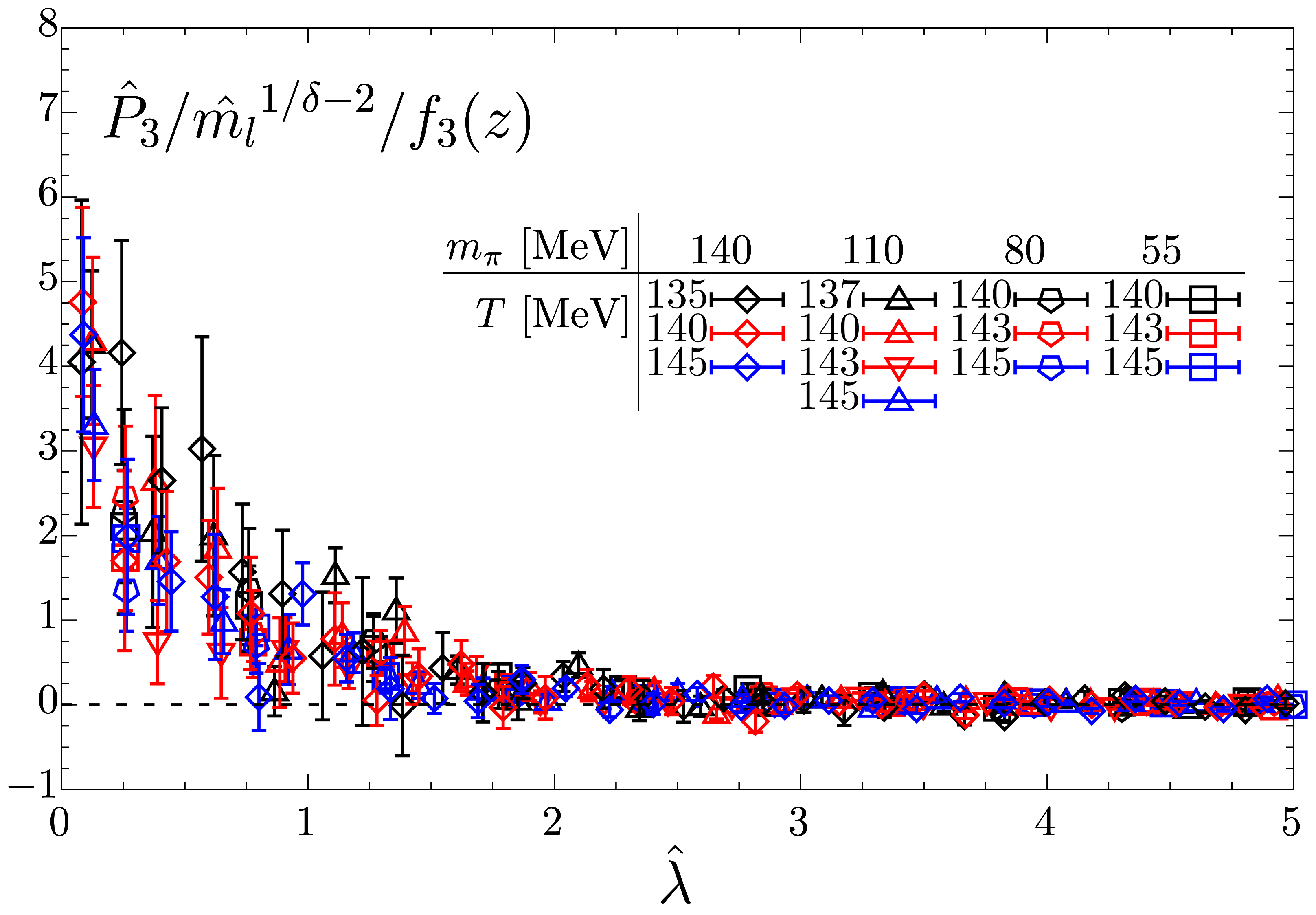
$\hat{P}_1(\hat{\lambda})/(m_l/m_s)^{1/\delta}/f_1(z)$



