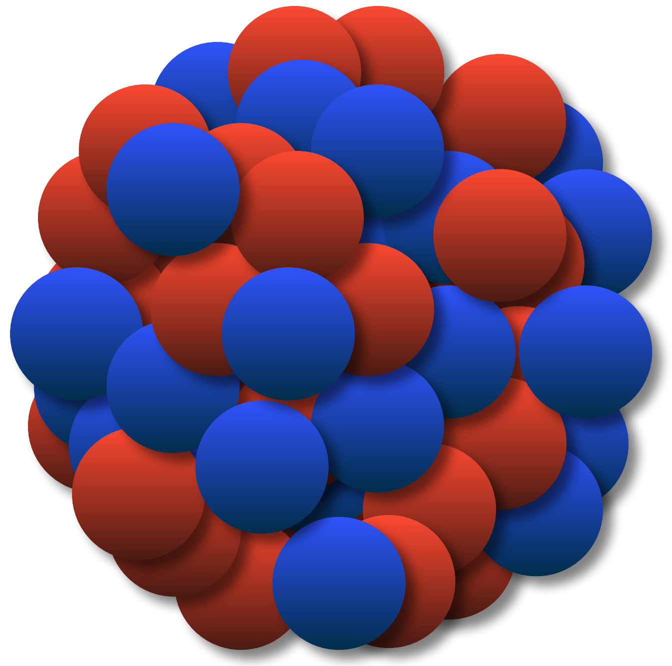
Perturbative Computations of Neutron-Proton Scattering Observables up to N^3LO using χEFT

The nuclear interaction: post-modern developments,
ECT*, Trento 2024

Oliver Thim | Theoretical Subatomic Physics | Chalmers University of Technology

The atomic nucleus

$\sim 10^{-15}$ m



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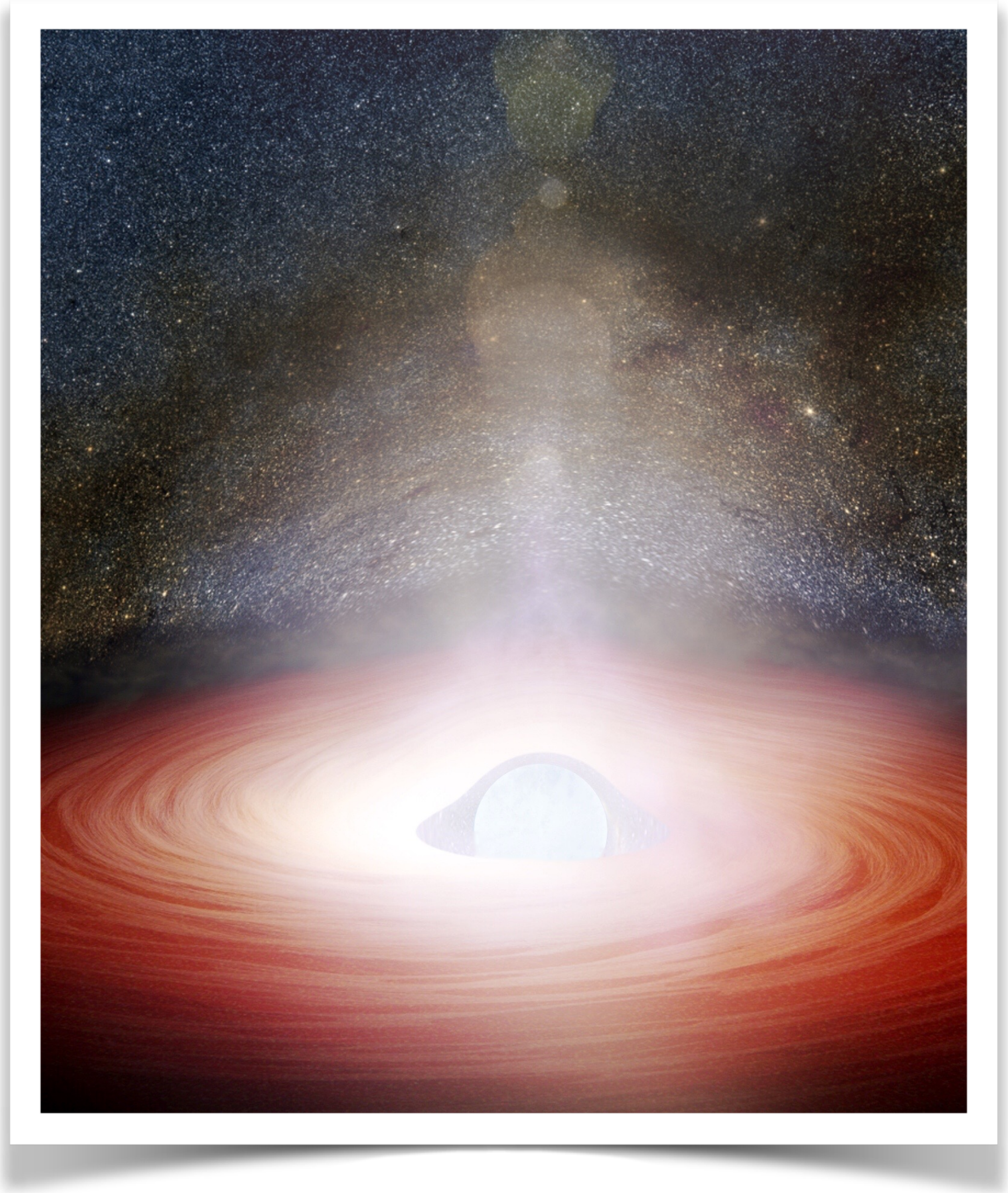
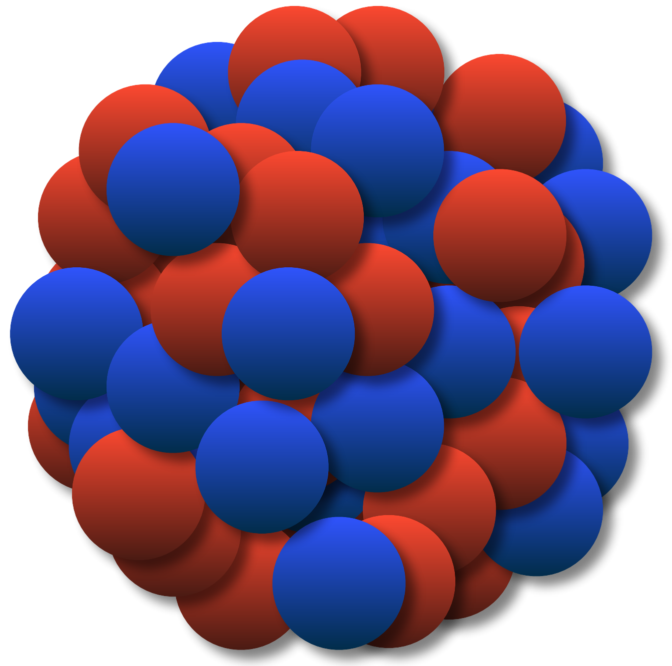


$\sim 10^4$ m

$\sim 10^{-15}$ m



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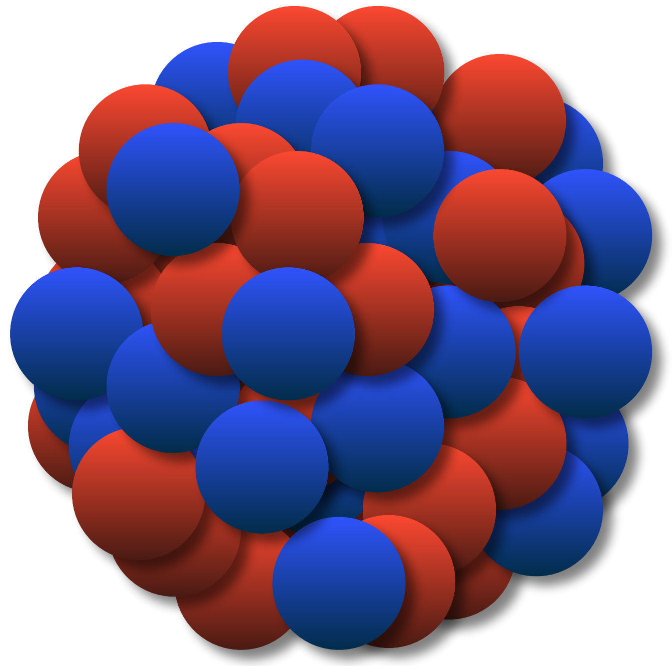

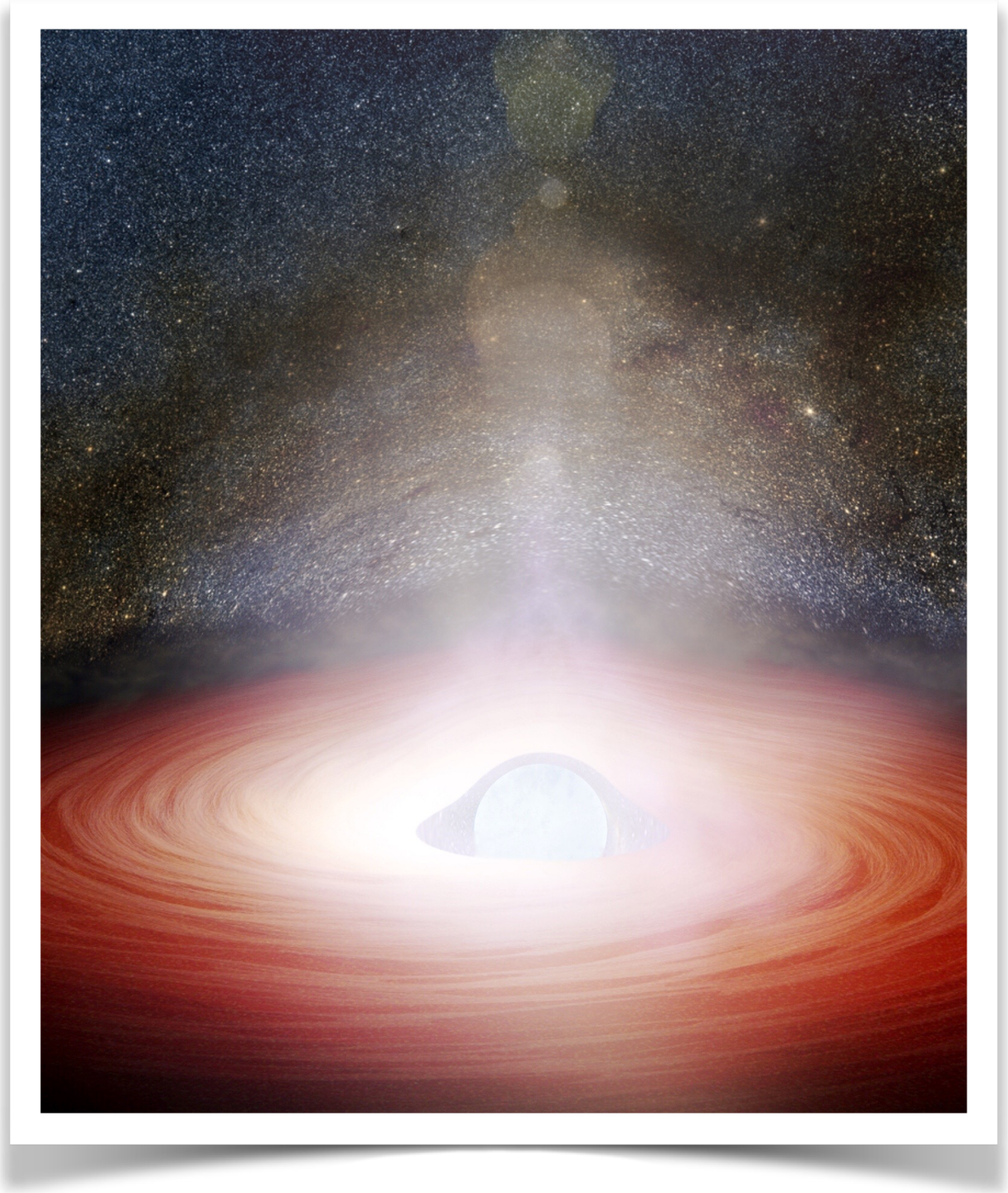

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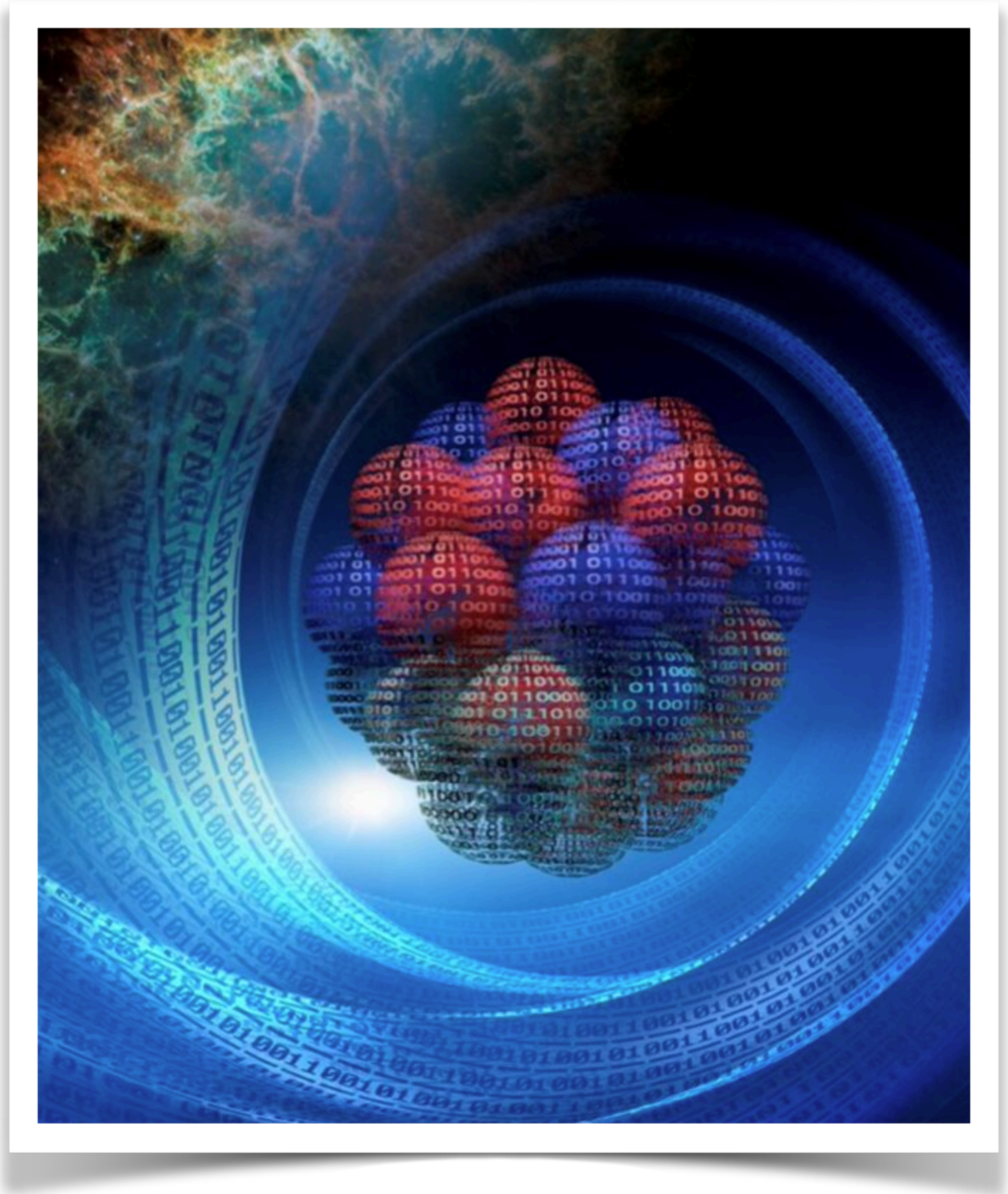

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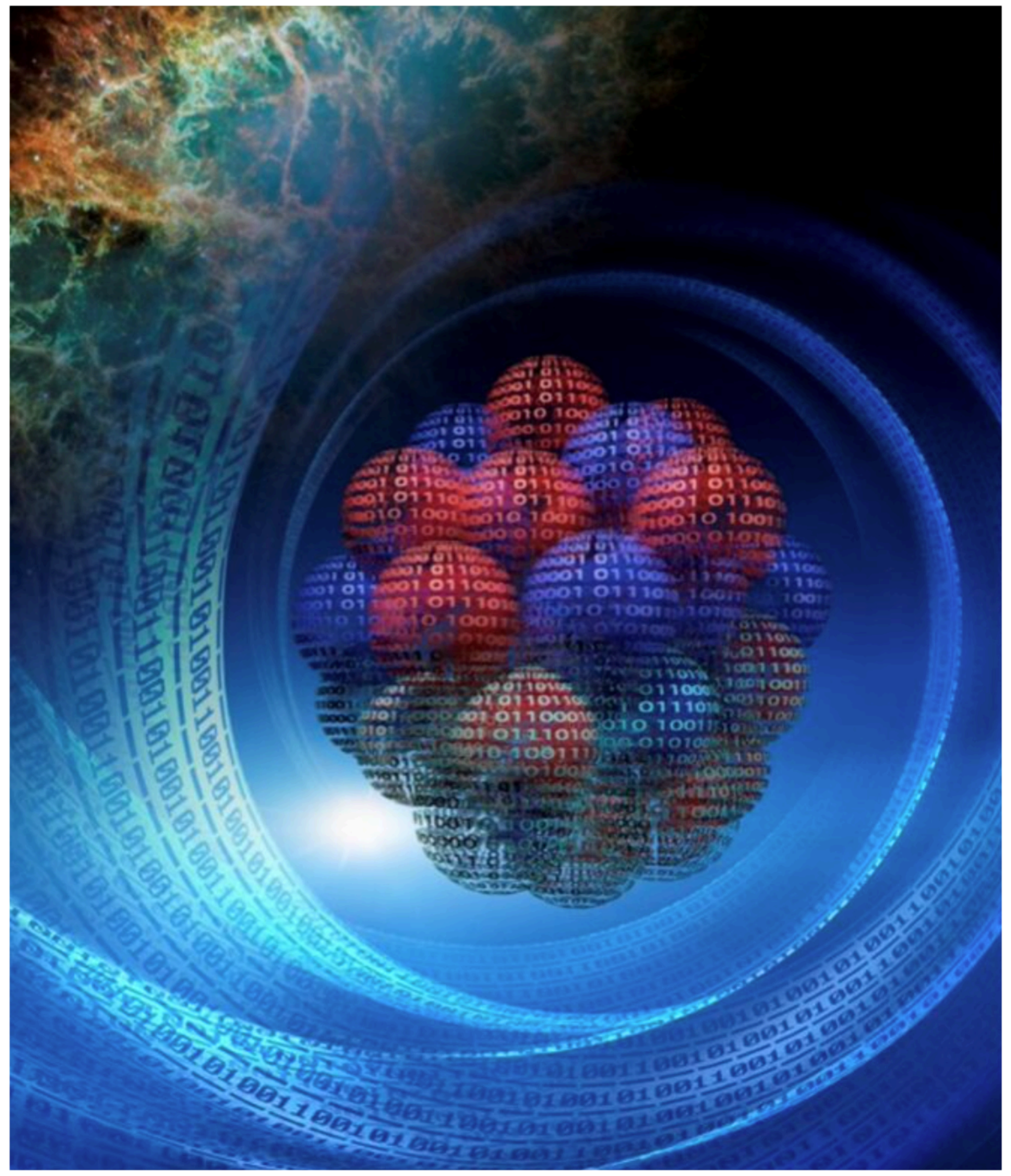

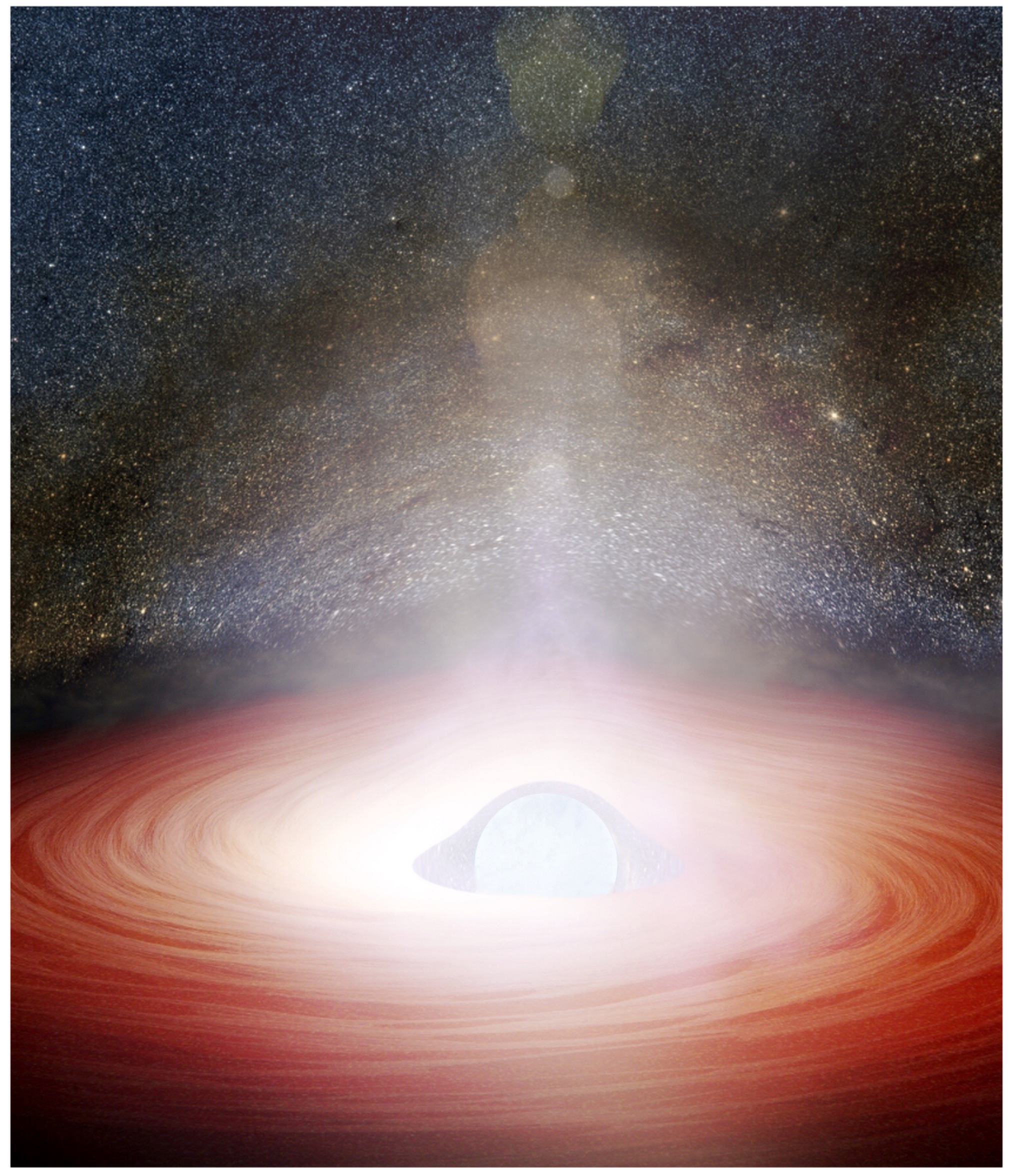

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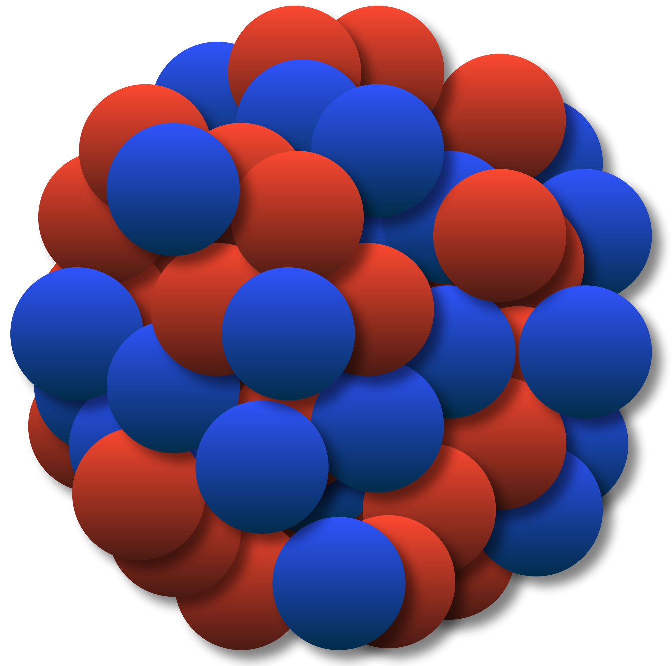

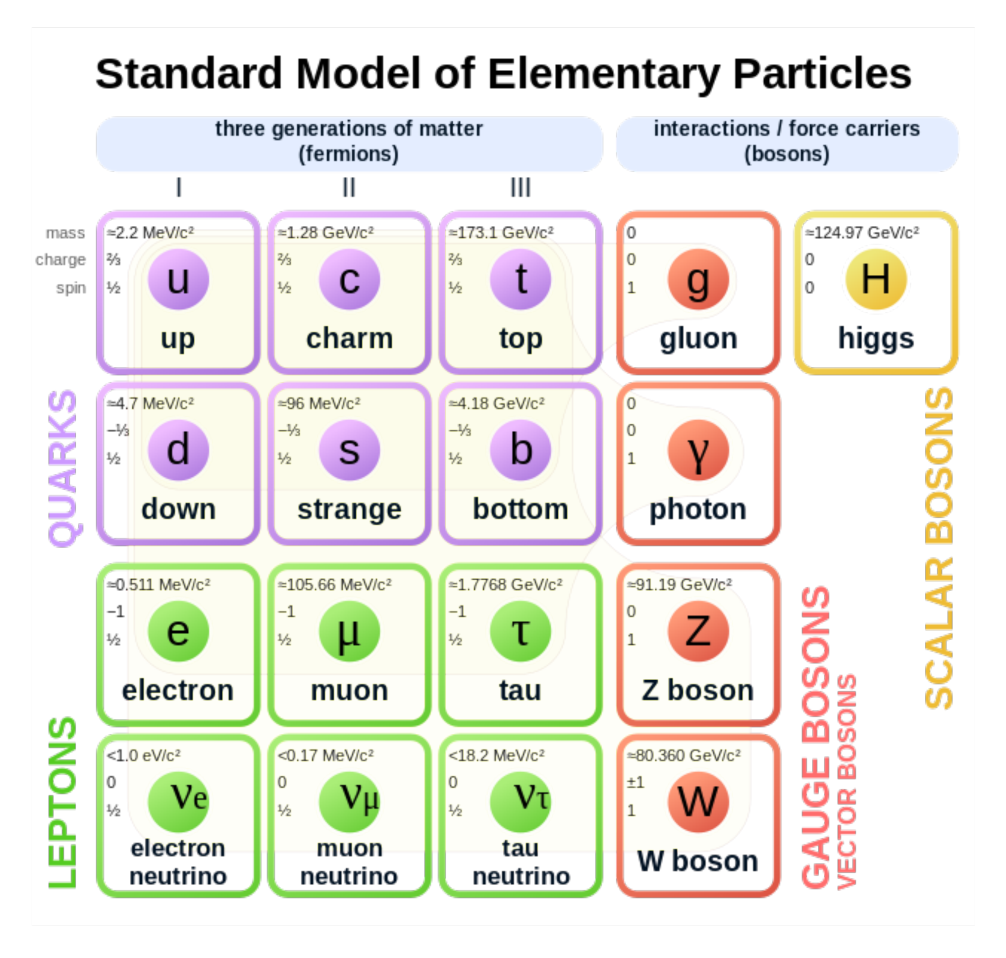

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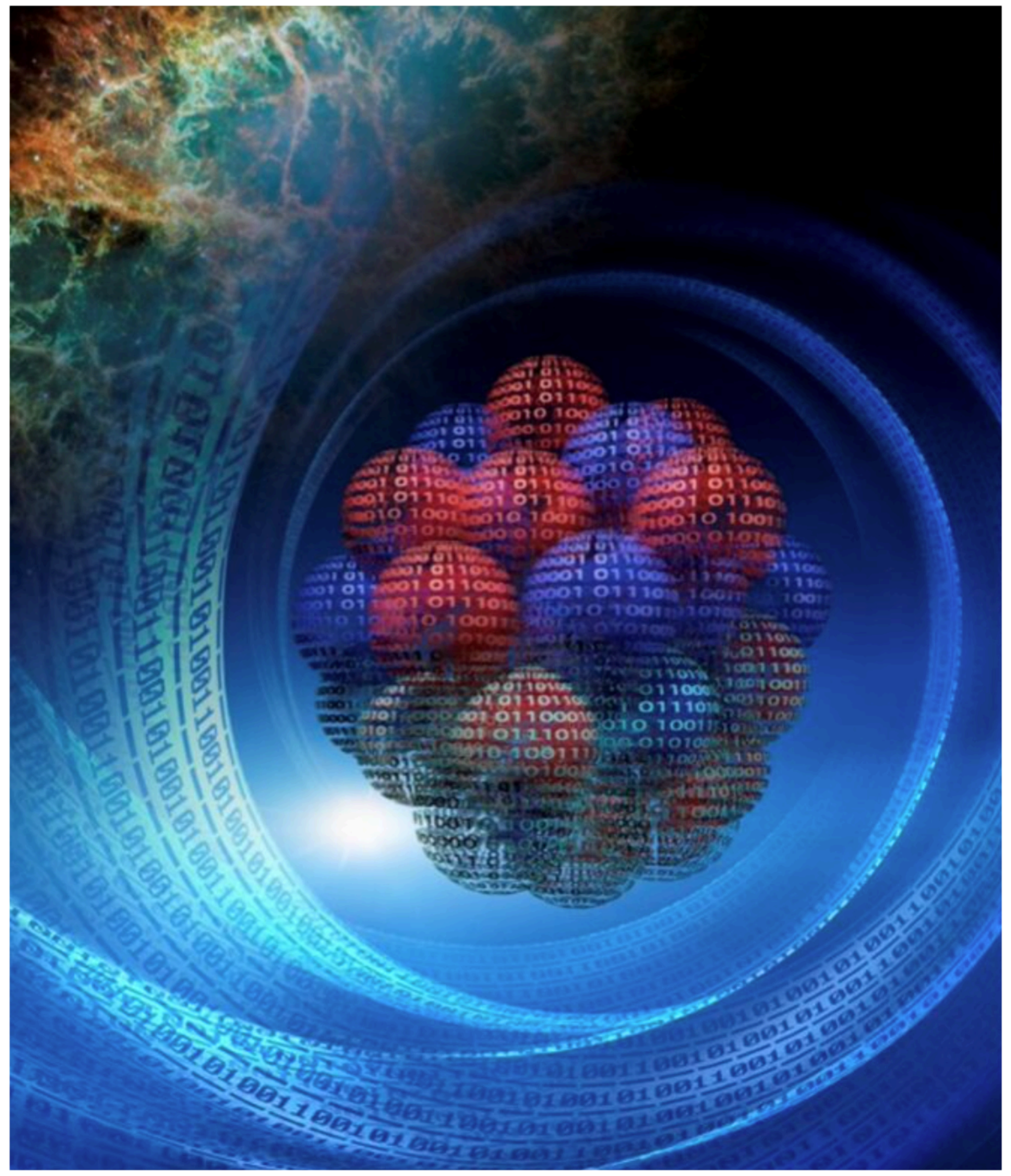

