The QCD phase diagram and Lee-Yang zeros

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[PRD 109 (2024) 11, 114516, arXiv: 2403.09390]
[PRD 105 (2022) 7, 074511, arXiv: 2202.09184]

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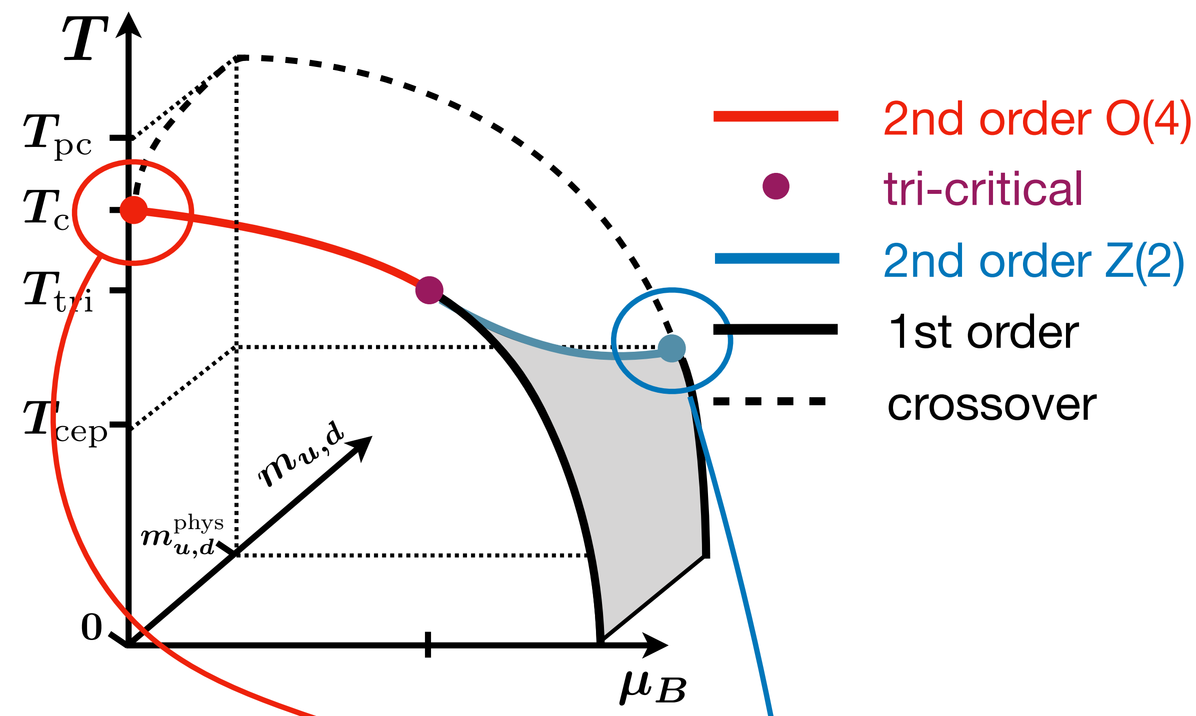
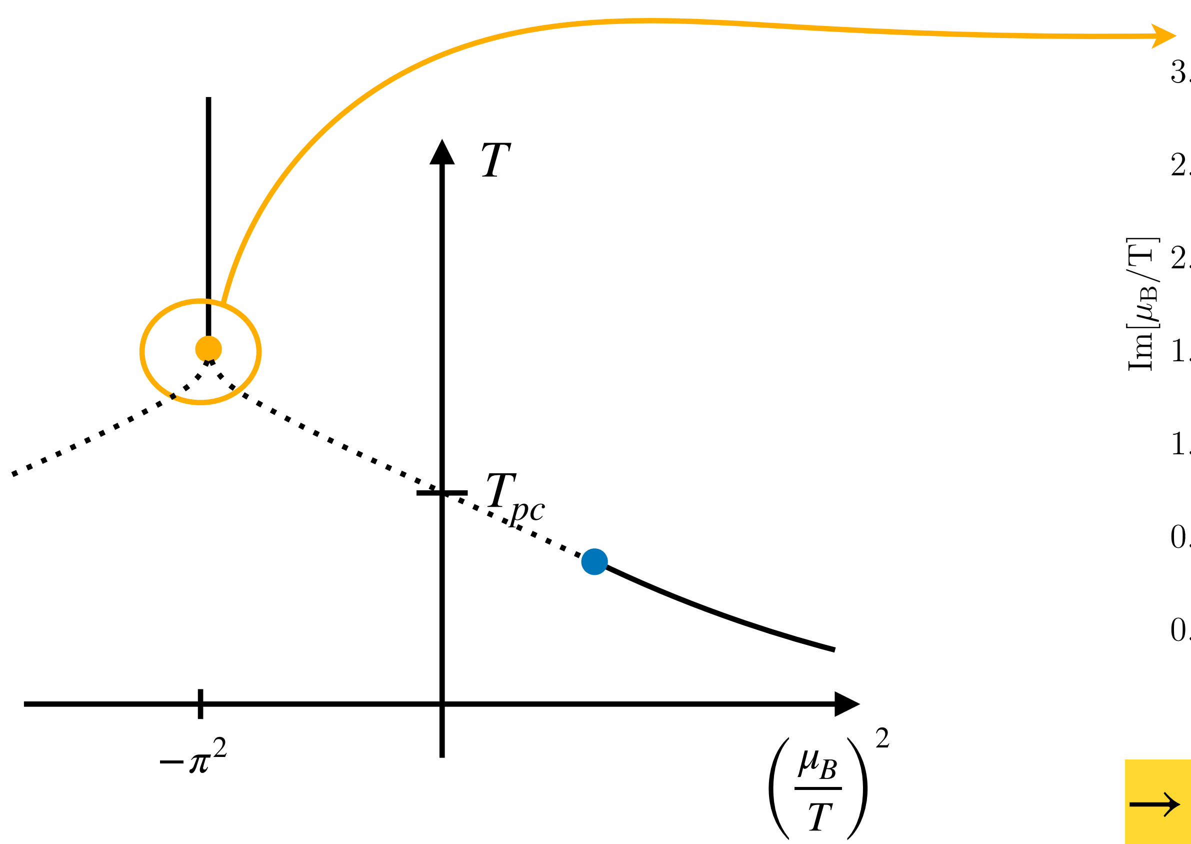
[arXiv: 2405.10196]
[PRD 105 (2022) 3, 034513, arXiv: 2110.15933]

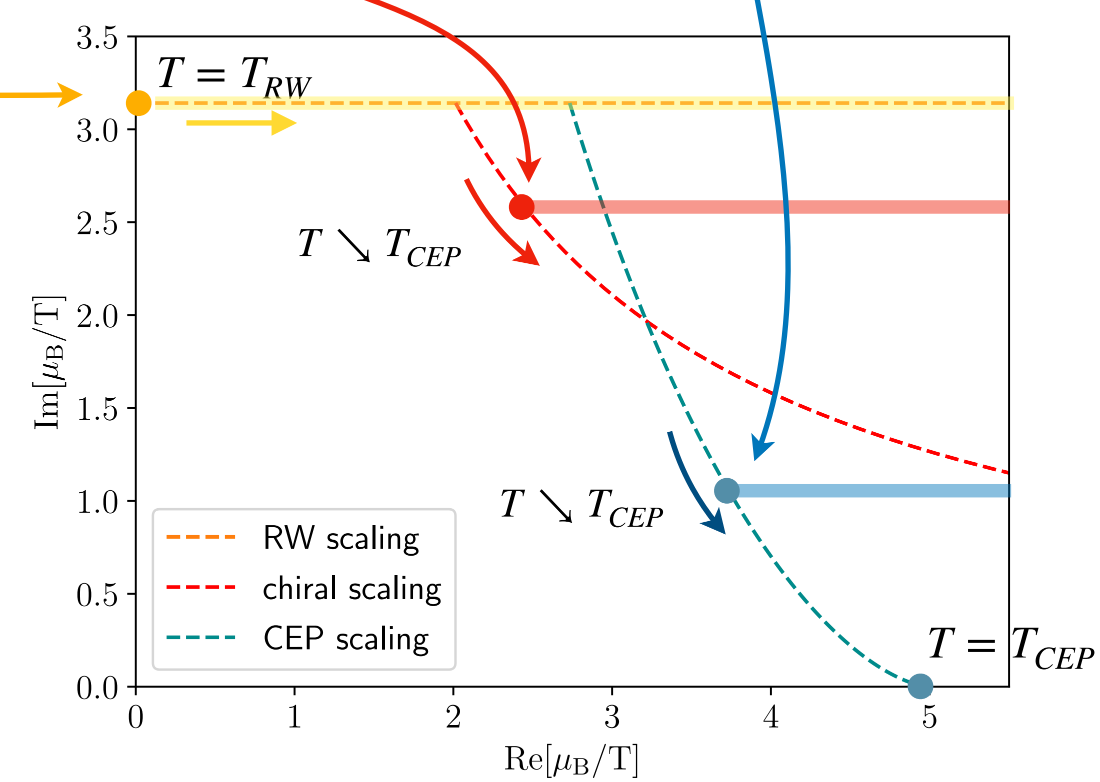
ECT,* Trient, September 9-13, 2024

Lee-Yang edge singularities in QCD

- Track Lee-Yang edge singularity in the complex $\frac{\mu_B}{T}$ -plane, as function of T
- We can think of three distinct critical points/ scaling regions: Roberge Weiss transition, chiral transition, QCD critical point
- Solve $t/h^{1/\beta\delta} \equiv z_{YL}$ for different scaling fields and non-universal constants.





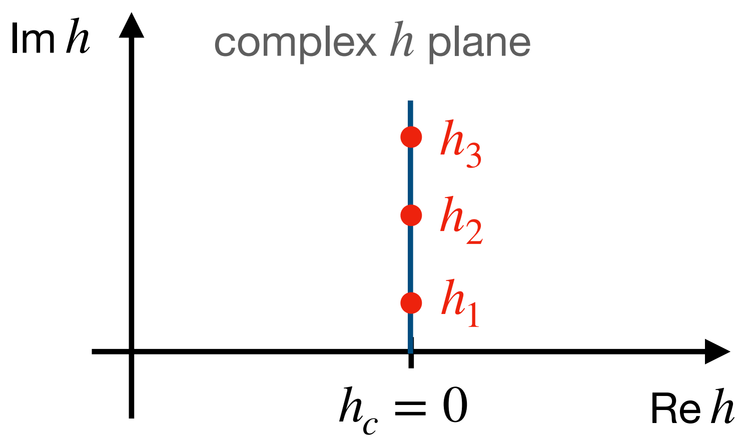


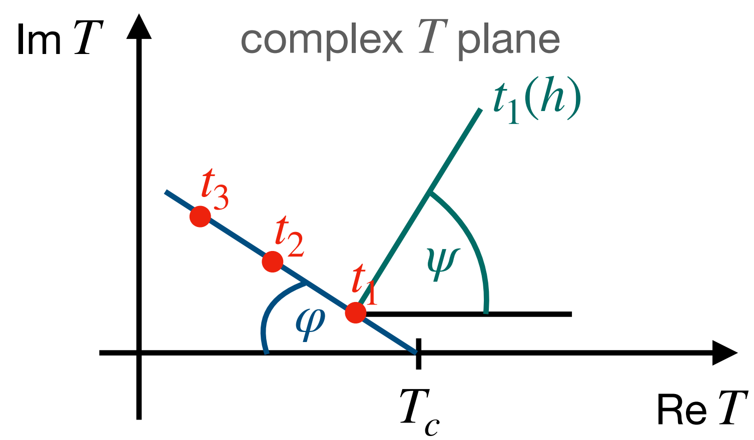
→ different temperature intervals are sensitive to different scaling of the Lee-Yang edge singularity

Plan

- Universal scaling of Lee-Yang zeros and Lee-Yang edge Singularity
  - ➡ Finite size vs infinite size scaling
- First Lee-Yang zero via Padé and multi-point Padé
  - ➡ Scaling evidence from Roberge-Weiss transition (and 2d-Ising model)
  - ➡ First Estimates of the QCD critical end-point
- Estimate via Fourier coefficients

Universal scaling of the zeros of the partition function





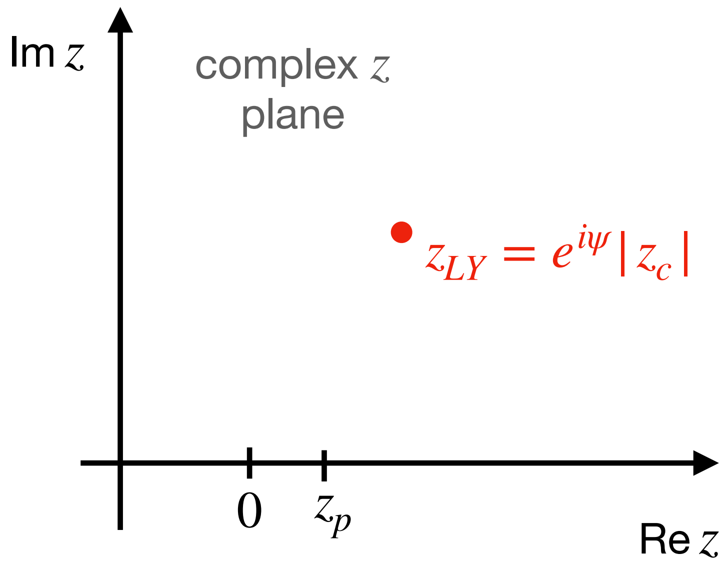
Finite size scaling function of the partition function

$$Q(tL^{1/\nu}, h^2 L^{2\beta\delta/\nu}) = 0$$

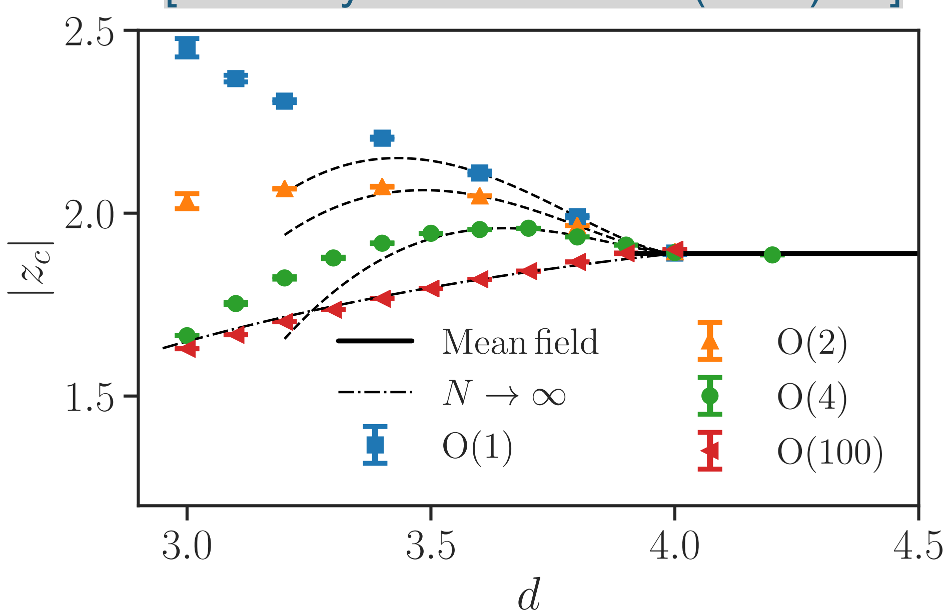
[tzykson et al. NPB 220 (1983) 415]

- **Finite size scaling:** scaling of the zeros is well understood, universal angels $\varphi, \psi = \pi/(2\beta\delta)$ indicate the universality class.