$\hat{P}_2(\hat{\lambda})/(m_l/m_s)^{1/\delta - 1}/f_2(z)$



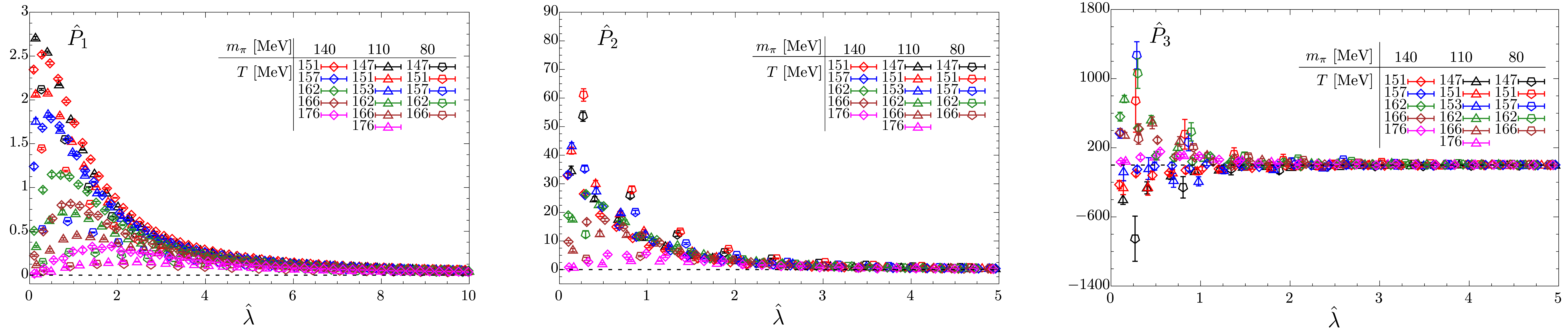
$\hat{P}_3(\hat{\lambda})/(m_l/m_s)^{1/\delta - 2}/f_3(z)$



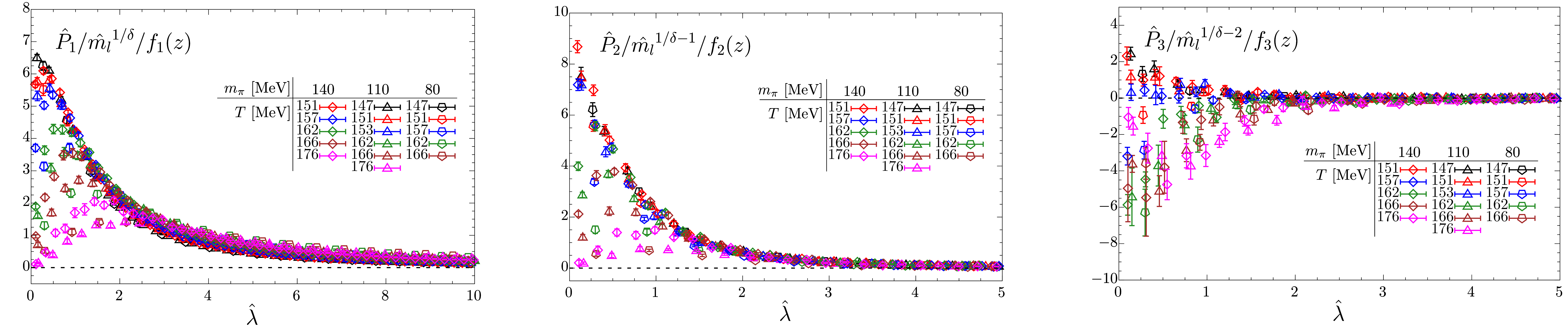
$z = z_0 (m_l/m_s)^{-\frac{1}{\beta\delta}} (T - T_c)/T_c$: $O(2)$ scaling parameters adopted from [S. Ejiri et al., Phys. Rev. D 80, 094505 (2009); D. A. Clarke et al., Phys. Rev. D 103, L011501 (2021)]