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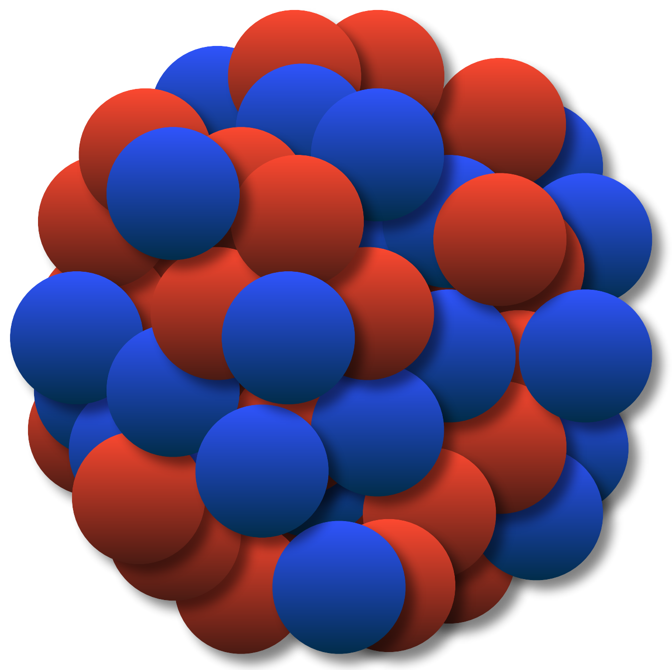
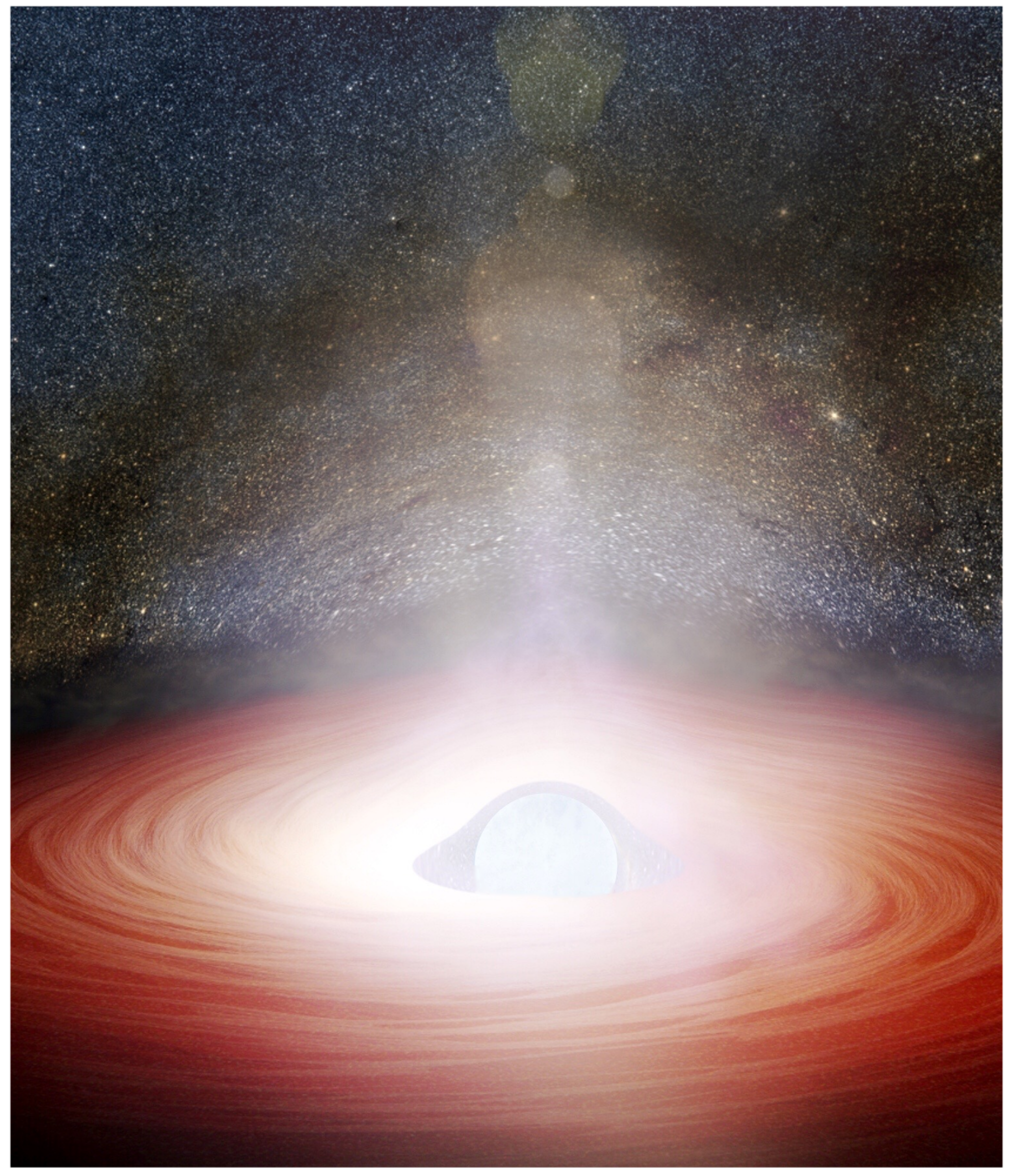


$$H\,|\psi\rangle = E\,|\psi\rangle$$



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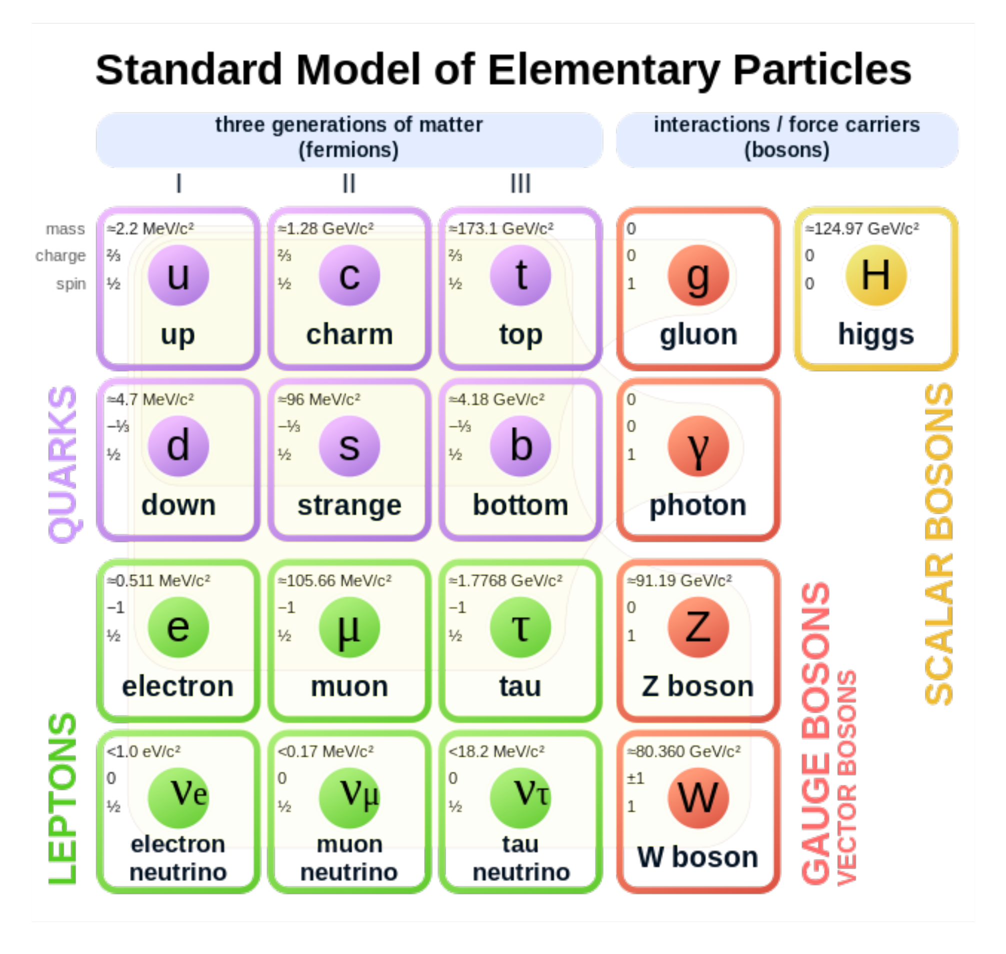
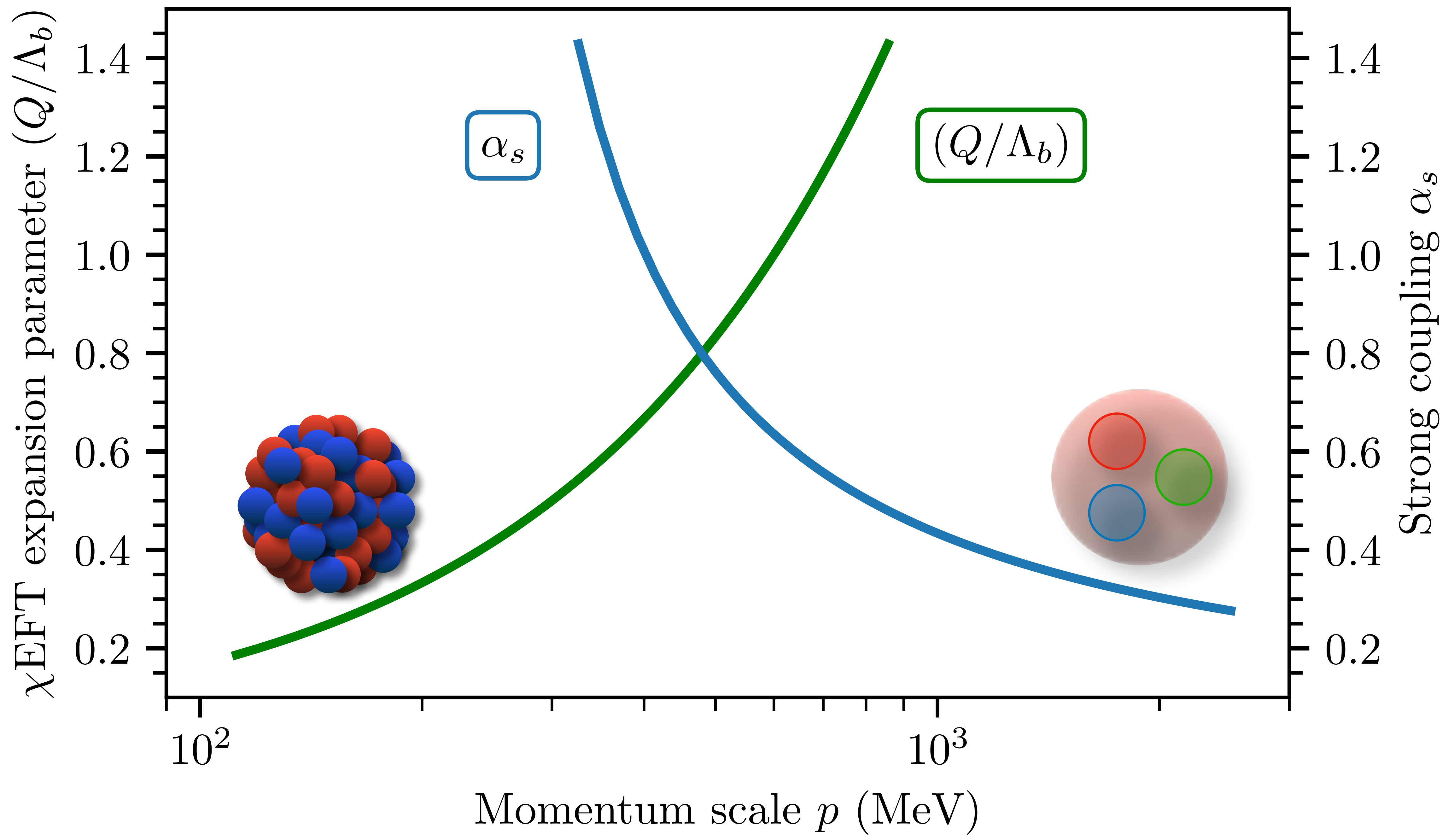


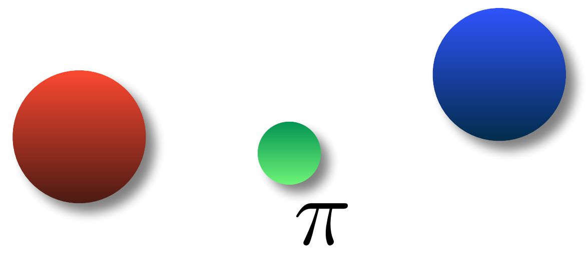
Key questions

- How to construct H to keep the connection to QCD?
- How to obtain precise predictions for nuclear observables with quantified theoretical error?

The nuclear force from EFT



- Weinberg, 90's: S. Weinberg, (1979), (1990), (1991)
  - Use protons, neutrons and pions as degrees of freedom.



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  - Perturbative expansion in (Q/Λ_b).

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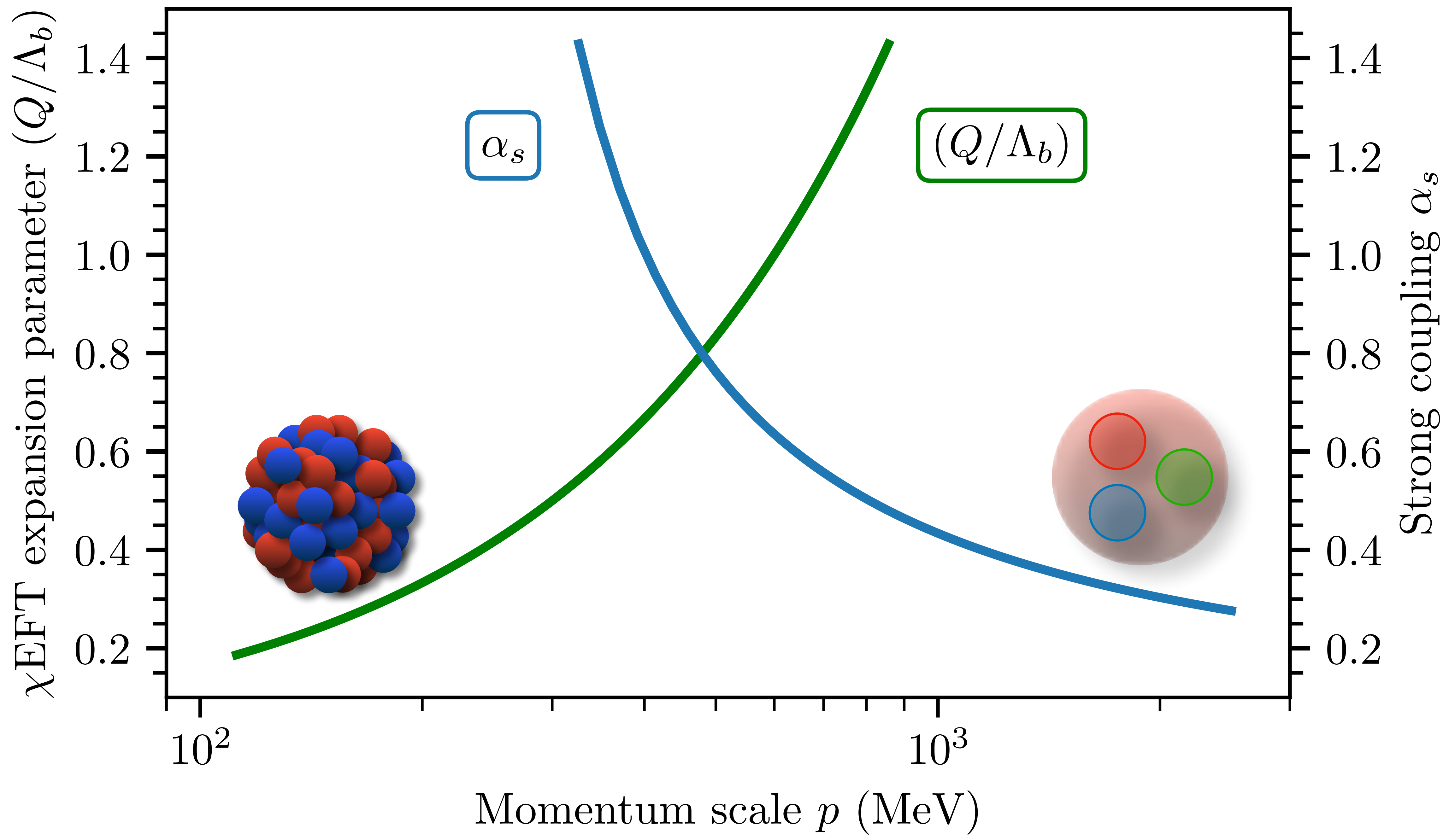

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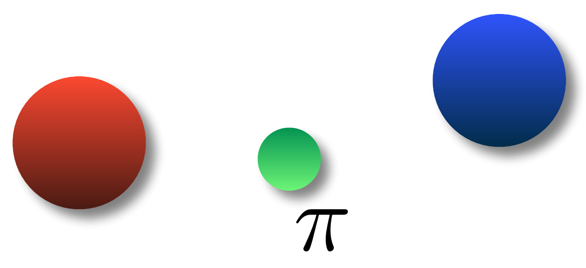

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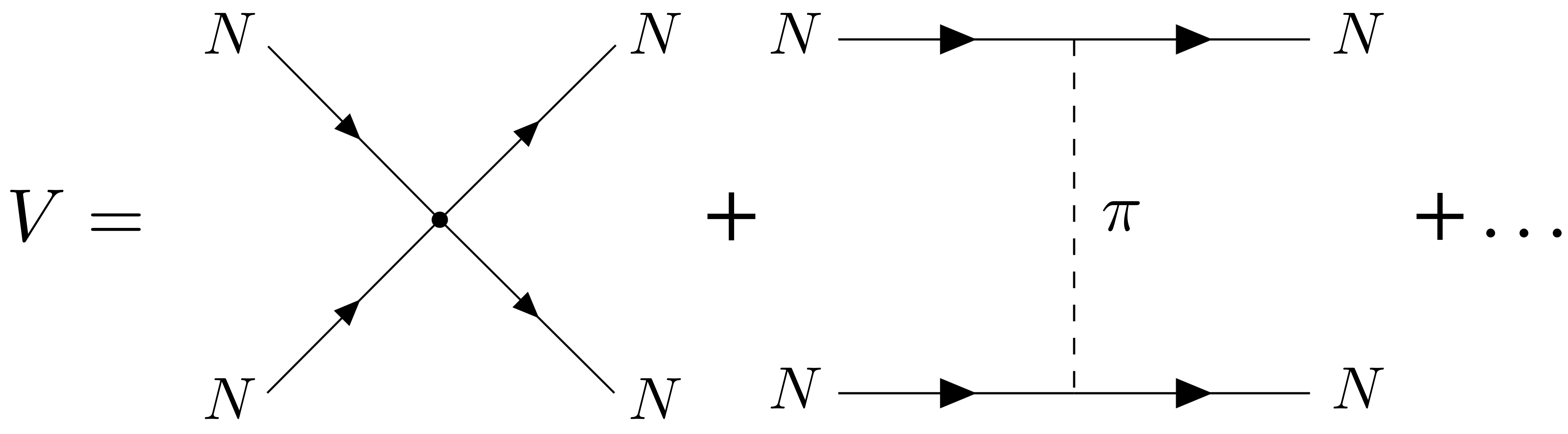
χEFT

The power of χEFT



✓ EFT description rooted in QCD.

✓ Systematic expansion with **quantifiable theoretical error**:

$$y_{\mathrm{th}}^{(n)} = y_{\mathrm{ref}} \sum_{k=0}^{n} b_k \left(\frac{Q}{\Lambda_\chi}\right)^k + y_{\mathrm{ref}} \left(\frac{Q}{\Lambda_\chi}\right)^{n+1} \mathcal{D}_{n+1}(\Lambda)$$

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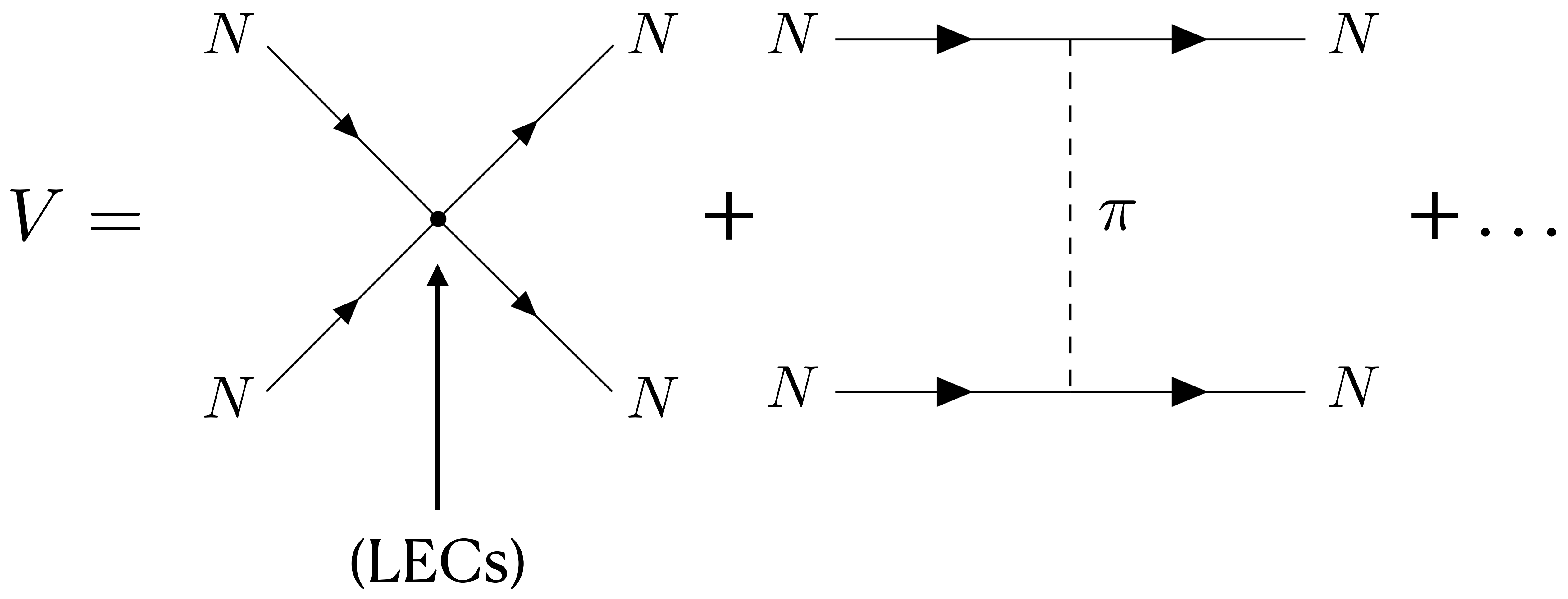

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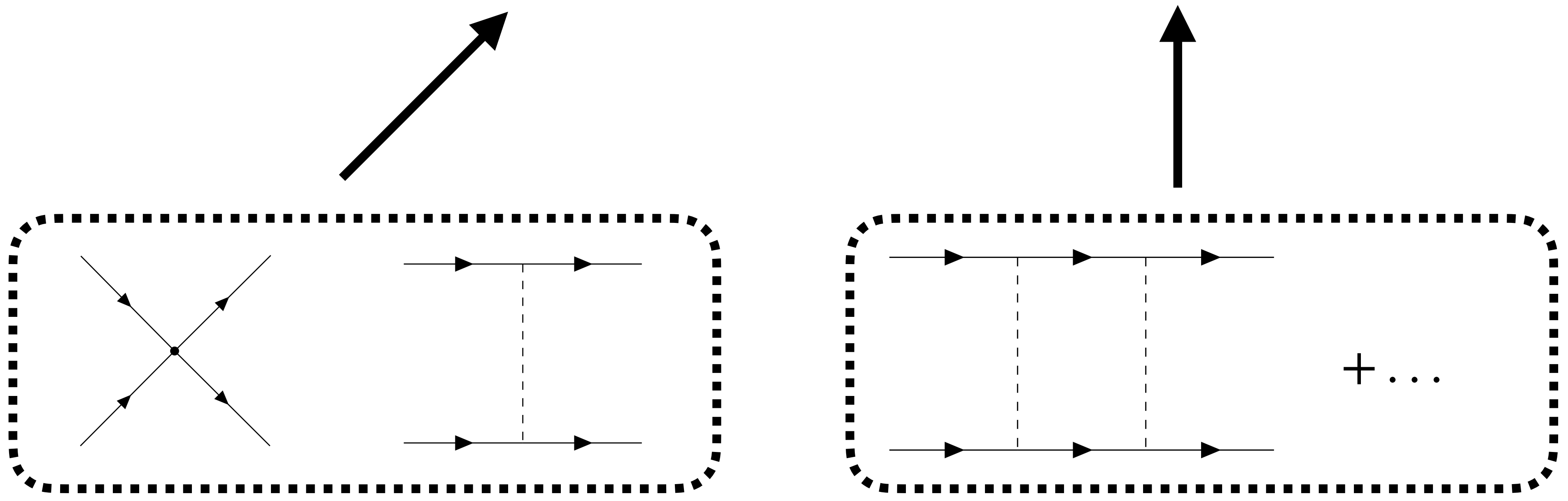
- Unknown values of low-energy constants (LECs).
- Importance of interactions: Power Counting (PC).

Weinberg PC

- Construct **potentials**:

$$V = V_{\mathrm{NN}}^{(0)}(\boldsymbol{\alpha}^{(0)}) + V_{\mathrm{NN}}^{(2)}(\boldsymbol{\alpha}^{(2)}) \ + \ \ldots$$



- Use dimensional analysis to organize diagrams.
- Resum potential nonperturbatively in LS-equation.