- **Infinite size scaling:** Lee-Yang condense to a branch cut in the infinite size limit. The edge singularity has a universal position in terms of $z = t/h^{1/\beta\delta}$



[Connelly et al. PRL 125 (2020) 19]



Scaling variable: $z = t/h^{1/\beta\delta}$

Use universal constant $z_{LY} = |z_c| e^{i\frac{\pi}{2\beta\delta}}$

Solve for $\mu_{LY}(T)$

Roberge-Weiss transition:

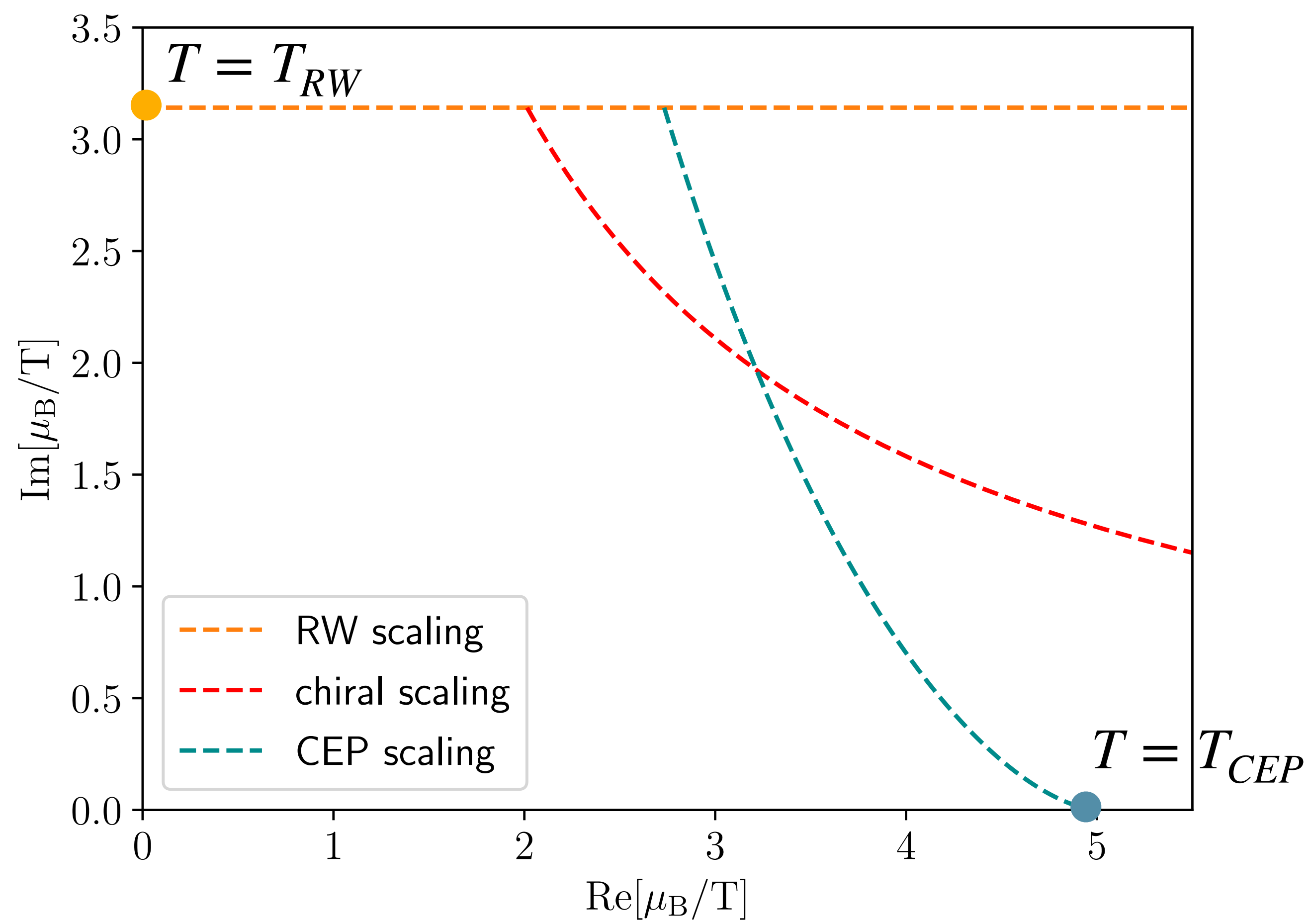
$$t = t_0 \left(\frac{T_{RW} - T}{T_{RW}} \right) \quad \text{and} \quad h = h_0 \left(\frac{\hat{\mu}_B - i\pi}{i\pi} \right)$$

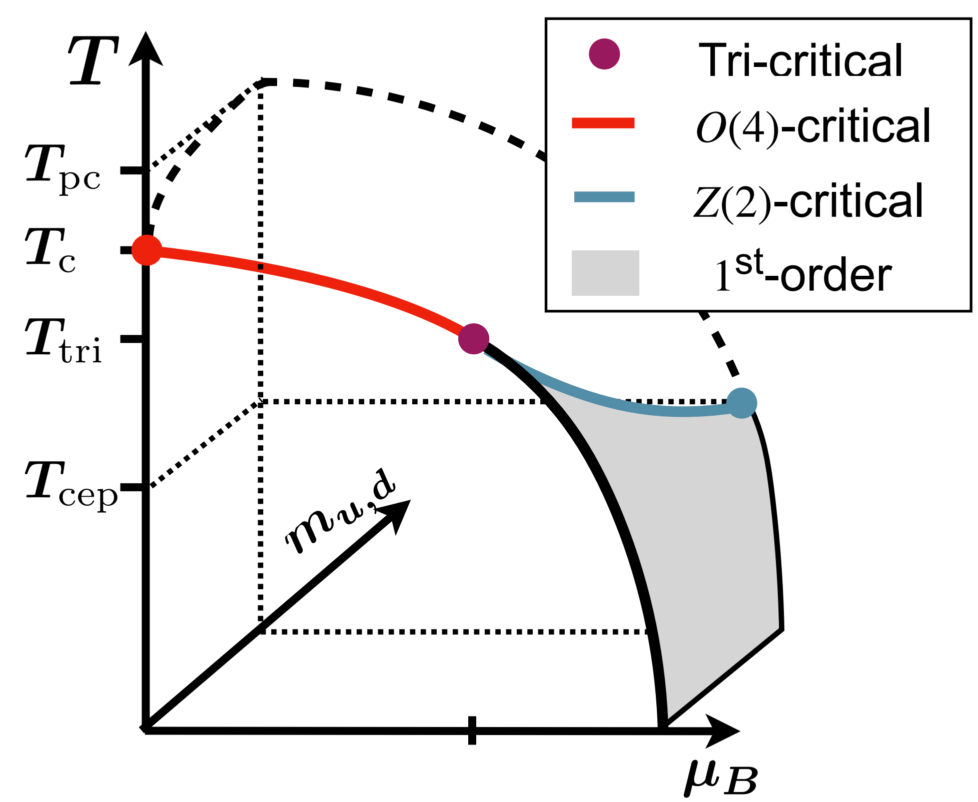
Chiral transition:

$$t = t_0 \left[\frac{T - T_c}{T_c} + \kappa_2^B \left(\frac{\mu_B}{T} \right)^2 \right] \quad \text{and} \quad h = h_0 \frac{m_l}{m_s^{\text{phys}}}$$

QCD critical point:

$$t = \alpha_t (T - T_{cep}) + \beta_t (\mu_B - \mu_{cep}) \quad \text{and}$$
$$h = \alpha_h (T - T_{cep}) + \beta_h (\mu_B - \mu_{cep})$$

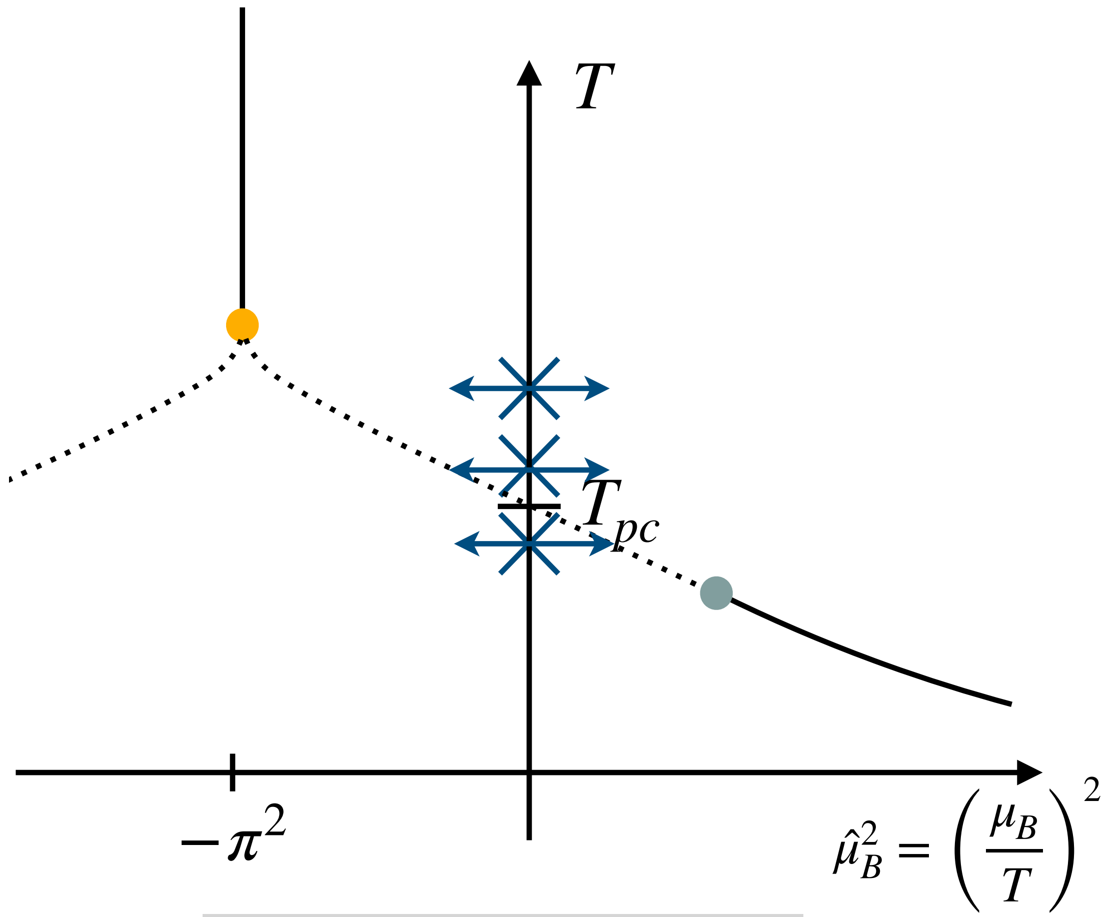




→ different temperature intervals exhibits different scaling of the Lee-Yang edge singularity

* Calculate derivatives of the pressure $\frac{p}{T^4} = \frac{\ln Z}{VT^3}$

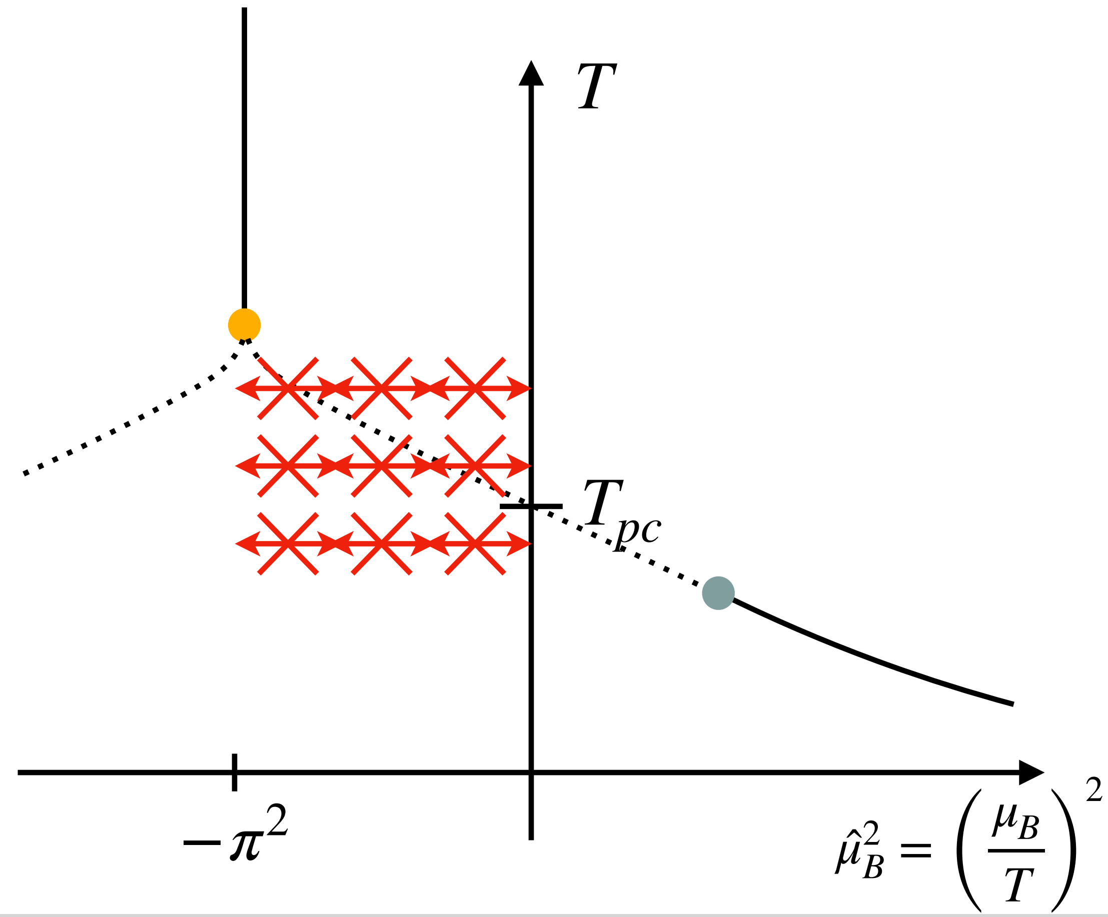
$(T, \mu_B = 0)$: **Taylor expansion in μ_B^2**



[Allton et al. PRD 66 (2002)]

⇒ perform a Padé resummation to obtain the complex singularity that limit the radius of convergence

$(T, \mu_B^2 < 0)$: **Taylor expansion in μ_B**

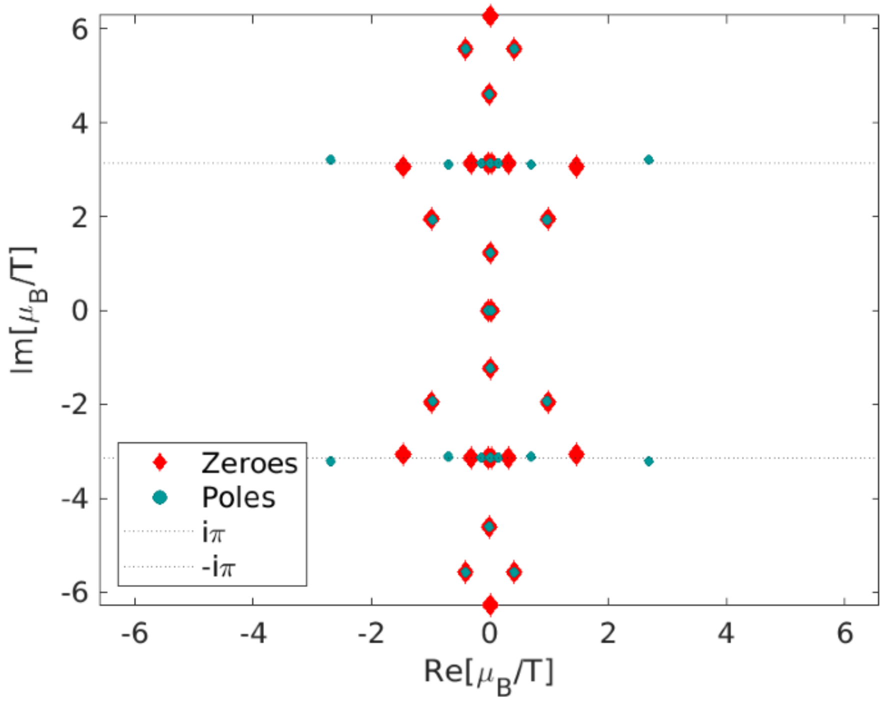


[De Frorcrand, Philipsen (2002); D'Elia, Lombardo (2003)]

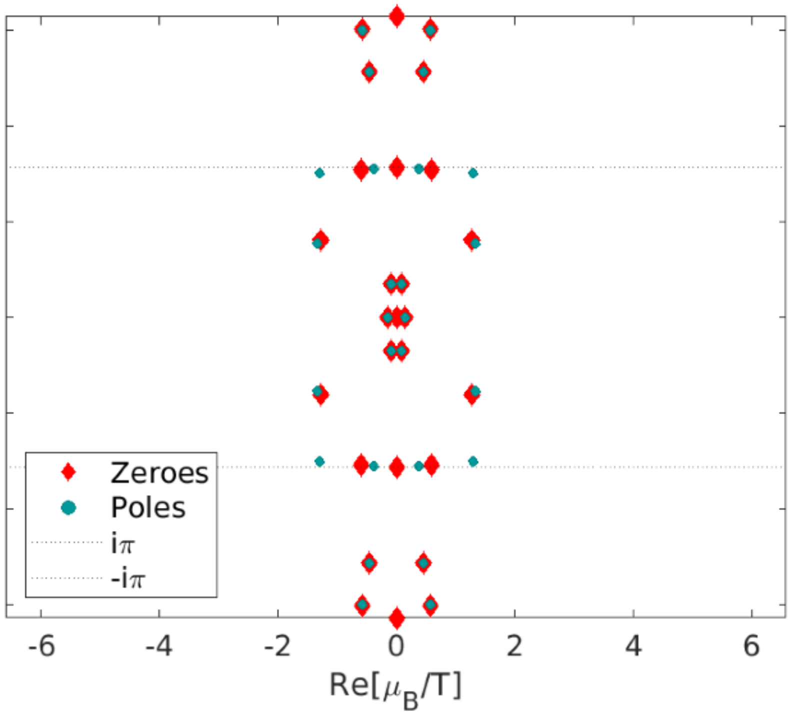
⇒ obtain a rational approximation of the data (e.g. by the multi-point Padé) to obtain the closest singularity

