Chiral symmetry breaking in hot QCD

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Symmetries of QCD and their realization

- partition function $Z = \int \mathscr{D}U \; \prod_f \det\left(D[U] + m_f\right) \cdot e^{-S_g[U]}$

- $m_u \approx m_d \approx 0$

- Symmetries: $SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A$
  - $U(1)_A$ anomalous
  - $SU(2)_A$ spontaneously broken below T_c

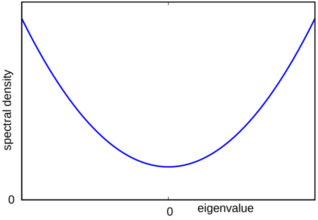
- Order parameter of $SU(2)_A$ (Banks-Casher formula):

$$\langle \bar{\psi}\psi \rangle \;\propto\; \frac{1}{V}\sum_i \frac{1}{\lambda_i + m} \;\propto\; \int_{-\Lambda}^{\Lambda} d\lambda \, \frac{m}{\lambda^2 + m^2}\, \rho(\lambda) \; \xrightarrow[m\to 0]{} \; \rho(0)$$

λ_i: eigenvalues of the Dirac operator, $\rho(\lambda)$: its spectral density

Below T_c

- Chiral symmetry broken
- Order parameter: $\rho(0) \neq 0$



The finite temperature transition

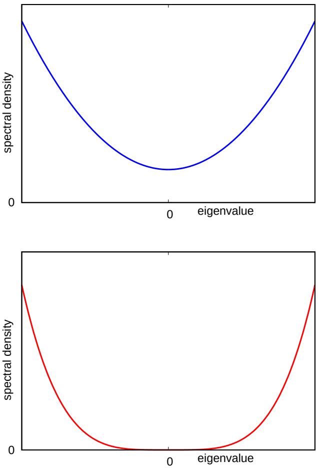
Standard picture

Below T_c

- Chiral symmetry broken
- Order parameter: $\rho(0) \neq 0$

Above T_c

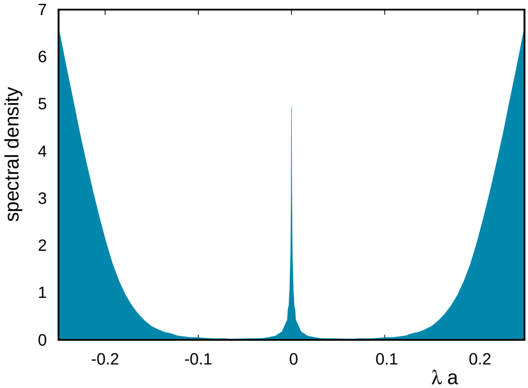
- Chiral symmetry restored
- Order parameter $\rho(0) = 0$
- (Pseudo)gap (lowest Matsubara mode)



spectral density at 0 $\Longleftrightarrow$ realization of chiral symmetry

Spectral density at $T = 1.1\,T_c$ on the lattice

quenched (quark back reaction omitted), exact zero modes not shown



$$Z = \int \mathscr{D}U \prod_f \mathrm{det}(D[U] + m_f) \cdot e^{-S_g[U]}$$

Peak at zero in the spectral density!

Edwards et al. PRD 61 (2000); Alexandru & Horvath, PRD 92 (2015); 2404.12298; Kaczmarek, Mazur, Sharma, PRD 104 (2021) 2021

Questions

- Why is there a peak at zero?
- How is it suppressed if the quark determinant is included?
- How does the peak influence chiral symmetry as $m \to 0$?

Instantons $\rightarrow$ zero eigenvalues of $D(A)$

- (Anti)instanton
  $\rightarrow$ zero eigenvalue of $D(A)$ with $(-)+$ chirality eigenmode

- High T:
  large instantons "squeezed out" in the temporal direction
  $\rightarrow$ dilute gas of instantons and antiinstantons

- Zero modes exponentially localized:

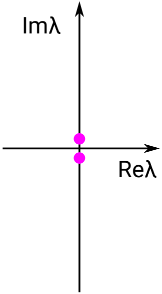
$$\psi(r) \propto e^{-\pi T r}$$

Instanton-antiinstanton pair

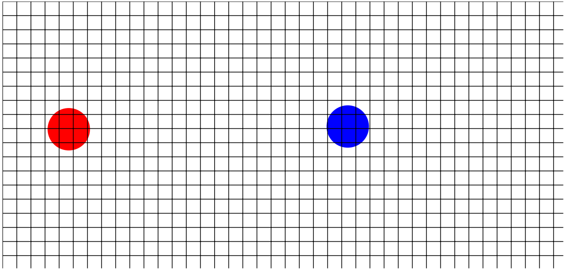
The Dirac operator in the subspace of zero modes

$$D(A) = \begin{pmatrix} 0 & iw \\ iw & 0 \end{pmatrix} \qquad w \propto e^{-\pi Tr}$$