R. Machleidt and D. R. Entem, Phys. Rep. **503** (2011)

E. Epelbaum, H.-W. Hammer, and U.-G. Meissner, Rev. Mod. Phys. **81,** (2009)

H.-W Hammer, S. König, and U. van Kolck, Rev. Mod. Phys. **92,** (2020)

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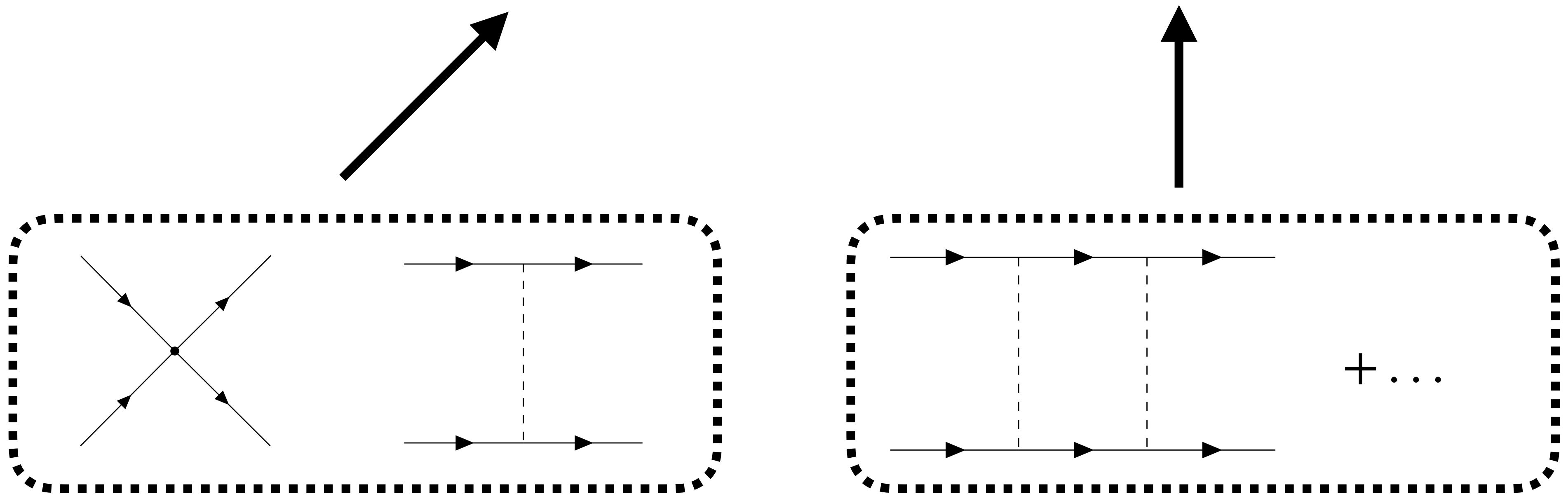

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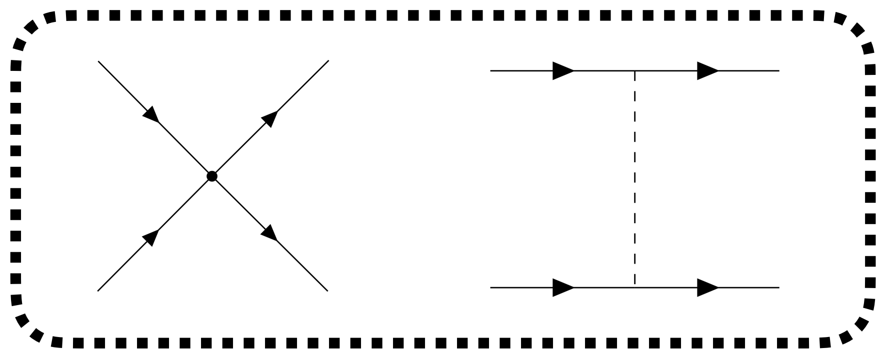

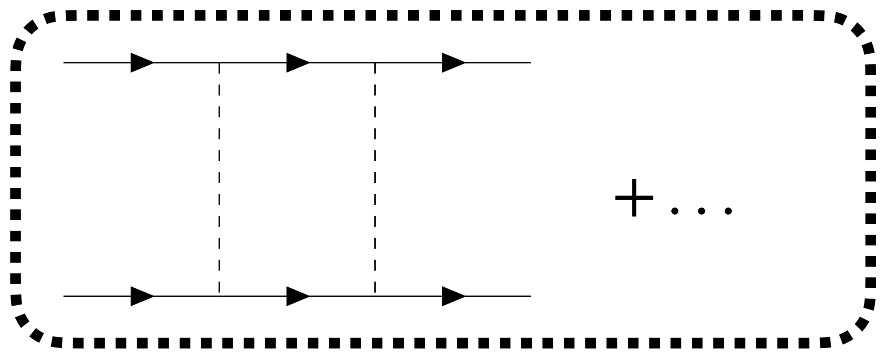

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- Calibrate unknown LECs using **data.**

- Compute **predictions.**

- χEFT with WPC: Successful description of many structure and reaction observables.

- Predictions of observables **depend on Λ** (= not RG invariant) A. Nogga *et al.,* Phys. Rev. C **72**, (2005)

Modified Weinberg PC

- What happens if we require Λ-independence and promote counterterms to lower orders to achieve this?

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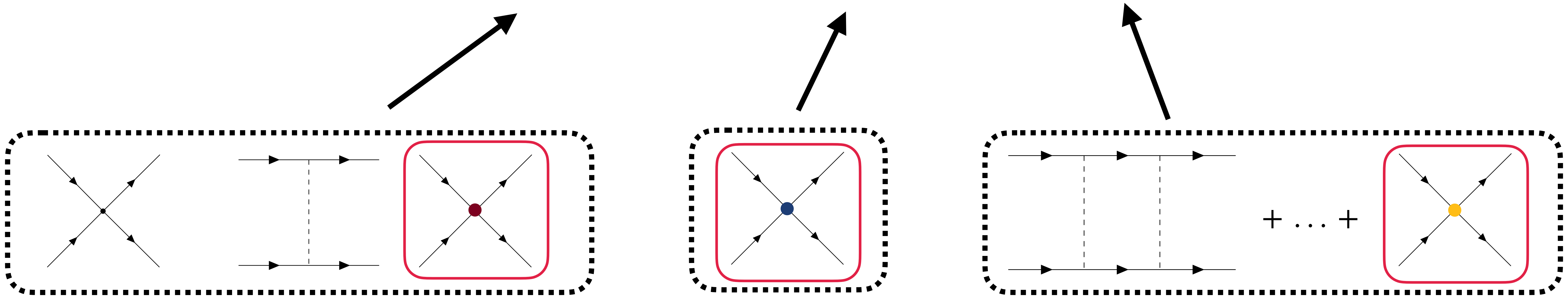

B. Long and U. van Kolck, Ann. Phys. **323**, (2008)

B. Long, C. J. Yang, Phys. Rev. C **84**, (2011), Phys. Rev. C **85**, (2012), Phys. Rev. C **86**, (2012)

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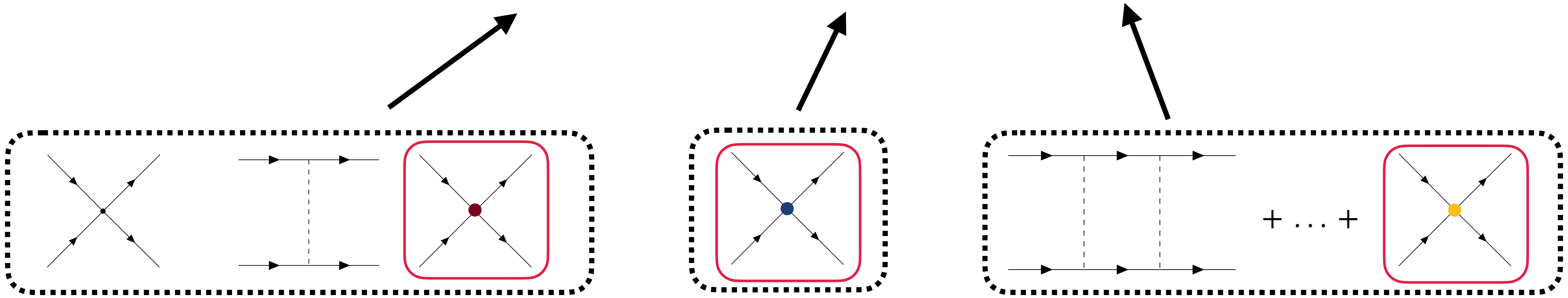
Modified Weinberg PC

- What happens if we require Λ-independence and promote counterterms to lower orders to achieve this?

Treated perturbatively

$$V = V_{\rm NN}^{(0)}(\boldsymbol{\alpha}^{(0)}) + V_{\rm NN}^{(1)}(\boldsymbol{\alpha}^{(1)}) + V_{\rm NN}^{(2)}(\boldsymbol{\alpha}^{(2)}) + V_{\rm NN}^{(3)}(\boldsymbol{\alpha}^{(3)}) + \ldots$$



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Modified Weinberg PC

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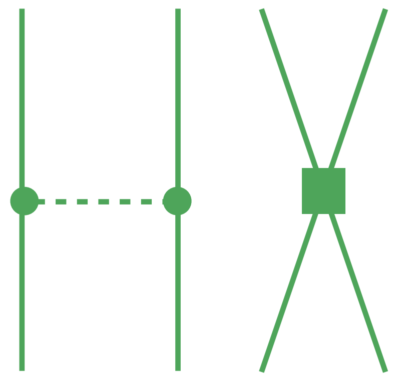
perturbative contritutions

Non-perturbative one-pion-exchange:

$^1S_0, ^3S_1 - ^3D_1, ^3P_{0,1}, ^3P_2 - ^3F_2, ^1P_1$

All other partial waves:

LO

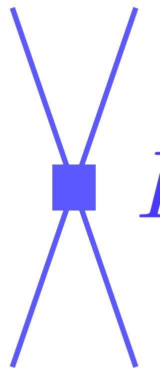
g_A  $C_{^1S_0}, C_{^3S_1}, C_{^3P_0}, C_{^3P_2}$

4 0

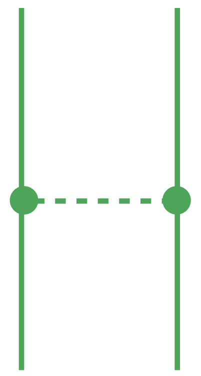
All other partial waves: —

Σ 4

NLO

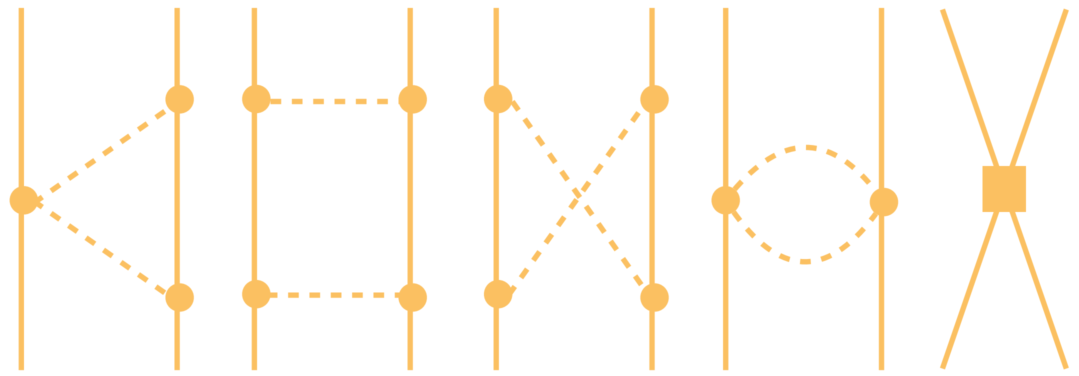
 $D_{^1S_0}$

1 1

All other partial waves: g_A 

Σ 6

N2LO

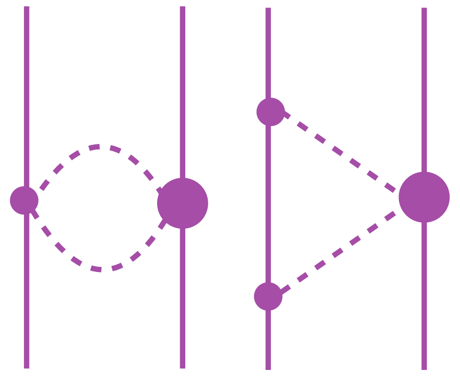
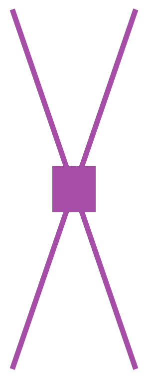
 $E_{^1S_0}, D_{^3S_1}, D_{^3S_1-^3D_1}, D_{^1P_1}, D_{^3P_1}, E_{^3P_0}, E_{^3P_2}, E_{^3P_2-^3F_2}$

8 5

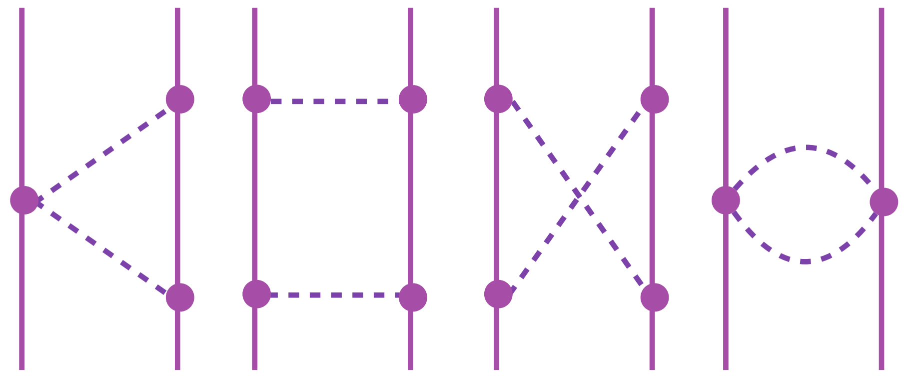
All other partial waves: —

Σ 19

N3LO

 $c_{1,3,4}$  $F_{^1S_0}$

1 13

All other partial waves: 

Σ 33

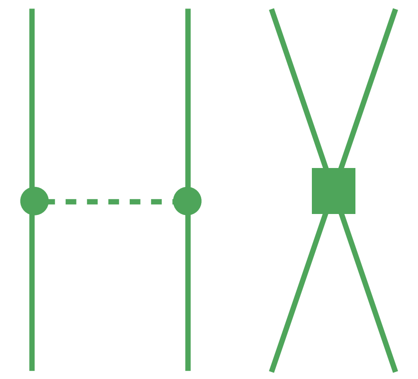
Modified Weinberg PC

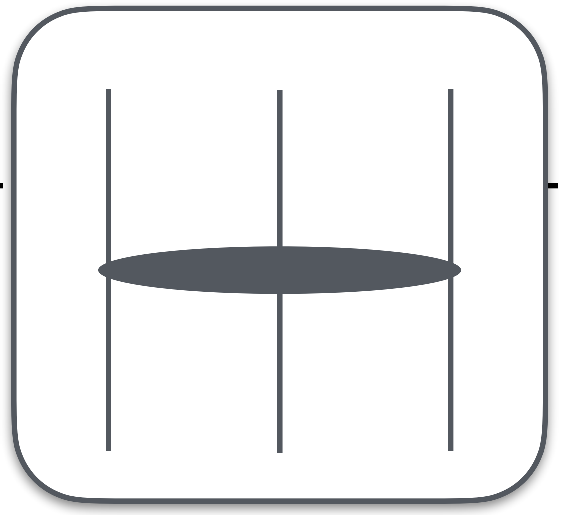
B. Long, C. J. Yang, Phys. Rev. C **84**, (2011), Phys. Rev. C **85**, (2012), Phys. Rev. C **86**, (2012)

Non-perturbative one-pion-exchange:
$^1S_0, ^3S_1 - ^3D_1, ^3P_{0,1}, ^3P_2 - ^3F_2, ^1P_1$

All other partial waves:

perturbative contritutions

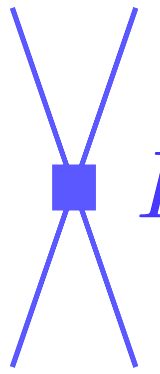
LO g_A  $C_{^1S_0}, C_{^3S_1}, C_{^3P_0}, C_{^3P_2}$

 ?

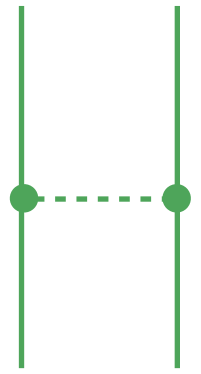
4 | 0

All other partial waves: –

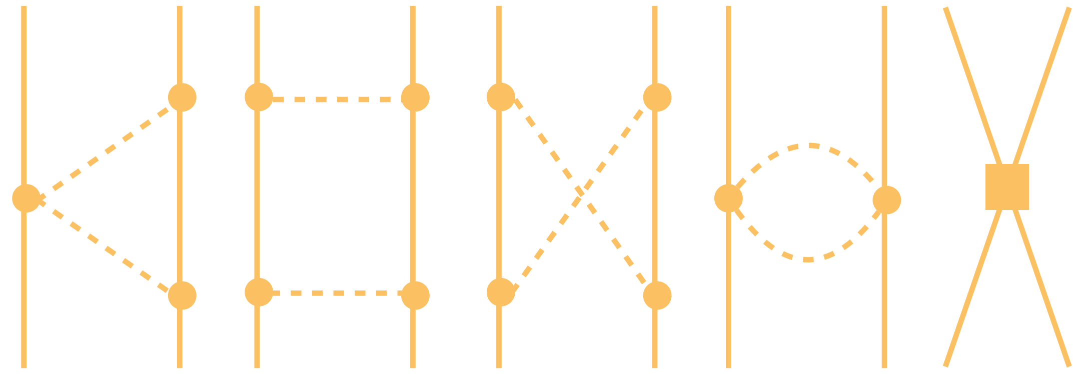
Σ 4

NLO  $D_{^1S_0}$

1 | 1

All other partial waves: g_A 

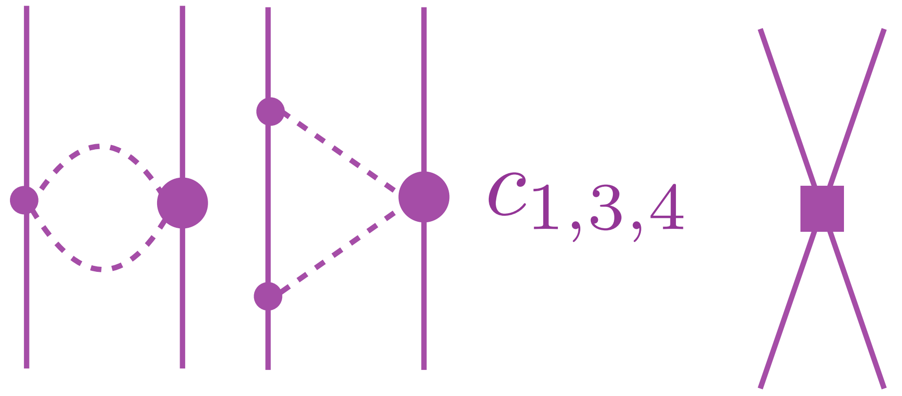
Σ 6

N2LO  $E_{^1S_0}, D_{^3S_1}, D_{^3S_1-^3D_1}, D_{^1P_1}, D_{^3P_1}, E_{^3P_0}, E_{^3P_2}, E_{^3P_2-^3F_2}$

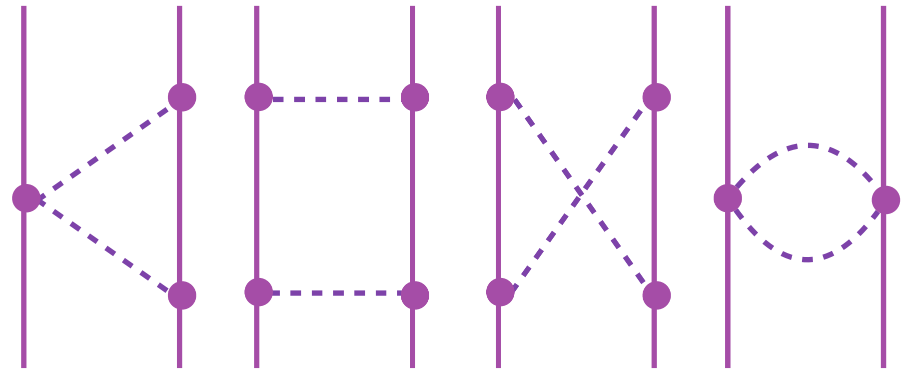
8 | 5

All other partial waves: –

Σ 19

N3LO  $c_{1,3,4}$ $F_{^1S_0}$

1 | 13

All other partial waves: 

Σ 33

Why do we study this PC?

- This modified PC is less studied than Weinberg PC.
- Yang *et al.* observed inaccurate predictions in nuclei with $A > 4$ using this PC at next-to-leading order (NLO). C. J. Yang *et al.*, Phys. Rev. C **103**, (2021)
- Studies of this modified PC can give us new insights about χEFT.
- **We want to:**
  - Infer LECs from observables with EFT error model.
  - Quantify prediction uncertainties with probability distributions.
  - Evaluate predictions for $A > 2$ systems beyond NLO.

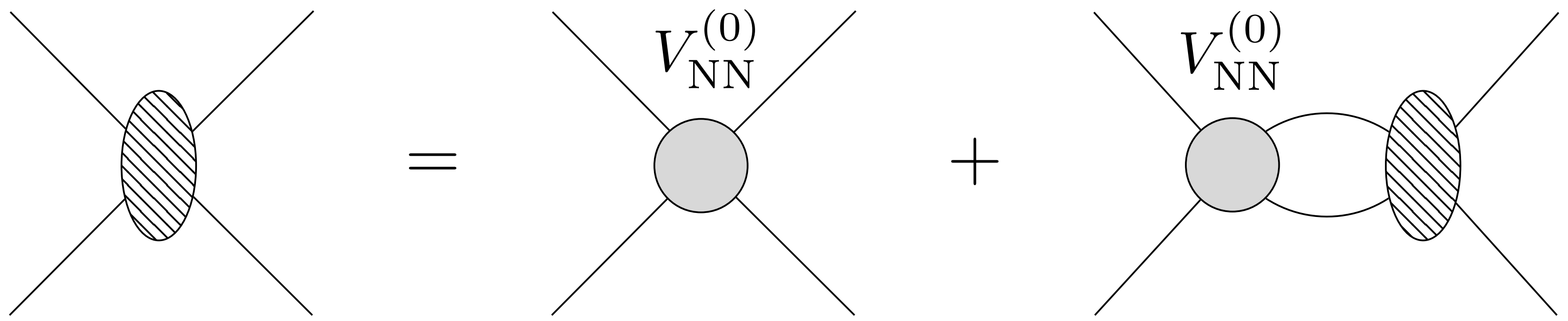
Computing *NN* amplitudes

$$V = V_{\rm NN}^{(0)}(\boldsymbol{\alpha}^{(0)}) + V_{\rm NN}^{(1)}(\boldsymbol{\alpha}^{(1)}) + V_{\rm NN}^{(2)}(\boldsymbol{\alpha}^{(2)}) + V_{\rm NN}^{(3)}(\boldsymbol{\alpha}^{(3)}) + \ldots$$

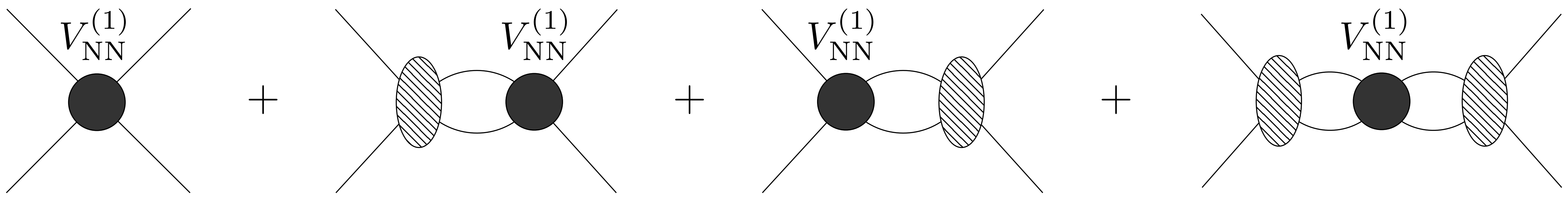
- Compute LO non-perturbatively.
- Use distorted-wave perturbation theory to add corrections beyond LO

$\Longrightarrow$ RG-invariance holds also at higher orders. B. Long and U. van Kolck, Ann. Phys. **323**, (2008)

LO: $T^{(0)}$ =



NLO: $T^{(1)}$ =



The starting point: LO

- Amplitudes computed perturbatively beyond LO:

$$T = T^{(0)} + T^{(1)} + T^{(2)} + T^{(3)} + \dots$$

$$T^{(0)} > T^{(1)} > T^{(2)} > T^{(3)}$$

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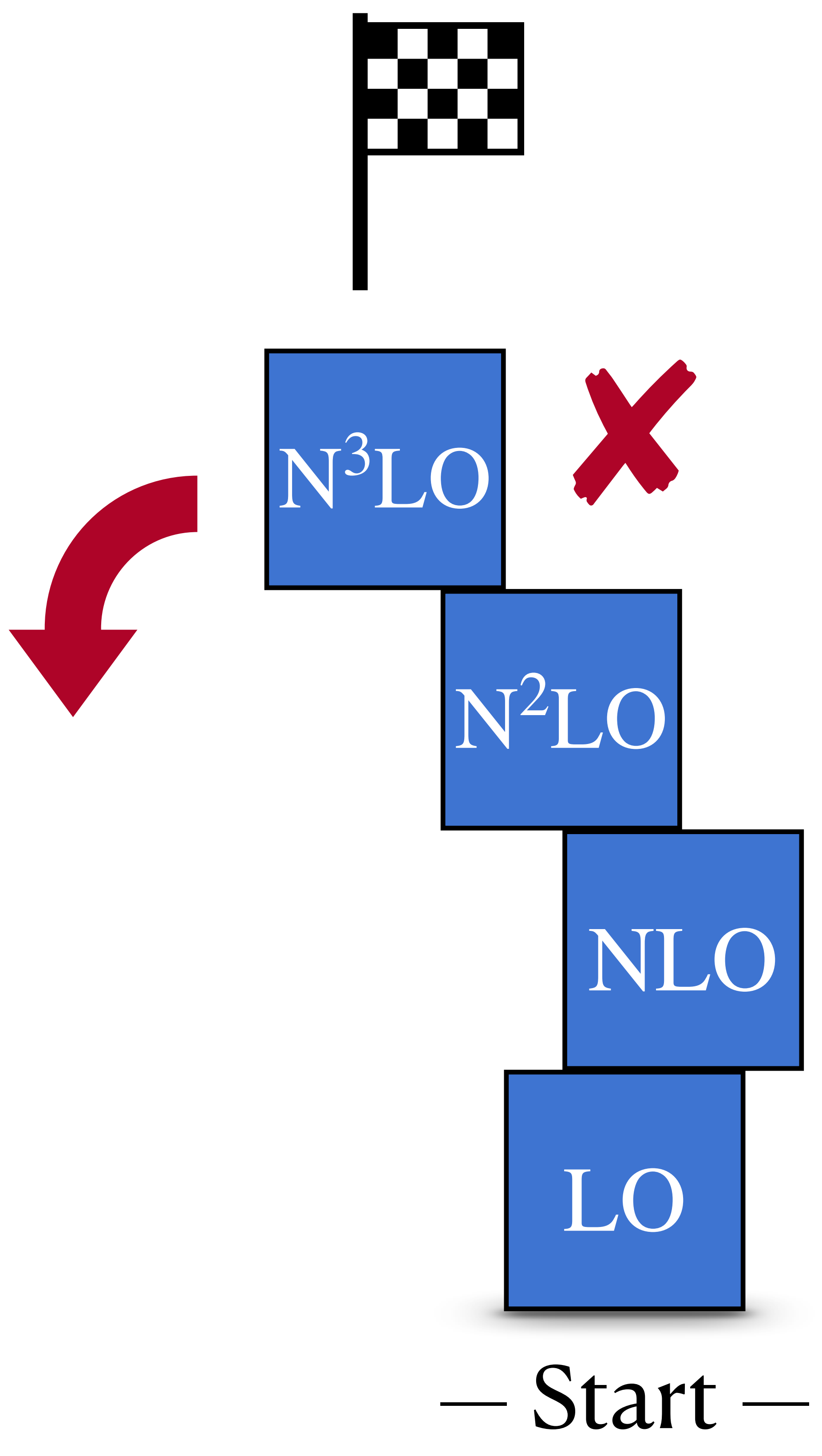

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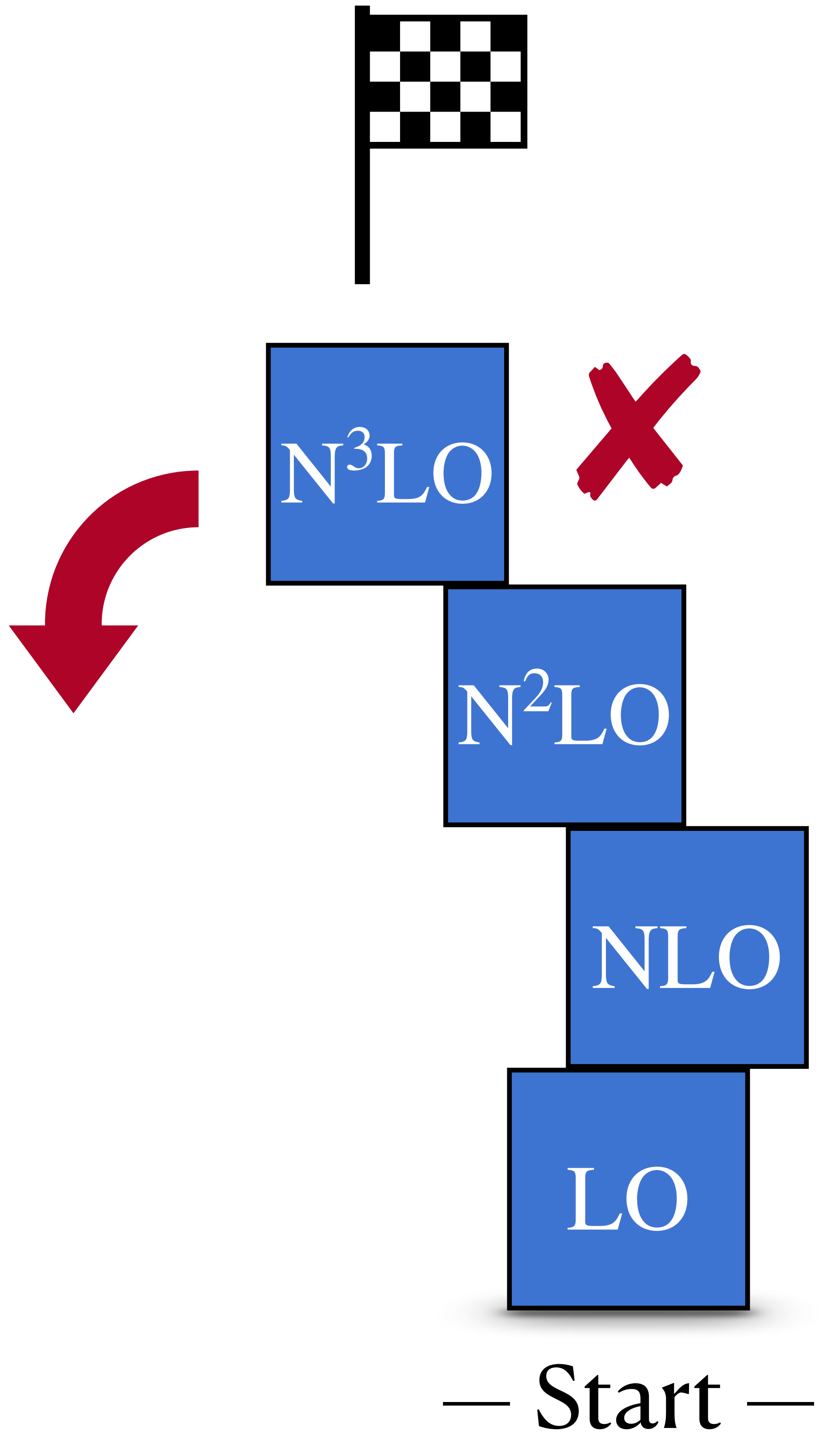