⇒ alternatively, analyse the (asymptotic) behaviour of the Fourier coefficients

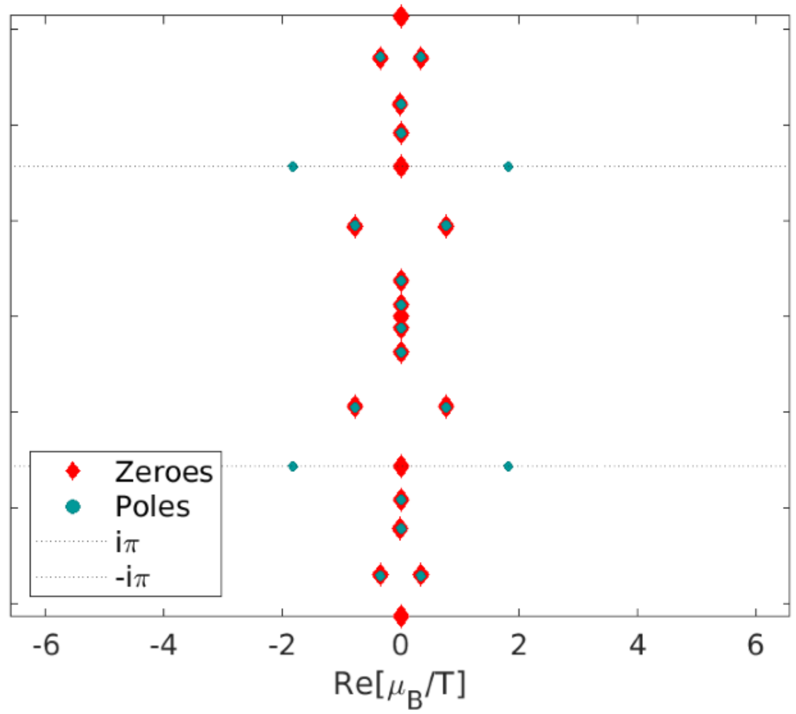
$T = 201$ MeV $= T_{RW}$



$T = 186$ MeV



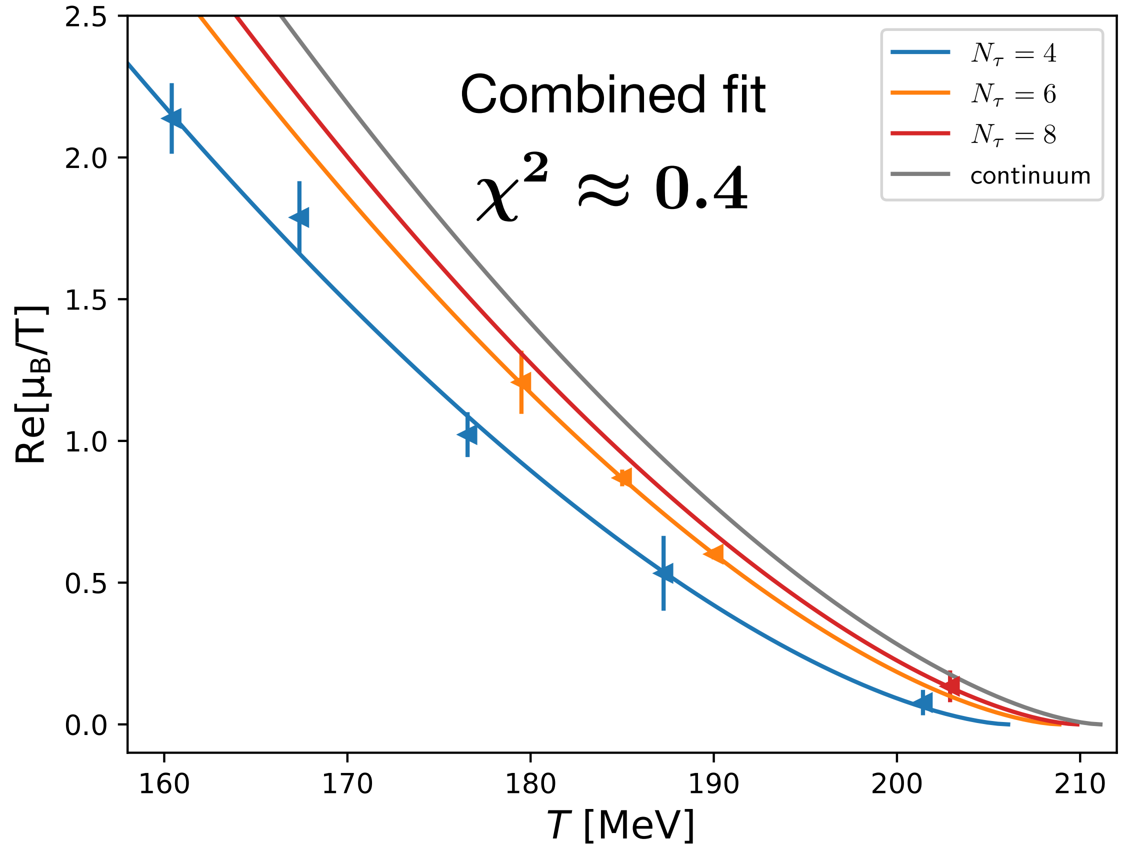
$T = 167$ MeV



→ find almost perfect cancelation of many zeros and poles

→ find signature for branch cut along $\mu_B/T = \mu_B^R \pm i\pi$ at $T = \{201{,}186\}$ MeV

[Dimopoulos et al., *PRD* 105 (2022)]



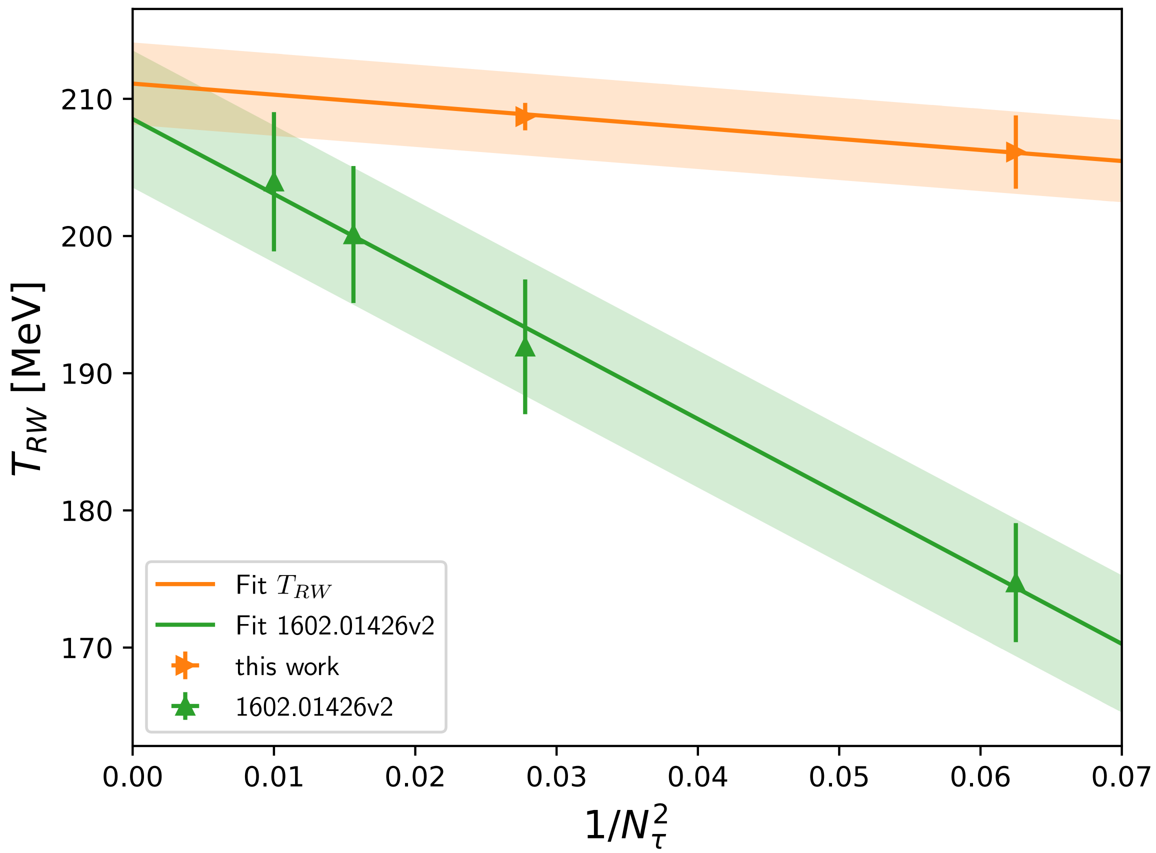
* The approach of the LY edge to the RW critical point: By solving $z = t/h^{1/\beta\delta} \equiv z_c$ we find

$$\hat{\mu}^R_{LY} = a(N_\tau)\left(\frac{T_{RW}(N_\tau) - T}{T_{RW}(N_\tau)}\right)^{\beta\delta}$$

with $\hat{\mu}^R_{LY} = \mathrm{Re}[\mu_B/T]$

We assume $T_{RW} = T^{(0)}_{RW} + T^{(2)}_{RW}/N^2_\tau$

$$a = a^{(0)} + a^{(2)}/N^2_\tau$$



* Obtain continuum result

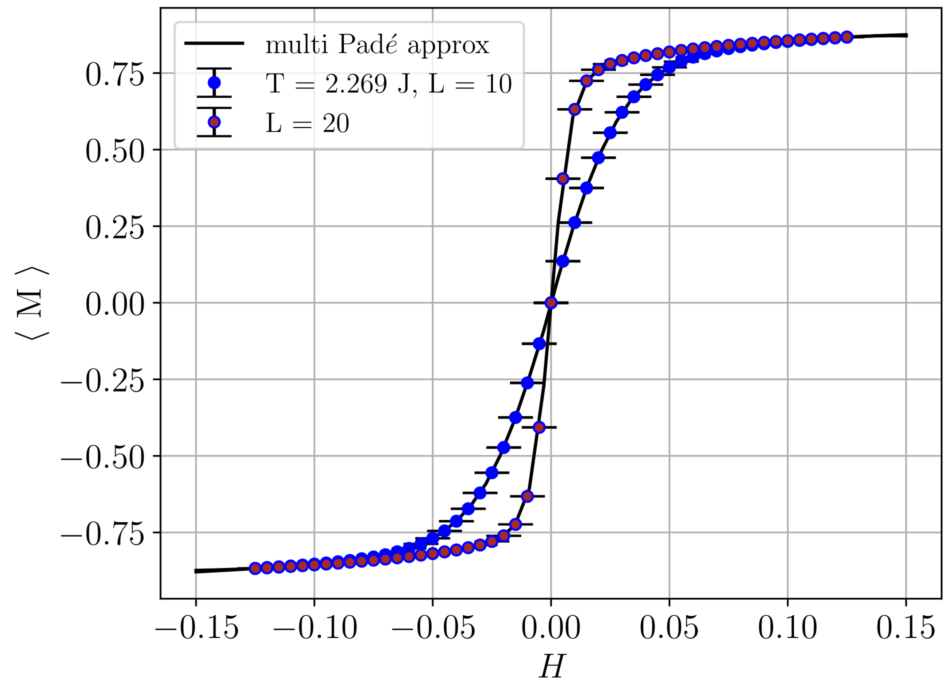
$$T^{(0)}_{RW} = 211.1 \pm 3.1\ \mathrm{MeV}$$

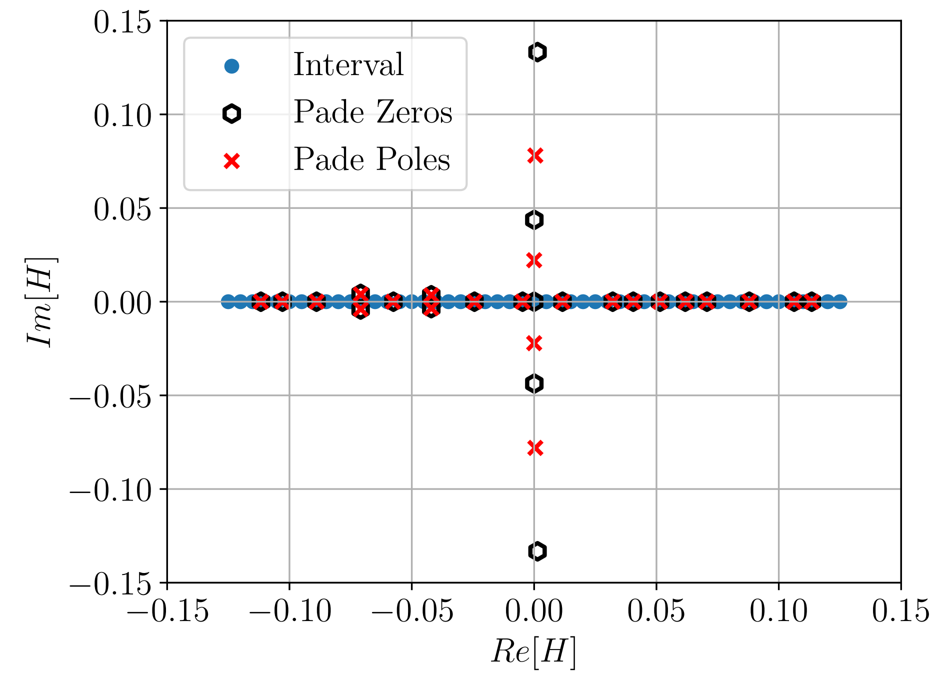
⇒ in good agreement with previous results from the Pisa group

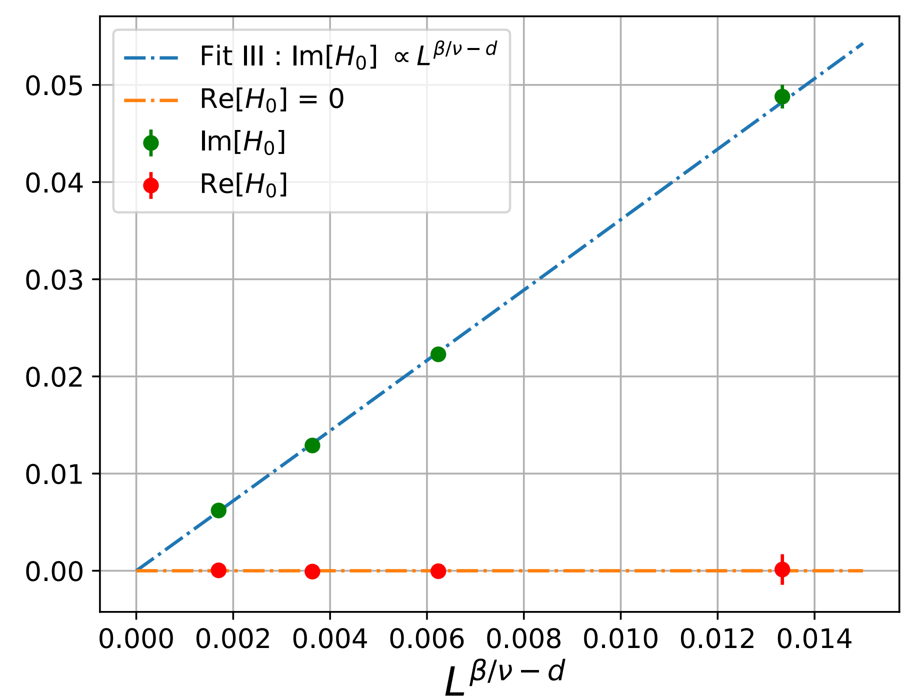
[Bonati et al., PRD 93 (2016) 074504]

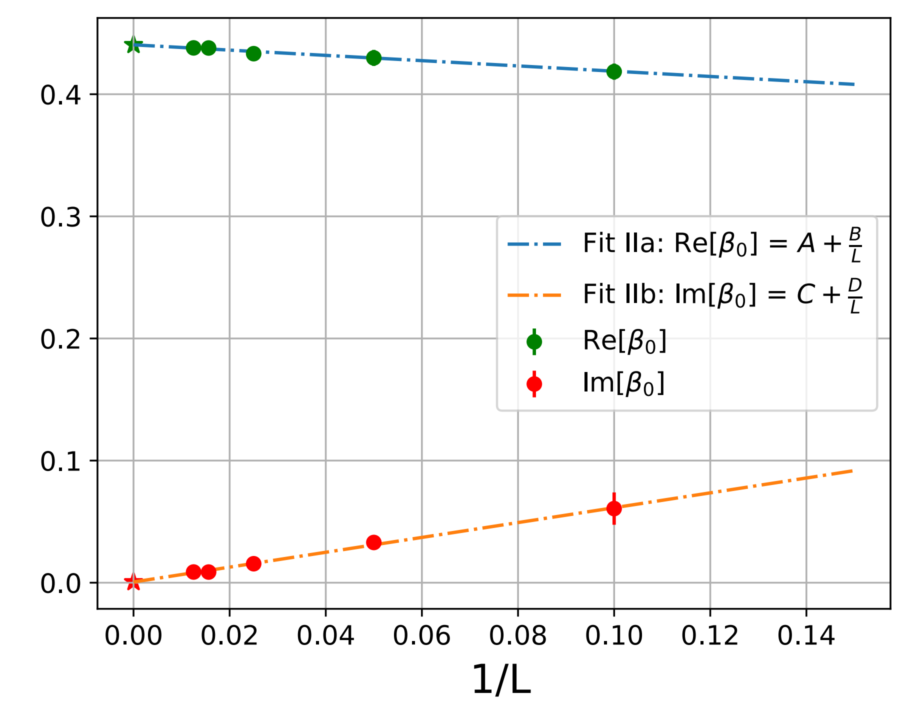
- The multi-point method works well in the Ising model when applied to the magnetisation or the specific heat.