Spectrum of $D(A)$



Instanton and antiinstanton



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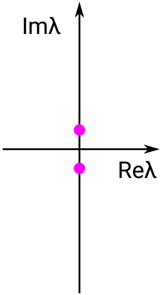

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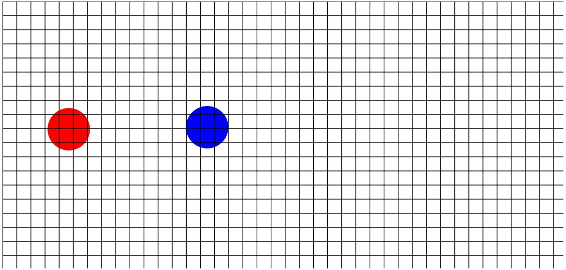

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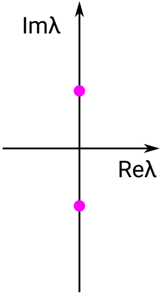

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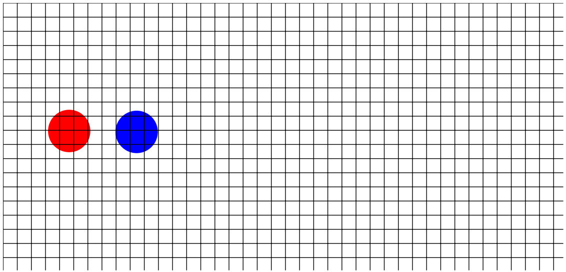

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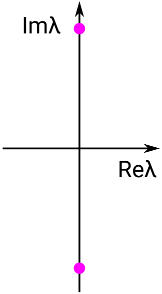

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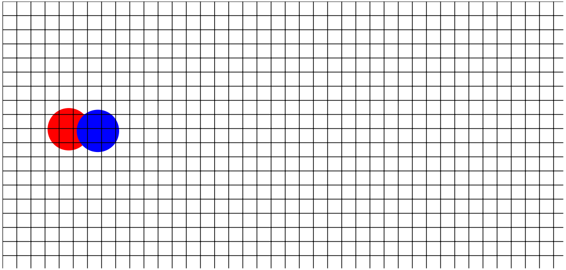
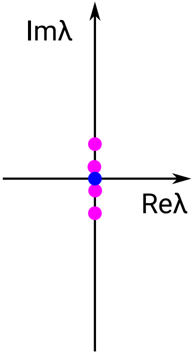


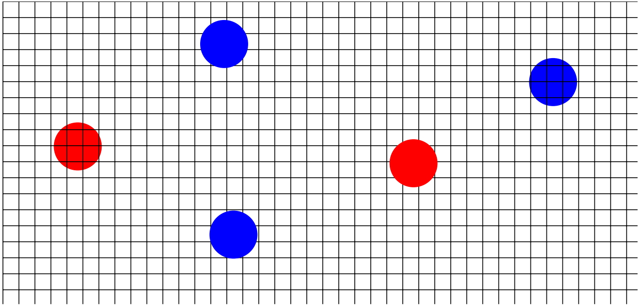
Spectrum of $D(A)$ in dilute gas of instantons

The Dirac operator in the subspace of zero modes

Spectrum of $D(A)$



Instantons and antiinstantons



n_i instantons n_a antiinstantons

$\rightarrow$ $|n_i - n_a|$ exact zero modes + mixing near zero modes

Dirac operator in the subspace of zero modes (ZMZ)

Work by E.V. Shuryak, J.J.M. Verbaarschot, T. Schäfer... (1990-2000)

- Given n_i instantons, n_a antiinstantons in 3d box of size L^3
- Construct $(n_i + n_a) \times (n_i + n_a)$ matrix:

$$D = \left(\begin{array}{c|c} \overbrace{0}^{n_i} & \overbrace{iW}^{n_a} \\ \hline iW^\dagger & 0 \end{array} \right)$$

- $w_{ij} = A \cdot \exp(-\pi T \cdot r_{ij})$ $\qquad$ r_{ij} is the distance of instanton i and antiinstanton j

Random matrix model of $D(A)$ in the zero mode zone

- How to choose instanton numbers (n_i, n_a) and locations?