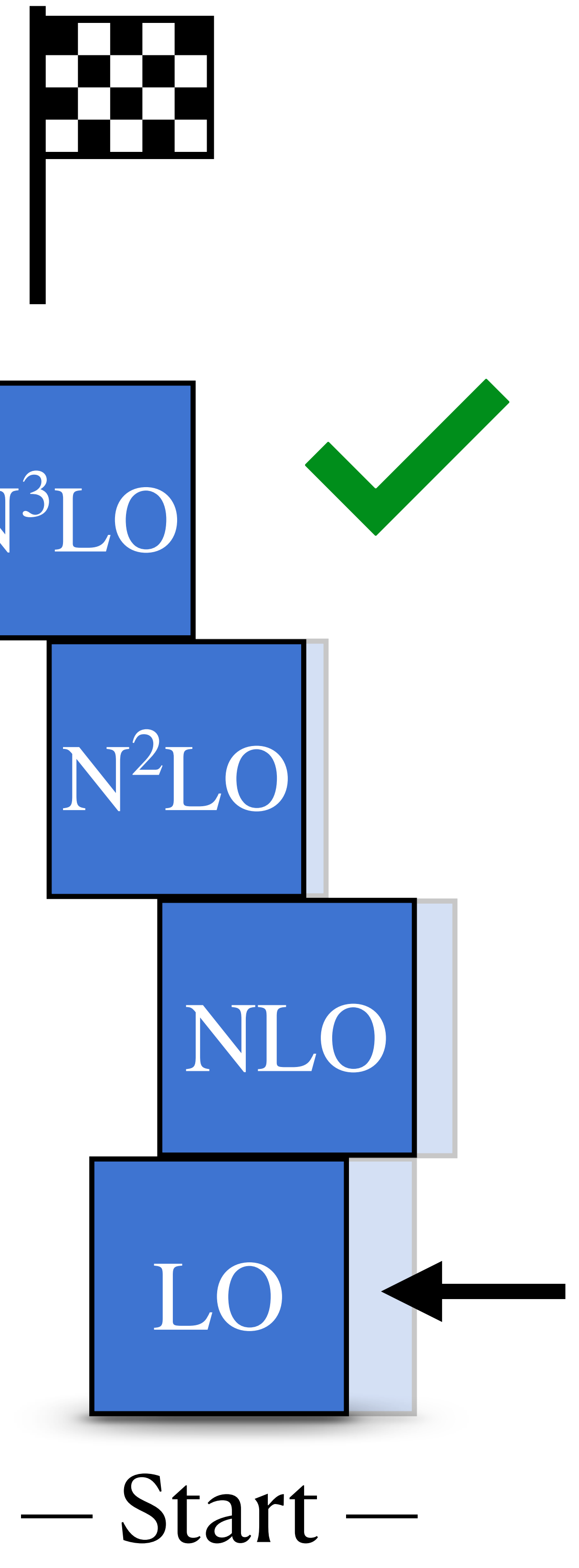

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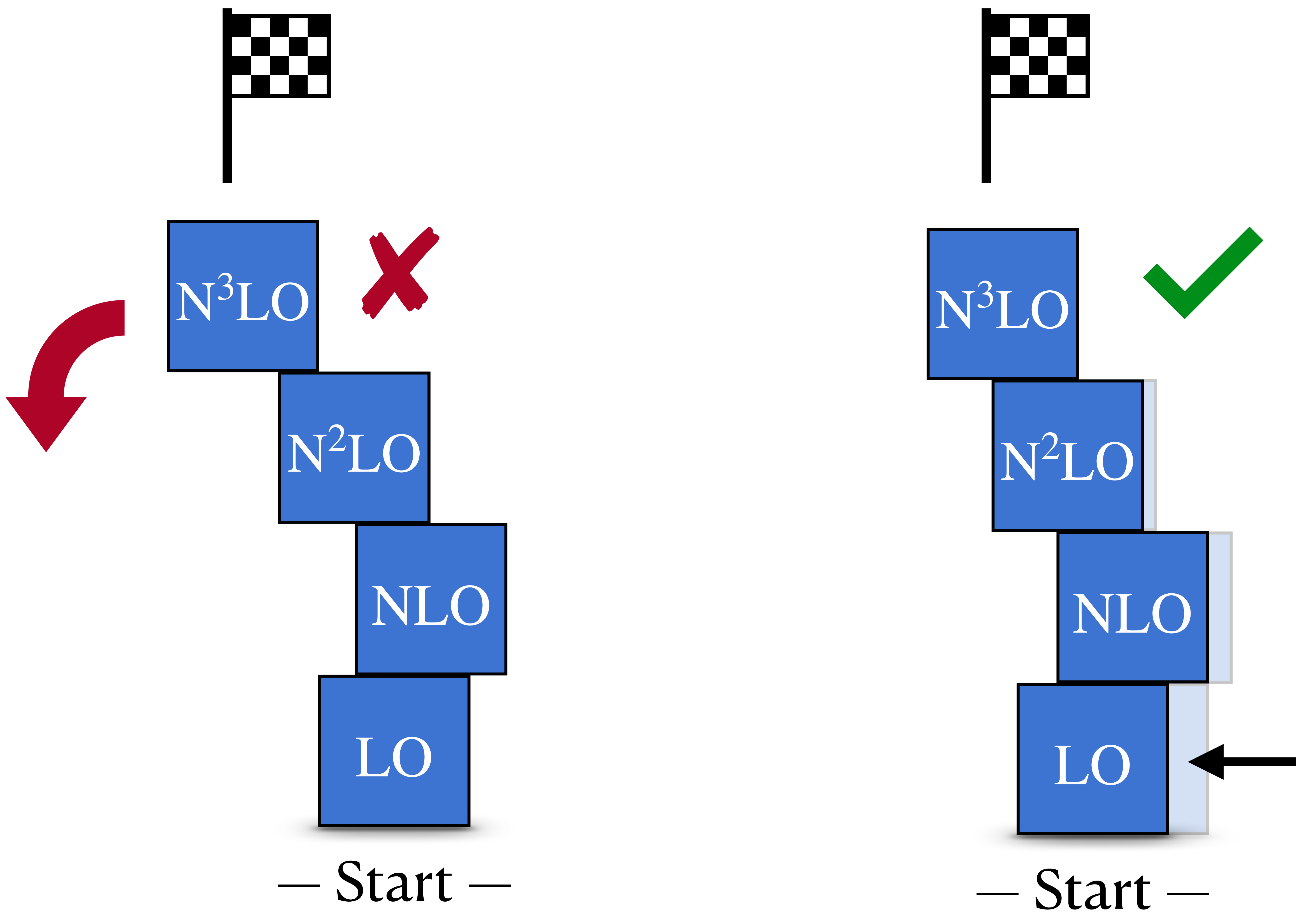
- Conclusion: The foundation (LO) is very important!

- What is the starting point?

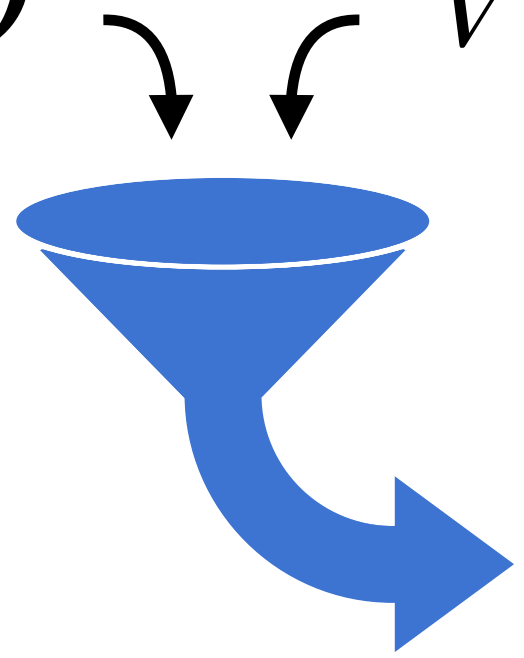


Calibrating the LO potential

- Use *NN* scattering observables to calibrate LO LECs.
- Bayesian inference: Treat LECs as random variables.

R. J. Furnstahl, N. Klco, D. R. Phillips, and S. Wesolowski, Phys. Rev. C **92** (2015)

D $\quad V = V_{\rm NN}^{(0)}(\boldsymbol{\alpha}^{(0)})$



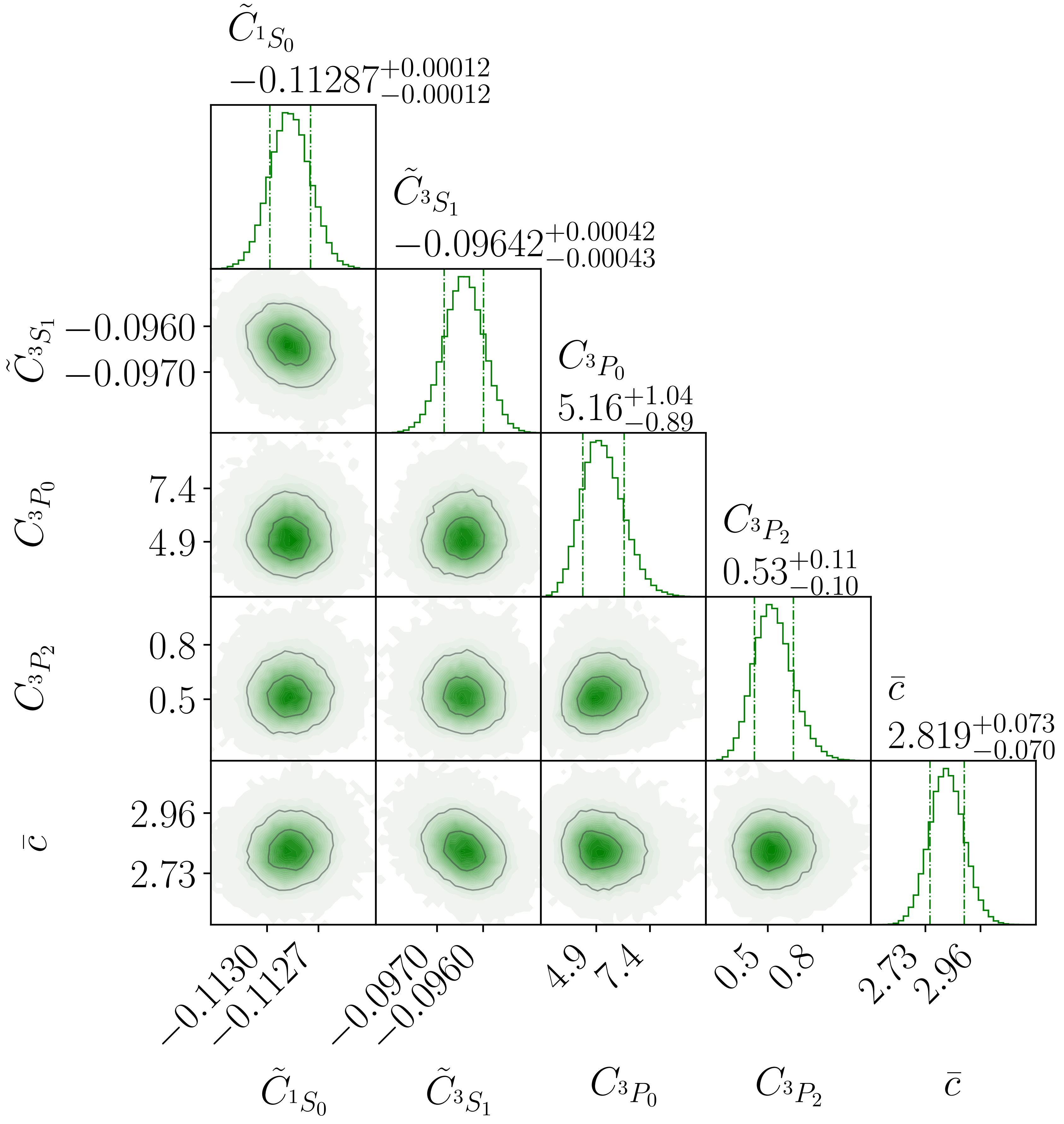
Likelihood, Prior, Posterior, Evidence

Bayes' rule: $$\frac{\mathrm{pr}(D|\boldsymbol{\alpha}^{(0)}, I)\cdot \mathrm{pr}(\boldsymbol{\alpha}^{(0)}|I)}{\mathrm{pr}(D|I)} = \mathrm{pr}(\boldsymbol{\alpha}^{(0)}|D, I)$$

$$\mathrm{pr}\left(\boldsymbol{\alpha}^{(0)}|D, I\right)$$

Calibrating LO potential

$\mathrm{pr}\left(\boldsymbol{\alpha}^{(0)}|D,I\right), \ \Lambda = 450 \text{ MeV}$



OT, E. May, A. Ekström, and C. Forssén, Phys. Rev. C **108**, (2023)

- Infer LECs for different cutoffs.
- LECs varies with the cutoff so the predictions do not.

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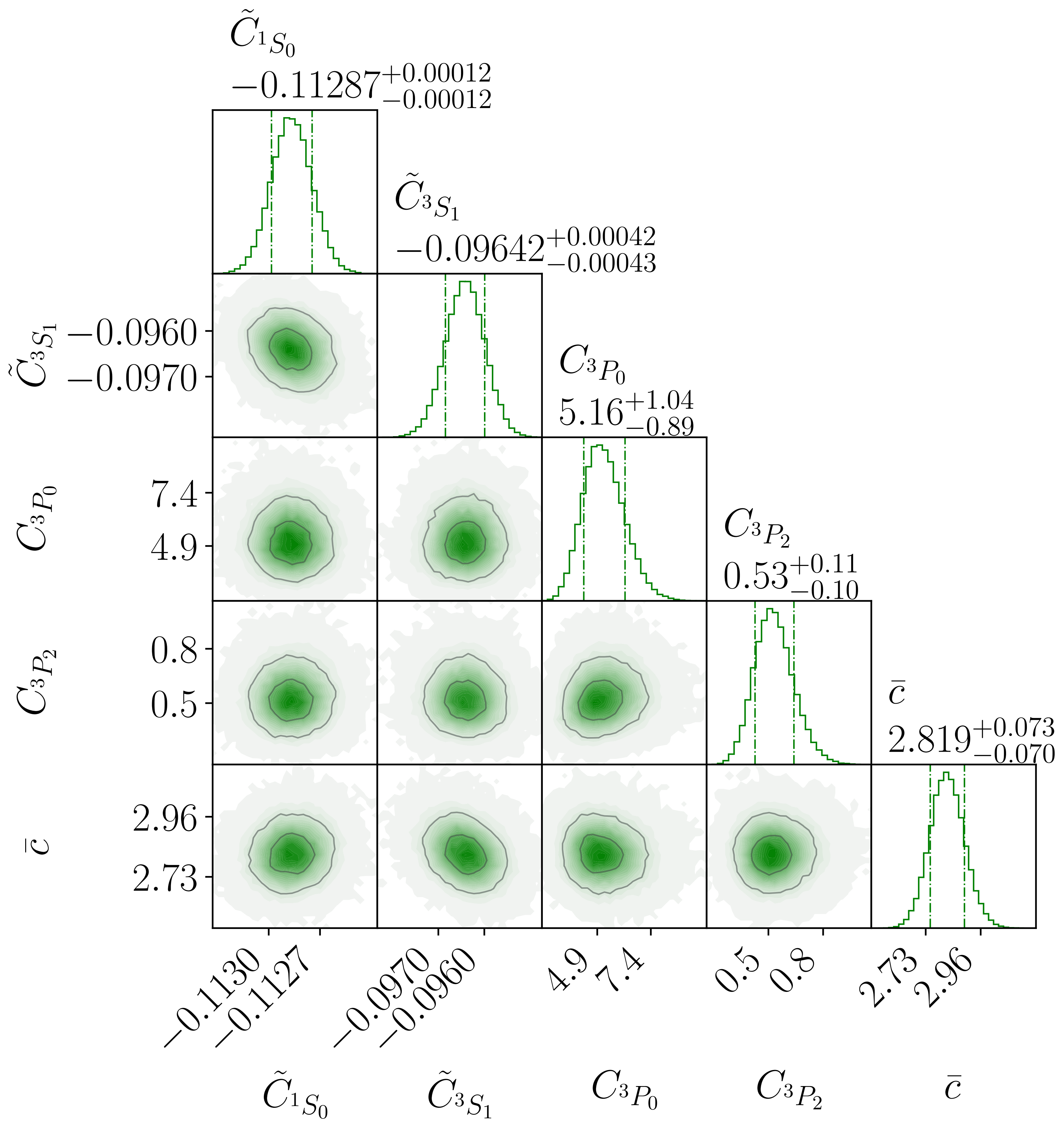

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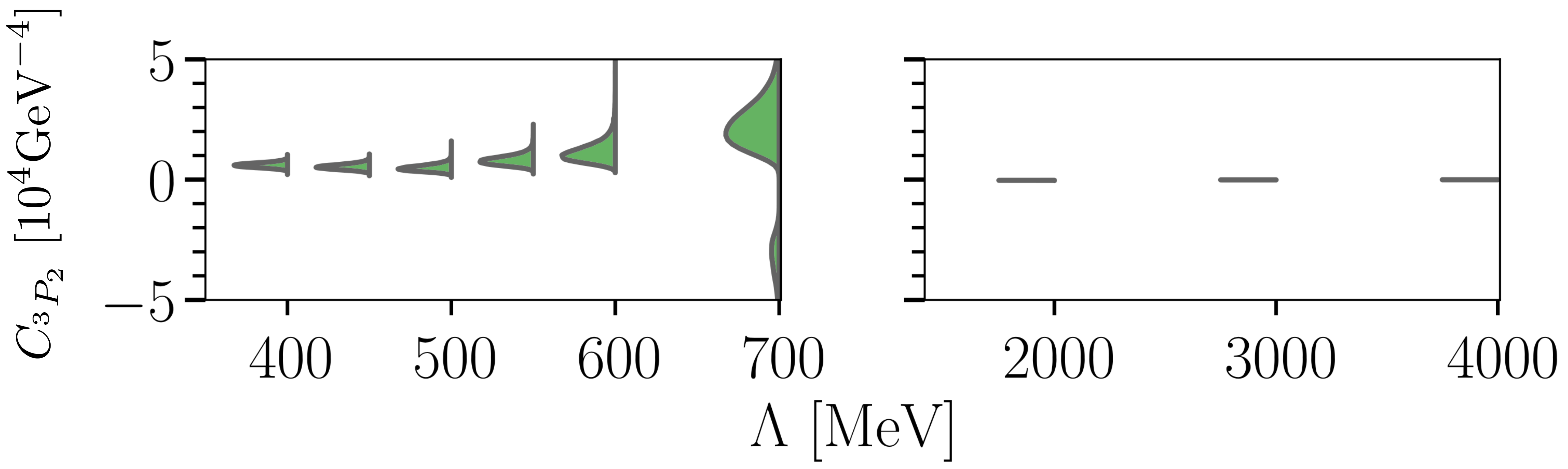

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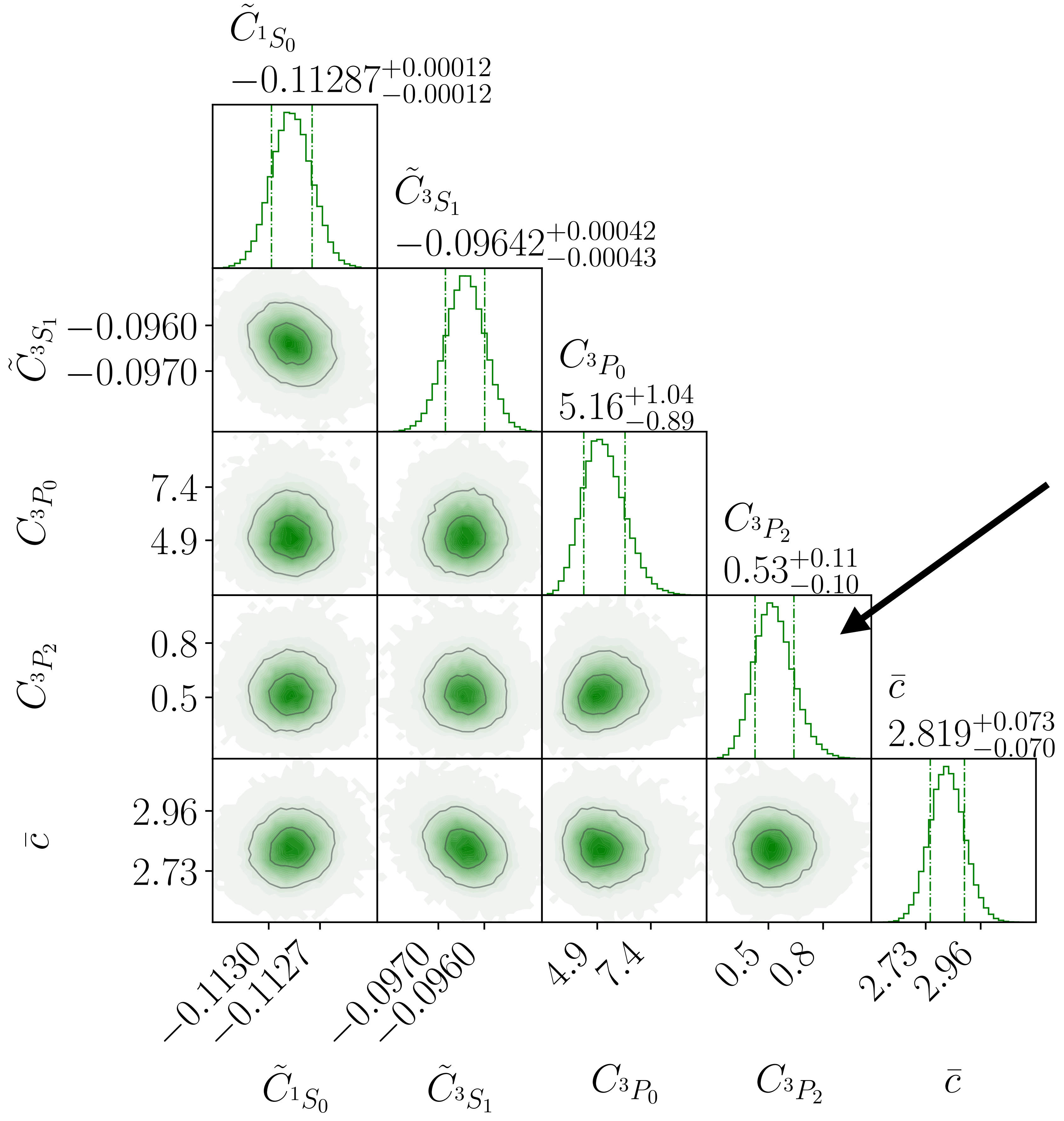

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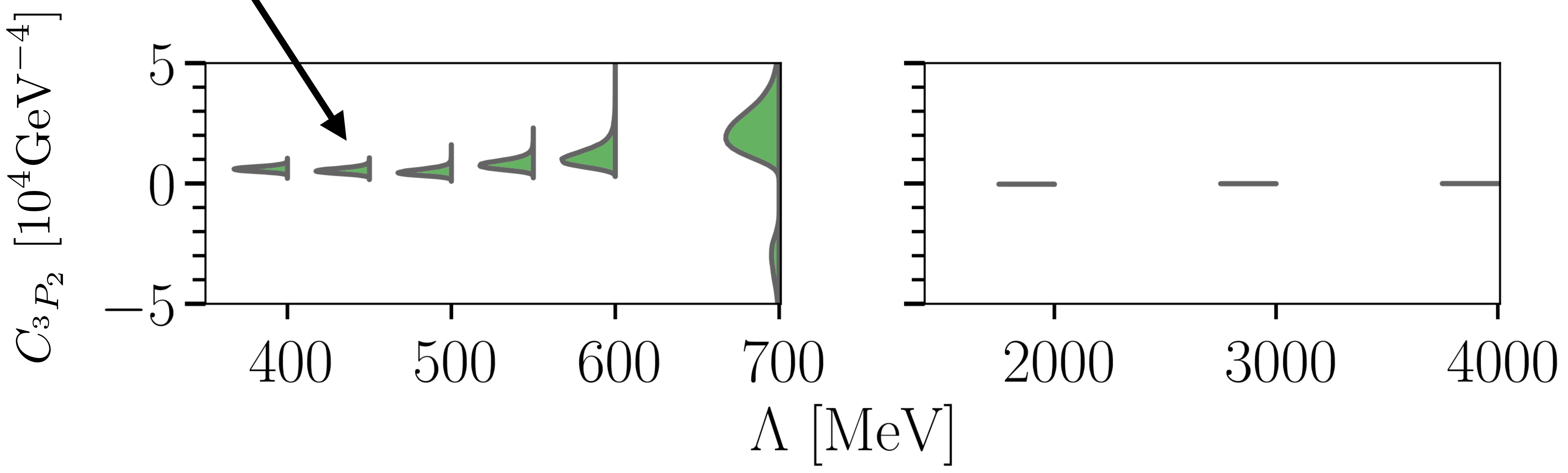
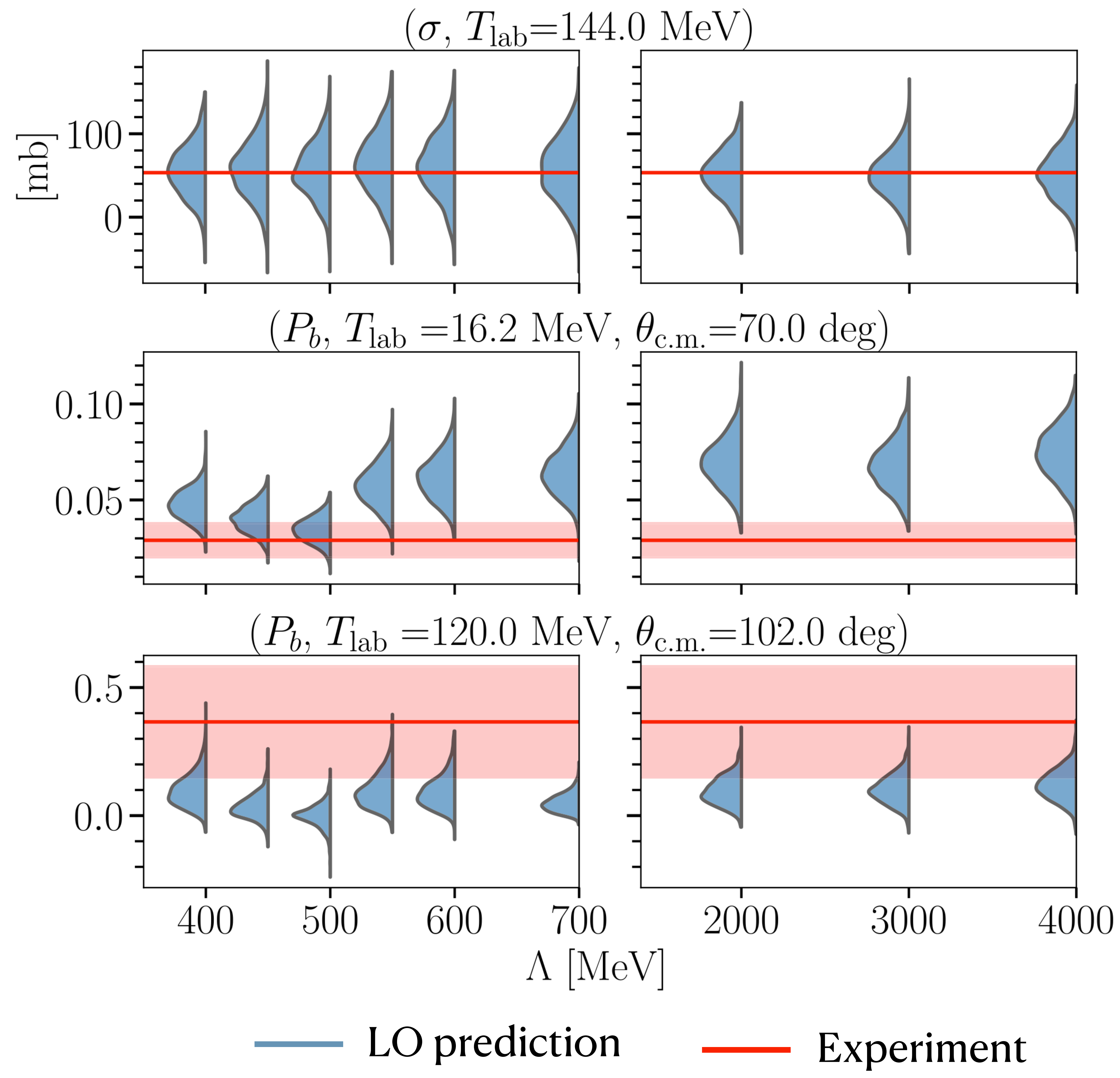