- A finite size scaling analysis reproduces the transition temperature β_c and the critical exponents ν and $\beta\delta$





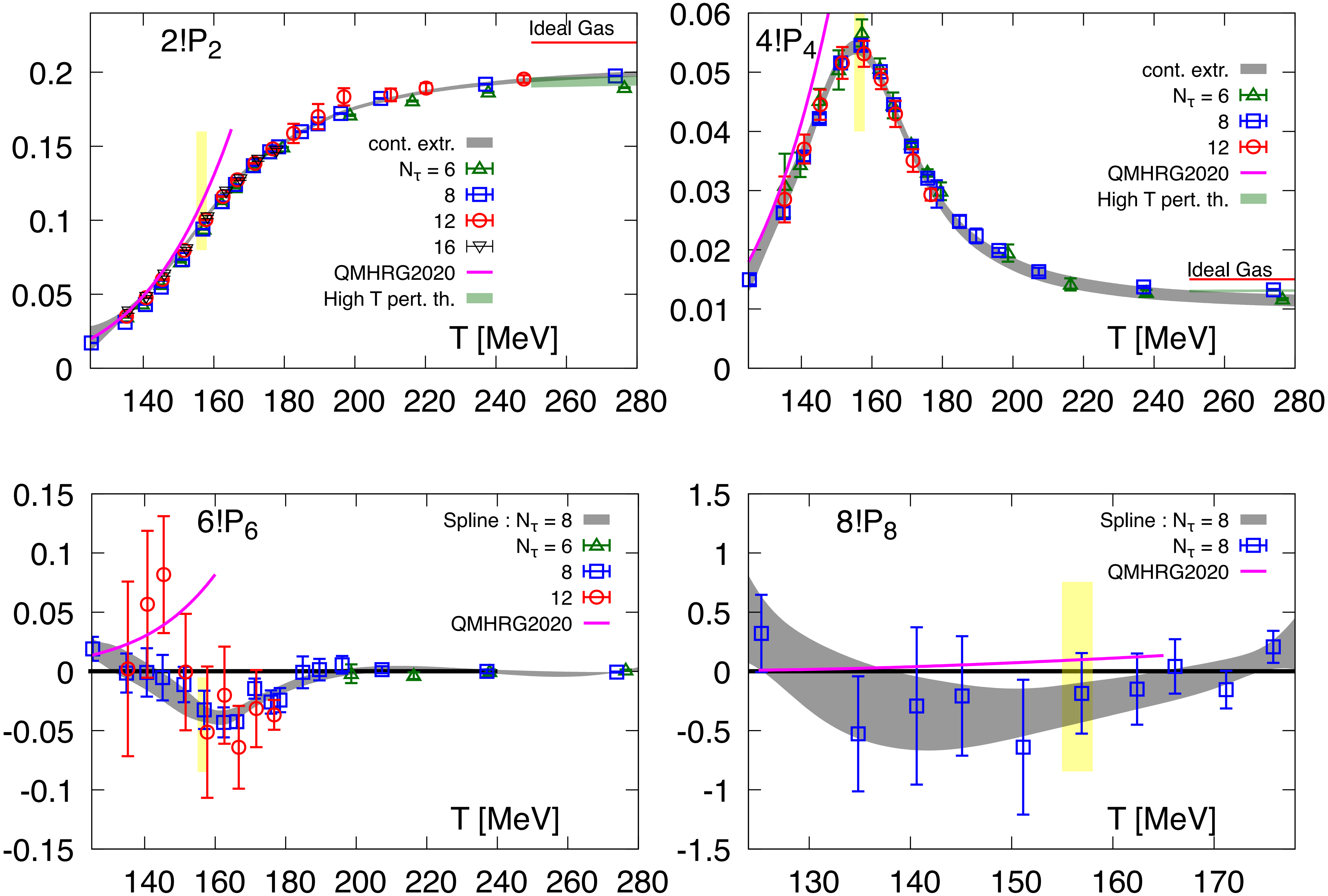




[Singh et al, *PRD* 109 (2024) 7, 07450, arXiv: 2312.03178]

$$\Delta\hat{p} \equiv \frac{p(T,\mu_B)}{T^4} - \frac{p(T,0)}{T^4} = \sum_{k=1}^{\infty} P_{2k}(T)\hat{\mu}_B^{2k}$$

- Detecting phase transitions via Padé and post-Padé approximants has a long history in statistical and high energy physics
- They are often used in combination with perturbation theory
- QCD is non-perturbative in the vicinity of the phase
- The numerical calculation of the pressure series in μ_B is difficult

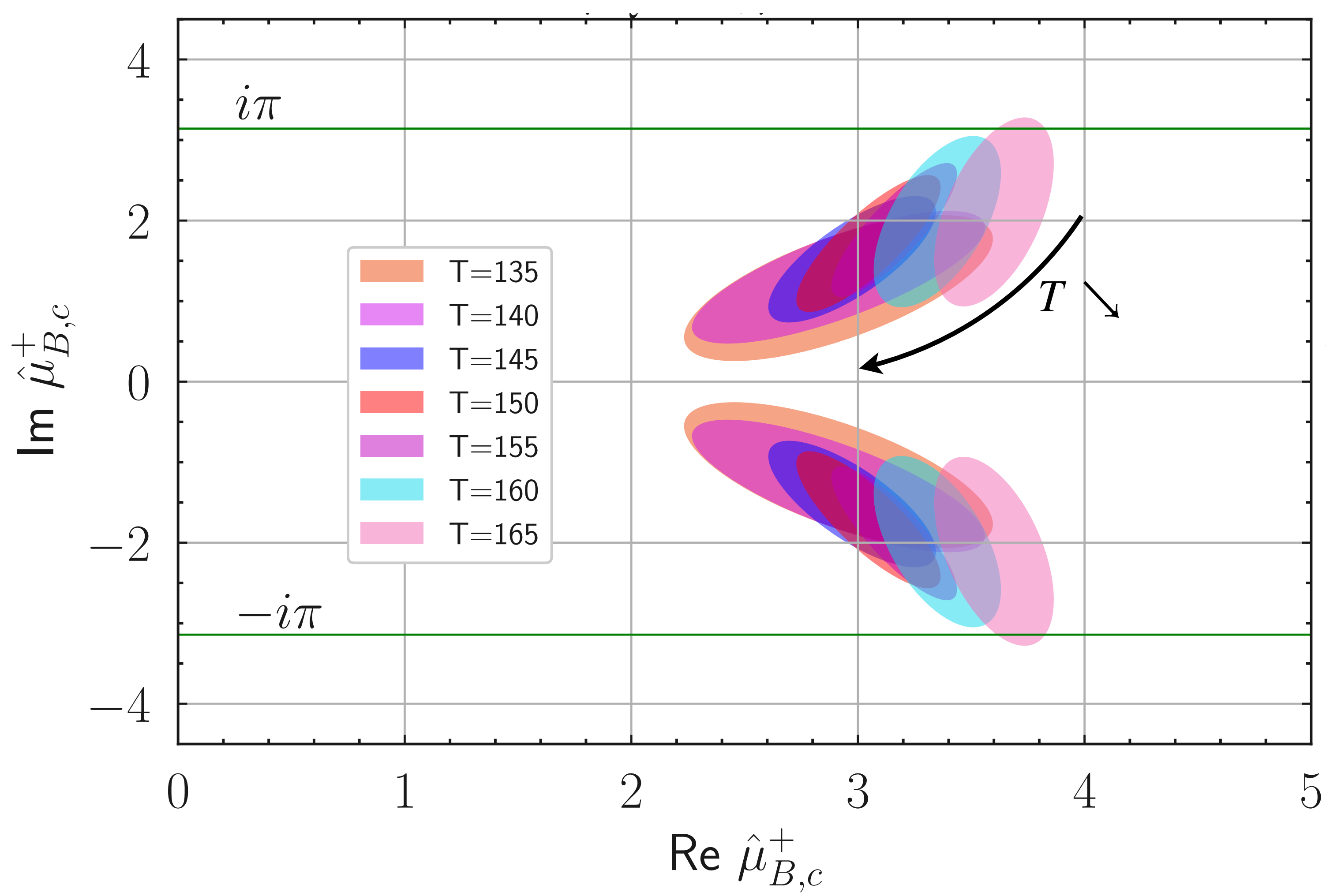


HotQCD, PRD 108 (2023) 1, 014510, arXiv: 2212.09043]

Padé resummation of the Taylor series about $\mu_B = 0$

$$P[4,4] = \frac{P_2\hat{\mu}_B^2 + (P_4 + (P_2^2 P_8)/P_4^2)\hat{\mu}_B^4}{1 + ((P_2 P_8)/P_4^2)\hat{\mu}_B^2 - (P_8/P_4)\hat{\mu}_B^4}$$

- Construct [4,4]-Padé from 8th order Taylor Expansion
- Calculate complex roots of the denominator
- Find apparent approach to the real axis with decreasing temperature
- Can also be combined with conformal maps **[Basar, 2312.06952]**



[PRD 105 (2022) 7, 074511, arXiv: 2202.09184]

Code:

- SIMULATeQCD by HotQCD **[Comp.Phys.Comm. 300 (2024) 109164]**

Simulation Parameters:

- Use (2+1)-flavor of *Highly Improved Staggered Quarks* (HISQ) with physical masses ($m_l/m_s = 1/27$).
- Lattice size: $36^3 \times 6$
- Use *Line of Constant Physics* (LCP) and scale setting from HotQCD
- Introduce non-zero imaginary chemical potential $\hat{\mu}_u = \hat{\mu}_d = \hat{\mu}_s = i\theta$, which corresponds to $\mu_B = 3\mu_u$ and $\mu_S = 0$

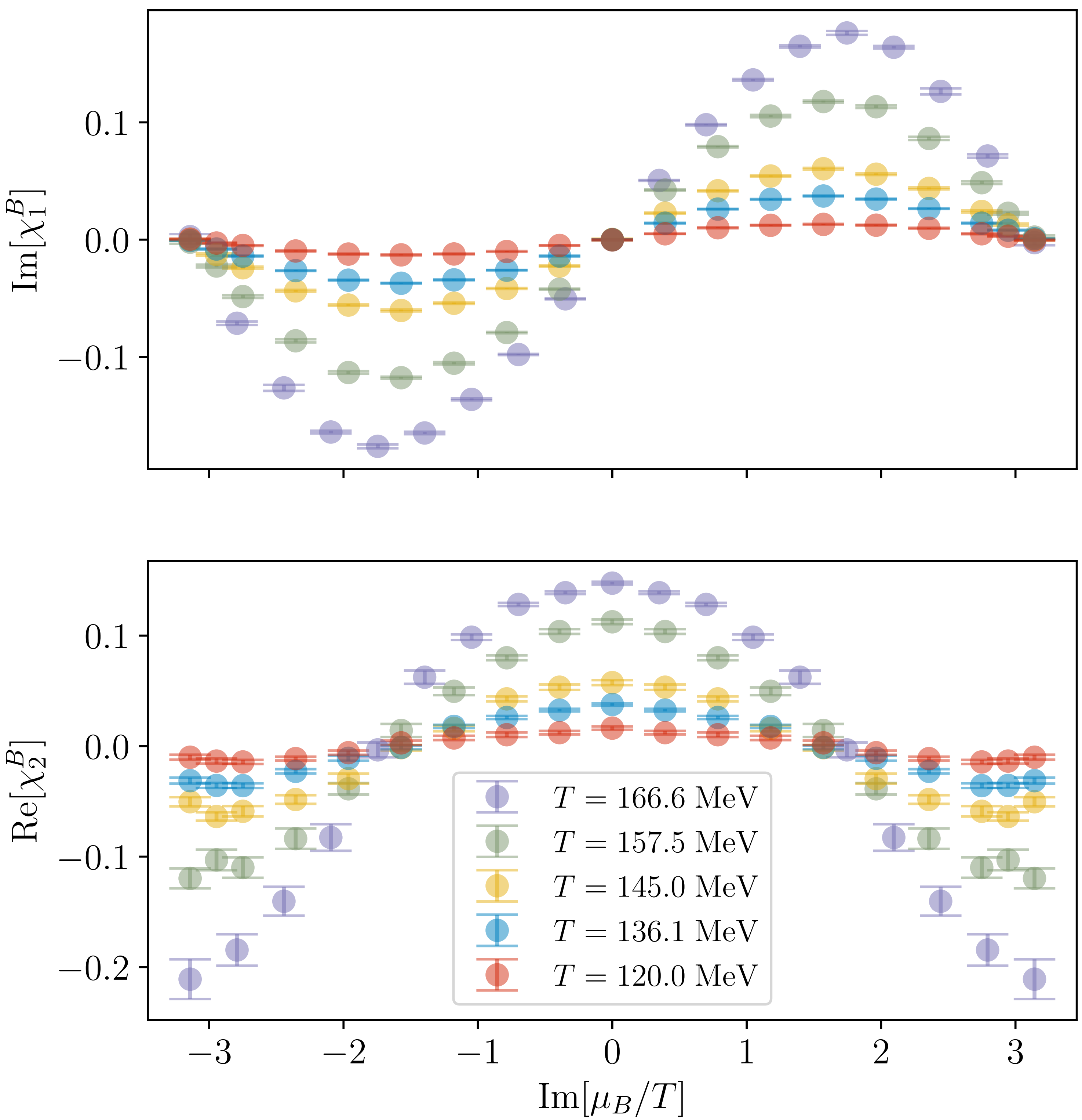
Statistics:

T [MeV]	N_μ	N_{conf}/N_μ
166.6	10	1800
157.5	10	4780
145.0	10	5300
136.1	10	6840
120.0	10	24000

Machines:

- Juwels-Booster @ JSC
- Marconi100 @ CINECA
- Leonardo @ CINECA

Lattice size: $36^3 \times 6$



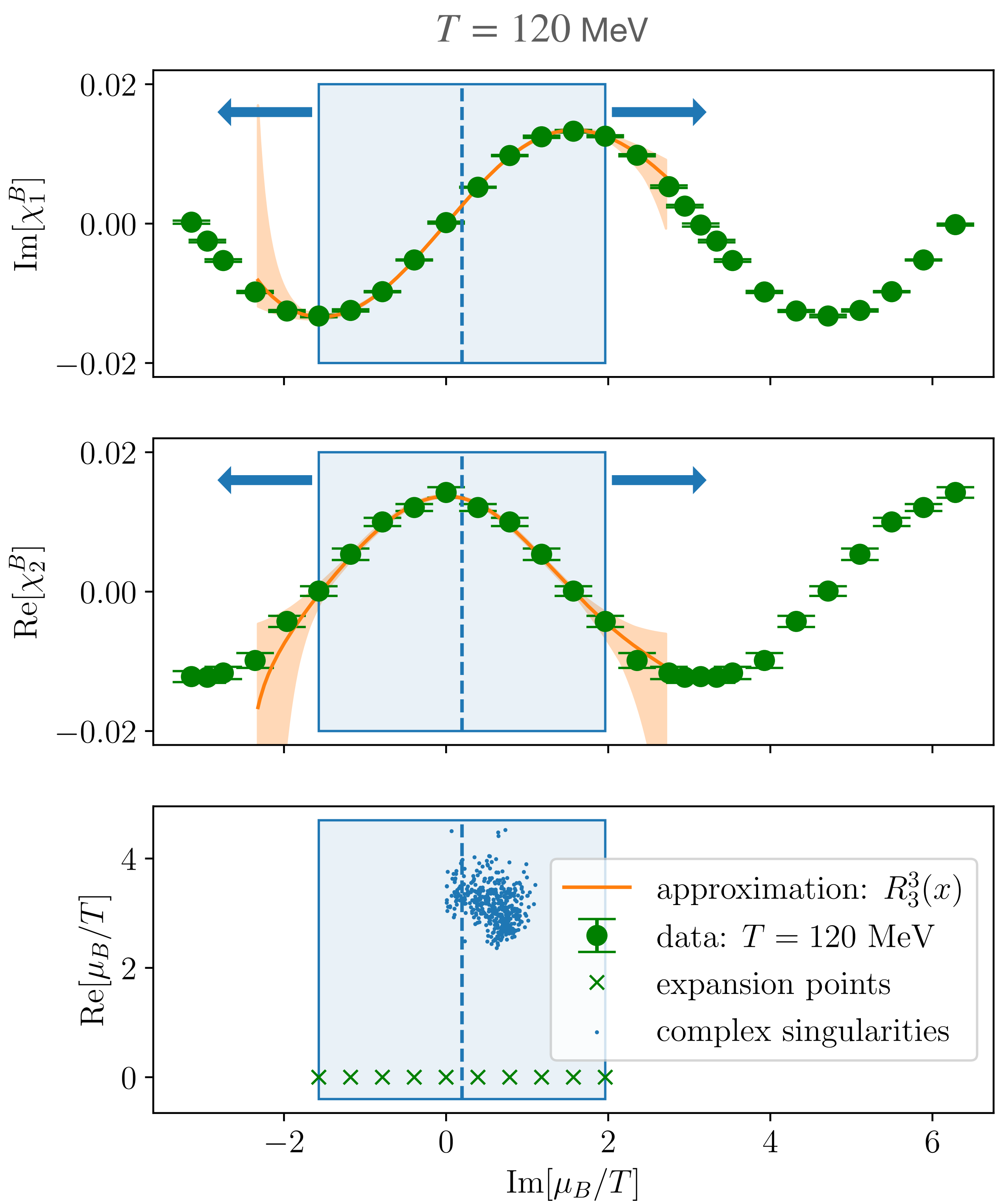
[arXiv: 2403.09390]

Observables:

- Derivatives of $\ln Z$, w.r.t $\hat{\mu}_B = \mu_B/T$

$$\chi_n^B(T) = \frac{V}{T^3}\left(\frac{\partial}{\partial \hat{\mu}_B}\right)^n \ln Z(T, \hat{\mu}_B)$$

- $\ln Z$ is even in $\hat{\mu}_B = i\theta$ and periodic, with periodicity 2π
- Choose 10 equidistant $\hat{\mu}_B$-points in $[0,i\pi]$, all further points are obtained by periodicity and parity
- Odd (even) derivatives are imaginary (real) at $\hat{\mu}_B = i\theta$



[arXiv: 2405.10196]

Procedure:

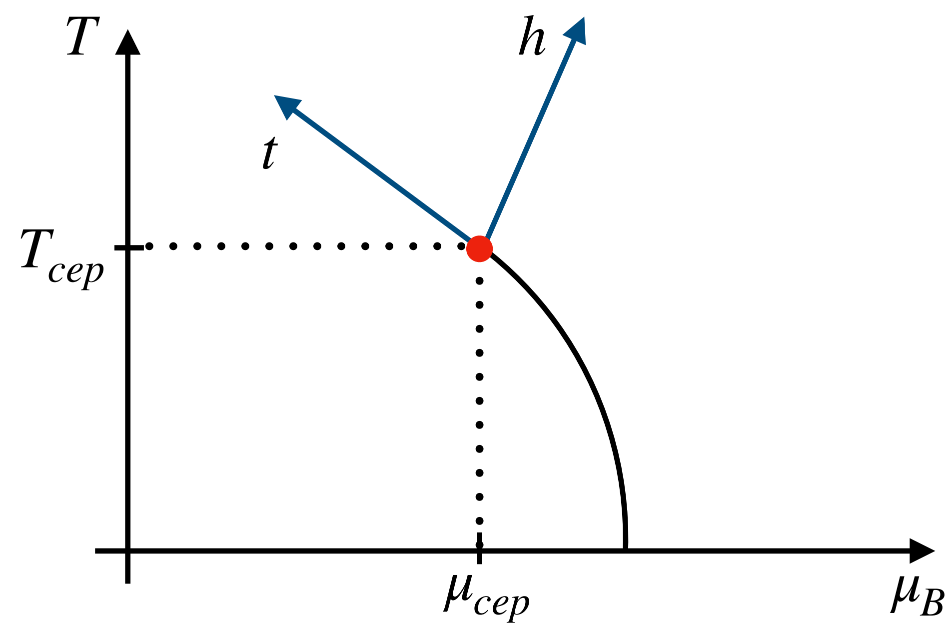
- Perform simultaneous fits to χ_1^B and χ_2^B for each temperature
- Use [3,3]-Padé
- Varry length of the fit interval in $[\pi, 2\pi]$ and the center of the interval in $[-\pi/2, +\pi/2]$
- bootstrap over the data by assuming independent and normal distributed errors
- Calculate roots of the denominator and keep only roots in the first quadrant
- Collect all the results for Lee-Yang scaling fits. We have 55 different intervals per temperature.