- In the vicinity of T_c, $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta - n + 1} f_n(z) g_n(\hat{\lambda})$
- Scaling behaviors in $\hat{P}_n(\hat{\lambda})$ extend up to physical light quark mass

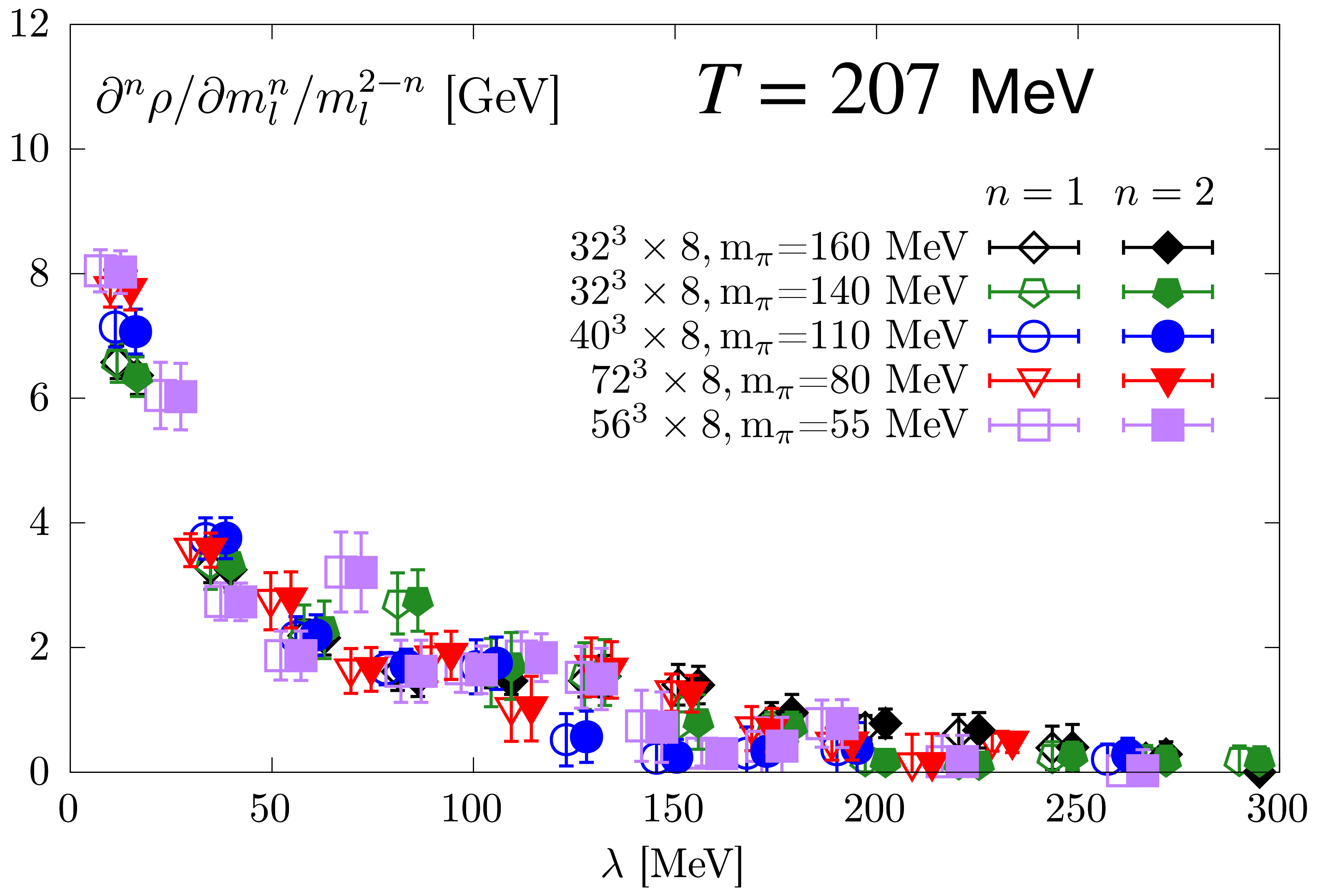
$P_n(\lambda)$ and Rescaled $P_n(\lambda)$ away from T_c