OT, E. May, A. Ekström, and C. Forssén, Phys. Rev. C **108**, (2023)

Predicted scattering observables



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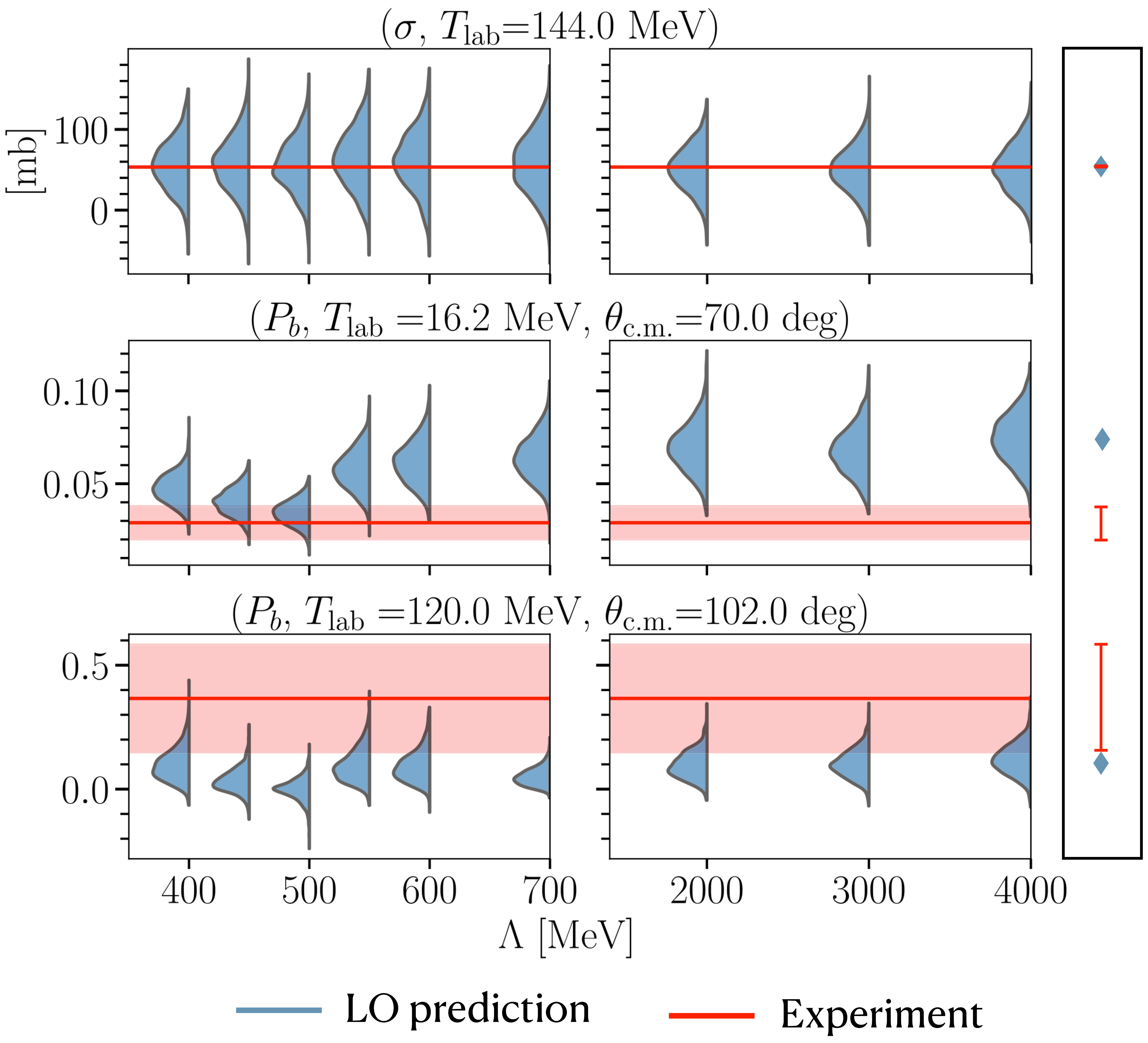

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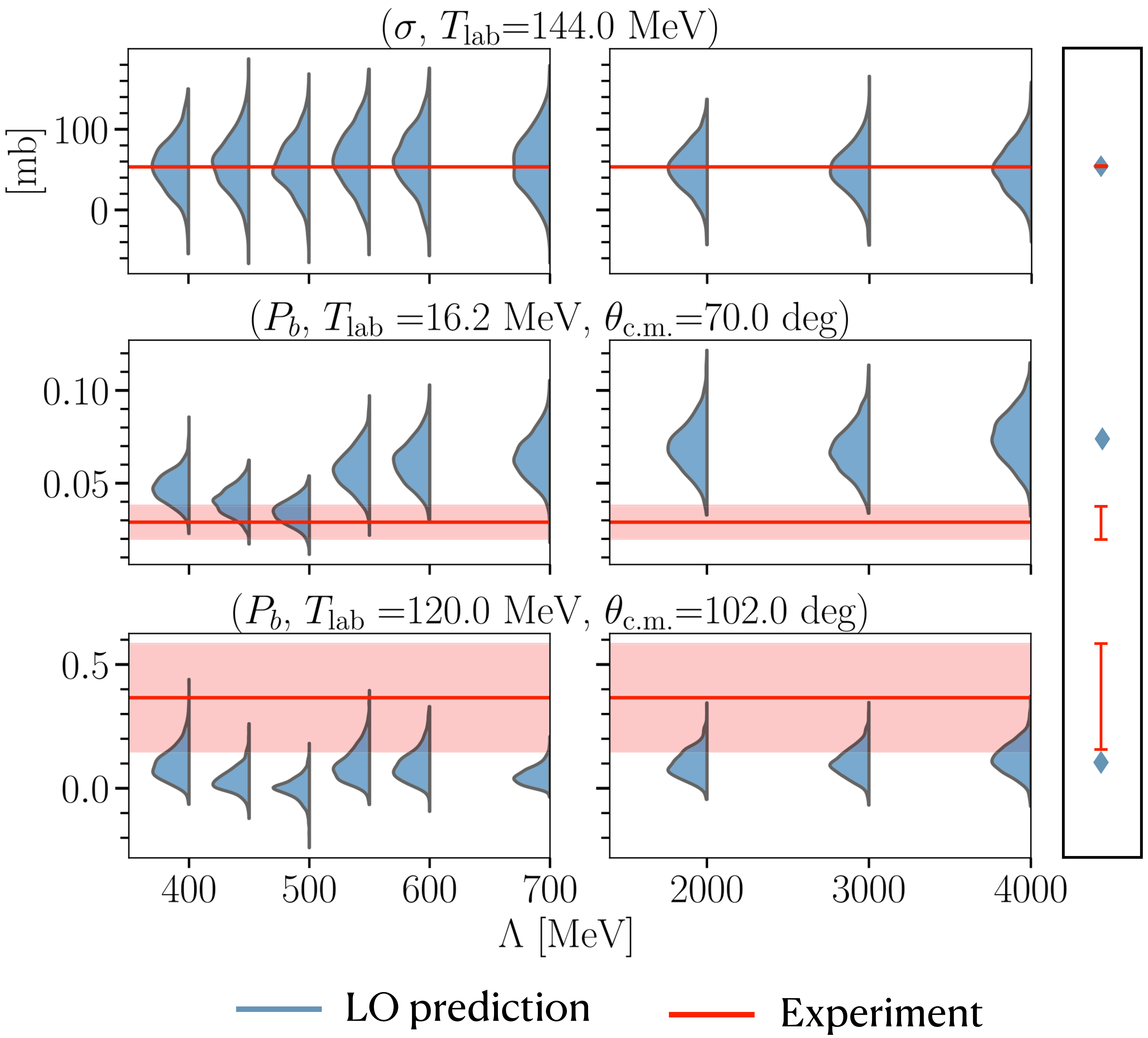

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- Accurate, but not very precise (high energy, LO).
- Not very accurate, but somewhat reasonable within LO uncertainty.
- Quite accurate, but the experimental error is large.

- Predictions are RG-invariant.
- Uncertainties are crucial for conclusions!
- The error model used is insufficient, higher orders are needed.

Adding perturbative corrections

$$V = \underbrace{V_{\rm NN}^{(0)}(\boldsymbol{\alpha}^{(0)})}_{\text{LO}} + \underbrace{V_{\rm NN}^{(1)}(\boldsymbol{\alpha}^{(1)}) + V_{\rm NN}^{(2)}(\boldsymbol{\alpha}^{(2)}) + V_{\rm NN}^{(3)}(\boldsymbol{\alpha}^{(3)})}_{\text{Perturbative corrections}} + \ldots$$

- More LECs: $\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \boldsymbol{\alpha}^{(3)}$.
- A first step: Calibrate LECs using **phase shifts** and compute predictions for **scattering observables**.

Perturbatively computed phase shifts

Sub-leading amplitudes using DWBA:

$$T^{(1)} = \Omega_-^\dagger V^{(1)} \Omega_+,$$

$$T^{(2)} = \Omega_-^\dagger \left(V^{(2)} + V^{(1)} G_1^+ V^{(1)} \right) \Omega_+,$$

$$T^{(3)} = \Omega_-^\dagger \Big(V^{(3)} + V^{(2)} G_1^+ V^{(1)} + V^{(1)} G_1^+ V^{(2)} + + V^{(1)} G_1^+ V^{(1)} G_1^+ V^{(1)} \Big) \Omega_+$$

$$\Omega_+ = \mathbb{1} + G_0^+ T^{(0)}$$

$$\Omega_-^\dagger = \mathbb{1} + T^{(0)} G_0^+$$

$$G_1^+ = \Omega_+ G_0^+$$

Phase shifts (uncoupled channels):

$$S = \exp(2i\delta)$$

$$S^{(0)} + S^{(1)} + S^{(2)} + S^{(3)} + \mathcal{O}(Q^3) = \exp(2i[\delta^{(0)} + \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \mathcal{O}(Q^3)])$$

$$S^{(0)} = \exp(2i\delta^{(0)}),$$

$$S^{(1)} = 2i\delta^{(1)} \exp(2i\delta^{(0)}),$$

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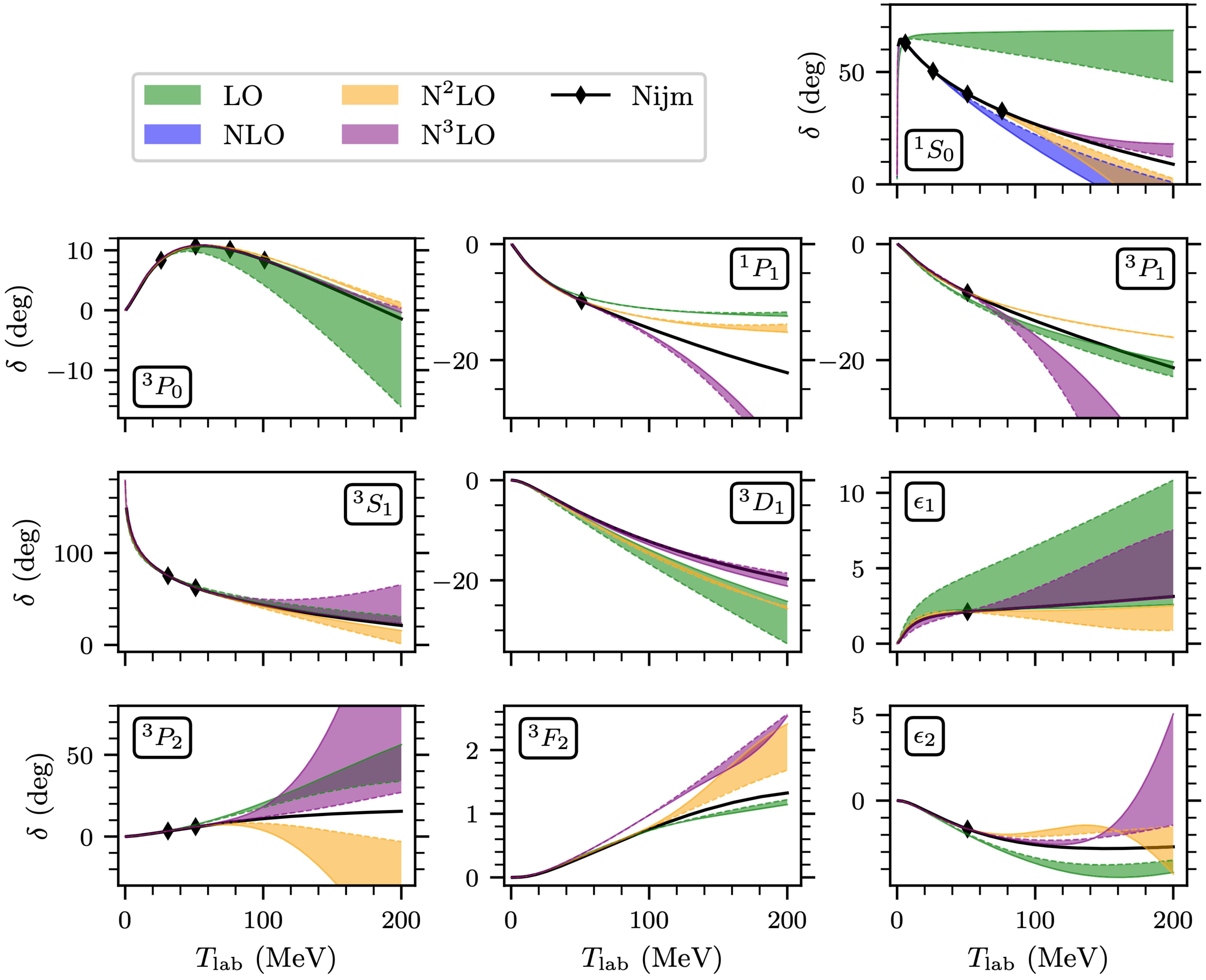
$$\delta_{\text{tot}}^{(\nu)} = \delta^{(0)} + \cdots + \delta^{(\nu)} \in \mathbb{R}$$

Calibrate LECs using *np* phase shifts



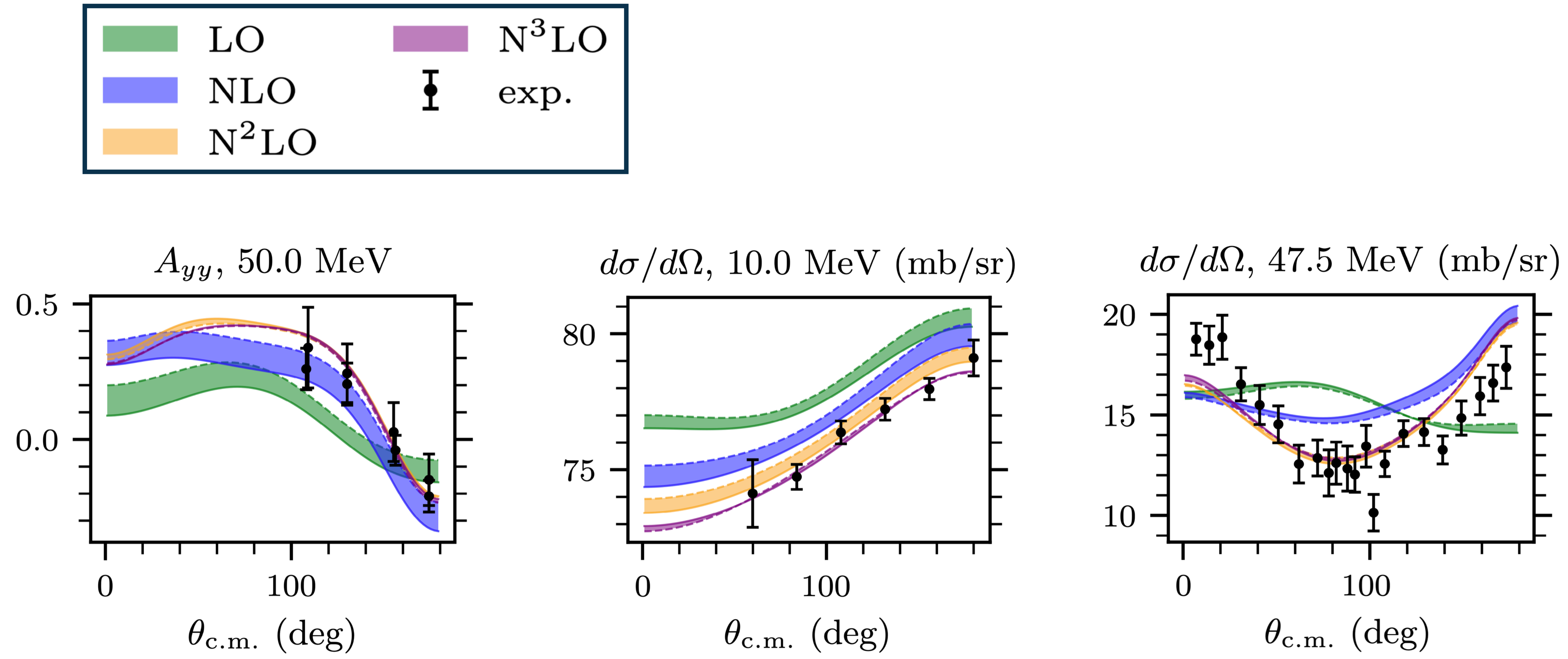
OT, A. Ekström, and C. Forssén, Phys. Rev. C **109**, (2024)

- Phase shifts are computed perturbatively.
- LECs are inferred by reproducing phase shifts at specific energies (♦).
- Two cutoffs:

$$\Lambda = 500 \text{ MeV}, \quad \Lambda = 2500 \text{ MeV}$$

- Note: NLO = LO except in $^1S_0$.

Predicted scattering observables



OT, A. Ekström, and C. Forssén, Phys. Rev. C **109**, (2024)

- Clear improvement order-by-order.
- **Sufficiently accurate** cross sections to use in inference of LECs.
- Energy-dependent accuracy.
- Hints that the breakdown scale can be as low as $\Lambda_b \sim 200 - 300$ MeV.

Computing phase shifts perturbatively

Uncoupled channels:

$$S = \exp(2i\delta)$$

$$S^{(0)} + S^{(1)} + S^{(2)} + S^{(3)} + \mathcal{O}(Q^3)$$
$$= \exp(2i[\delta^{(0)} + \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \mathcal{O}(Q^3)])$$

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$$S^{(3)} = \left[2i\delta^{(3)} - 4\delta^{(1)}\delta^{(2)} - \frac{4i}{3}(\delta^{(1)})^3\right] \exp(2i\delta^{(0)})$$

$$\implies \delta^{(\nu)}_{\text{tot}} = \delta^{(0)} + \cdots + \delta^{(\nu)} \in \mathbb{R}$$

Note: All $\delta^{(\nu)} \in \mathbb{R}$ by construction:

$$\mathbb{R} \ni \delta = \frac{1}{2i} \ln(S) \equiv f(V), \quad V \in \mathbb{R}$$

$$V(x) = \sum_{\nu=0}^{3} x^{\upsilon} V^{(\nu)} \implies \delta(x) = f(V(x))$$

$\Longrightarrow$Taylor expansion of $\delta(x)$ must be real!

Coupled channels:

$$S = \begin{pmatrix} \cos(2\epsilon)e^{2i\delta_1} & i\sin(2\epsilon)e^{i(\delta_1+\delta_2)} \\ i\sin(2\epsilon)e^{i(\delta_1+\delta_2)} & \cos(2\epsilon)e^{2i\delta_2} \end{pmatrix}$$

$$S = \sum_{\nu=0}^{\infty} = S^{(\nu)}, \quad \delta_1 = \sum_{\nu=0}^{\infty} \delta_1^{(\nu)}, \quad \delta_2 = \sum_{\nu=0}^{\infty} \delta_2^{(\nu)}, \quad \epsilon = \sum_{\nu=0}^{\infty} \epsilon^{(\nu)}$$

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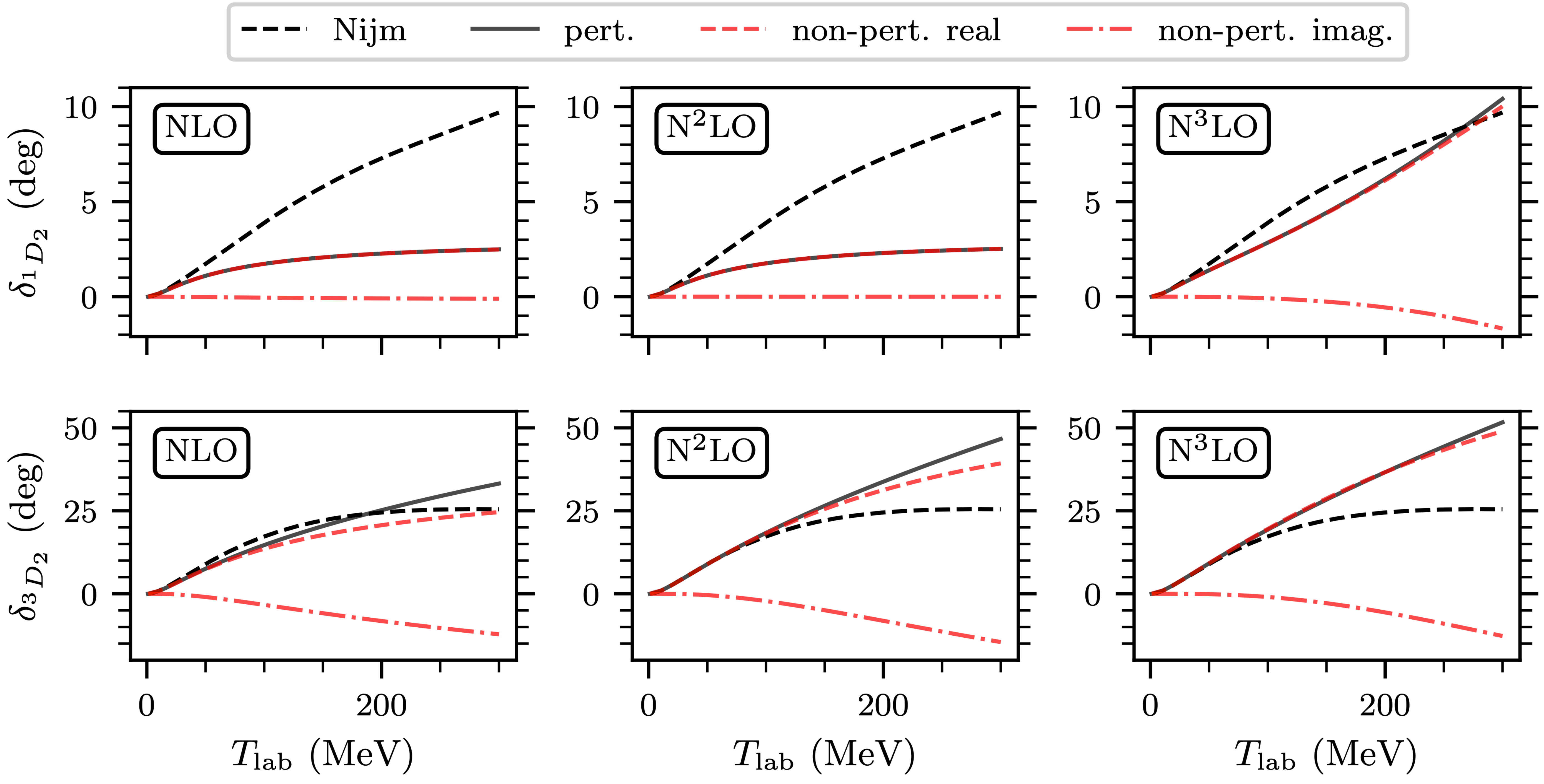

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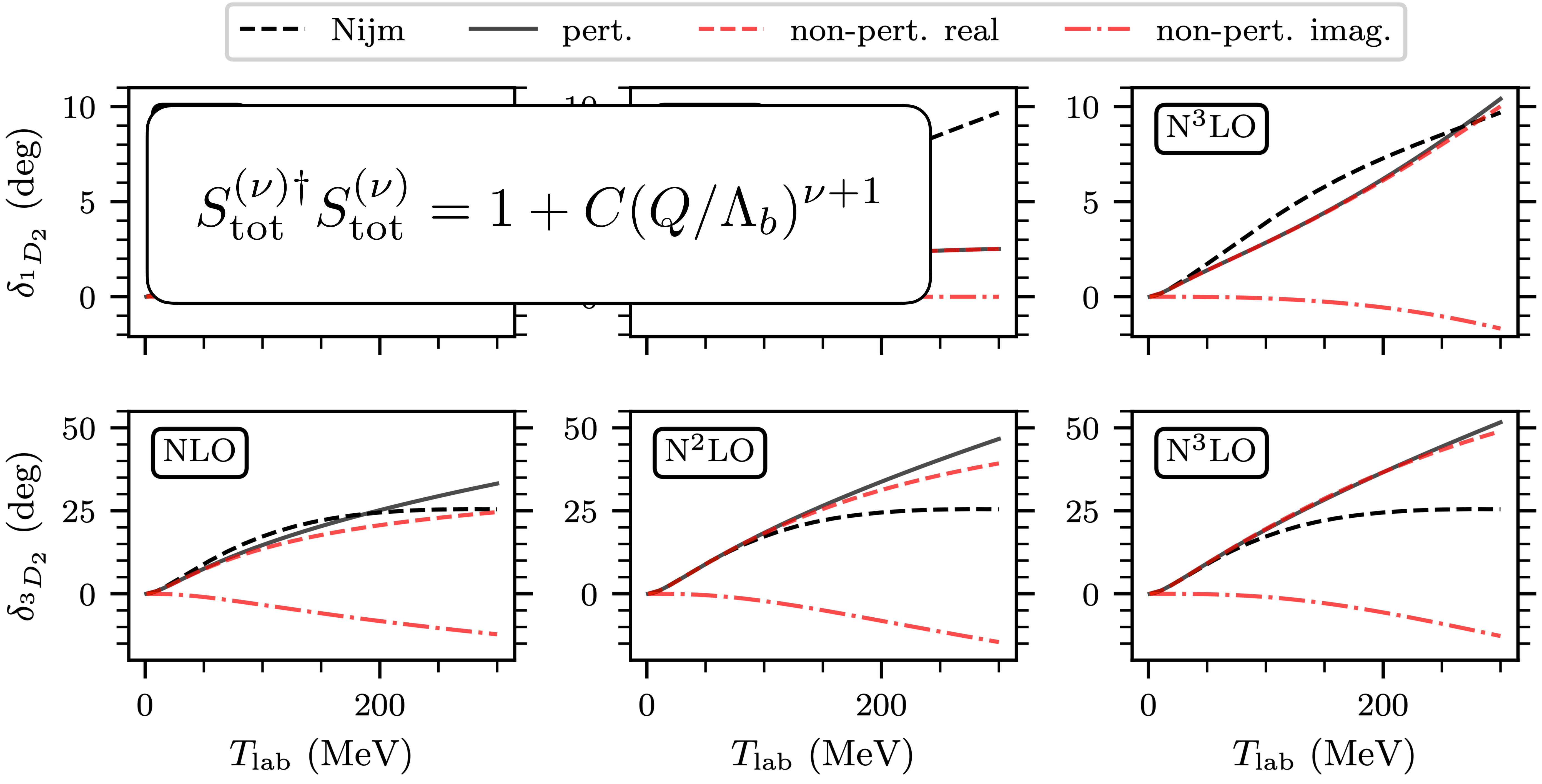
$$S^{(\nu)\dagger}_{\text{tot}} S^{(\nu)}_{\text{tot}} = 1 + C(Q/\Lambda_b)^{\nu+1}$$



Perturbative unitarity breaking

In each channel:

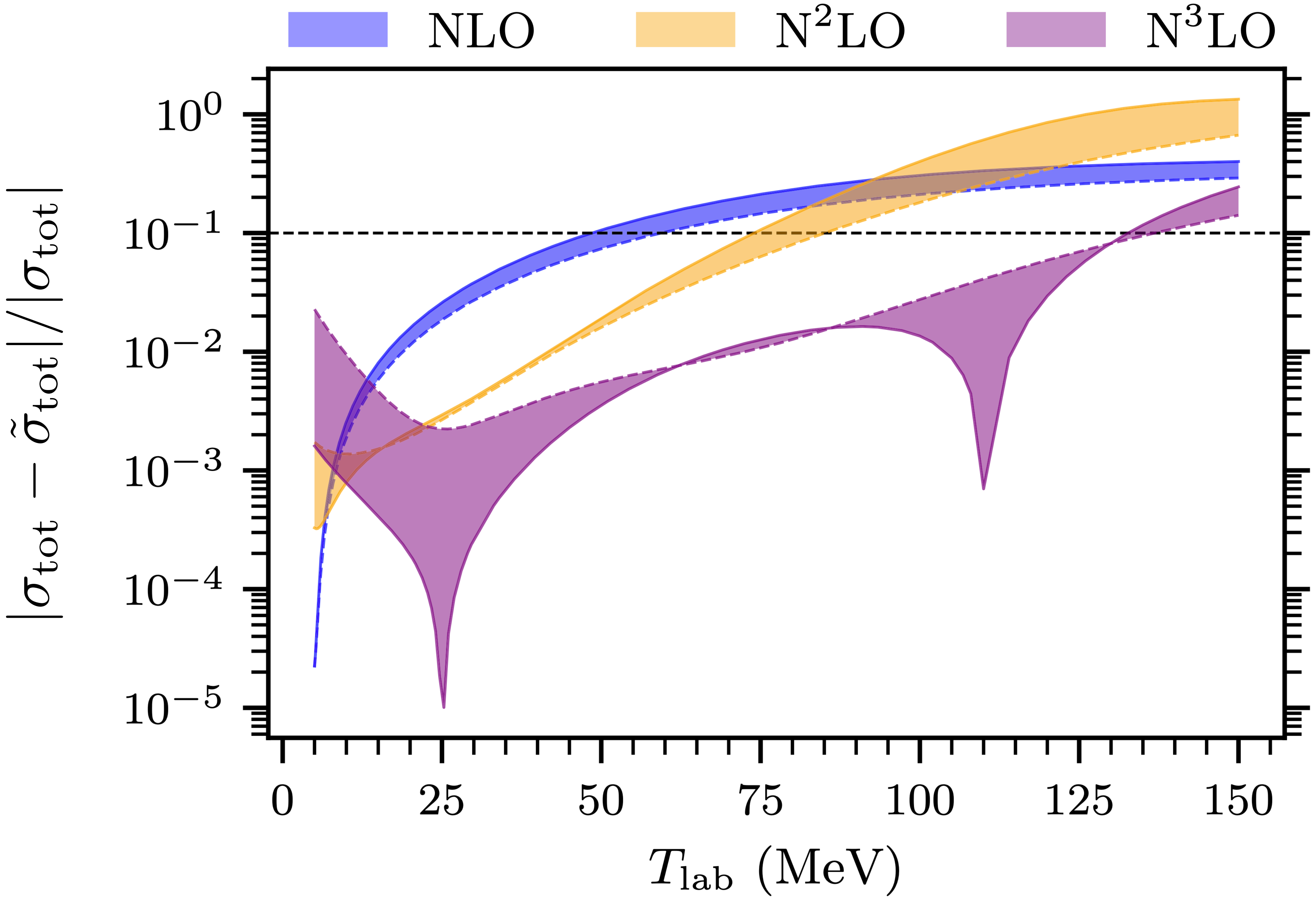
$$S^{(\nu)\dagger}_{\rm tot} S^{(\nu)}_{\rm tot} = 1 + C(Q/\Lambda_b)^{\nu+1}$$