Mixing of scaling fields:

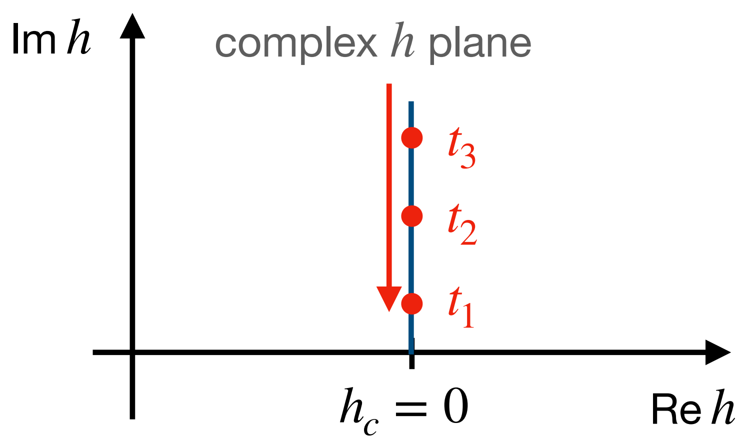
* Scaling fields are unknown, a frequently used ansatz is given by a linear mixing of T, μ_B

$$t = A_t \Delta T + B_t \Delta \mu_B,$$
$$h = A_h \Delta T + B_h \Delta \mu_B,$$

with $\Delta T = T - T^{\mathrm{CEP}}$ and $\Delta \mu_B = \mu_B - \mu_B^{\mathrm{CEP}}$



Lee-Yang edge:



* Poles approach critical point along imaginary h-axis **[Yang,Lee'59]**
* $t/h^{1/\beta\delta} = z_c$ is const. and universal

Fit Ansatz:

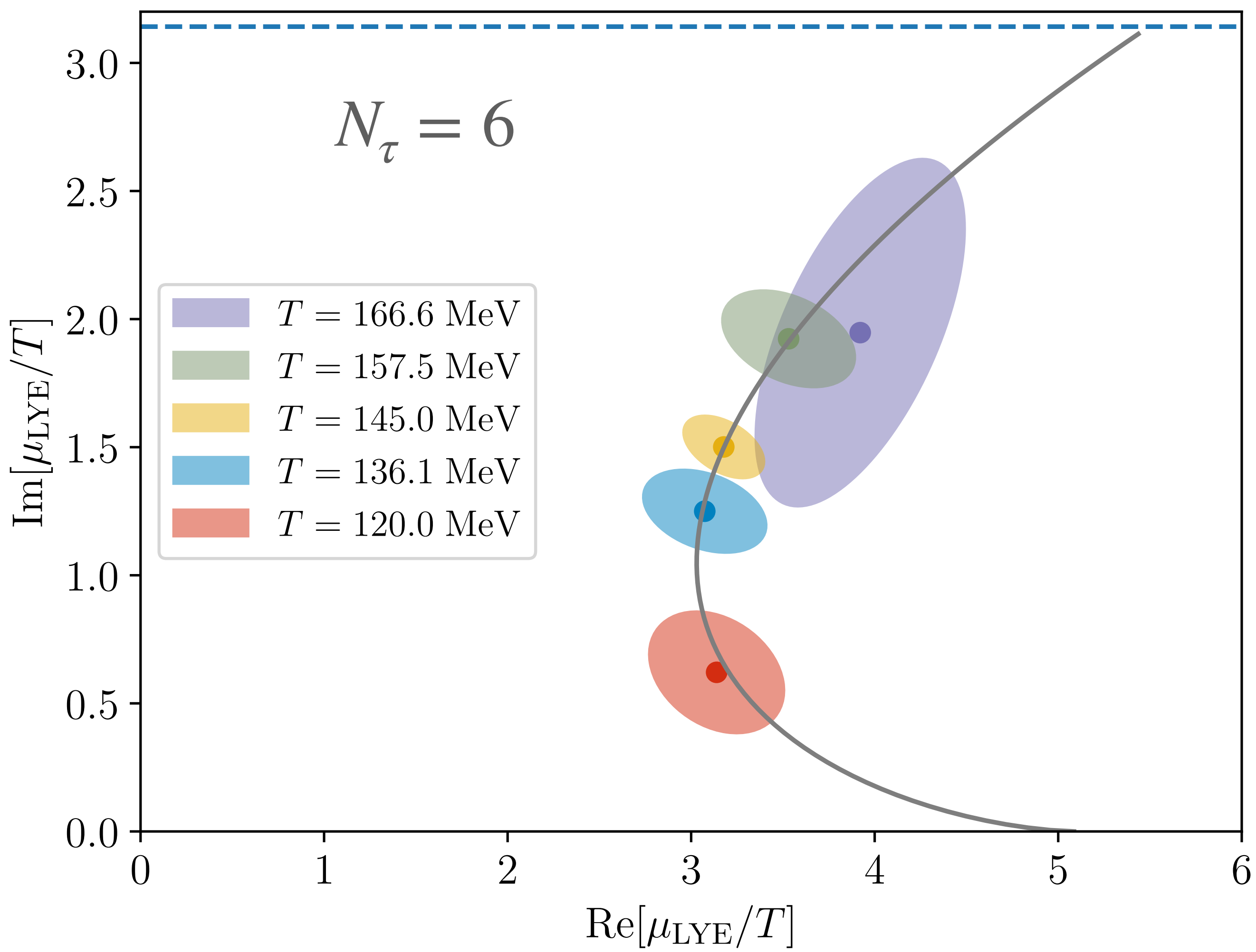
* For a constant $z = z_c$ we obtain

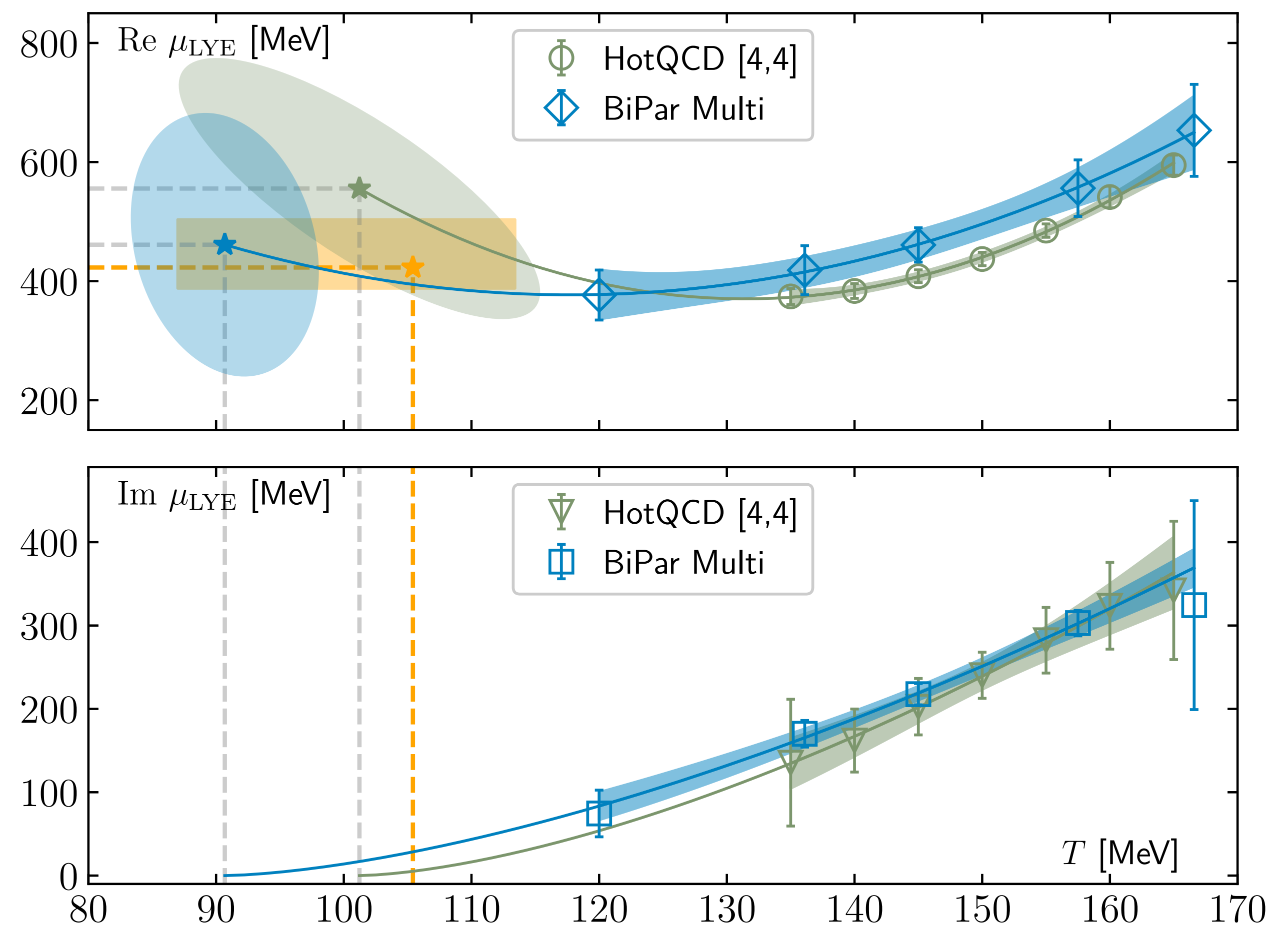
$$\mathrm{Re}[\mu_{\mathrm{LYE}}] = \mu_B^{\mathrm{CEP}} + c_1 \Delta T + c_2 \Delta T^2 + O(\Delta T^3)$$
$$\mathrm{Im}[\mu_{\mathrm{LYE}}] = c_3 \Delta T^{\beta\delta},$$

[Stephanov, Phys. Rev. D, 73.9, 094508 (2006)]

* The fit parameter c_1 gives the (inverse) slope of the 1st order line at the critical point: $c_1 = -A_h/B_h$

* Perform one fit for $N_\tau = 8$ and $\mathcal{O}(10^5)$ fits for $N_\tau = 6$



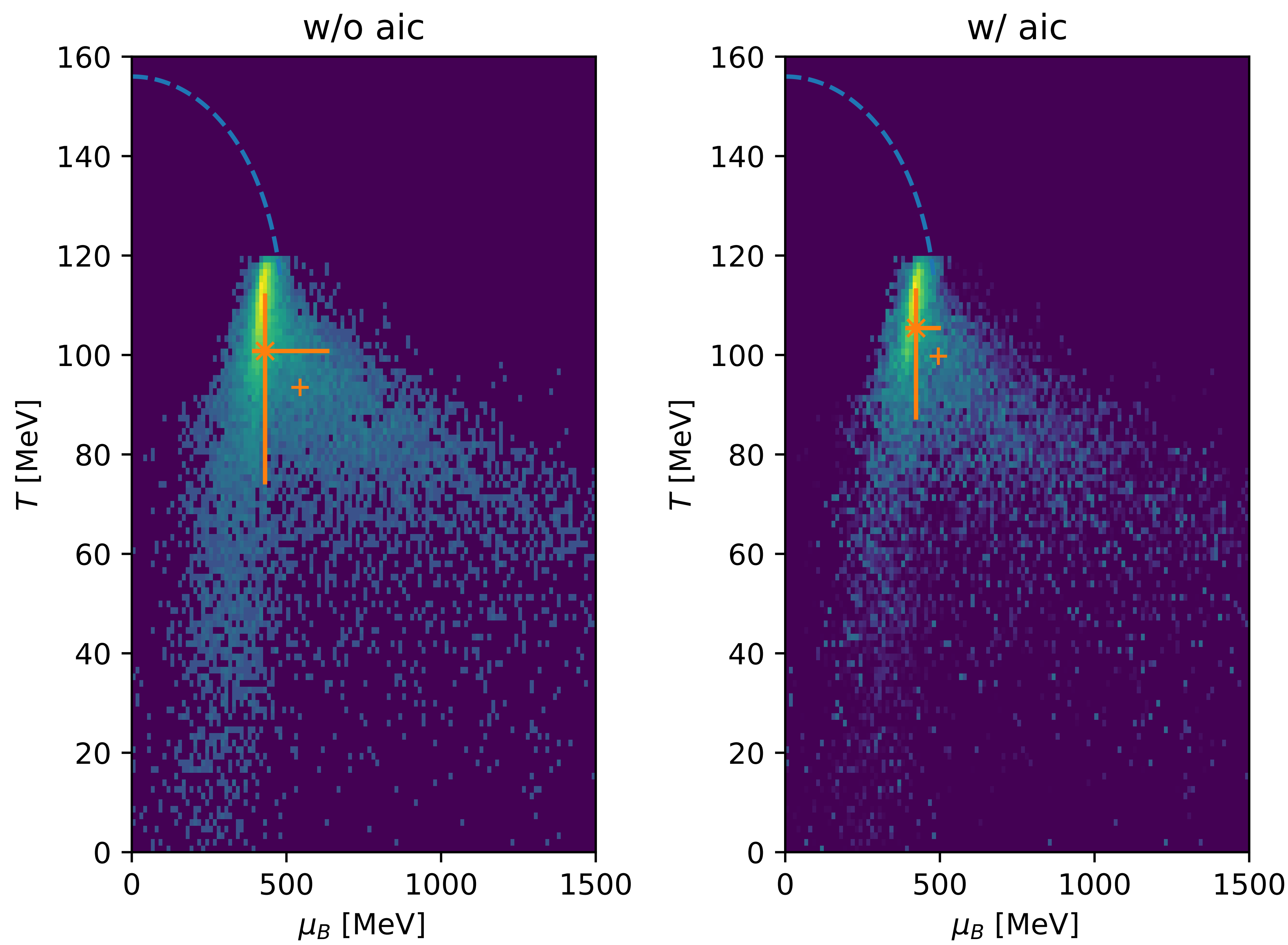


$$\mathrm{Re}[\mu_{\mathrm{LYE}}] = \mu_B^{\mathrm{CEP}} + c_1 \Delta T + c_2 \Delta T^2$$
$$\mathrm{Im}[\mu_{\mathrm{LYE}}] = c_3 \Delta T^{\beta\delta},$$

* Ellipses show 1σ confidence region, using the Pearson correlation coefficient

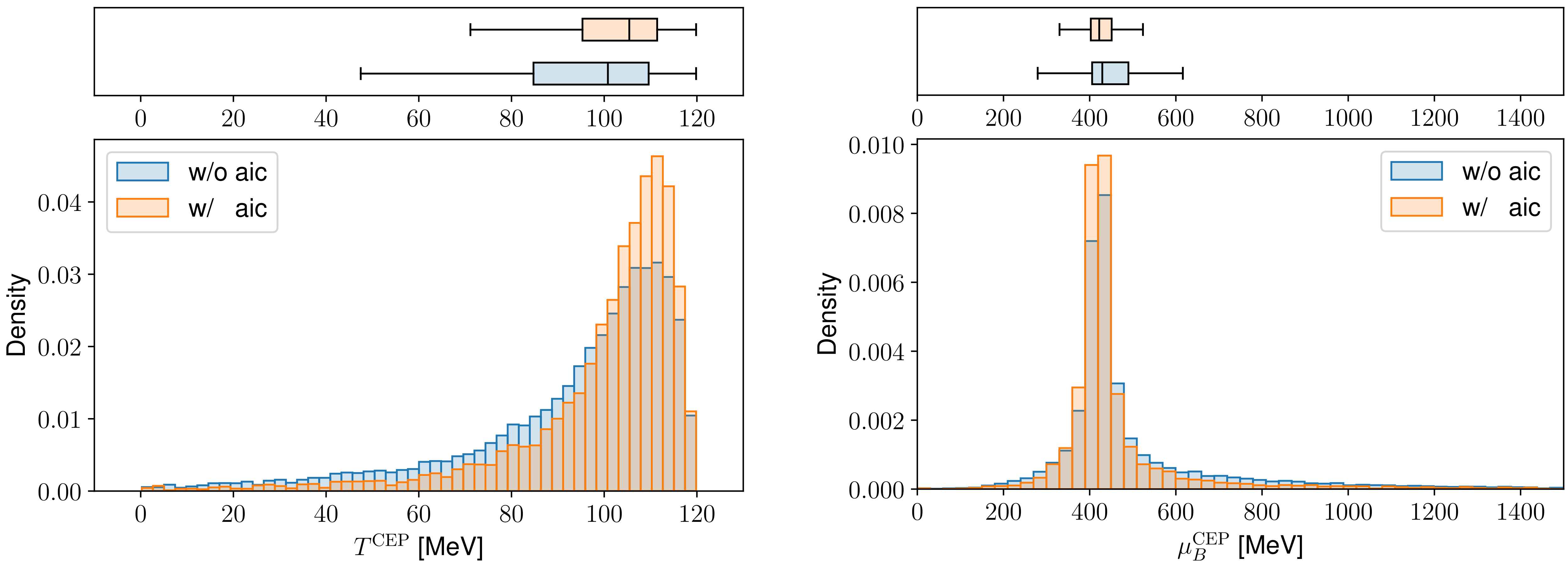
* $N_\tau = 6$ singularities show here are chosen on the basis of the χ^2 of the scaling fit (“best fit”)

* Orange box shows the AIC weighted result for $N_\tau = 6$, based on $\mathcal{O}(10^5)$ scaling fits



- Dashed line indicates the continuum extrapolated crossover line

- Histogram over the T^{CEP} and μ_B^{CEP} from the $\mathcal{O}(10^5)$ fits
- Error bars are based on the inner 68-percentile
- Observe interesting structure