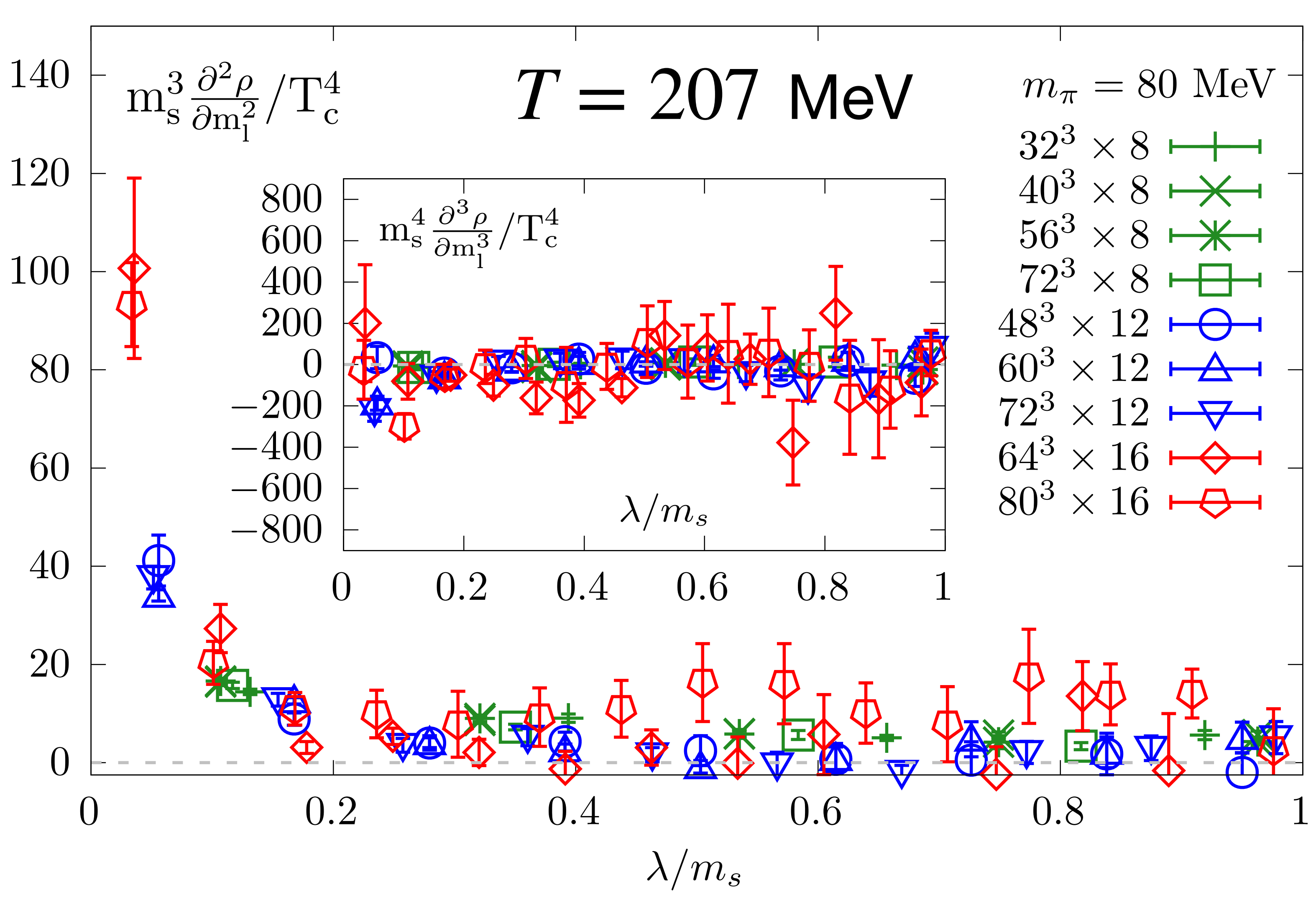


Away from T_c , no scaling behaviors are observed in $\hat{P}_n(\hat{\lambda})$