Gauge the effect on observables:

$$\sigma_{\rm tot}(p_0) = 2\pi \int_{-1}^{1} d(\cos\theta_{\rm cm})\, \frac{d\sigma}{d\Omega}(p_0, \theta_{\rm cm})$$

Compare to using the optical theorem, which is exact when the amplitude is unitary:

$$\tilde{\sigma}_{\rm tot}(p_0) = \frac{2\pi}{p_0}\, \mathrm{Im}\left[a(\theta_{\rm cm}=0) + b(\theta_{\rm cm}=0)\right]$$



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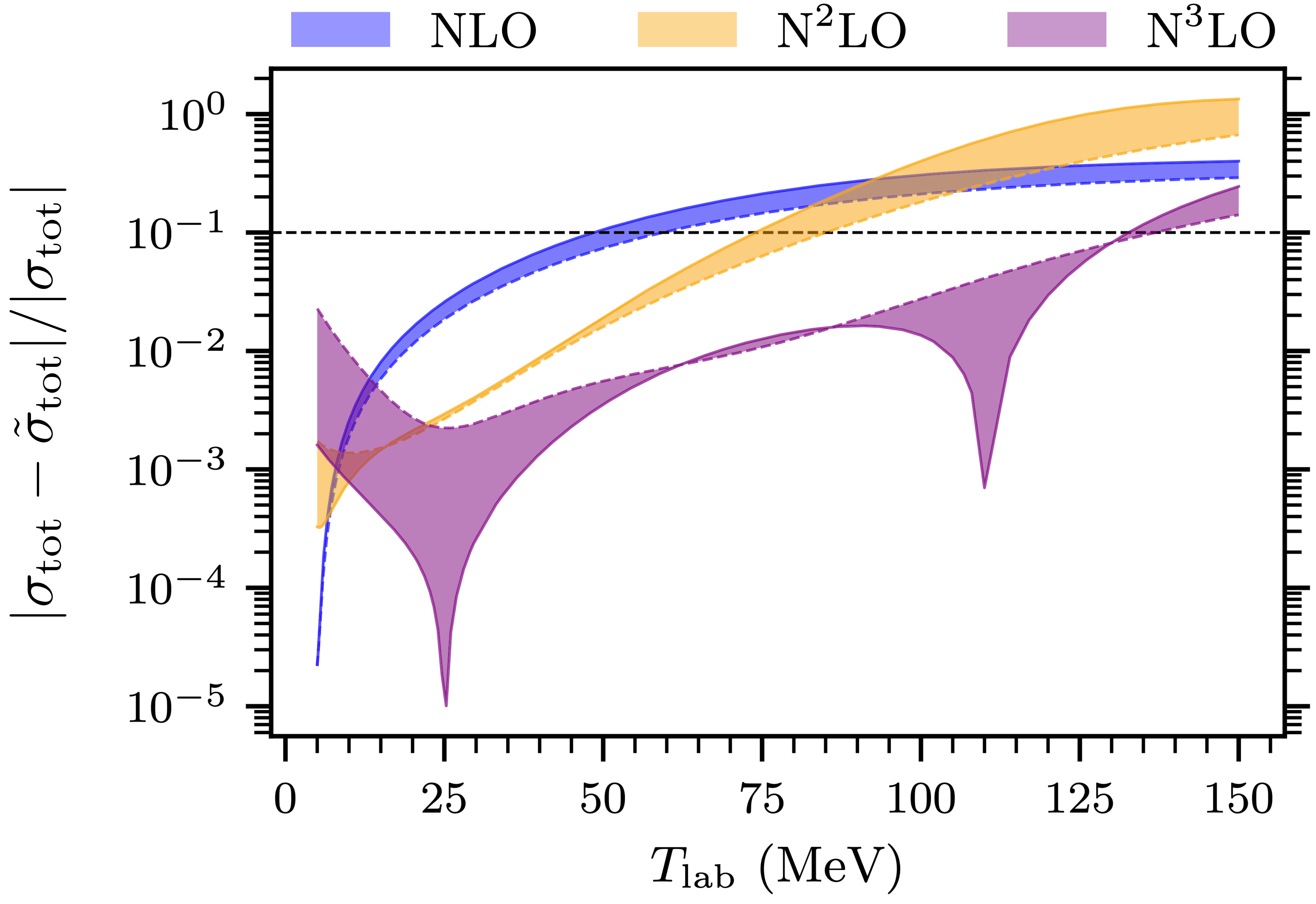

10 % error at energies:

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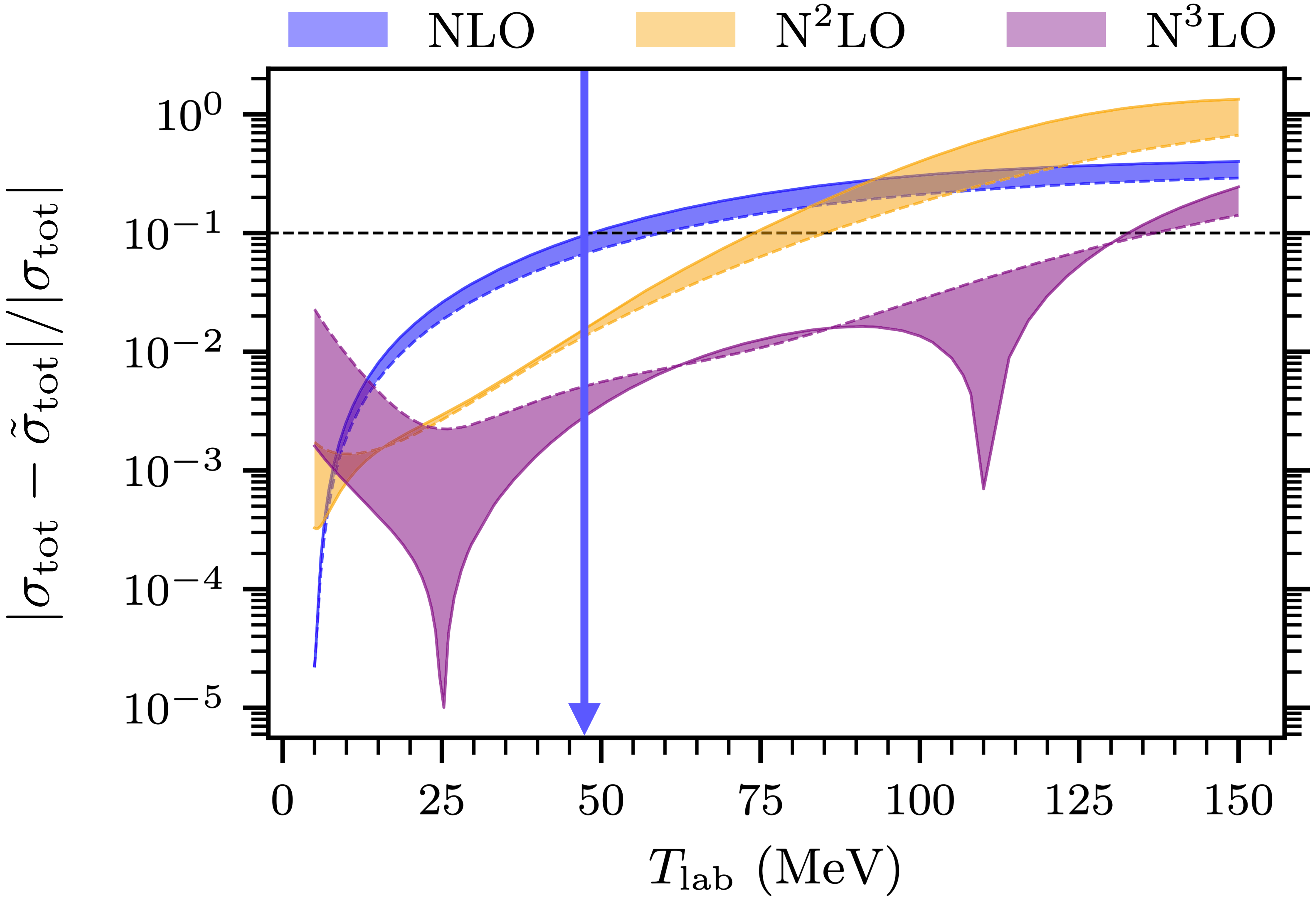

(NLO)

10 % error at energies: 45 MeV

Perturbative unitarity breaking

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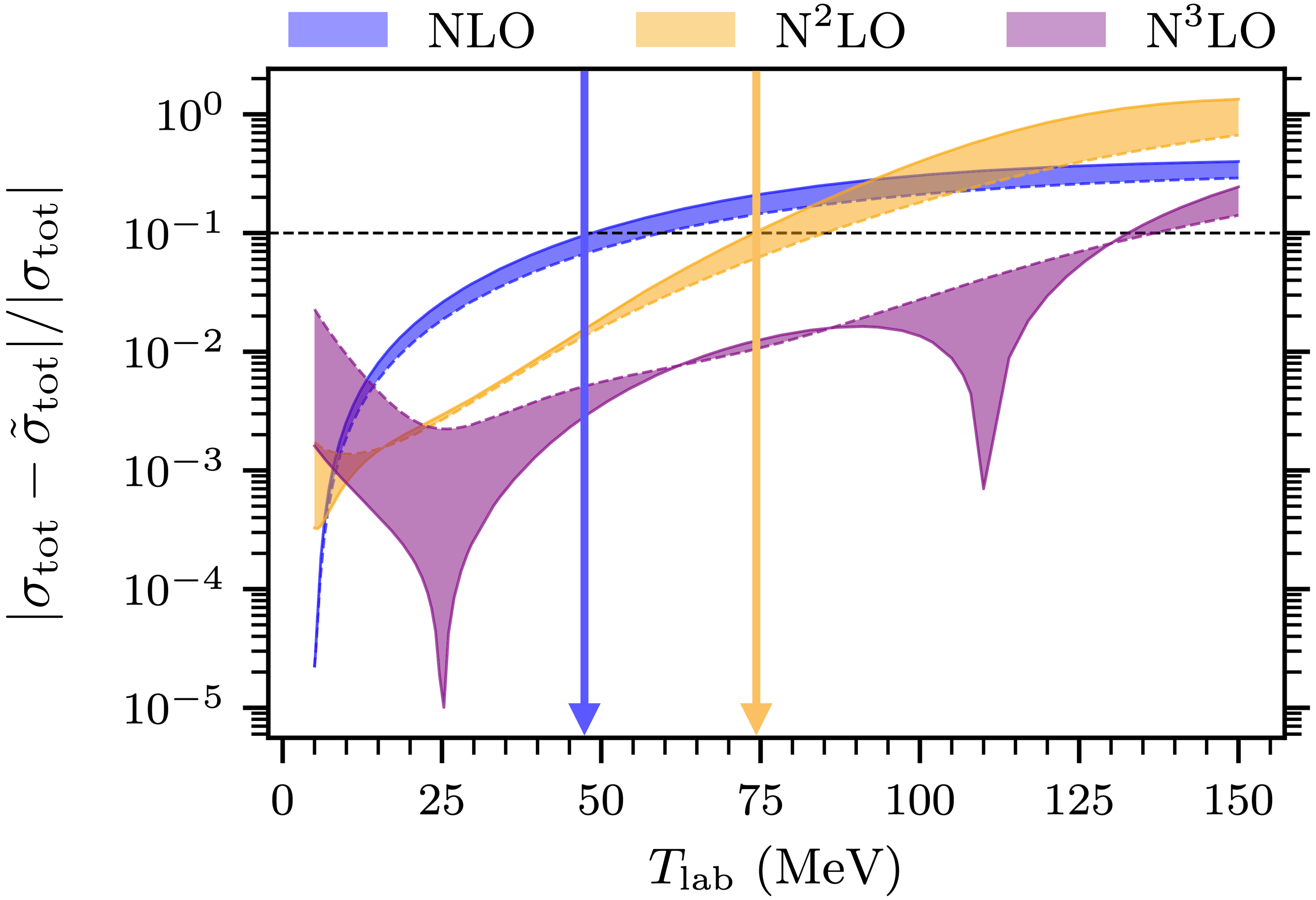

	(NLO)	(N2LO)
10 % error at energies:	45 MeV	75 MeV

Perturbative unitarity breaking

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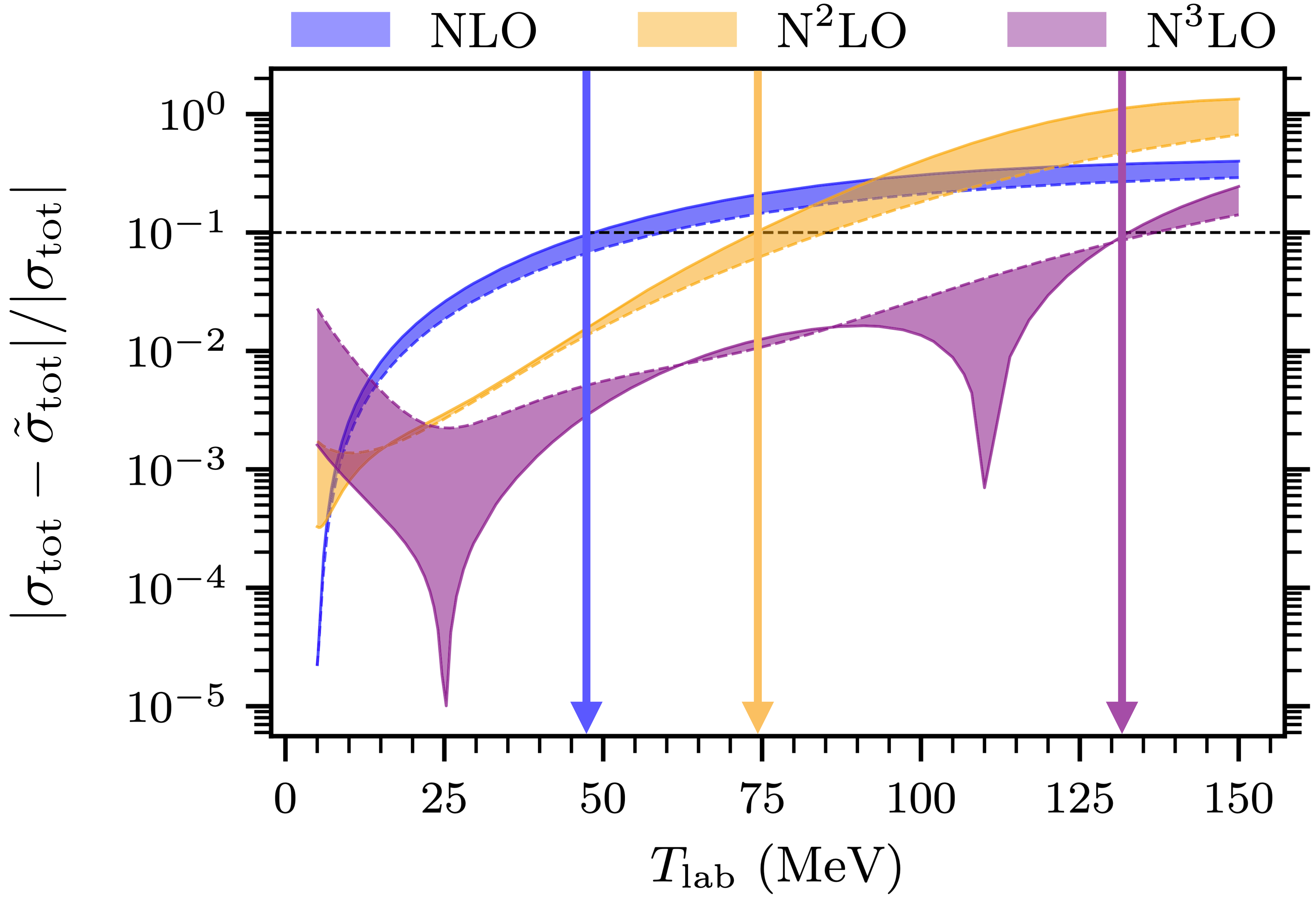

	(NLO)	(N2LO)	(N3LO)
10 % error at energies:	45 MeV	75 MeV	135 MeV

Perturbative unitarity breaking

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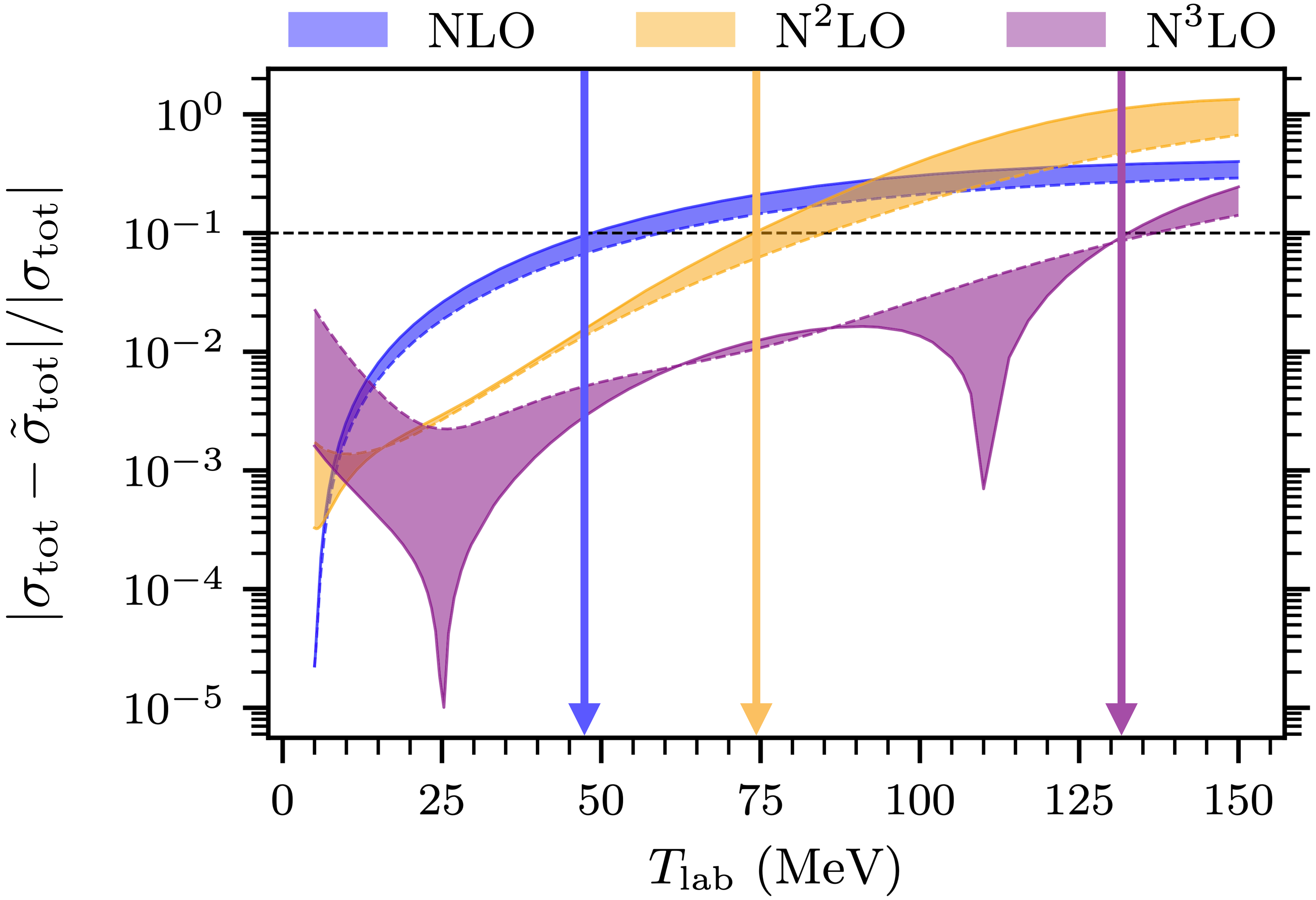

	(NLO)	(N2LO)	(N3LO)
10 % error at energies:	45 MeV	75 MeV	135 MeV

- What can be gained from connecting perturbative unitarity breaking and EFT errors?

Low-energy theorems (LETs)

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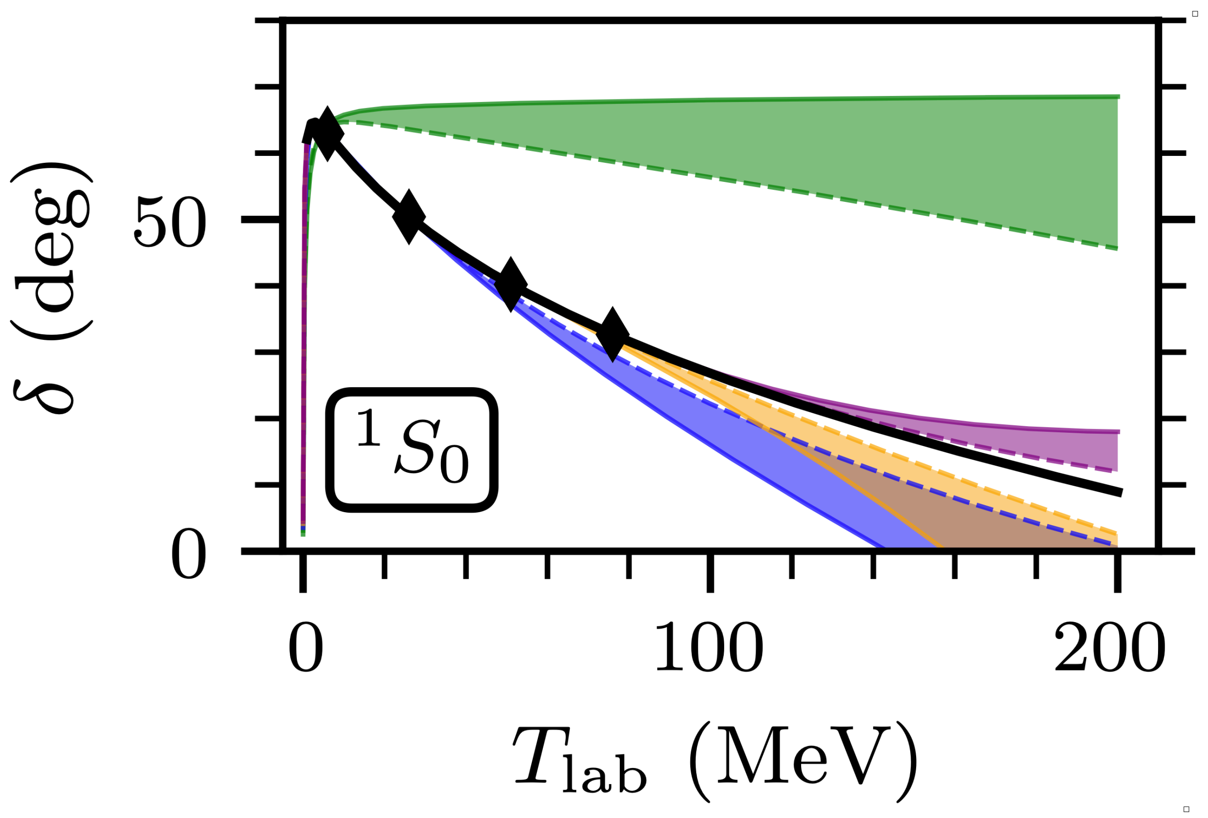
- Is pion dynamics being treated properly?
- LET: Predicted higher-order coefficients in the effective-range expansion.

$$F(k) \equiv k \cot \delta(k) = \underbrace{-\frac{1}{a} + \frac{1}{2} r k^2}_{\text{Fit}} + \underbrace{v_2 k^4 + v_3 k^6 + v_4 k^8}_{\text{Predict}} + \mathcal{O}\left(k^{10}\right)$$

- Predictions are clear indicators of correctly captured pion dynamics.