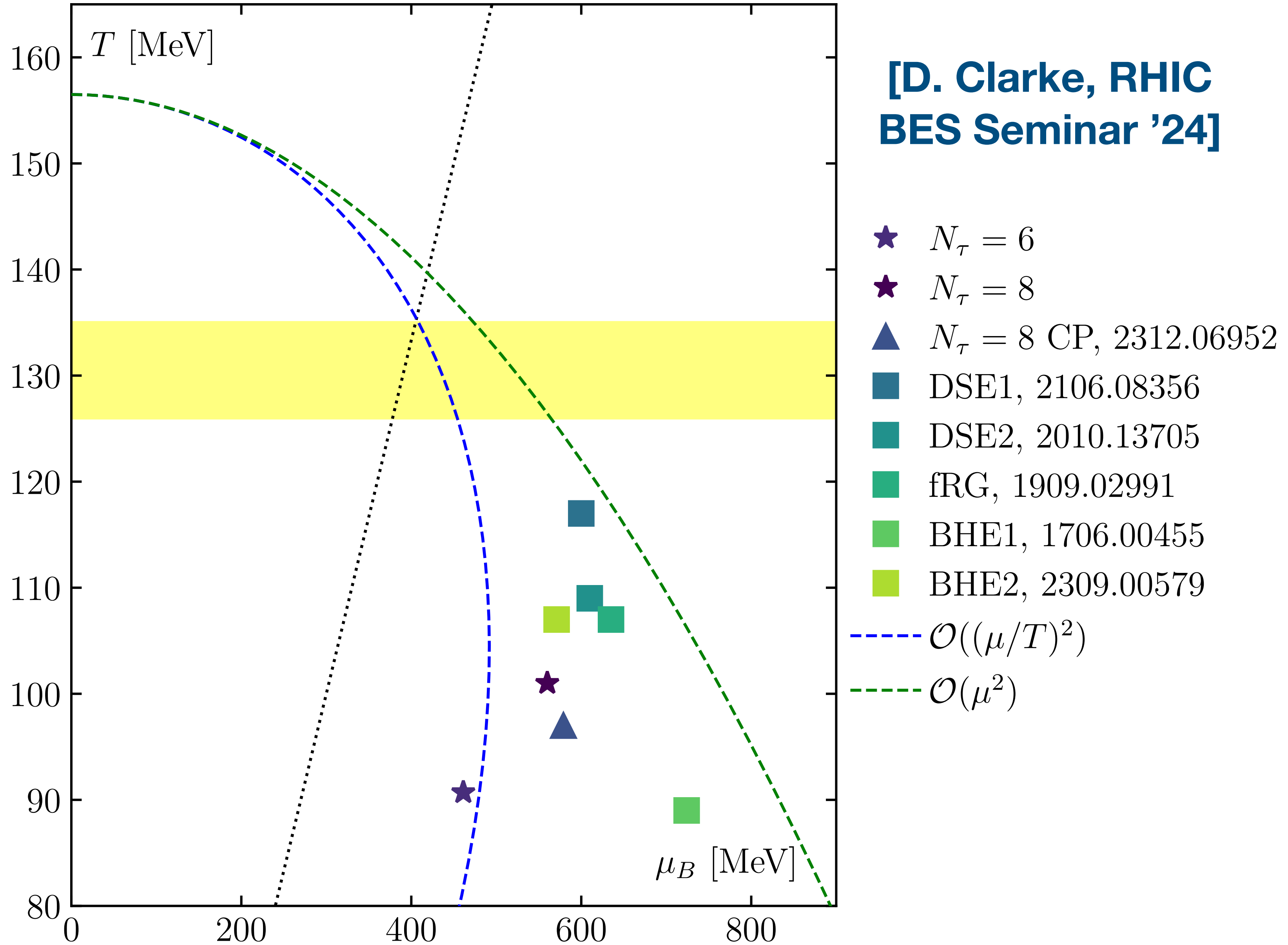
	$N_\tau = 6$ multi-point Padé			$N_\tau = 8$ [4,4]-Padé		
	T^{CEP} [MeV]	μ_B^{CEP} [MeV]	μ_B/T	T^{CEP} [MeV]	μ_B^{CEP} [MeV]	μ_B/T
best fit	90.7 ± 7.7	461.2 ± 220	5.09 ± 0.68	101 ± 15	560 ± 140	5.5 ± 1.7
weight-1	$105.4 + 8.0 - 18.4$	$422.9 + 80.5 - 34.9$	$3.92 + 1.52 - 0.24$			
weight-2	$100.8 + 11.6 - 26.8$	$430.9 + 208.2 - 42.2$	$4.20 + 4.13 - 0.47$			
	c_1	c_2	c_3	c_1	c_2	c_3
best fit	-6.2 ± 9.2	0.115 ± 0.090	0.424 ± 0.086	-12.3 ± 8.1	0.203 ± 0.059	0.55 ± 0.25

* For $N_\tau = 8$: similar results by **[Basar, arXiv: 2312.06952]**

* Continuum estimate might suffer from large systematic effects (Padé vs multi-point Padé)

* $\kappa_2 = \bar{\kappa}_2 = -0.015(1)$
  [HotQCD, 2403.09390]

* Many results seem to favour a small $\bar{\kappa}_4 \approx -0.0002(1)$



[D. Clarke, RHIC BES Seminar '24]

Parametrizations of the crossover line:

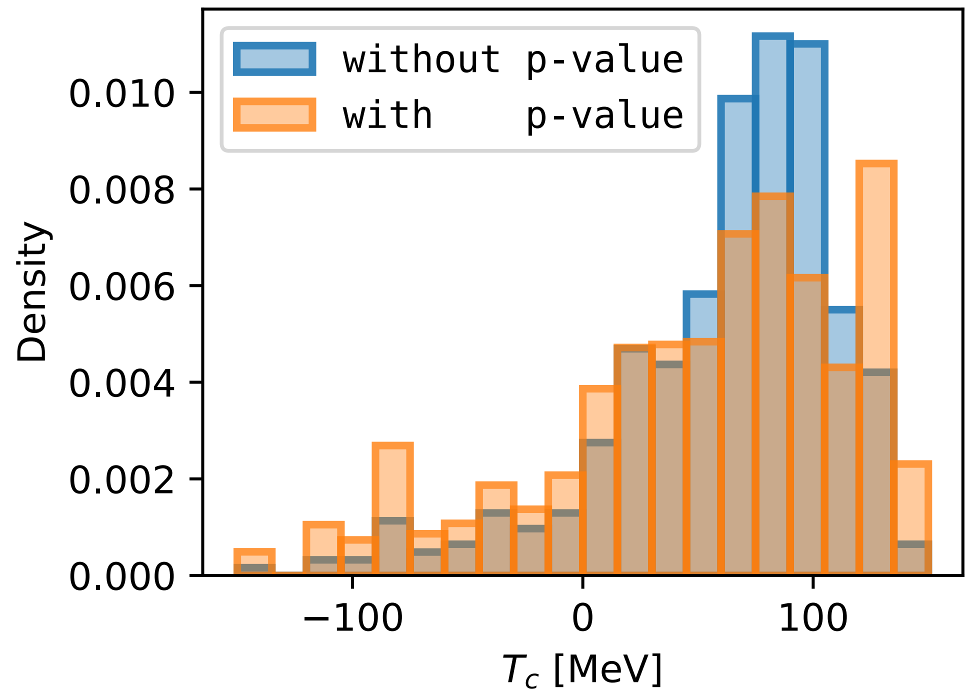
* 1.) $$T_{\rm pc}(\mu_B) = T_{\rm pc}(0)\left[1 + \kappa_2^B\left(\frac{\mu_B}{T}\right)^2 + \kappa_4^B\left(\frac{\mu_B}{T}\right)^4\right]$$

* 2.) $$T_{\rm pc}(\mu_B) = T_{\rm pc}(0)\left[1 + \bar{\kappa}_2^B\left(\frac{\mu_B}{T_{\rm pc}(0)}\right)^2 + \bar{\kappa}_4^B\left(\frac{\mu_B}{T_{\rm pc}(0)}\right)^4\right]$$

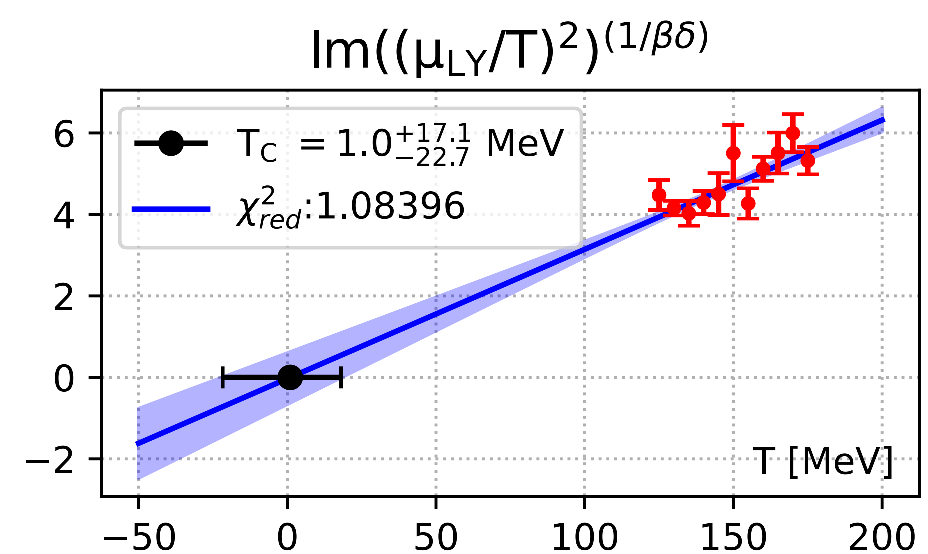
➡ **Talk by A. Adam@Lattice24: single point Padé approximation of the pressure based on $\chi_2^B, \chi_4^B, \chi_6^B, \chi_8^B$ (LT=2)**

- Interesting to check results with the Budapest-Wuppertal data
- Preliminary results for single Point Padé analysis on $16^3 \times 8$ lattices, multi-Point is work in progress
- Preliminary result on the transition temperature based on extrapolations 432 on different approximations and fit ranges
- T^{CEP} around 90 MeV, in agreement with **[BiePar, 2405.10196]**
- Results are very sensitiv to noise

Histogram of the T^{CEP} results

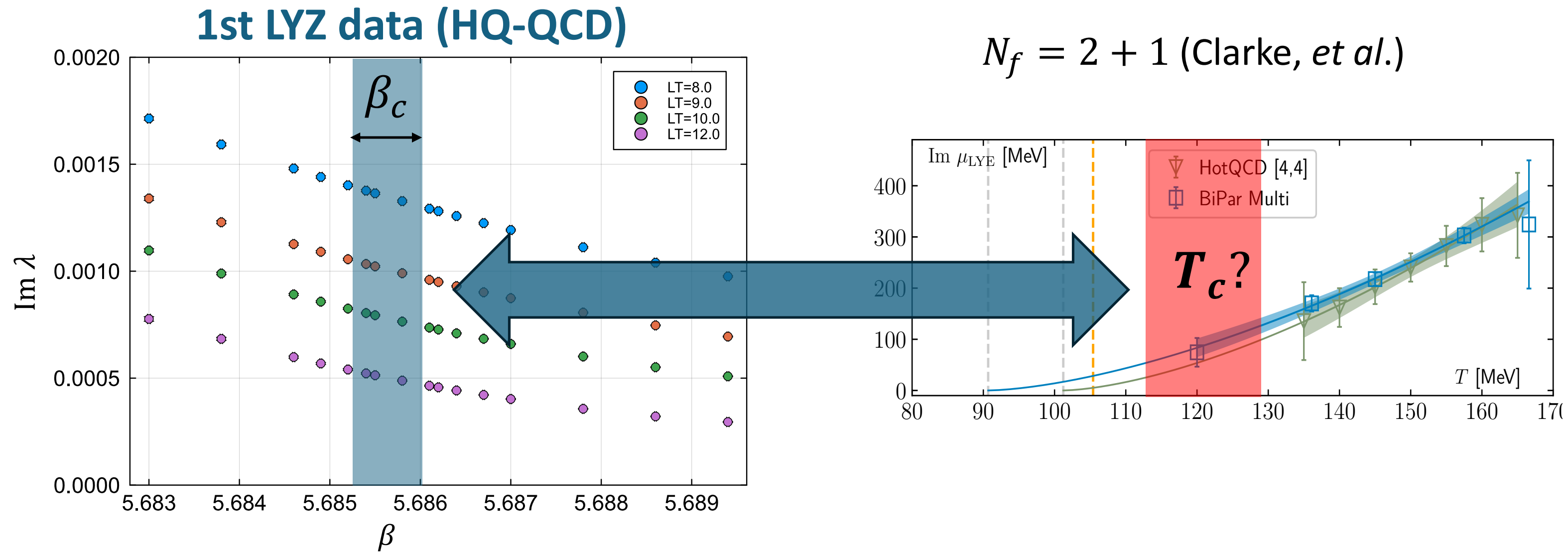


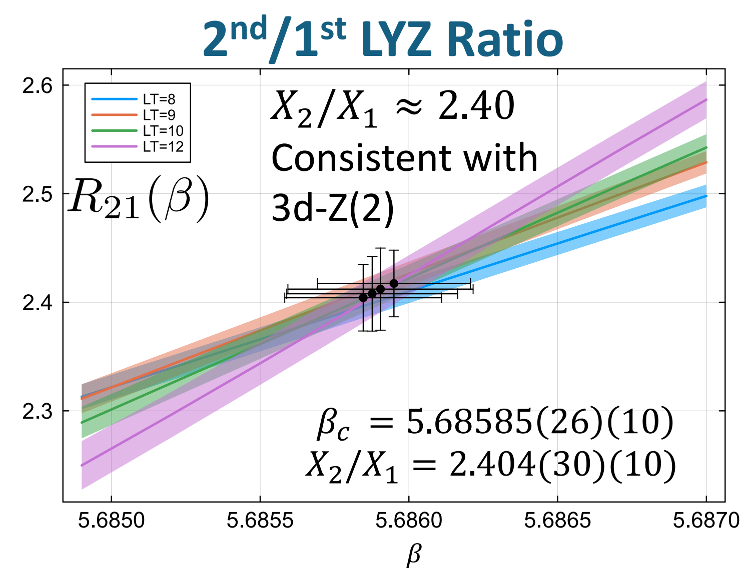
One example of the extrapolations



➡ **Talk by T. Wada@Lattice24: Finite size scaling of Lee-Yang zeros in 3d Pots model and heavy-quark QCD**

- Construct ratios of the LYZ locations
- Scaling is in accordance with a ratio of scaling function: non-universal pre-factors cancel, intersection point of different volumes is universal.
- Ratios show reduced corrections to scaling and regular parts
- T^{CEP} is shifted to higher values, results from extrapolation of first LYZ can serve as a lower bound





$$R_{12}(t) = \frac{h^{(2)}_{LY}(t)}{h^{(1)}_{LY}(t)}$$

$$= \frac{X_1}{X_2}(1 + C(tL^{y_t}))\underbrace{(1 + D(tL^{2(y_t - y_h)}))}$$

+ higher orders

mixing with temperature like

Definition:

$\mu_B/T = i\theta_B$, with $\theta_B \in \mathbb{R}$

$$b_k(T) = \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\theta_B \, \mathrm{Im}\,\chi_1^B(T, i\theta_B)\,\sin(k\theta_B)$$

A Fourier interpolation of the data is periodic by construction!

Data only defined on a discrete set of points

highly oscillatory for large k

Method:

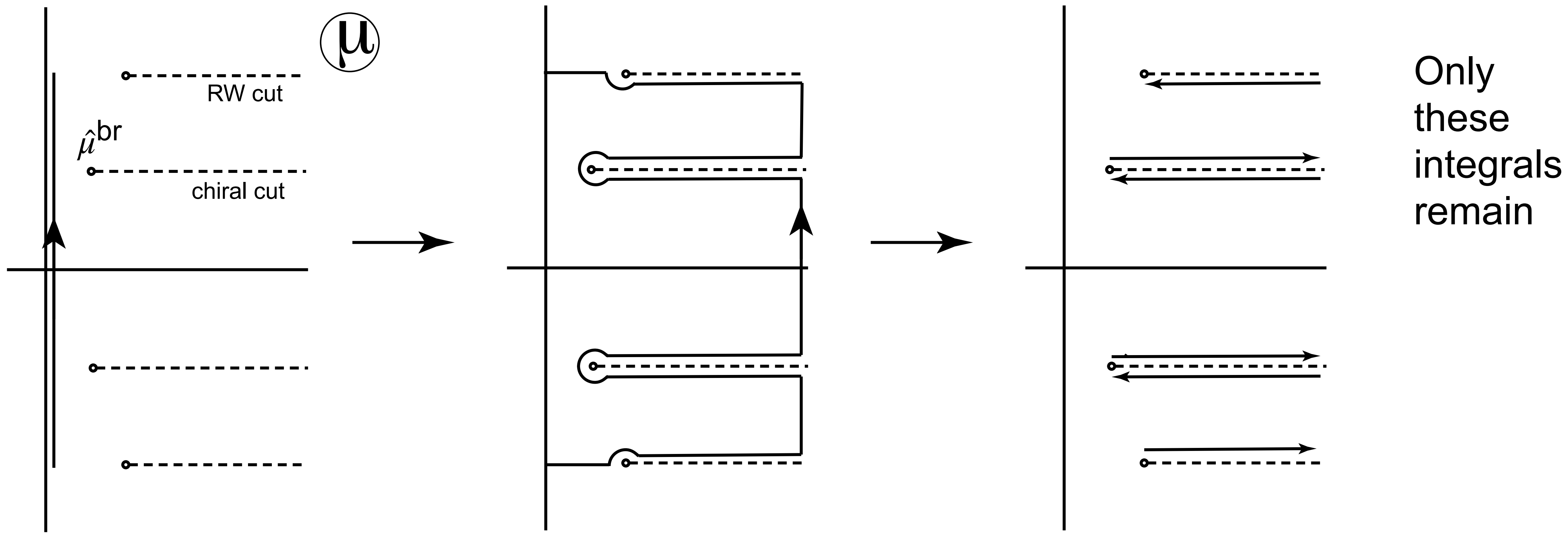
- Interpolate Im χ_1^B, take also it's derivative Re χ_2^B and eventually higher derivatives up to order s into account → Hermite-interpolation (spline)
- Piecewise integration can be done analytically

$$b_k = \frac{2}{\pi}\sum_{i=0}^{N-1}\int_{\theta_B^{(i)}}^{\theta_B^{(i+1)}} \mathrm{d}\theta_B \, p(\theta_B)\sin(k\theta_B) \quad \text{with} \quad 0 = \theta_B^{(0)} < \theta_B^{(1)} < \cdots < \theta_B^{(N)} = \pi$$

→ variant of a Filon-type quadrature: error decreases as $\mathcal{O}(k^{-s-2})$ (for exact data)

- Statistical error is estimated by bootstrapping over the error of Im χ_1^B and Re χ_2^B.