- Quenched lattice $T > 1.05\,T_c \quad \rightarrow \quad$ free instanton gas

  Bonati et al. PRL 110 (2013); Vig R. & TGK, PRD 103 (2021)

  - n_i and n_a independent identical Poisson-distributed

  $$p(n_i, n_a) = e^{-\chi V} \cdot \frac{(\chi V/2)^{n_i}}{n_i!} \cdot \frac{(\chi V/2)^{n_a}}{n_a!}$$

  χ is the topological susceptibility

  - Locations random (uniform)

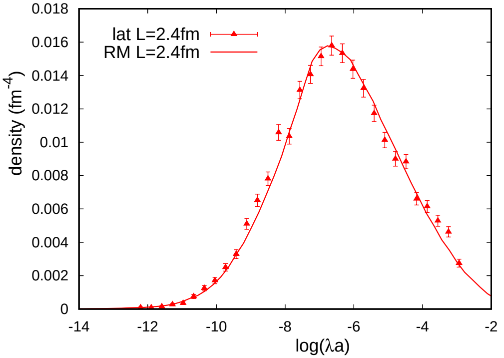
- $\rightarrow$ D(A) in quenched QCD: ensemble of random matrices

Fit parameters to quenched lattice Dirac spectrum

$T = 1.1\,T_c$ overlap Dirac spectrum

- Two parameters:
  - χ – topological susceptibility: from exact zero modes $\rightarrow\ \chi = \langle Q^2 \rangle / V$
  - A – prefactor of the exponential mixing between zero modes
- Fit A to distribution of Dirac eigenvalues (lowest eigenvalue)

$L = 2.4$fm fit



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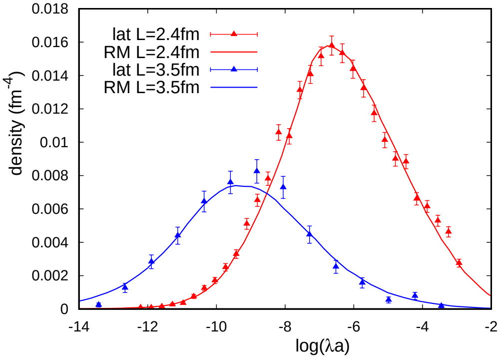
$L = 2.4$fm fit

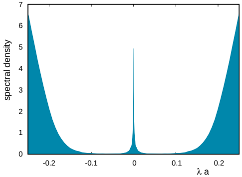
$L = 3.5$fm prediction



Random matrix model of full QCD zero mode zone

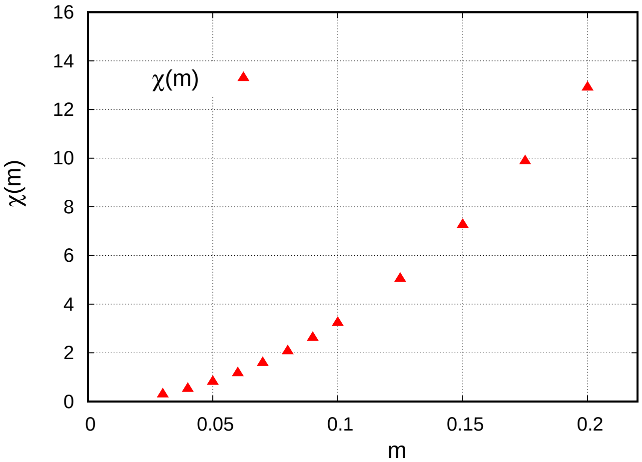
- Include $\det(D+m)^{N_\mathrm{f}}$ in Boltzmann weight
- $\det(D+m) = \prod_{\mathrm{zmz}}(\lambda_i + m) \times \prod_{\mathrm{bulk}}(\lambda_i + m)$
- Bulk weakly correlated with zero mode zone
- Approximate det with $\prod_{\mathrm{zmz}}(\lambda_i + m)$
- Consistently included in RM model:

$$P(n_\mathrm{i}, n_\mathrm{a}) = \underbrace{\mathrm{e}^{-\chi_0 V} \frac{1}{n_\mathrm{i}!} \frac{1}{n_\mathrm{a}!} \left(\frac{\chi_0 V}{2}\right)^{n_\mathrm{i}+n_\mathrm{a}}}_{\text{free instanton gas with random locations}} \times \det(D+m)^{N_\mathrm{f}}$$