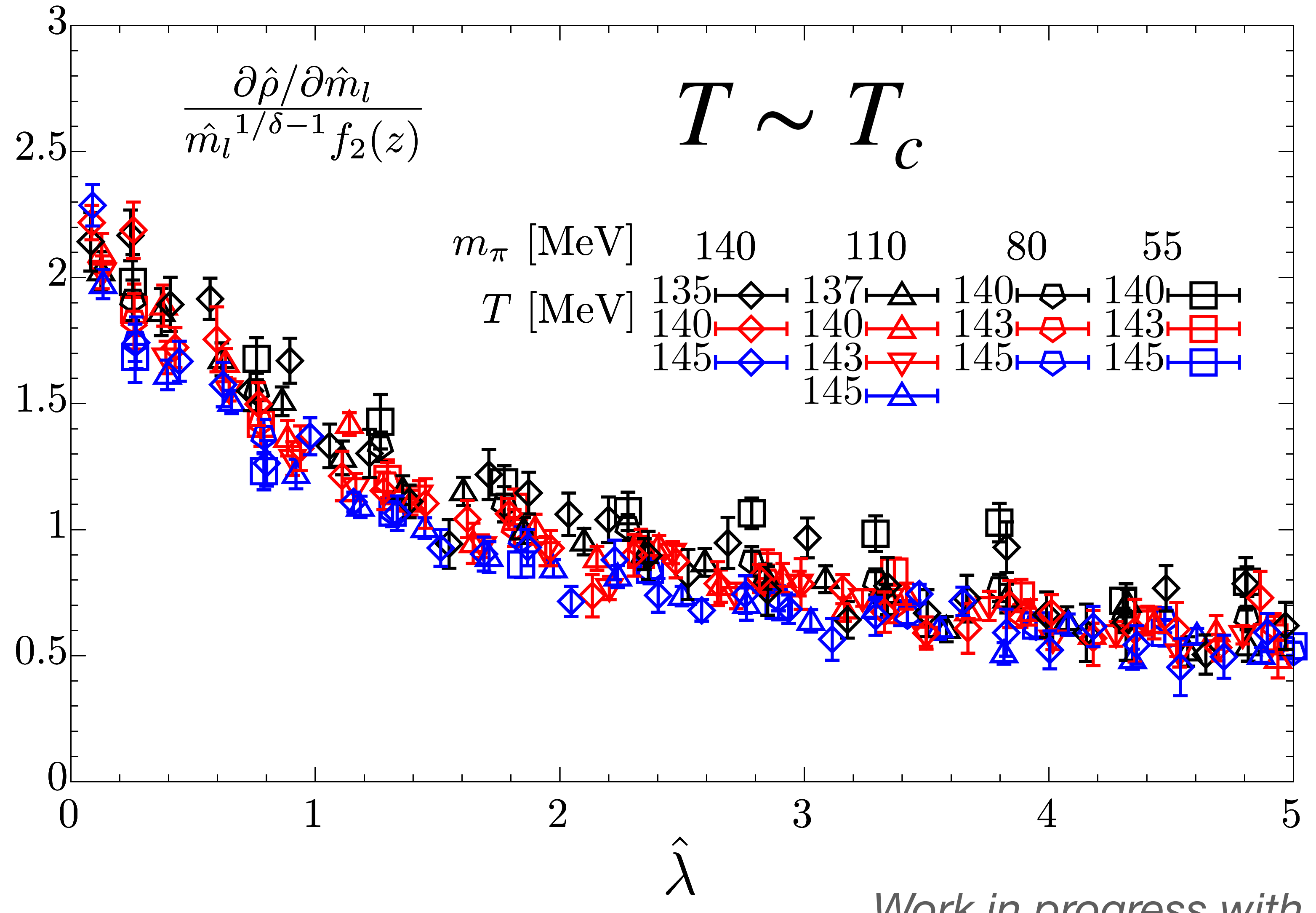
Recap

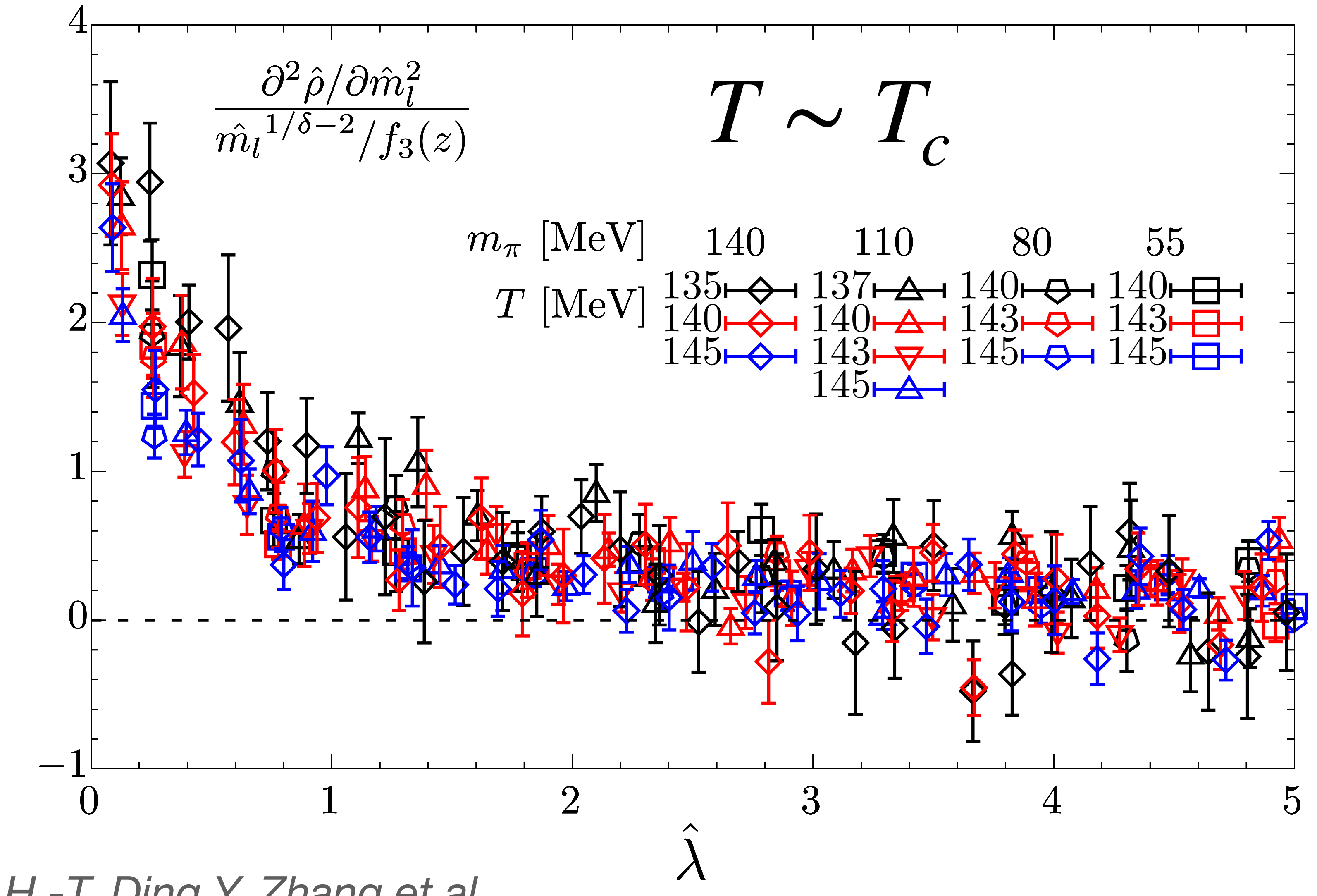




H.-T. Ding et al., PRL 126 (2021), 082001

$\partial^n\rho/\partial m_l^n$ ∝ combinations of P_n





Work in progress with H.-T. Ding, Y. Zhang et al.