- We can deform the integration contour to integrate along the cuts



- Assume that we can express the density along the cuts as

edge coefficient, $\sigma > -1$

$$n_B(\hat{\mu}) = \underbrace{A(\hat{\mu} - \hat{\mu}^{\rm br})^{\sigma}}_{\text{Leading order non-analytic part}}(1 + B(\hat{\mu} - \hat{\mu}^{\rm br})^{\theta_c} + \ldots) + \underbrace{\sum_{n=0}^{\infty} a_n (\hat{\mu} - \hat{\mu}^{\rm br})^n}_{\text{analytic part}}$$

Leading order
non-analytic part

analytic part

- The final result for one cut is

$$b_k = \frac{e^{-\mu^{\mathrm{br}}k}}{i\pi} A \frac{\Gamma(1+\sigma)}{k^{1+\sigma}} \left(1 - e^{i2\pi\sigma} + \frac{B}{k^{\theta_c}} \left[1 - e^{i2\pi(\sigma+\theta_c)}\right] \frac{\Gamma(1+\sigma+\theta_c)}{\Gamma(1+\sigma)} + \dots\right)$$

Absorbing k-independent factors into A and B we get

$$b_k = \tilde{A} \frac{e^{-\hat{\mu}^{\mathrm{br}}k}}{k^{1+\sigma}} \left(1 + \frac{\tilde{B}}{k^{\theta_c}} + \dots\right)$$

Note that the regular part cancels completely

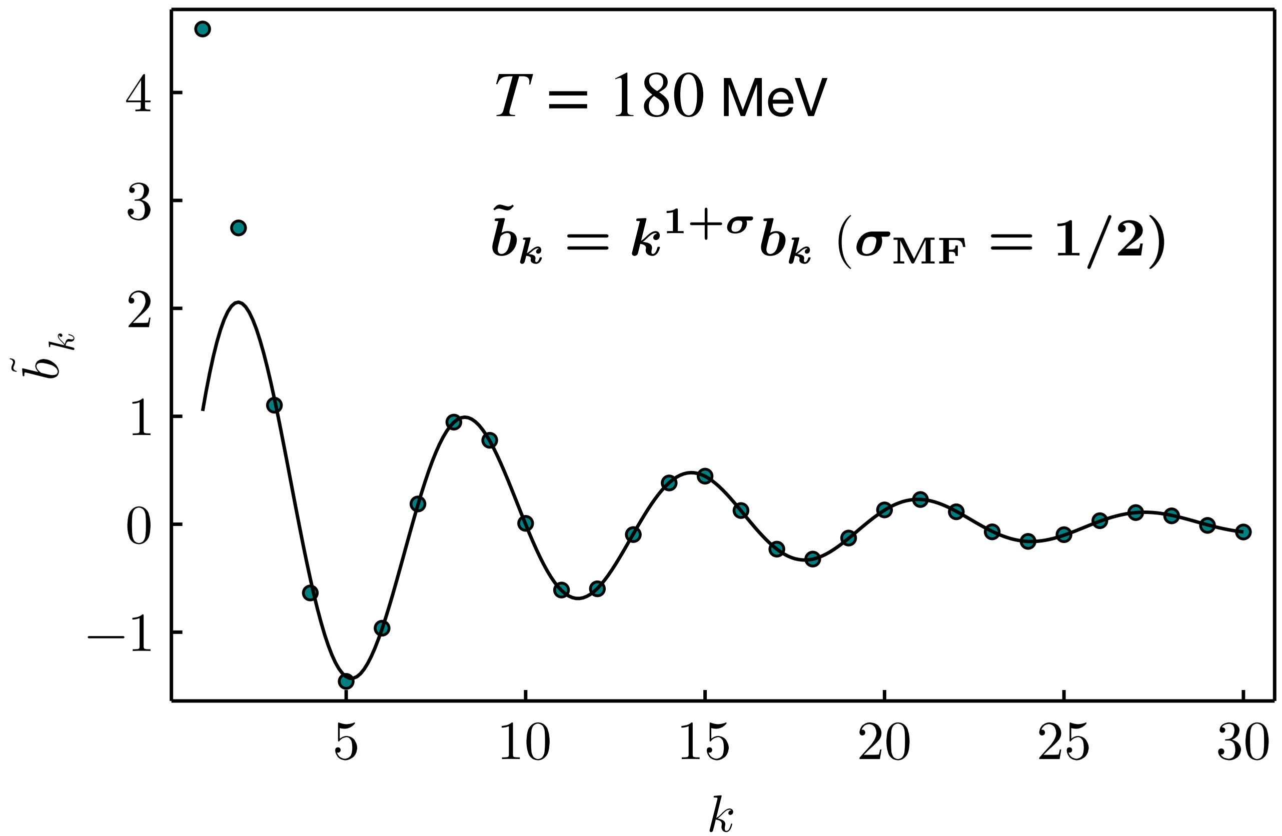
- The final result for both cuts is (dropping NLO)

$$b_k = |\tilde{A}_{\mathrm{YLE}}| \frac{e^{-\hat{\mu}_r^{\mathrm{YLE}}k}}{k^{1+\sigma}} \cos(\hat{\mu}_i^{\mathrm{YLE}}k + \phi_a^{\mathrm{YLE}}) + |\hat{A}_{\mathrm{RW}}| (-1)^k \frac{e^{-\hat{\mu}_r^{\mathrm{RW}}k}}{k^{1+\sigma}}$$

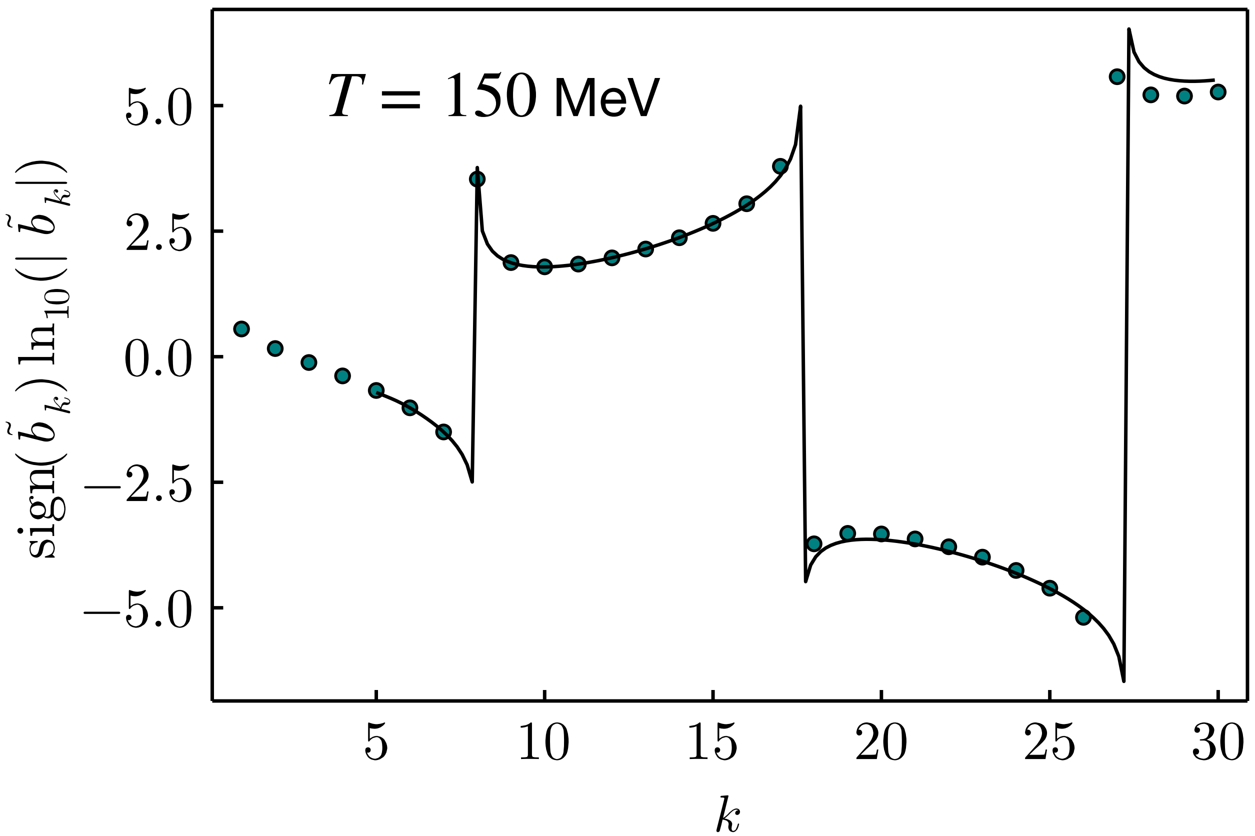
$\hat{\mu}_i^{RW} = \pi$

- The analytic form fits the Fourier coefficients from the quark-meson model well. Details of the Model can be found here [Skokov et al., PRD 82 (2010) 034029]

- In Mean-Field and LAP approximation fits to the Fourier coefficients reproduce the correct location of the LY edge up to (5 – 7)%.



$$\hat{\mu}^{\mathrm{fit}}_{\mathrm{YLE}} = 0.1156(6) + i\,0.9952(5)$$
$$\hat{\mu}_{\mathrm{YLE}} = 0.118657 + i\,1.00256$$



$$\hat{\mu}^{\mathrm{fit}}_{\mathrm{YLE}} = 0.441(2) + i\,0.325(3)$$
$$\hat{\mu}_{\mathrm{YLE}} = 0.412884 + i\,0.342187$$

Summary

* Universal scaling is a very powerful tool if the scaling fields and the universality class are known.

* Pseudo-critical lines correspond (asymptotically) to a constant real $z = t/h^{1/\beta\delta}$, the Lee-Yang edge to a universal complex z_c

* New Strategy: Determine the QCD critical point by the temperature scaling of the Lee-Yang edge singularity

* Technically this requires Pade or multi-point Pade analysis of $\ln Z$ derivatives. The later eliminates the need for the calculation of high order expansion coefficients but introduces some interval dependence.

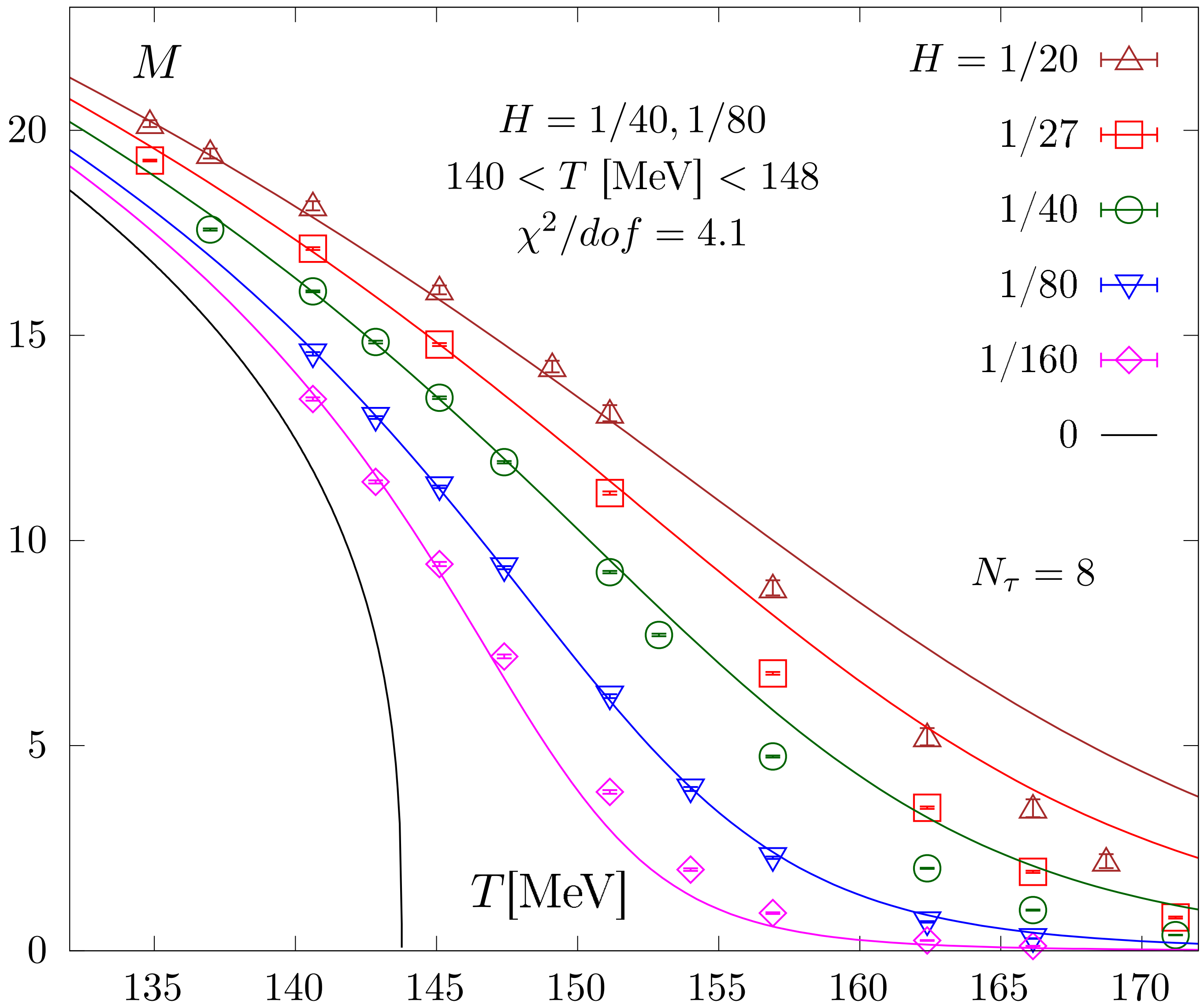
* Find encouraging results for $N_\tau = 6$: $(T^{\rm CEP}, \mu_B^{\rm CEP}) = (105^{+8}_{-18}, 422^{+80}_{-35})$ MeV .

* No continuum result yet

* Current estimates of the cutoff effects increase $\mu_B^{\rm CEP}$ towards $\mu_B^{\rm CEP} \approx 650$ MeV

Back Up Slides

$$M = h_0^{-1/\delta} H^{1/\delta} \left(f_G(z) - f_\chi(z) \right)$$



[PRD 109 (2024) 11, 114516, arXiv: 2403.09390]

* $N_\tau = 8$ (with updated statistics)
* Corresponding pion masses: $m_\pi \simeq$ 180 MeV, 140 MeV, 110 MeV, 80 MeV, 55 MeV.
* Use O(2) scaling functions and exponents due to staggered fermions
* Fit results for $N_\tau = 8$

$$T_c = 143.7(2) \text{ MeV} ,$$
$$z_0 = 1.42(6) ,$$
$$h_0^{-1/\delta} = 39.2(4) .$$

* Continuum estimate: $T_c = 132^{+2}_{-6}$ MeV

[PRL 123 (2019) 6, 062002, arXiv: 1903.04801]

Scaling fields

$$h = \frac{1}{h_0} H = \frac{1}{h_0} \frac{m_l}{m_s}$$

light quark mass

strange quark mass

$$t = \frac{1}{t_0} \left(\Delta T + \kappa_2^l \hat{\mu}_l^2 + \kappa_2^s \hat{\mu}_s^2 + 2\kappa_{11}^{ls} \hat{\mu}_l \hat{\mu}_s \right)$$

$$\Delta T = \frac{T - T_c}{T_c}$$

* chiral condensates couple to the temperature-like scaling field

Magnetic equation of state

$$M = h^{1/\delta} \left(f_G(z) - f_\chi(z) \right)$$

* $f_G(x)$ and $f_\chi(z)$ are universal function of a single scaling variable $z = t/h^{1/\beta\delta}$

Order parameter

Remove multiplicative UV divergences

$$M_l = \frac{m_s}{f_K^4} \frac{T}{V} \frac{\partial \ln Z}{\partial m_l}$$

$$\frac{\partial}{\partial m_l} = \frac{\partial}{\partial m_u} + \frac{\partial}{\partial m_d}$$

