Non-perturbative renormalization of NN singular interactions

The Nuclear Interaction: post-modern developments

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In loving memory of Ruprecht Machleidt

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Weinberg's proposal

- **Chiral EFT has been extensively used to study the NN system**
  For me, more than 20 years collaborating with Ruprecht
- **Evidences of non-perturbative nature: large scattering lengths and a bound state in NN**
- **Iterative diagrams breaks the Chiral expansion (non-perturbative)**

$$\int \frac{d^3q}{(2\pi)^3} V(p',q) \frac{m_N}{k^2 - q^2 + i\epsilon} V(q,p)$$

If $V(p',p) = C_0$

$$\int \frac{d^3q}{(2\pi)^3} C_0 \frac{m_N}{k^2 - q^2 + i\epsilon} C_0 \quad = \quad -iC_0^2 \frac{m_N k}{4\pi}$$

- **Compute the potential using χPT and include it in a Lippmann-Schwinger Equation to account for the non-perturbative contribution**
- **The infinities of perturbative loop diagrams are polynomial in the external momenta $\Rightarrow$ naive dimensional power counting allows renormalization of the irreducible terms**

The perturbative amplitude

- All amplitudes can be evaluated using dimensional regularization and $\overline{MS}$.
- The ampitude is organize as

$$
\begin{aligned}
\mathcal{V}_{LO} &= \mathcal{V}_{ct}^{(0)} + \mathcal{V}_{1\pi}^{(0)} \\
\mathcal{V}_{NLO} &= \mathcal{V}_{LO} + \mathcal{V}_{ct}^{(2)} + \mathcal{V}_{1\pi}^{(2)} + \mathcal{V}_{2\pi}^{(2)} \\
\mathcal{V}_{NNLO} &= \mathcal{V}_{NLO} + \mathcal{V}_{1\pi}^{(3)} + \mathcal{V}_{2\pi}^{(3)}
\end{aligned}
$$

- In order to include it in the Lippman-Schwinger equation a regularization is needed
  - Introduce a regularization with a cut-off $V(p', p) \to f(p', p; \Lambda) V(p', p)$
- Renormalization of the EFT implies regularization independence. Two points of view
  - Lepage plots point of view $\Lambda < \Lambda_\chi$, works but cutoff artifacts
  - Usual prescription in QFT $\Lambda \gg \Lambda_\chi$ does not work (with renormalization with one counter term, substractive renormalization and renormalization with boundary conditions which are the same)

The N/D method

We write the T matrix as

$$T(A) = \frac{N(A)}{D(A)}$$

with $A = p^2$ (p the on-shell momentum) and so

- $N(A)$ **has a left hand cut (LHC) (for us** $A < -\frac{m_\pi^2}{4} \equiv L$**)**
- $D(A)$ **has a right hand cut (RHC) (**$A > 0$**)**
- **Unitarity and Analitycal properties are used to get the equations and, if necessary, substractions**
  - **Unitarity on the RHC** $\mathrm{Im}[D(A)] = -N(A)\rho(A)$
  - **On the LHC** $D(A)$ **is real** $\mathrm{Im}[N(A)] = D(A)\mathrm{Im}[T(A)] \equiv D(A)\Delta(A)$
- **Dispersion relations (without substractions)**

$$\begin{aligned} N(A) &= \frac{1}{\pi}\int_{-\infty}^{L} d\omega \frac{\mathrm{Im}[N(\omega)]}{\omega - A} = \frac{1}{\pi}\int_{-\infty}^{L} d\omega \frac{\Delta(\omega) D(\omega)}{\omega - A} \\ D(A) &= \frac{1}{\pi}\int_{0}^{\infty} d\omega \frac{\mathrm{Im}[D(\omega)]}{\omega - A} = -\frac{1}{\pi}\int_{0}^{\infty} d\omega \frac{N(\omega)\rho(\omega)}{\omega - A} \end{aligned}$$

The N/D method

The general Equations with $2n$ substractions and a substraction point C given in Z.-H. Guo, J.A. Oller and G. Ríos, Phys. Rev. C **89**, 014002 (2014)

$$D(A) = \sum_{i=1}^{n} \delta_i (A-C)^{n-i} - \frac{(A-C)^n}{\pi} \int_0^\infty d\omega_R \frac{\rho(\omega_R) N(\omega_R)}{(\omega_R - A)(\omega_R - C)^n}$$

$$N(A) = \sum_{i=1}^{n} \nu_i (A-C)^{n-i} - \frac{(A-C)^n}{\pi} \int_{-\infty}^{L} d\omega_L \frac{\Delta(\omega_L) D(\omega_L)}{(\omega_L - A)(\omega_R - C)^n}$$

- We need as input the LHC discontinuity $2i\Delta(A)$
- Usually this is computed perturbatively using χPT
- The contact terms does not give any contribution to $\Delta(A)$
- Contacts are taken into account through substraction constants
- In principle any number of substractions can be used, however for singular interactions not always a solution exists.
- To solve the Equations
  - Including the Eq. for N in D one gets an integral Eq. only for D
  - Once D is known in the LHC, D, N and T are known in all the complex plane

The N/D$_{01}$

- We use the N/D equations with one substraction in D
- Always needed due to the invariance $N \to \alpha N$ and $D \to \alpha D$
- Results for non-singular interactions should be the same as the result of Lippman-Schwinger Equation
- The usual prescription is $D(0) = 1$

The Equations are

$$
\begin{aligned}
D(A) &= 1 - \frac{A}{\pi} \int_0^\infty d\omega_R \frac{\rho(\omega_R) N(\omega_R)}{(\omega_R - A)\omega_R} \\
N(A) &= \frac{1}{\pi} \int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{(\omega_L - A)}
\end{aligned}
$$

where $\rho(A) = \frac{M_N \sqrt{A}}{4\pi}$ is the phase space and $2i\Delta(A)$ is the LHC discontinuity of the T-matrix.