Random matrix simulation: results for $N_f = 2$

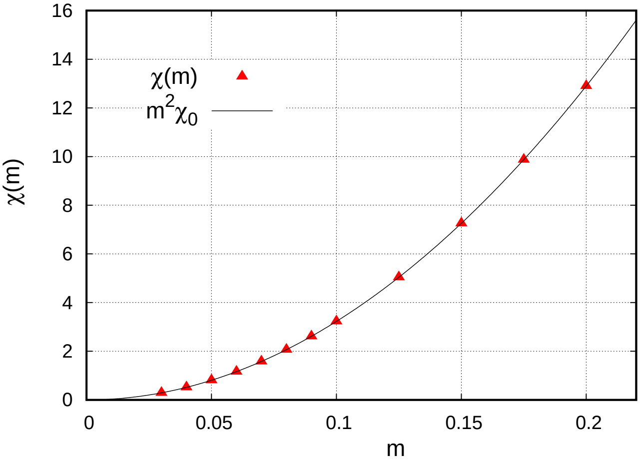
Topological susceptibility:



Random matrix simulation: results for $N_f = 2$

Topological susceptibility: $\chi(m) = m^2 \chi_0$ not a fit!

↑ quenched susceptibility



Explanation: free instanton gas

- Quark determinant for n_i instantons and n_a antiinstantons:

$$\det(D+m)^{N_f} = \prod_{n_i,n_a} (\lambda_i + m)^{N_f} \approx m^{N_f(n_i+n_a)}$$

$$\text{if } |\lambda_i| \ll m$$

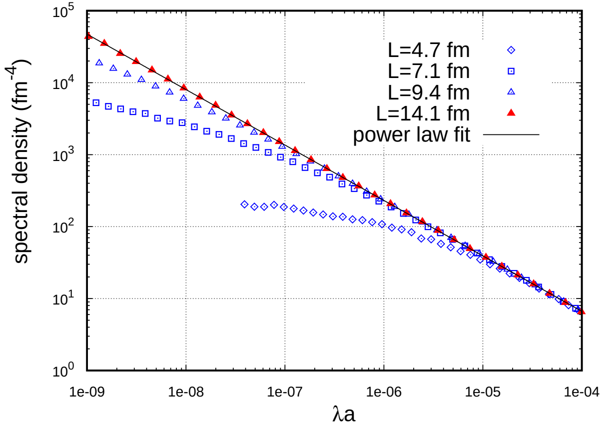
- Reweighting depends on number of topological objects, not on their type or location

$$P(n_i, n_a) \propto \left(\frac{\chi_0 V}{2}\right)^{n_i+n_a} \times \det(D+m)^{N_f} \approx \left(\frac{m^{N_f}\chi_0 V}{2}\right)^{n_i+n_a}$$

- Free gas, but susceptibility suppressed as $\chi_0 \to m^{N_f}\chi_0$
- As $m \to 0$ instanton gas more dilute $\Rightarrow$ $|\lambda_i|$ smaller
- Even in the chiral limit $|\lambda_i| \ll m \implies$ free instanton gas

Spectral density singular at the origin for $V \to \infty$

RM model simulation, parameters from quenched $T = 1.1\,T_c$ overlap spectrum



$$\rho(\lambda) \propto \lambda^{\alpha}$$

fit: $\alpha = -0.770(5)$

Singular spectral density from similar instanton model:

Sharan and Teper, hep-ph/9910216

Banks-Casher for a singular spectral density?

“Banks-Casher” for singular spectral density

$$\langle \bar{\psi}\psi \rangle \;\propto\; \Big\langle \sum_i \frac{m}{m^2+\lambda_i^2} \Big\rangle \;\approx\; \underbrace{\begin{pmatrix}\text{avg. number of in-}\\ \text{stantons in free gas}\end{pmatrix}}_{m^{N_\mathrm{f}}\chi_0 V} \cdot \frac{1}{m} \;=\; m^{N_\mathrm{f}-1}\chi_0 V$$

$$|\lambda_i| \ll m$$

“Banks-Casher” for singular spectral density

$$\langle\bar{\psi}\psi\rangle \propto \langle\sum_i \frac{m}{m^2+\lambda_i^2}\rangle \approx \underbrace{\begin{pmatrix}\text{avg. number of in-}\\ \text{stantons in free gas}\end{pmatrix}}_{m^{N_\mathrm{f}}\chi_0 V} \cdot \frac{1}{m} = m^{N_\mathrm{f}-1}\chi_0 V$$

$$|\lambda_i| \ll m$$

$U(1)_\mathrm{A}$ breaking susceptibility $\chi_\pi - \chi_\delta$

$$\langle\sum_i \frac{m^2}{(m^2+\lambda_i^2)^2}\rangle \approx \underbrace{\begin{pmatrix}\text{avg. number of in-}\\ \text{stantons in free gas}\end{pmatrix}}_{m^{N_\mathrm{f}}\chi_0 V} \cdot \frac{1}{m^2} = m^{N_\mathrm{f}-2}\chi_0 V$$

$$\rightarrow \quad \lim_{m\rightarrow 0}(\chi_\pi - \chi_\delta) \neq 0 \qquad \text{for } N_f = 2$$