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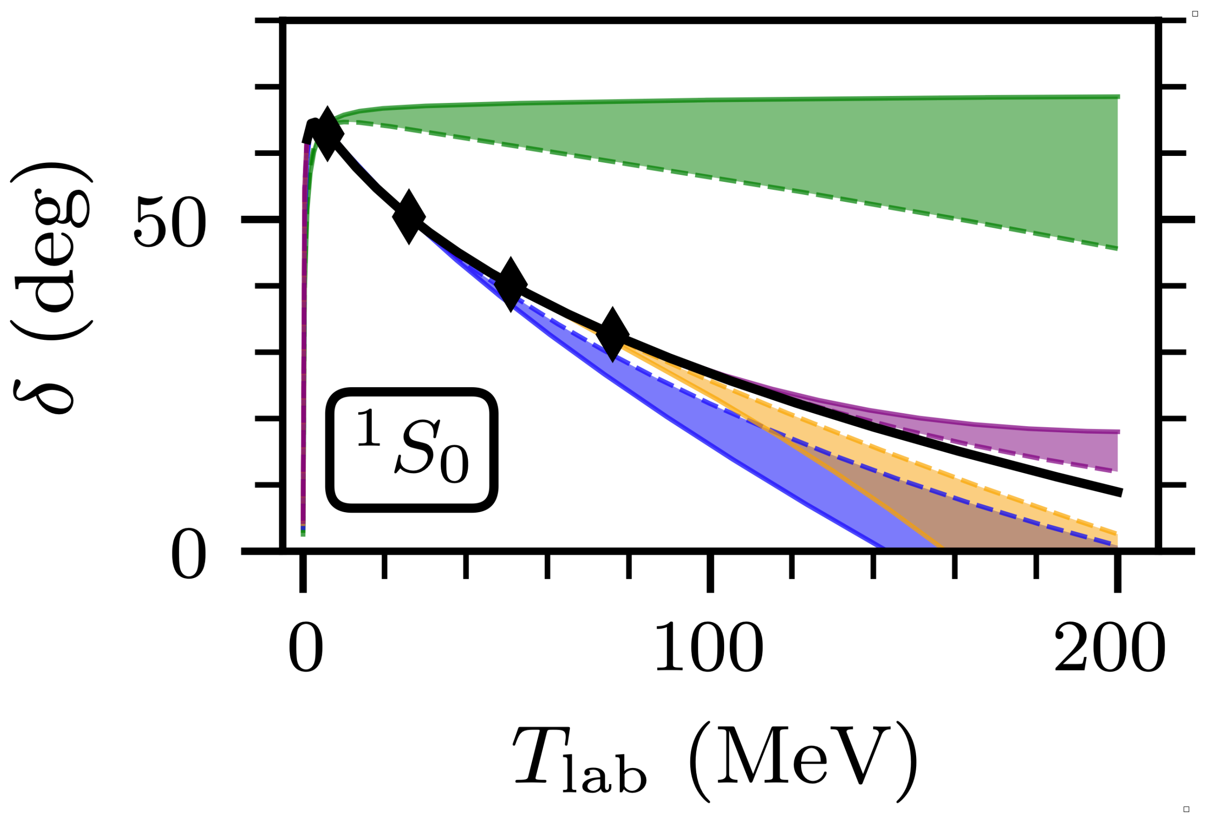
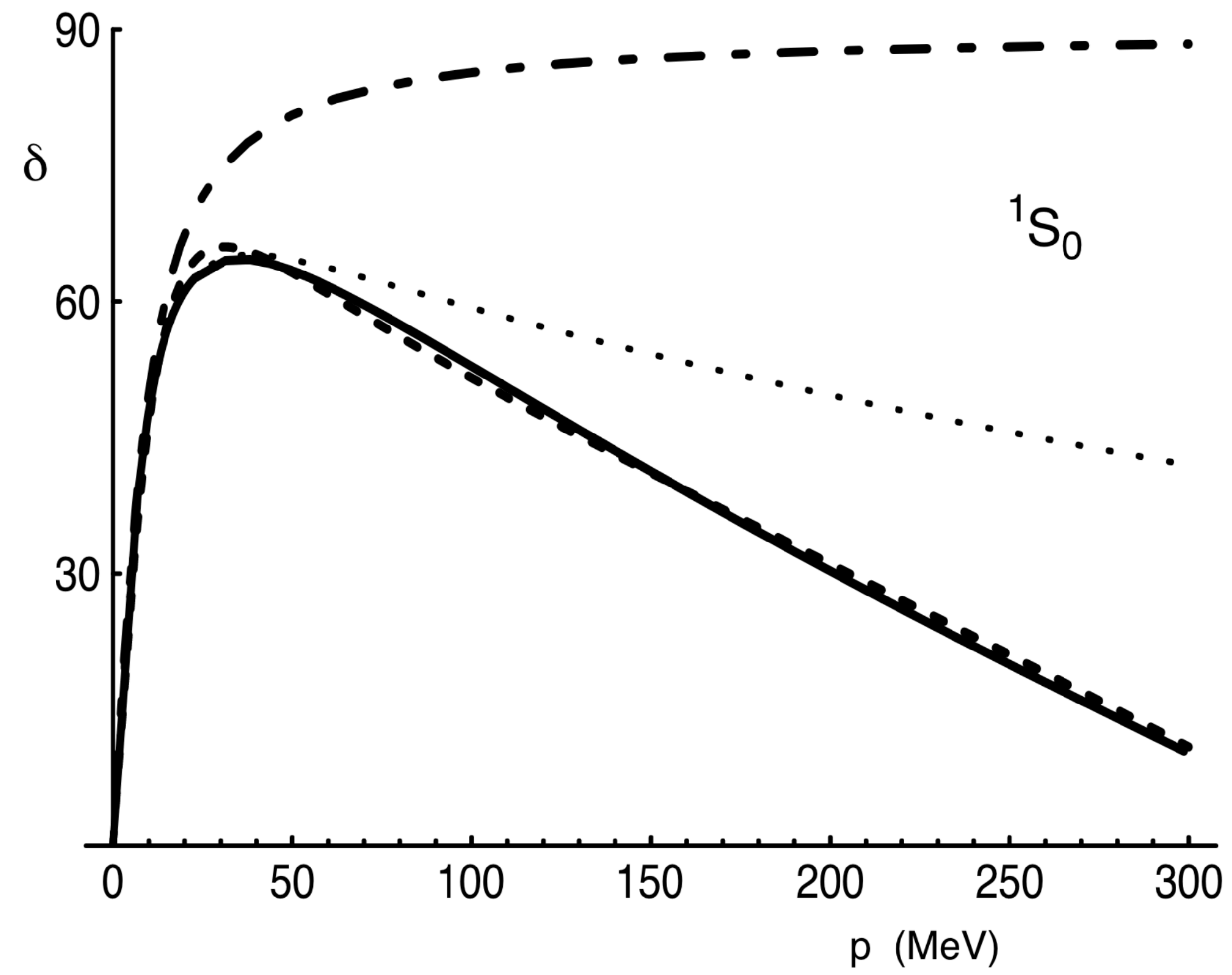


- Predictions are clear indicators of correctly captured pion dynamics.
  - Cohen and Hansen: Analytical expressions for $v_{2,3,4}$.

$$v_2 = \frac{g_A^2 M}{16\pi f_\pi^2}\left(-\frac{16}{3a^2 m_\pi^4} + \frac{32}{5a\, m_\pi^3} - \frac{2}{m_\pi^2}\right)$$



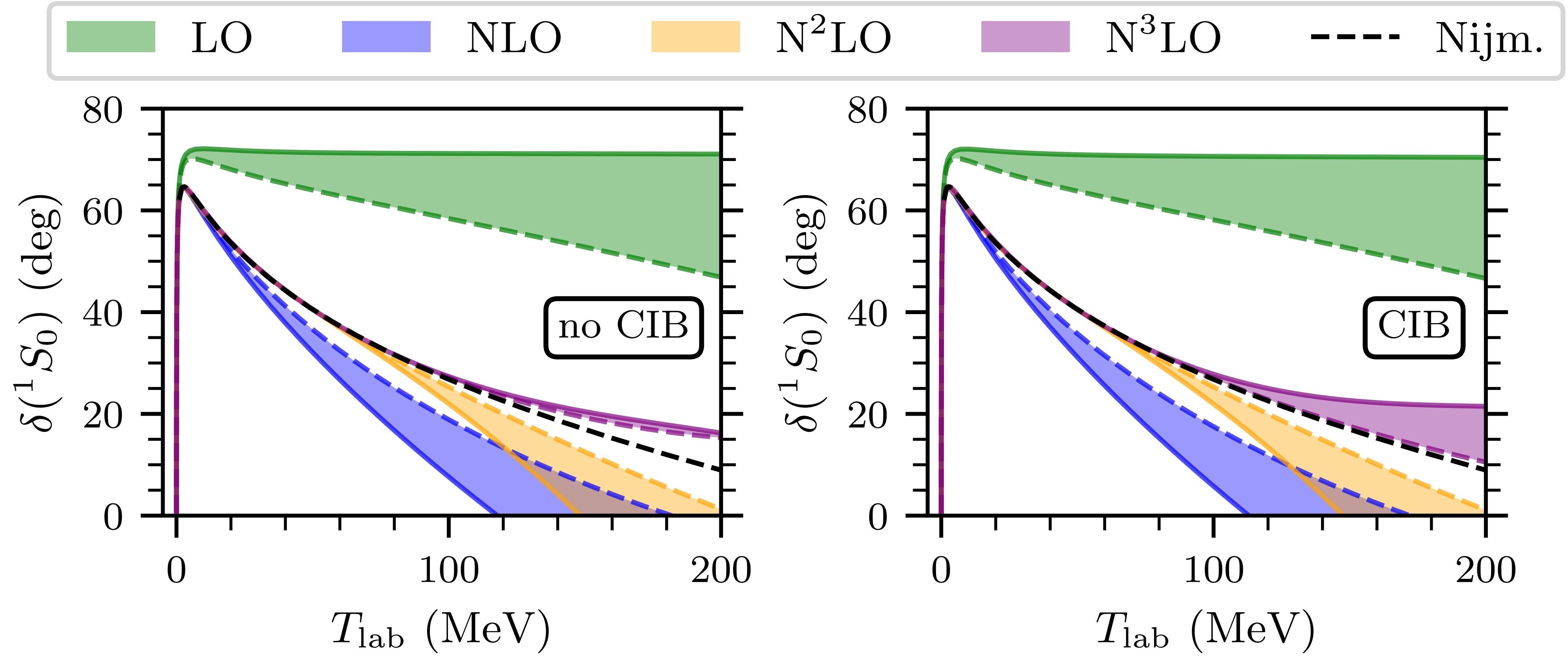
D.B. Kaplan *et al.*, Nucl. Phys. B **534** (1998)

δ ($^1S_0$ channel)	v_2 (fm^3)	v_3 (fm^5)	v_4 (fm^7)
low energy theorem	-3.3	17.8	-108.0
partial wave analysis	-0.48	3.8	-17.0
δ ($^3S_1$ channel)	v_2 (fm^3)	v_3 (fm^5)	v_4 (fm^7)
low energy theorem	-0.95	4.6	-25.0
partial wave analysis	0.04	0.67	-4.0
ϵ ($^3S_1$–$^3D_1$ mixing)	g_1 (fm^3)	g_2 (fm^5)	g_3 (fm^7)
low energy theorem	3.9	-86.0	1.8×10^3
partial wave analysis	1.7	-26.0	2.2×10^2

T.D. Cohen, J.M. Hansen, Phys. Rev. C **59**, (1999)

Low-energy theorems: $^1S_0$

Phase shifts in $^1S_0$



$$F(k) \equiv k \cot \delta(k) = -\frac{1}{a} + \frac{1}{2} r k^2 + v_2 k^4 + v_3 k^6 + v_4 k^8 + \mathcal{O}(k^{10})$$

$$F(k) - ik = -\frac{2}{\pi m_N T^{(0)}} \left[1 - \frac{T^{(1)}}{T^{(0)}} + \left(\left[\frac{T^{(1)}}{T^{(0)}} \right]^2 - \frac{T^{(2)}}{T^{(0)}} \right) + \left(2 \frac{T^{(1)} T^{(2)}}{(T^{(0)})^2} - \frac{T^{(3)}}{T^{(0)}} - \left[\frac{T^{(1)}}{T^{(0)}} \right]^3 \right) + \mathcal{O}\left(\frac{Q^4}{\Lambda_b^4} \right) \right].$$

Predicted effective range parameters (LETs)

$^1S_0$ partial wave	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
Empirical (Ref. [27])	−23.735(16)	2.68(3)	−0.48(2)	3.9(1)	−19.6(5)
$\Lambda = 500$ MeV, (no CIB)					
LO	*	1.71(0)	−1.77(0)	8.54(0)	−47.0(3)
NLO	*	*	−0.64(0)	4.79(0)	−29.9(2)
N^{2}LO	*	2.72(0)	−0.71(0)	5.05(0)	−29.3(2)
N^{3}LO	*	2.69(0)	−0.66(0)	5.42(0)	−31.0(2)
$\Lambda = 500$ MeV, (CIB)					
LO	*	1.68(0)	−1.55(0)	6.63(0)	−31.64(8)
NLO	*	*	−0.45(0)	3.42(0)	−18.95(8)
N^{2}LO	*	2.70(0)	−0.55(0)	3.77(0)	−18.8(2)
N^{3}LO	*	2.68(0)	−0.50(0)	4.02(0)	−19.8(2)

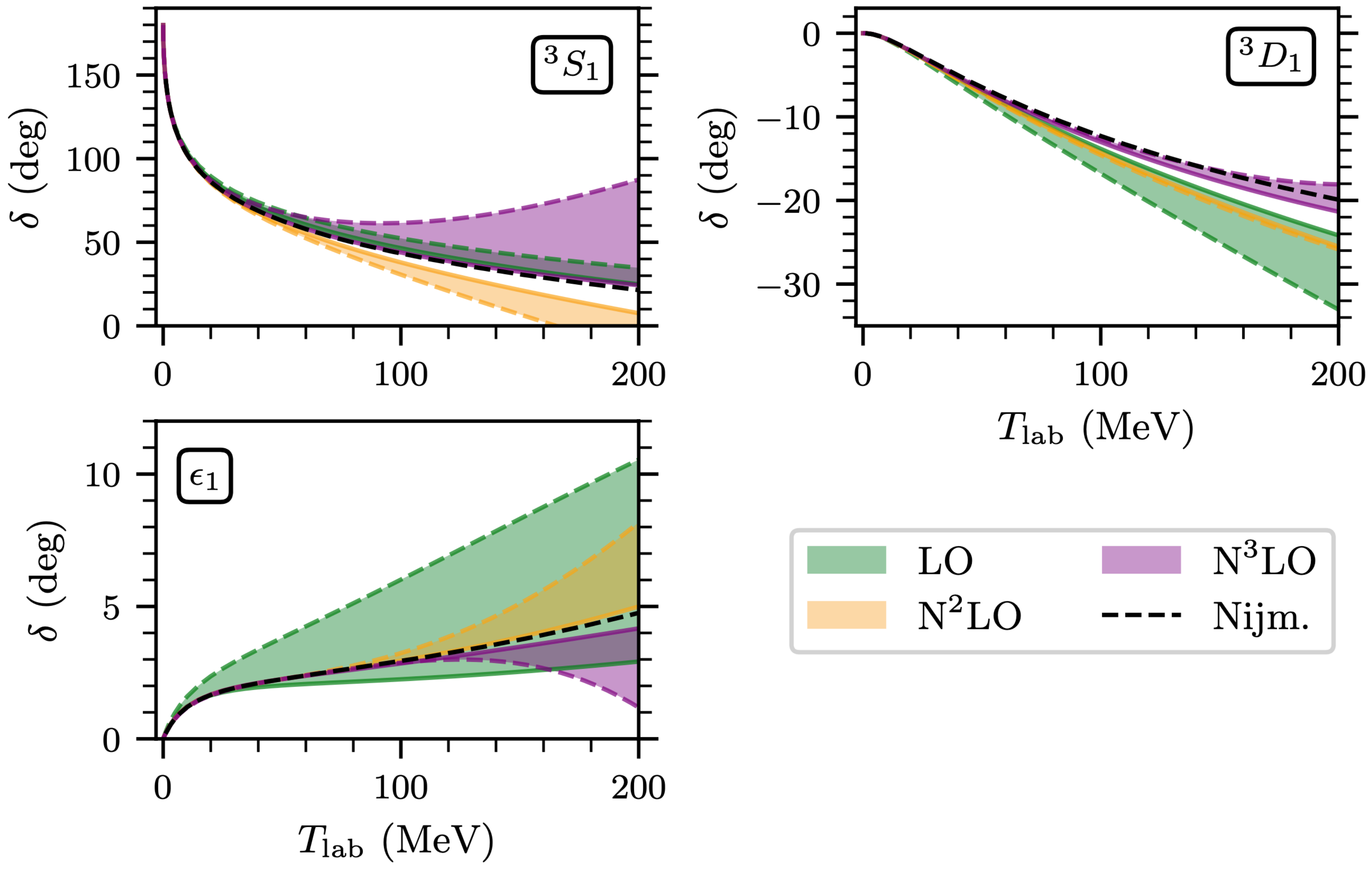
- CIB in one-pion exchange is **significant** in $^1S_0$.

✓ Both phase shift and LETs are accurate.

OT, Few-Body Syst. 65, 69 (2024)

Low-energy theorems: $^3S_1$

Phase shifts in $^3S_1$— $^3D_1$



Predicted effective range parameters (LETs)

$^3S_1$ partial wave	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
Empirical (Ref. [27])	5.42	1.75	0.045	0.67	−3.94
$\Lambda = 500$ MeV					
LO	*	1.58(0)	−0.10(0)	0.89(0)	−5.5(2)
N^{2}LO	*	*	0.14(0)	0.80(0)	−4.2(2)
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N^{2}LO	*	*	0.09(0)	0.74(0)	−4.2(7)
N^{3}LO	*	*	0.04(0)	0.67(2)	−4.0(9)

- CIB in one-pion exchange is **not** significant in $^3S_1$.
- Cutoff independence for $\Lambda \gtrsim 750$ MeV.

✓ Both phase shift and LETs are accurate, and improved for high cutoffs.

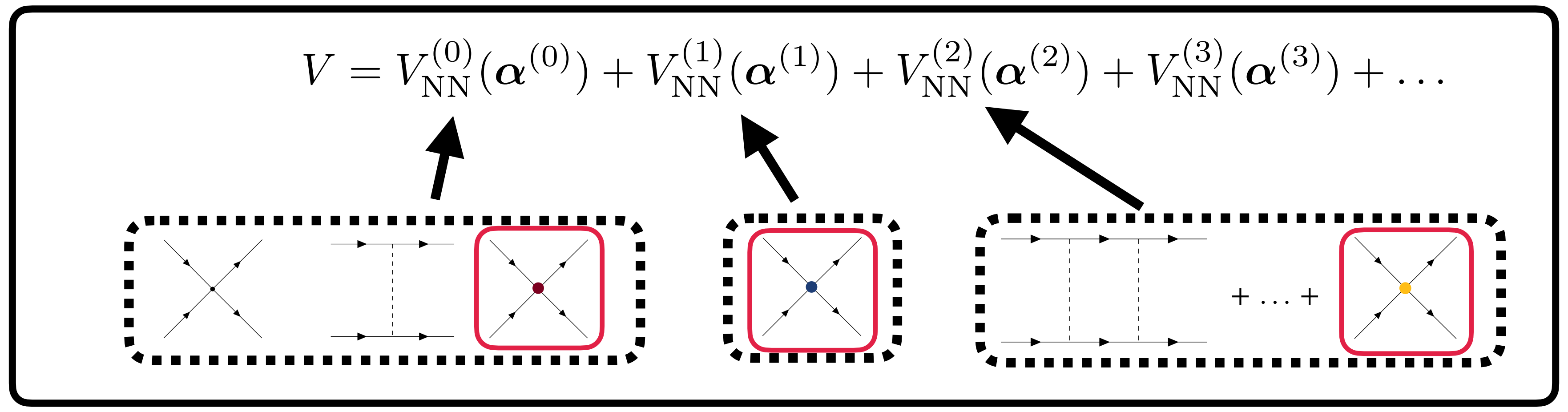
OT, Few-Body Syst. 65, 69 (2024)

Summary

Summary

- Modified PC:

Summary



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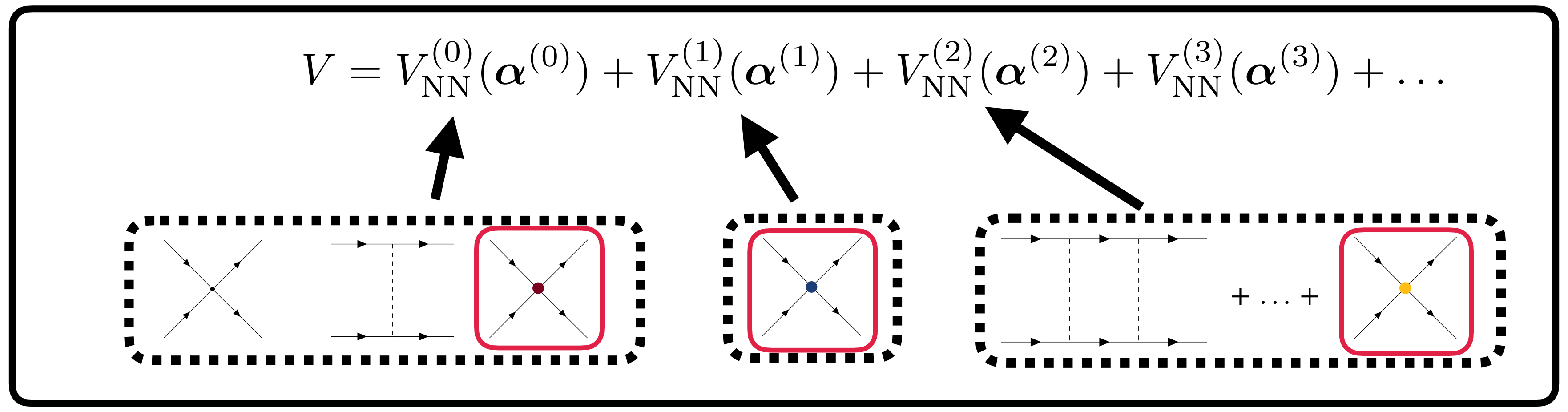

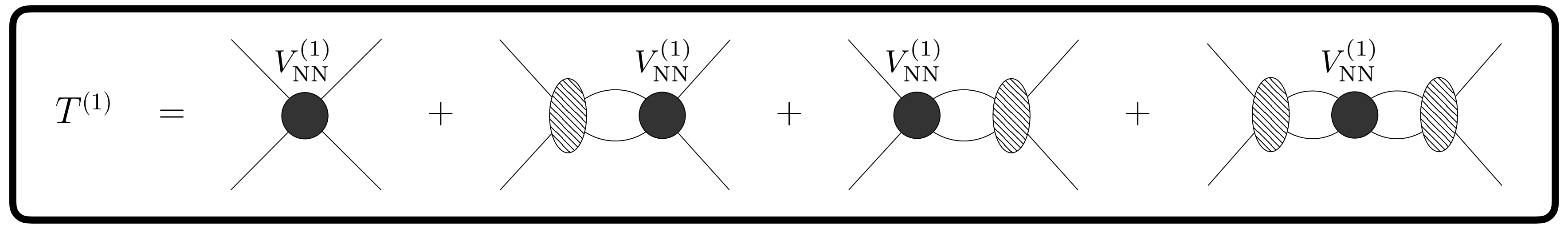

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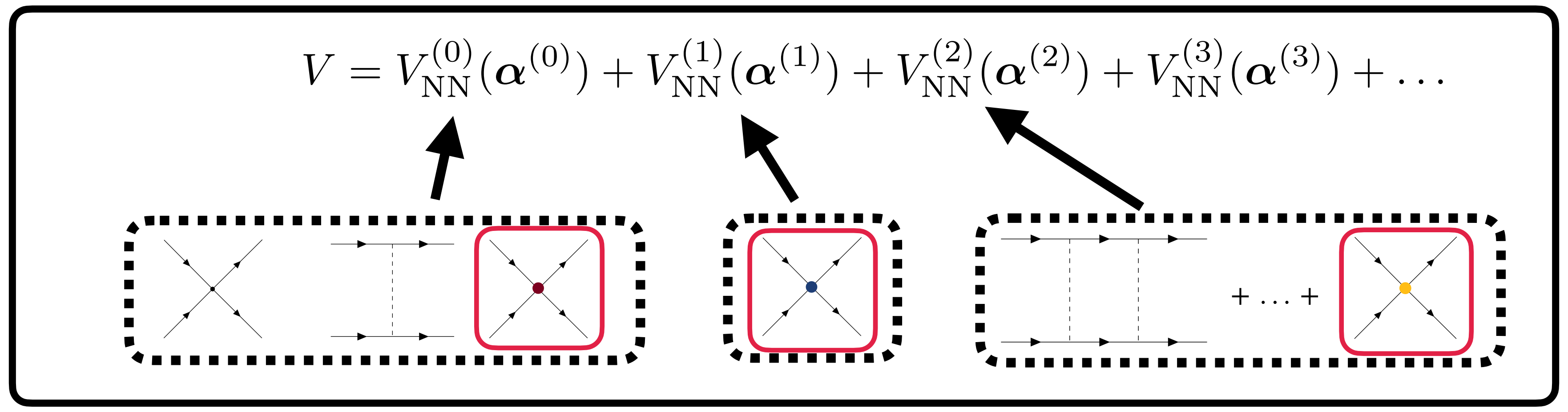

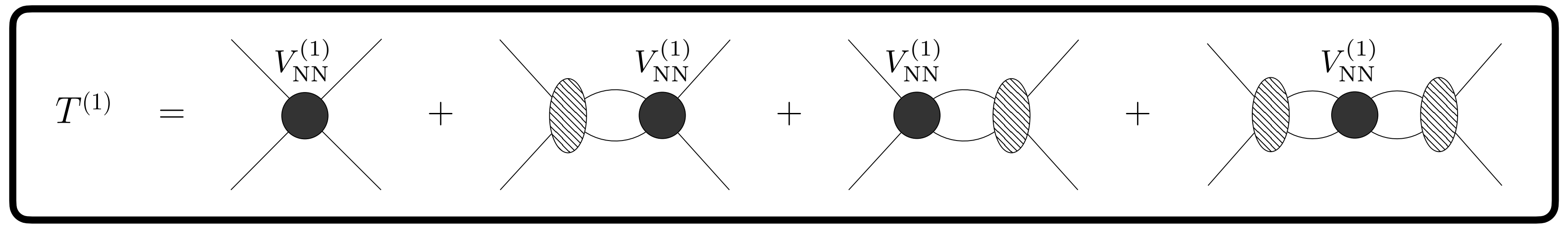

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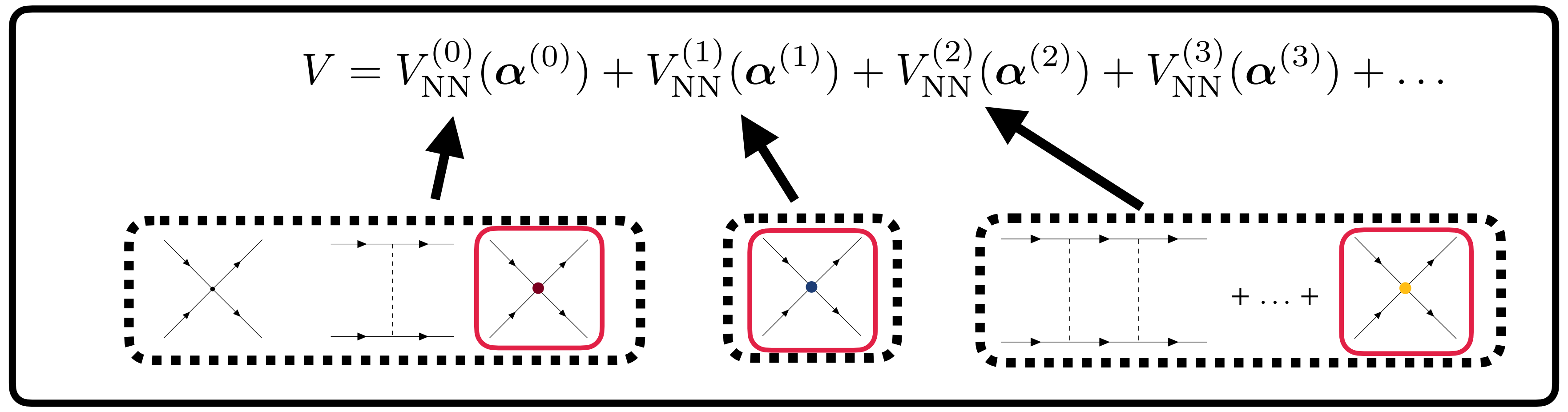