For high $T \sim 1.6T_c$:
Consistent with dilute instanton gas picture

$$m_l^{-1}\,\partial\rho/\partial m_l \approx \partial^2\rho/\partial m_l^2 \;\&\; \partial^3\rho/\partial m_l^3 \approx 0$$

$$\Rightarrow \rho(\lambda\to 0,\, m_l\to 0) \propto m_l^2\delta(\lambda)$$

What happens in between ?

For $T \sim T_c$:
Governed by scaling behaviors

Transition from Scaling to Dilute Instanton Gas Behaviors

$\partial^n \rho / \partial m_l^n$ at $T \in [166, 207]$ MeV at physical point

- $m_l^{-1} \partial\rho/\partial m_l \approx \partial^2\rho/\partial m_l^2$ gradually recovers as approaching to high temperature
- $\partial^3\rho/\partial m_l^3 \approx 0$ recovers at some higher temepratures

Other kind of mass dependence besides m^2 here? Hidden mechanism?

Work in progress with H.-T. Ding,Y. Zhang et al.

Summary

- [x] At $1.6T_c$ axial anomaly remains manifested ⇒ 2nd $O(4)$ chiral transition, driven by weakly interacting (quasi-) instanton gas $\rho(\lambda \to 0,\ m_l \to 0) \propto m_l^2 \delta(\lambda)$ at micro level.

- [x] We establish a novel relation

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1; m), P_U(\lambda_2; m), \cdots, P_U(\lambda_n; m)] \prod_{i=1}^{n} d\lambda_i \equiv \int_0^\infty P_n(\lambda)\ d\lambda\,.$$

n-th order cumulant of the chiral condensate

n-point correlation of the quark energy spectra

- [x] A generalization of the Banks-Casher relation is obtained:

$$\lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)]\,.$$

- [x] In the vicinity of T_c : Microscopic encoding of macroscopic criticality

$$P_n(\lambda) = m_l^{1/\delta - n + 1} f_n(z) g_n(\lambda)\,.$$

- [ ] Transitioning from the dilute instanton gas picture to criticality in chiral phase transition … ?

Backup

Calculation of Massless Dirac Eigenspectra $\rho_U(\lambda)$

- Commonly used method: Lanczos algorithm to calculate the individual low-lying eigenvalues

- Here we utilized the Chebyshev filtering technique combined with a stochastic estimate of the mode number

Mode number : $n_{[s,t]} \approx \frac{1}{n_v} \sum_{k=1}^{n_v} \left[\sum_{j=0}^{p} g_j^p \gamma_j v_k^T T_j(A) v_k \right]$

T_j : Chebyshev polynomial

γ_j & g_j^p : expansion coefficients

n_v : number of random vectors

p : number of polynomial orders

eigenvalue spectrum : $\rho_U(\lambda) = \frac{1}{4} \frac{n_{[\lambda-\delta/2,\lambda+\delta/2]}}{2\delta\lambda}$

$1/4$: Staggered Fermion Discretization Scheme

$1/2$: positive and negative eigenvalue pairs

$\delta\lambda$: bin-size

H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)

Yu Zhang, Lattice 19', arXiv: 2001.05217

Cossu et al., arXiv: 1601.00744

Reproduction of Cumulants $\mathbb{K}_n[\bar{\psi}\psi]$ via $P_n(\lambda)$

$$\mathbb{K}_1[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_1[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_1(\lambda)\ d\lambda \qquad \mathbb{K}_2[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_2[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_2(\lambda)\ d\lambda \qquad \mathbb{K}_3[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_3[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_3(\lambda)\ d\lambda$$

Open symbols: computation via inversions of the fermion matrix $\mathrm{Tr}M^{-1}$

Filled symbols: reconstructed from $P_n(\lambda)$

Cumulants related to $P_n(\lambda)$ can successfully reproduce their corresponding results from inverse fermion matrix