Related developments & outlook

- RM model @ small m $\rightarrow$ also has instanton-antiinstanton molecules

  do not contribute to $\langle\bar{\psi}\psi\rangle$ and $\chi_\pi - \chi_\delta$ in the chiral limit

- Constraints on the Dirac spectrum from chiral symmetry restoration

  $\rightarrow$ consistent with free instanton gas M. Giordano, 2404.03546 (2024)

- Localization properties of eigenmodes in ZMZ

  M. Giordano and TGK, Universe 7 (2021); A. Alexandru and I Horvath, PRL 127 (2021), PLB 833 (2022)

- What is the lowest temperature where the instanton gas is ideal?

  $\rightarrow$ dynamical simulation with chiral quark action talk by A. Kotov on Wed.

Conclusions for high enough $T < \infty$

- Breaking of chiral symmetry controlled by ideal instanton gas

- $N_f = 2$: $U(1)_A$ broken even in the chiral limit at any $T < \infty$
  Symmetry is not spontaneously broken, but order of the $m \to 0$ and $V \to \infty$ limit still matters.

- Dirac spectral density has singular peak at zero

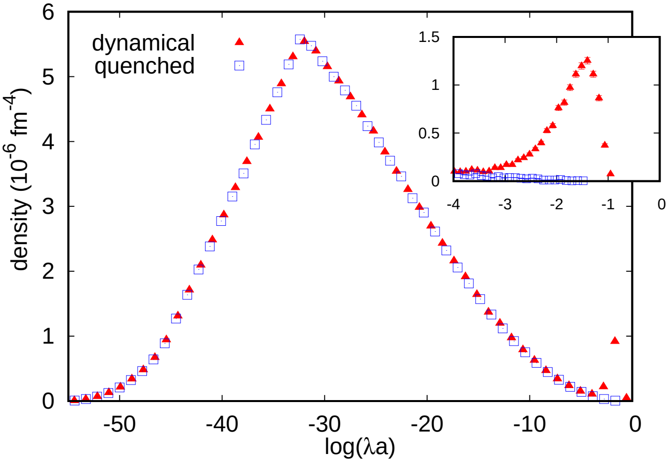
$$\rho(\lambda) \propto \lambda^{-p}$$

  - $p < 1$ (integrable!)
  - Smaller m_q or higher T $\to$ p increases (peak more singular)
  - Conjecture: if $m \to 0$ or $T \to \infty$, then $p \to 1$

BACKUP SLIDES

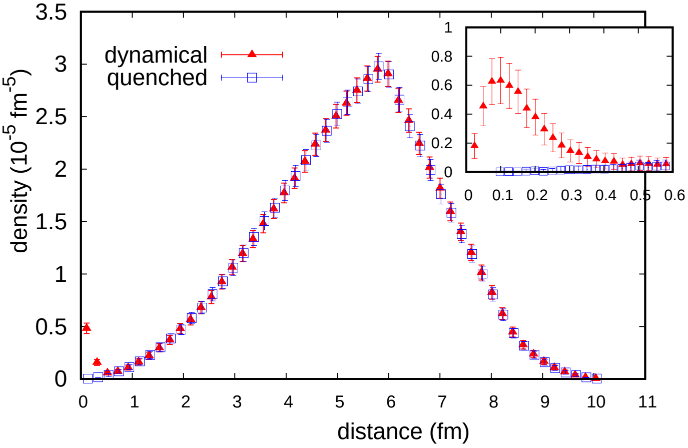
Spectral density – full QCD vs. ideal instanton gas

random matrix model, same topological susceptibility



Instanton-antiinstanton molecules

density of closest opposite charge pairs at given distance



Direct lattice simulations?

- Important to resolve small Dirac eigenvalues
  $\rightarrow$ chiral action needed JLQCD, PRD 103 (2021)
- To see spectral peak: large volume, close to T_c needed
- $\dfrac{\chi_\pi - \chi_\delta}{\chi_{\text{top}}} \propto m^{-2}$ instanton contribution independent of T
- Explore how far down in T free instanton gas persists
  - Compare eigenvalue statistics to prediction of free instanton gas
  - Can be done in each topological sector separately