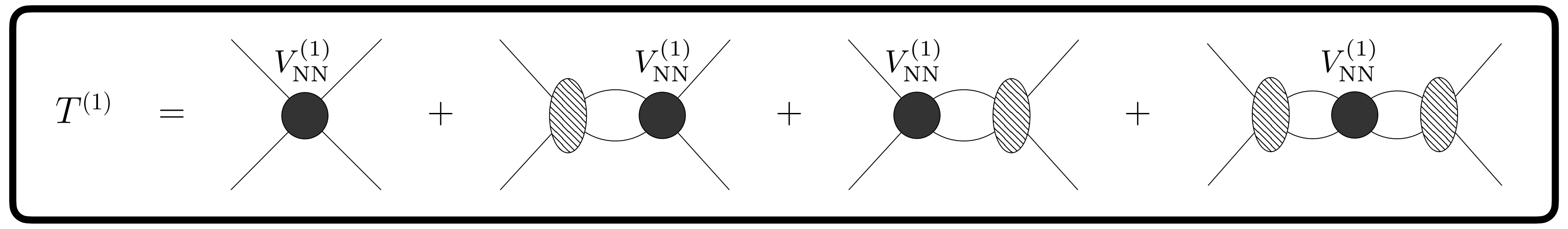

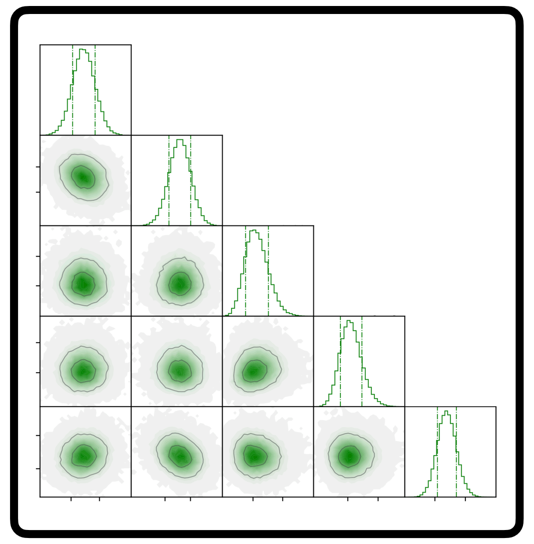

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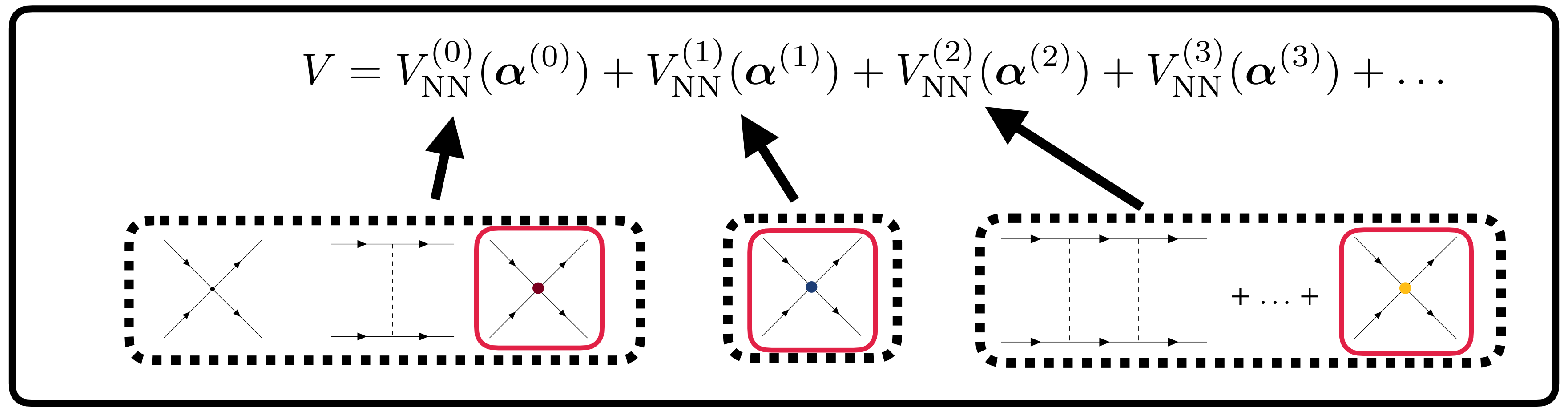
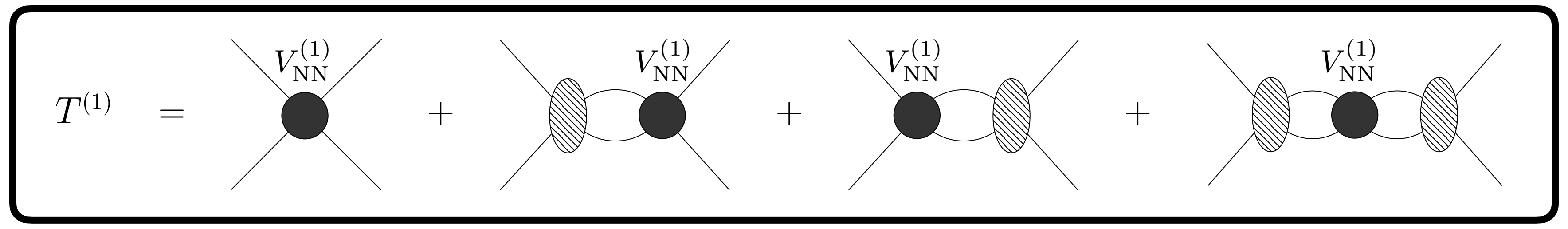
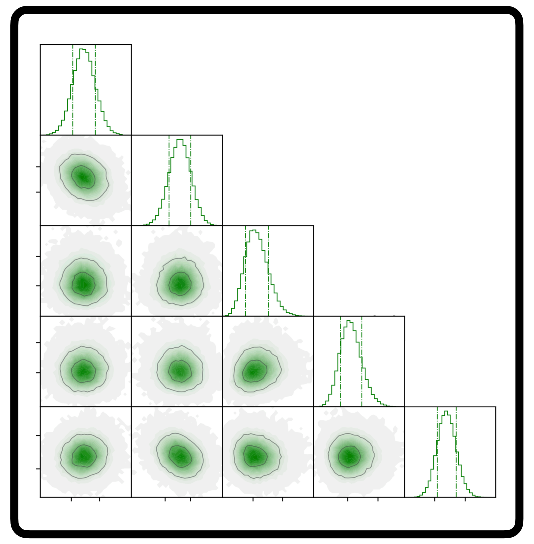
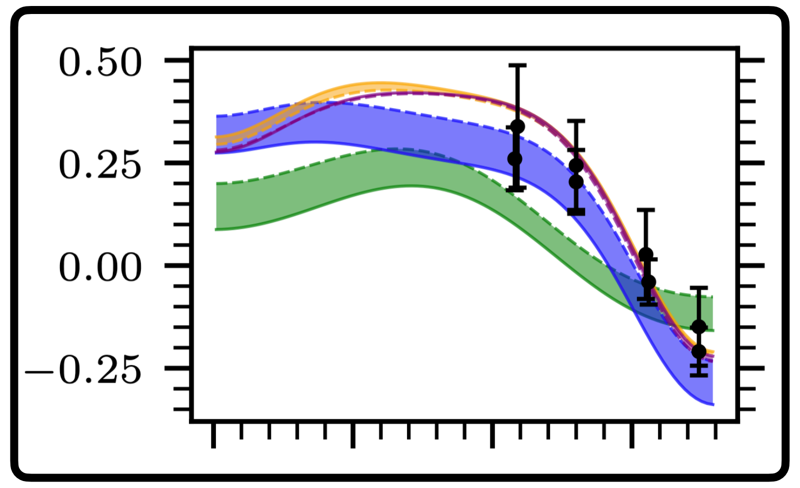
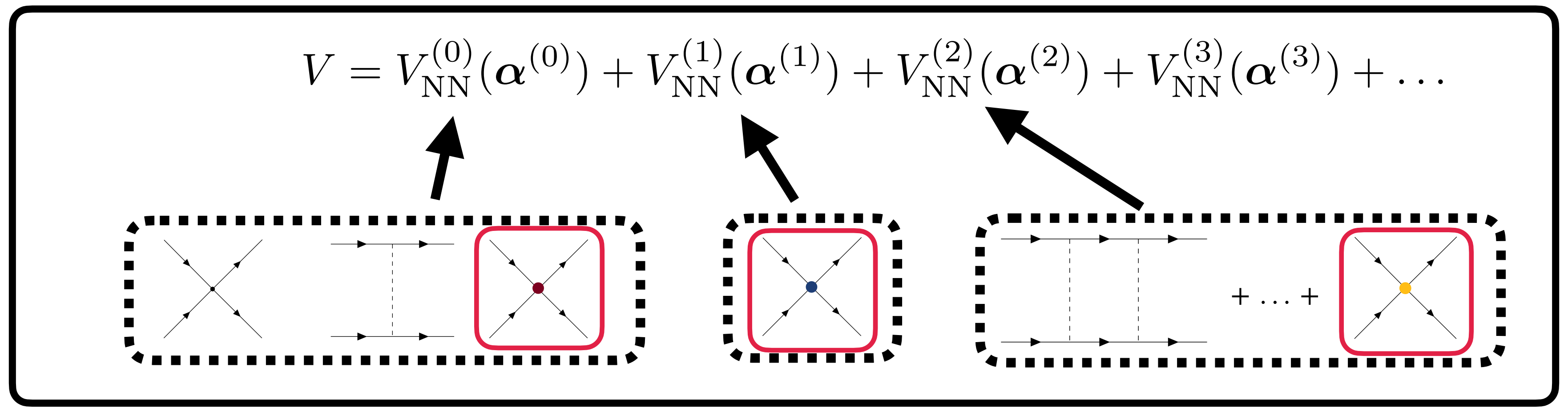








Summary



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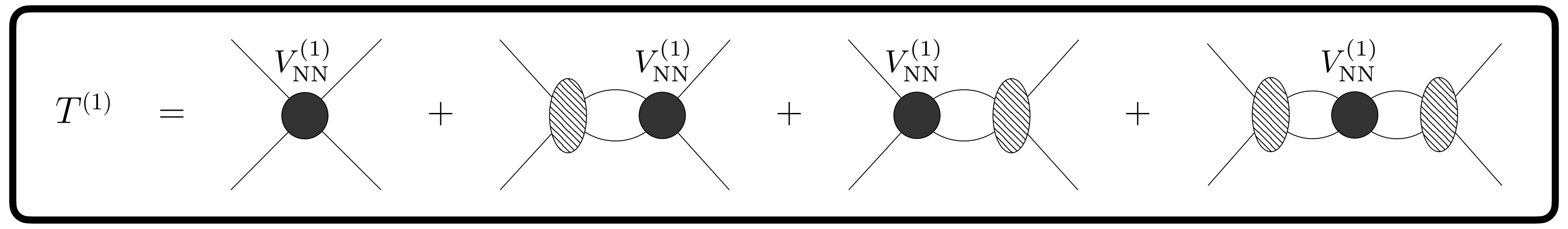

- We have found:
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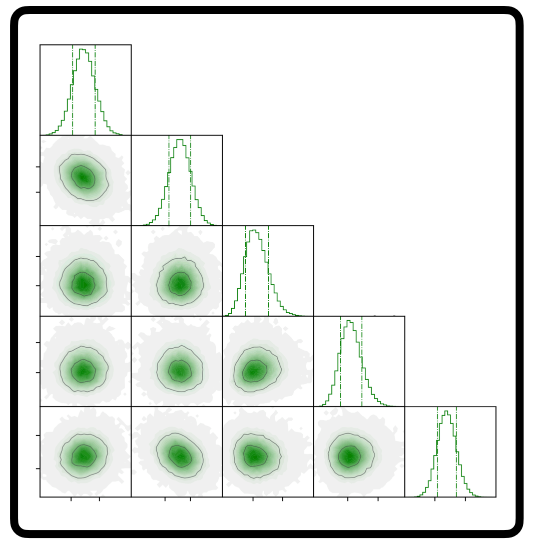
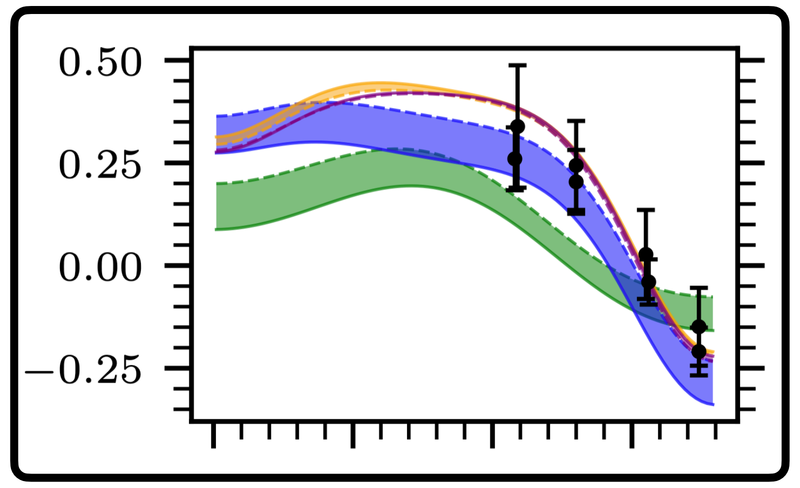
  - Accurate description of *np* scattering up to 100 MeV at N^3LO.
  - Satisfactory low-energy behavior of amplitudes.





$$F(k) \equiv k \cot \delta(k) = -\frac{1}{a} + \frac{1}{2} r k^2 + v_2 k^4 + v_3 k^6 + v_4 k^8 + \mathcal{O}\left(k^{10}\right)$$

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Thank you!

Discussion points

1. What are the advantages/disadvantages of treating sub-leading interactions in perturbation theory for light and heavy systems respectively?

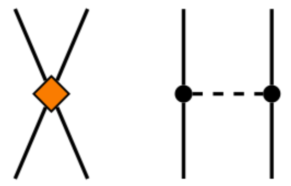
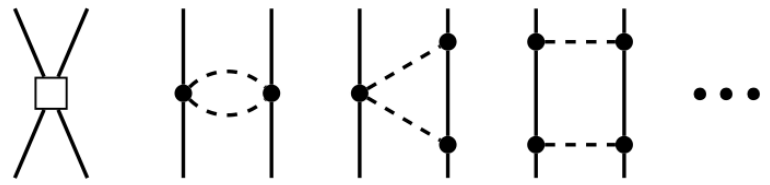
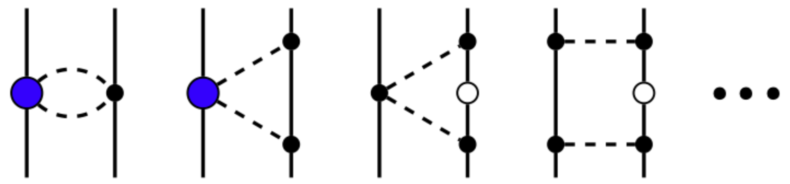
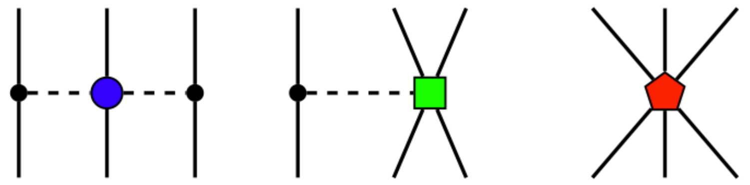
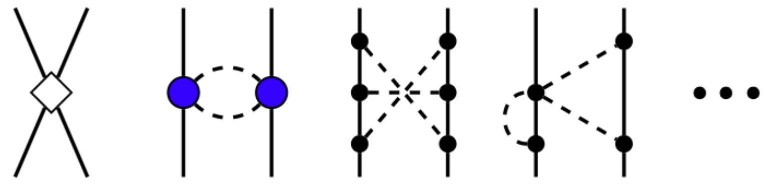
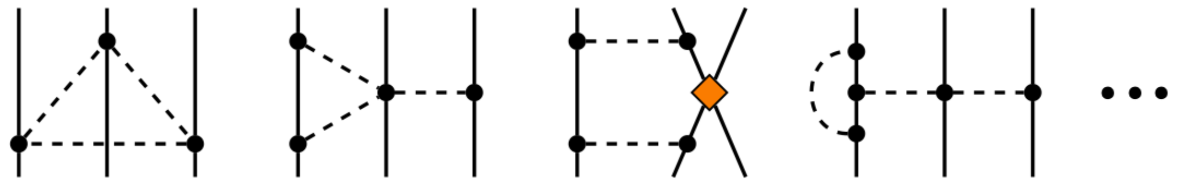
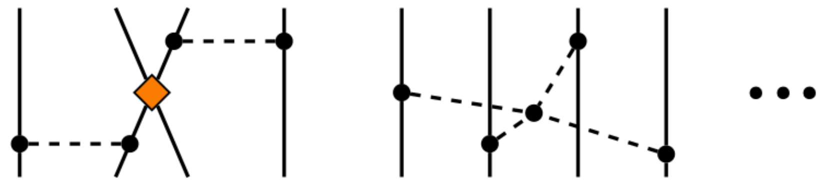
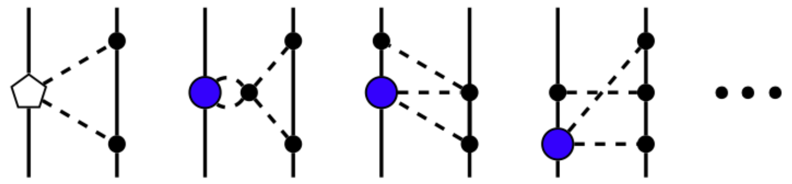
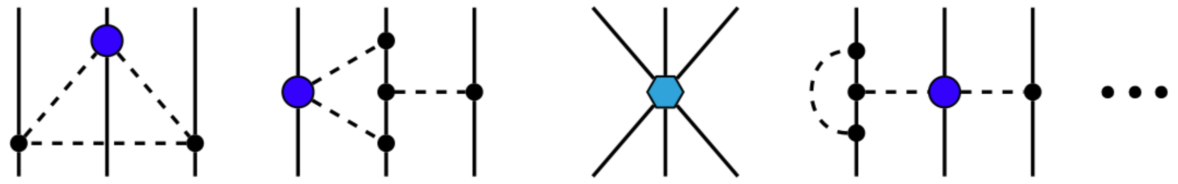
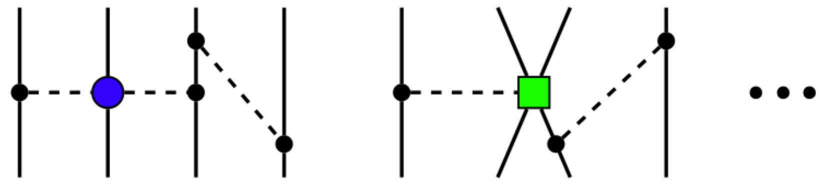
2. Can perturbative computations of observables contribute to learning about the LO interaction?

3. What can be gained from connecting perturbative unitarity breaking and EFT errors?

4. How much should including Δ's affect the breakdown scale in χEFT?

Extra slides

Weinberg PC

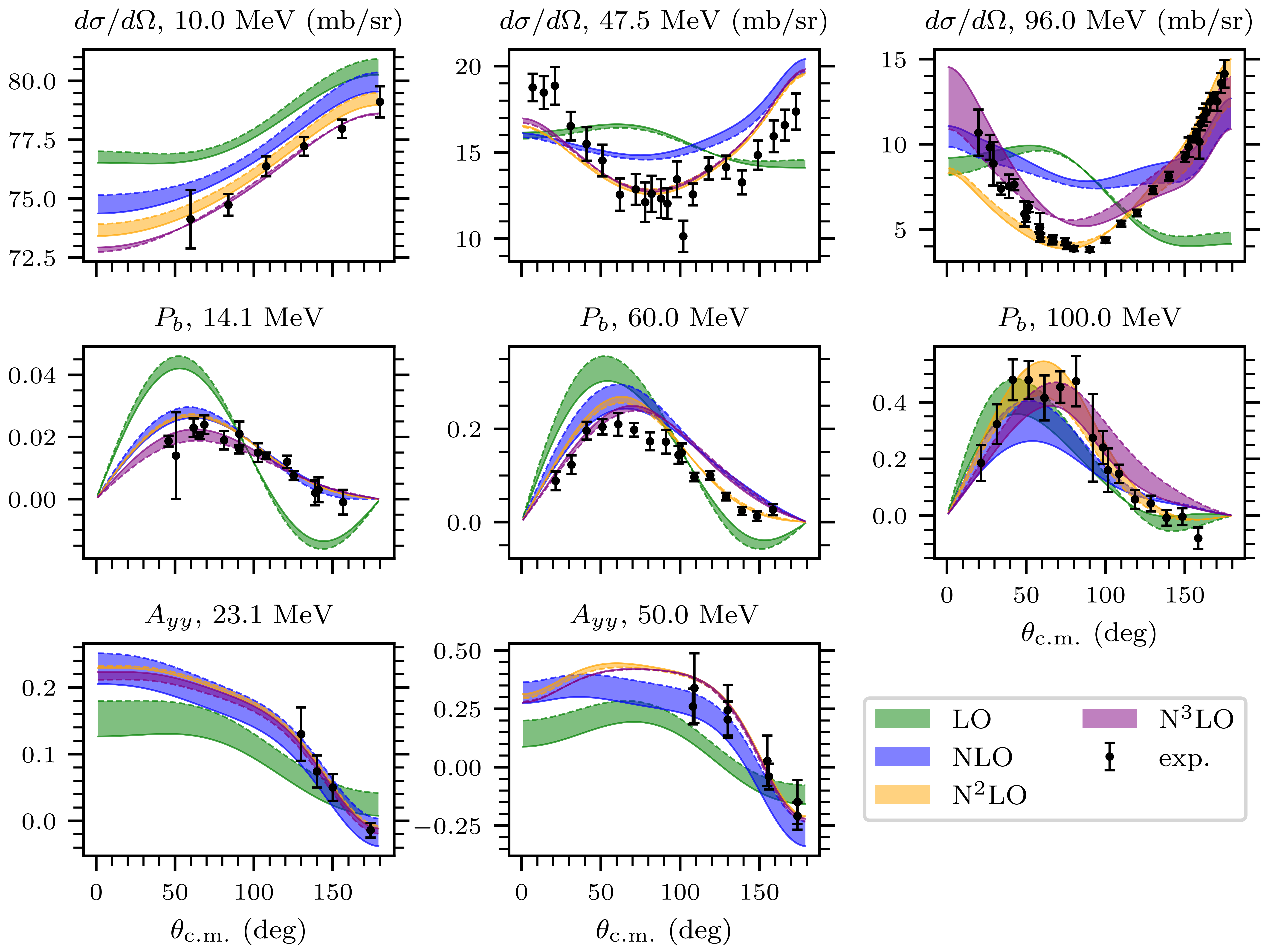
	NN	3N	4N
LO $\mathcal{O}(Q^0/\Lambda_b^0)$	1990 [151,152] 2 	—	—
NLO $\mathcal{O}(Q^2/\Lambda_b^2)$	1992 [164,165] 7 	1992,1994 [166-169] —	—
N^{2}LO $\mathcal{O}(Q^3/\Lambda_b^3)$	1992 [164,165] 0 	1994 [167,170] 2 	—
N^{3}LO $\mathcal{O}(Q^4/\Lambda_b^4)$	2000–2002 [179-182] 12 	2008–2011 [183-185] 0 	2006 [186] 0 
N^{4}LO $\mathcal{O}(Q^5/\Lambda_b^5)$	2015 [188,189] 0 	2011– [190-192] ? 	? 

K. Hebeler, Phys. Rept. **890** (2021)

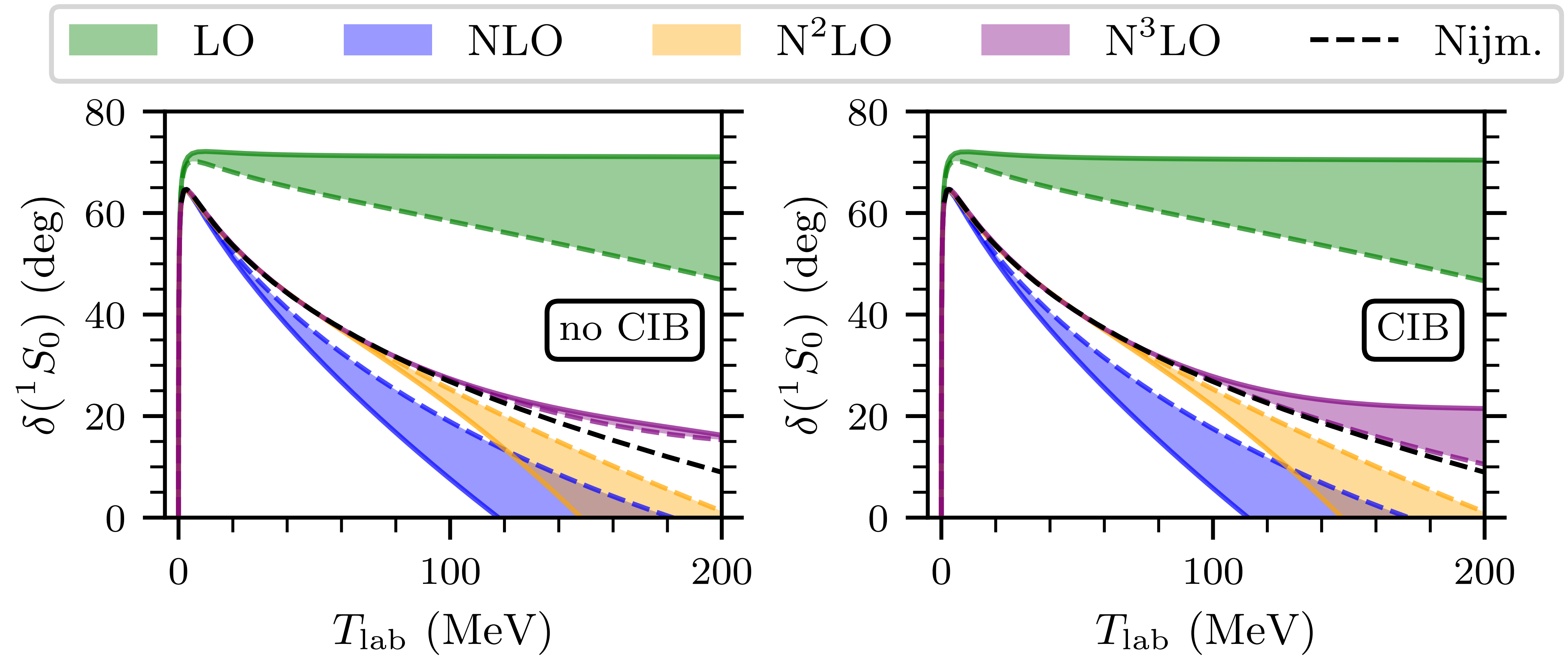
MWPC by Long and Yang

Order	Pion contribution	Contact terms
LO	$V^{(0)}_{1\pi}$	$V^{(0)}_{\rm ct}$: $C^{(0)}_{{}^1S_0}$, $\begin{pmatrix} C^{(0)}_{{}^3S_1} & 0 \\ 0 & 0 \end{pmatrix}$, $D^{(0)}_{{}^3P_0} p'p$, $\begin{pmatrix} D^{(0)}_{{}^3P_2} p'p & 0 \\ 0 & 0 \end{pmatrix}$
NLO	-	$V^{(1)}_{\rm ct}$: $D^{(0)}_{{}^1S_0}(p'^2+p^2)$, $C^{(1)}_{{}^1S_0}$
$\mathrm{N^2LO}$	$V^{(2)}_{2\pi}$	$V^{(2)}_{\rm ct}$: $E^{(0)}_{{}^1S_0} p'^2 p^2$, $D^{(1)}_{{}^1S_0}(p'^2+p^2)$, $C^{(2)}_{{}^1S_0}$, $\begin{pmatrix} D^{(0)}_{{}^3S_1}(p'^2+p^2) & D^{(0)}_{SD} p^2 \\ D^{(0)}_{SD} p'^2 & 0 \end{pmatrix}$, $\begin{pmatrix} C^{(1)}_{{}^3S_1} & 0 \\ 0 & 0 \end{pmatrix}$, $E^{(0)}_{{}^3P_0} p'p(p'^2+p^2)$, $D^{(1)}_{{}^3P_0} p'p$, $p'p \begin{pmatrix} E^{(0)}_{{}^3P_2}(p'^2+p^2) & E^{(0)}_{PF} p^2 \\ E^{(0)}_{PF} p'^2 & 0 \end{pmatrix}$, $\begin{pmatrix} D^{(1)}_{{}^3P_2} p'p & 0 \\ 0 & 0 \end{pmatrix}$, $D^{(0)}_{{}^1P_1} p'p$, $D^{(0)}_{{}^3P_1} p'p$
$\mathrm{N^3LO}$	$V^{(3)}_{2\pi}$, (include πN LECs: c_1, c_3, c_4)	$V^{(3)}_{\rm ct}$: $F^{(0)}_{{}^1S_0} p'^2 p^2 (p'^2+p^2)$, $E^{(1)}_{{}^1S_0} p'^2 p^2$, $D^{(2)}_{{}^1S_0}(p'^2+p^2)$, $C^{(3)}_{{}^1S_0}$, $\begin{pmatrix} D^{(1)}_{{}^3S_1}(p'^2+p^2) & D^{(1)}_{SD} p^2 \\ D^{(1)}_{SD} p'^2 & 0 \end{pmatrix}$, $\begin{pmatrix} C^{(2)}_{{}^3S_1} & 0 \\ 0 & 0 \end{pmatrix}$, $E^{(1)}_{{}^3P_0} p'p(p'^2+p^2)$, $D^{(2)}_{{}^3P_0} p'p$, $p'p \begin{pmatrix} E^{(1)}_{{}^3P_2}(p'^2+p^2) & E^{(1)}_{PF} p^2 \\ E^{(1)}_{PF} p'^2 & 0 \end{pmatrix}$, $\begin{pmatrix} D^{(2)}_{{}^3P_2} p'p & 0 \\ 0 & 0 \end{pmatrix}$, $D^{(1)}_{{}^1P_1} p'p$, $D^{(1)}_{{}^3P_1} p'p$

Predicted scattering observables



Low-energy behavior: $^1S_0$



$$F(k) - ik = -\frac{2}{\pi m_N T^{(0)}}\left[1 - \frac{T^{(1)}}{T^{(0)}} + \left(\left[\frac{T^{(1)}}{T^{(0)}}\right]^2 - \frac{T^{(2)}}{T^{(0)}}\right) + \right.$$
$$\left. + \left(2\frac{T^{(1)}T^{(2)}}{\left(T^{(0)}\right)^2} - \frac{T^{(3)}}{T^{(0)}} - \left[\frac{T^{(1)}}{T^{(0)}}\right]^3\right) + \mathcal{O}\left(\frac{Q^4}{\Lambda_b^4}\right)\right].$$

$^1S_0$ partial wave	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
Empirical (Ref. [27])	−23.735(16)	2.68(3)	−0.48(2)	3.9(1)	−19.6(5)
$\Lambda = 500$ MeV, (no CIB)					
LO	*	1.71(0)	−1.77(0)	8.54(0)	−47.0(3)
NLO	*	*	−0.64(0)	4.79(0)	−29.9(2)
N^{2}LO	*	2.72(0)	−0.71(0)	5.05(0)	−29.3(2)
N^{3}LO	*	2.69(0)	−0.66(0)	5.42(0)	−31.0(2)
$\Lambda = 2500$ MeV, (no CIB)					
LO	*	1.49(0)	−2.06(0)	9.34(0)	−50.7(3)
NLO	*	*	−0.55(0)	4.70(0)	−30.1(2)
N^{2}LO	*	2.75(0)	−0.75(0)	4.80(0)	−28.1(2)
N^{3}LO	*	2.70(0)	−0.69(0)	5.52(0)	−30.6(5)
$\Lambda = 500$ MeV, (CIB)					
LO	*	1.68(0)	−1.55(0)	6.63(0)	−31.64(8)
NLO	*	*	−0.45(0)	3.42(0)	−18.95(8)
N^{2}LO	*	2.70(0)	−0.55(0)	3.77(0)	−18.8(2)
N^{3}LO	*	2.68(0)	−0.50(0)	4.02(0)	−19.8(2)
$\Lambda = 2500$ MeV, (CIB)					
LO	*	1.47(0)	−1.81(0)	7.27(0)	−34.23(8)
NLO	*	*	−0.36(0)	3.35(0)	−19.13(8)
N^{2}LO	*	2.72(0)	−0.59(0)	3.56(0)	−17.7(3)
N^{3}LO	*	2.67(0)	−0.52(0)	4.26(2)	−20.0(7)

OPE no CIB: $$V_{1\pi}^{(0)} = -\frac{g_A^2}{4f_\pi^2}\frac{(\boldsymbol{\sigma}_1\cdot\boldsymbol{q})(\boldsymbol{\sigma}_2\cdot\boldsymbol{q})}{\boldsymbol{q}^2+m_\pi^2}\left[2I(I+1)-3\right]$$

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Low-energy behavior: ${}^3S_1$ (no CIB)

$$F^{(0)}(k) = k\cot\left(\delta^{(0)}_{3S1}\right),$$

$$F^{(1)}(k) = k\frac{d\cot\left(\delta^{(0)}_{3S1}\right)}{d\delta}\times\delta^{(1)}_{3S1},$$

$$F^{(2)}(k) = k\left[\frac{d\cot\left(\delta^{(0)}_{3S1}\right)}{d\delta}\times\delta^{(2)}_{3S1} + \frac{1}{2}\frac{d^2\cot\left(\delta^{(0)}_{3S1}\right)}{d\delta^2}\times\left(\delta^{(1)}_{3S1}\right)^2\right],$$

$$F^{(3)}(k) = k\left[\frac{d\cot\left(\delta^{(0)}_{3S1}\right)}{d\delta}\times\delta^{(3)}_{3S1} + \frac{d^2\cot\left(\delta^{(0)}_{3S1}\right)}{d\delta^2}\times\delta^{(1)}_{3S1}\delta^{(2)}_{3S1} + \right.$$
$$\left. + \frac{1}{6}\frac{d^3\cot\left(\delta^{(0)}_{3S1}\right)}{d\delta^3}\times\left(\delta^{(1)}_{3S1}\right)^3\right].$$

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LO	*	1.69(0)	0.01(0)	0.77(0)	−4.5(4)
N^{2}LO	*	*	0.10(0)	0.77(0)	−4.2(4)
N^{3}LO	*	*	0.01(0)	0.62(0)	−4.0(4)
$\Lambda = 1000$ MeV					
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N^{2}LO	*	*	0.09(0)	0.75(0)	−4.2(7)
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