Magnetic susceptibility

$$\chi_l = m_s \frac{\partial}{\partial m_l} M_l$$

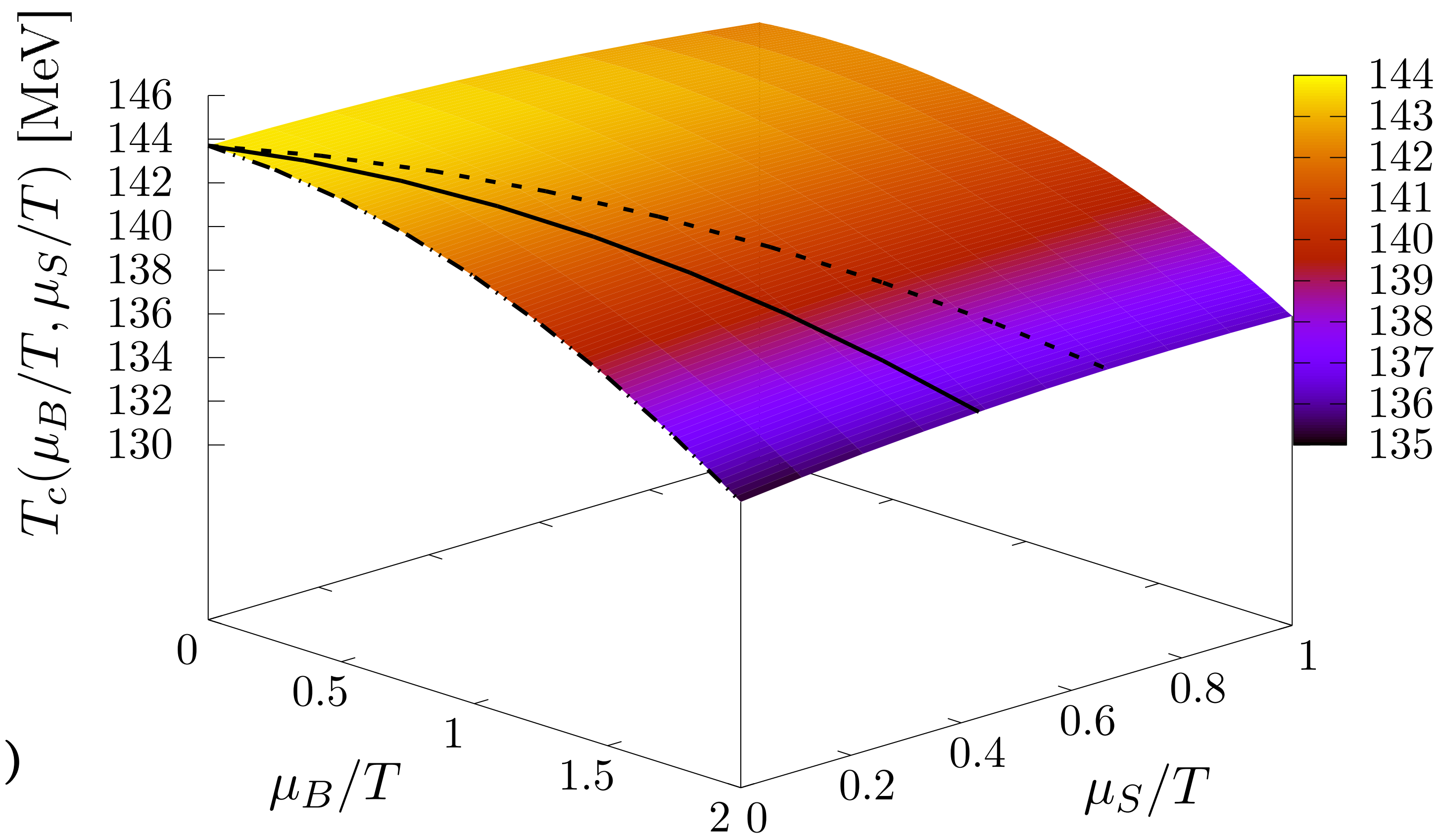
"Improved" order parameter

$$M = M_l - H\chi_l$$

[PRD 109 (2024) 11, 114516, arXiv: 2403.09390]

- Remember:

$$t = \frac{1}{t_0}\left(\Delta T + \kappa_2^l \hat{\mu}_l^2 + \kappa_2^s \hat{\mu}_s^2 + 2\kappa_{11}^{ls}\hat{\mu}_l\hat{\mu}_s\right)$$



- Ratio of mixed susceptibilities are related to the curvature coefficients

$$\kappa_2^l = \frac{1}{2T_c}\left(\frac{\partial^2 M_l/\partial\hat{\mu}_l^2}{\partial M_l/\partial T}\right)\Bigg|_{(T=T_c,\vec{\mu}=0)}$$

$$\kappa_{11}^{ls} = \frac{1}{2T_c}\left(\frac{\partial^2 M_l/\partial\hat{\mu}_l\hat{\mu}_s}{\partial M_l/\partial T}\right)\Bigg|_{(T=T_c,\vec{\mu}=0)}$$

[PRD 109 (2024) 11, 114516, arXiv: 2403.09390]

- results may be transformed to the hadronic basis

$$\kappa_2^{B,\hat{\mu}_S=0} \equiv \kappa_2^B = 0.015(1)$$

$$\kappa_2^{B,n_S=0} = 0.893(35)\kappa_2^B$$

$$\kappa_2^{B,\hat{\mu}_s=0} = 0.968(23)\kappa_2^{n_S=0}$$

Mixed susceptibilities

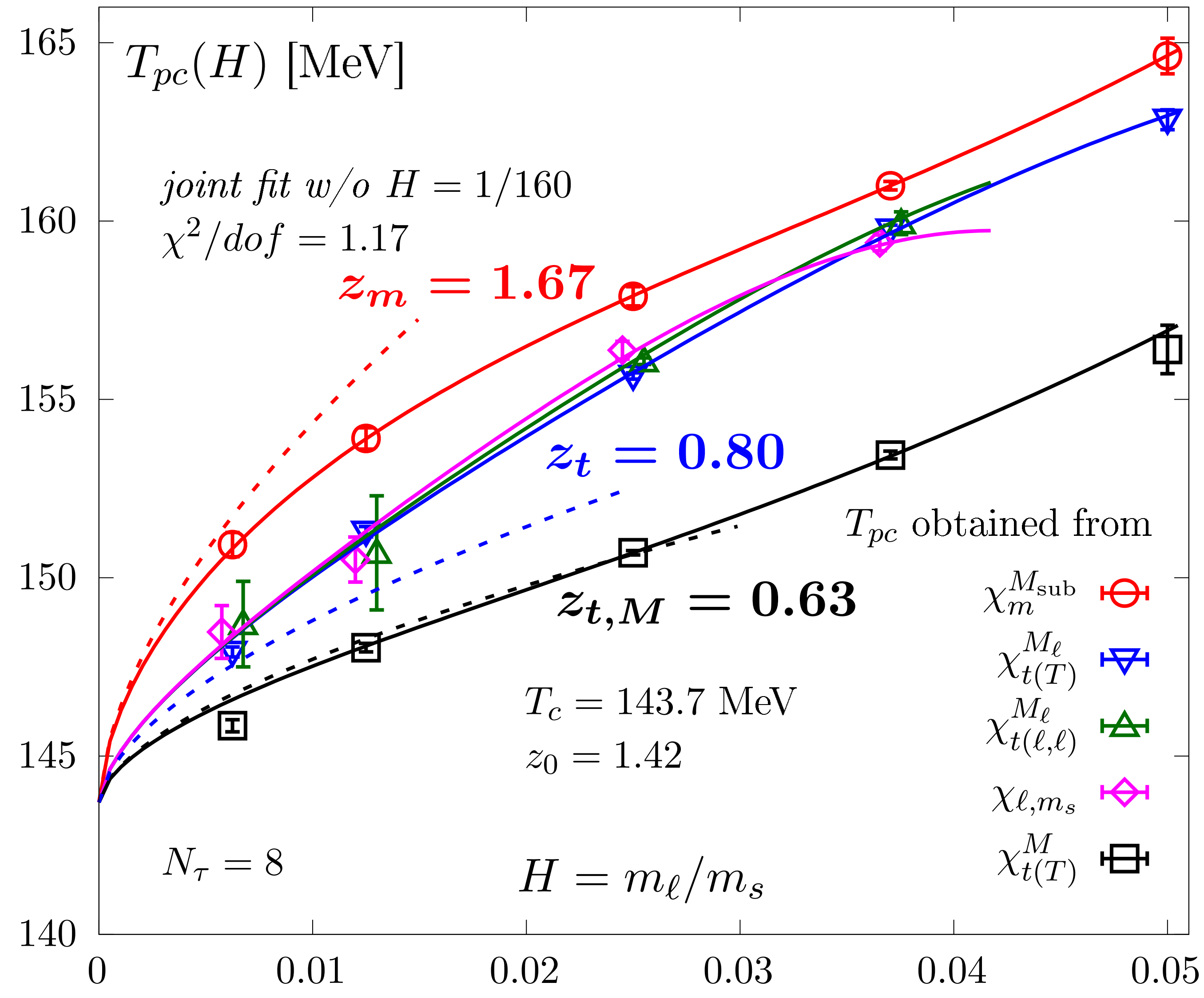
- Temperature-like derivatives of the order parameter

$$\chi_{t(T)}^{M_\ell} = -T_c \frac{\partial M_\ell}{\partial T} ,$$

$$\chi_{t(f,f)}^{M_\ell} = -\frac{\partial^2 M_\ell}{\partial \hat{\mu}_f^2} ,$$

$$\chi_{t(\ell,s)}^{M_\ell} = -\frac{\partial^2 M_\ell}{\partial \hat{\mu}_\ell \partial \hat{\mu}_s} ,$$

- Peak-position of susceptibilities determine a pseudo critical line (constant z_x)



[PRD 109 (2024) 11, 114516, arXiv: 2403.09390]

$$T_{pc,x} = T_c \left(1 + \frac{z_x}{z_0} H^{1/\beta\delta} + \text{corrections to scaling}\right)$$

- Perform joined fit to peak positions of mixed susceptibilities, including corrections to scaling → results for T_c, z_0 are in good agreement with EoS fits.

Standard Padé:

- Starting point is a power series
$$f(x) = \sum_{i=0}^{L} c_i\, x^i + \mathcal{O}(x^{L+1})\,.$$
- A Padé approximation is constructed such that the expansion of the Padé is identical to the Taylor series about $x = 0$
- We denote the [m/n]-Padé as
$$R_n^m(x) = \frac{P_m(x)}{\tilde{Q}_n(x)} = \frac{P_m(x)}{1 + Q_n(x)} = \frac{\sum_{i=0}^{m} a_i\, x^i}{1 + \sum_{j=1}^{n} b_j\, x^j}$$
- One possibility to solve for the coefficients a_i, b_j, is by solving the tower of equations
$$P_m(0) - f(0)Q_n(0) = f(0)$$
$$P'_m(0) - f'(0)Q_n(0) - f(0)Q'_n(0) = f'(0)$$
$$\vdots$$

→ Linear system of size $m + n + 1$, need $m + n$ derivatives of $f(x)$

Multipoint Padé:

- We have power series at several points x_k
- We demand that at all points x_k the expansion of the Padé is identical to the Taylor series about $x = x_k$
- One possibility (method I) to solve for the coefficients a_i, b_j, is by solving the tower of equations
$$P_m(x_0) - f(x_0)Q_n(x_0) = f(x_0)$$
$$P'_m(x_0) - f'(x_0)Q_n(x_0) - f(x_0)Q'_n(x_0) = f'(x_0)$$
$$\vdots$$
$$P_m(x_1) - f(x_1)Q_n(x_1) = f(x_1)$$
$$P'_m(x_1) - f'(x_1)Q_n(x_1) - f(x_1)Q'_n(x_1) = f'(x_1)$$
$$\vdots$$

→ again a linear system of size $m + n + 1$, need much less derivatives, we have
$$m + n + 1 = \sum_k (L_k + 1)$$

[Dimpopoulos et al. Phys.Rev.D 105 (2022) 3, 034513]

$$\begin{cases} M = m_0 R^{\beta}\theta \\ t = R(1-\theta^2) \\ h = h_0 R^{\beta\delta} h(\theta) \end{cases}$$

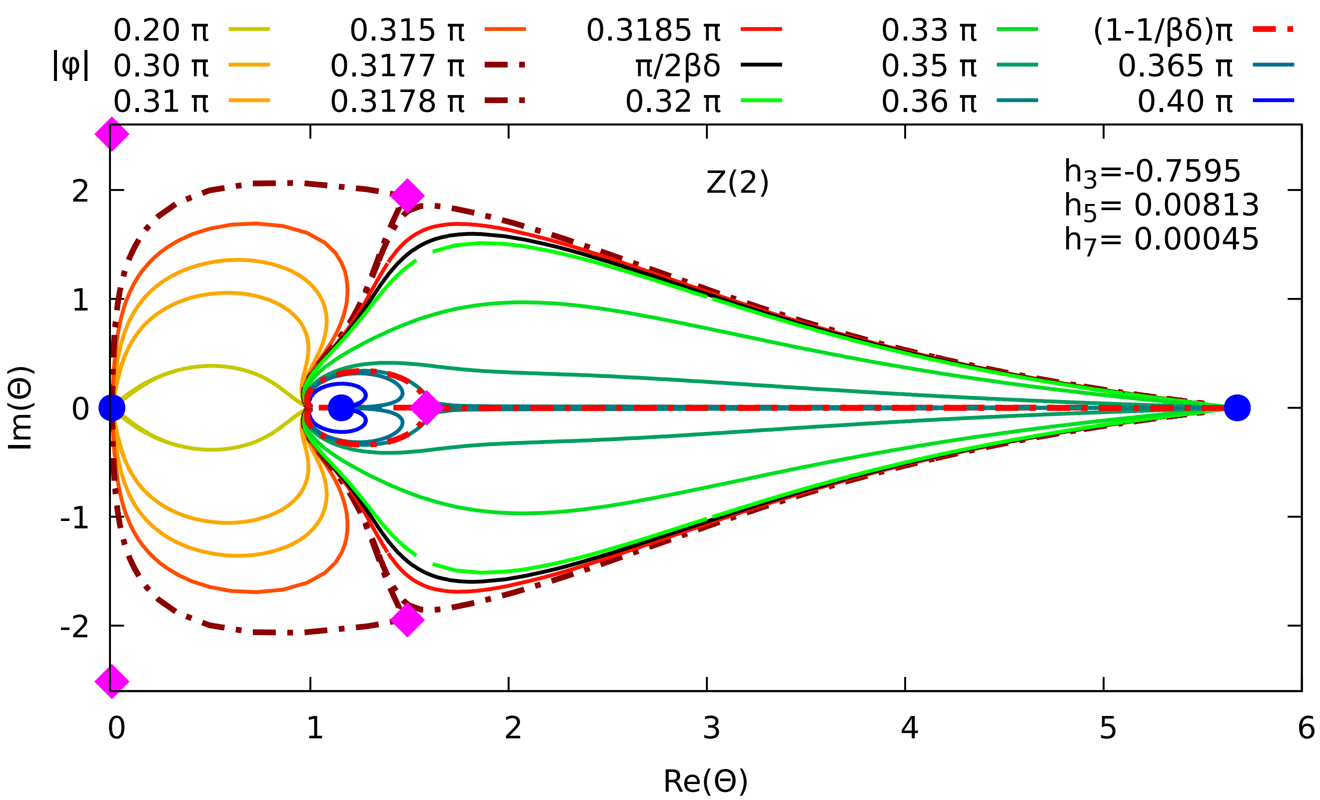
$$h(\theta) = \theta\left(1 + h_3\theta^2 + h_5\theta^4 + h_7\theta^6 + \cdots\right)$$

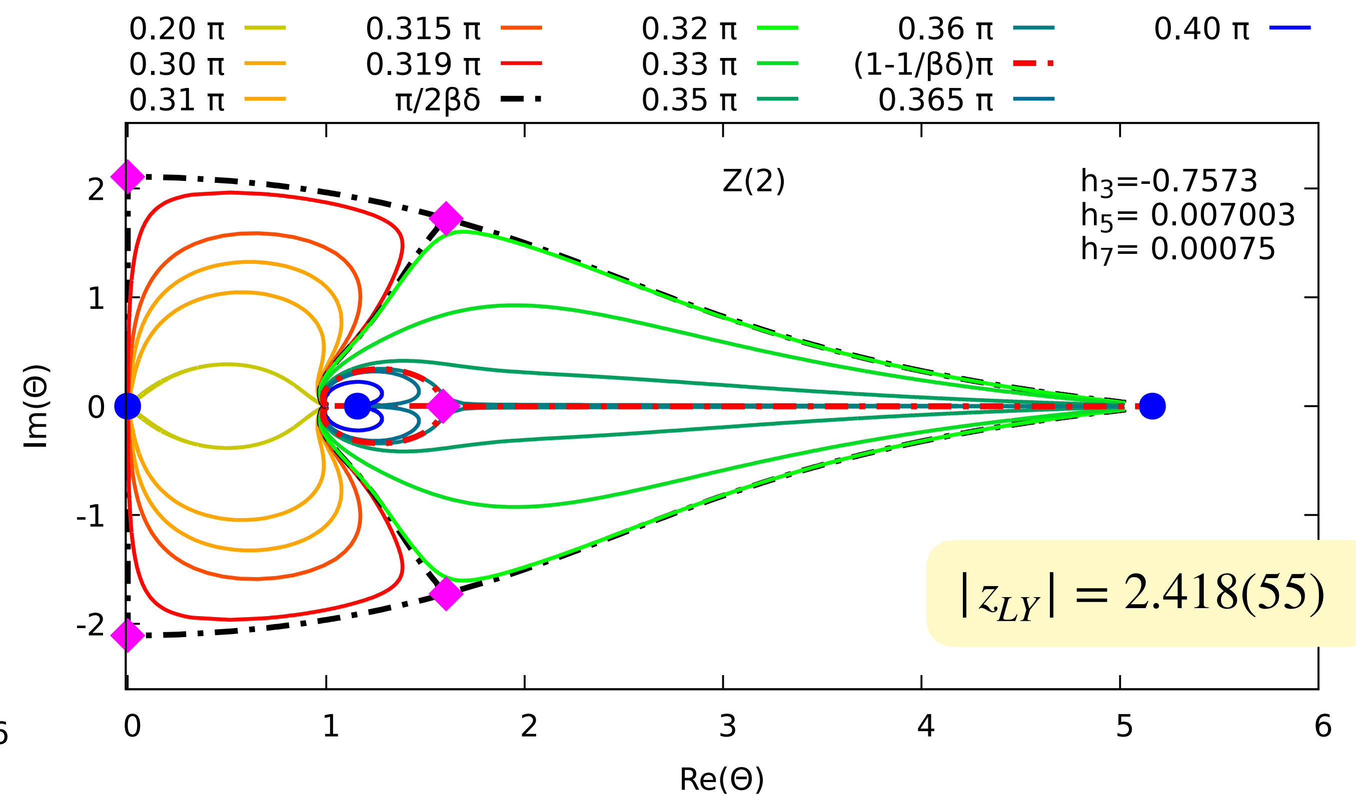
- The coefficients h_3, h_5 can be determined perturbatively and non-perturabively

[Guida, Zinn-Justin, NPB 489 (1997)]

[Karsch et al. PRD 108 (2023) 014505]

$$z(\theta) = \frac{1-\theta^2}{\theta_0^2 - 1}\theta_0^{1/\beta}\left(\frac{h(\theta)}{h(1)}\right)^{-1/\beta\delta}$$





- the coefficient h_7 is not known precisely (zero within current precision)
- $h_7 \neq 0$ introduces an additional pair of singularities (imaginary if $h_7 > 0$, real if $h_7 < 0$)
- h_7 can be used to tune the phase of the LY edge singularity

[Karsch, CS, Singh, arXiv:2311.13530]