The N/D$_{11}$

- We perform an additional substraction in N

$$D(A) = 1 - \frac{A}{\pi}\int_0^\infty d\omega_R \frac{\rho(\omega_R)N(\omega_R)}{(\omega_R - A)\omega_R}$$

$$N(A) = \nu_1 + \frac{A-C}{\pi}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{(\omega_L - A)(\omega_L - C)}$$

- We use the Effective range expansion to fix the substraction constant and the substraction point

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2}rk^2 + \sum_{i=2} v_i k^{2i}$$

- We get the Equations

$$D(A) = 1 + ia\sqrt{A} + i\frac{M_N}{4\pi^2}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{\omega_L}\frac{A}{\sqrt{A}+\sqrt{\omega_L}}$$

$$N(A) = -\frac{4\pi a}{M_N} + \frac{A}{\pi}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{(\omega_L - A)\omega_L}$$

- They are independent of the substraction constant and point

The N/D$_{12}$

- We perform an additional substraction now in D
- We fix now the Effective Range parameter r
- We get the Equations

$$
\begin{aligned}
D(A) &= 1 + ia\sqrt{A} - \frac{ar}{2}A - i\frac{M_N A}{4\pi^2}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{\omega_L} \\
&\quad \left[\frac{\sqrt{A}}{(\sqrt{\omega_L}+\sqrt{A})\sqrt{\omega_L}} - \frac{i}{a\omega_L}\right] \\
N(A) &= -\frac{4\pi a}{M_N} + \frac{A}{\pi}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{(\omega_L - A)\omega_L}
\end{aligned}
$$

- They are independent of the substraction constant and point

The N/D$_{nd}$

- **We perform n substractions in N and d in D**
- **We fix the $n + d - 1$ first coefficients in the effective range expansion**
- **We get Equations that only depends on a, r, v_2, $\ldots$, v_{n+d-2}.**
- **They are independent of the substraction constant and point**
- **The input of the method is**
  - **$\Delta(A)$ for the finite range interaction**
    The contact terms don't give any contribution to $\Delta(A)$
  - **The effective range expansion parameters fitted through the substractions in the N/D method**
    Substractions are equivalent to contacts

Leading Order - LO

The potential is

$$V(\vec{q}) = \frac{g_A^2}{4f_\pi^2}\frac{\vec{\sigma}_1\cdot\vec{q}\vec{\sigma}_2\cdot\vec{q}}{q^2+m_\pi^2}\vec{\tau}_1\cdot\vec{\tau}_2 + C_S + C_T\vec{\sigma}_1\cdot\vec{\sigma}_2$$

for the singlet $^1S_0$ partial wave

$$V(p',p) = \frac{g_A^2}{4f_\pi^2}\frac{m_\pi^2}{2p'p}Q_0(z) + C_0$$

$$Q_0(z) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$$

$$z = \frac{p'^2+p^2+m_\pi^2}{2p'p}$$

The LHC discontinuity is ($L \equiv -\frac{m_\pi^2}{4}$, $A = p^2 \sim -k^2$, $p = ik+\epsilon$, $k, \epsilon > 0$)

$$\Delta_{1\pi}(A) = \theta(L-A)\frac{g_A^2}{4f_\pi^2}\frac{\pi m_\pi^2}{4A}$$

Iterative diagrams

- **2π exchange**

$$\Delta_{2\pi}(A) = -\theta(4L-A)\left(\frac{g_A^2 m_\pi^2}{16 f_\pi^2}\right)^2 \frac{M_N}{k^3}\log\left(\frac{2k}{m_\pi}-1\right)$$

- **3π exchange**

$$\begin{aligned}\Delta_{3\pi}(A) &= -\theta(9L-A)\left(\frac{g_A^2 m_\pi^2}{4 f_\pi^2}\right)^3 \left(\frac{M_N}{4\pi}\right)^2 \frac{\pi}{4k^2}\int_{2m_\pi}^{2k-m_\pi} d\mu_1 \frac{1}{\mu_1(2k-\mu_1)} \\ &\quad \theta(\mu_1-2m_\pi)\int_{m_\pi}^{\mu_1-m_\pi} d\mu_2 \frac{1}{\mu_2(2k-\mu_2)}\end{aligned}$$

- **4π exchange**

$$\begin{aligned}\Delta_{4\pi}(A) &= -\theta(16L-A)\left(\frac{g_A^2 m_\pi^2}{4 f_\pi^2}\right)^4 \left(\frac{M_N}{4\pi}\right)^3 \frac{\pi}{4k^2}\int_{3m_\pi}^{2k-m_\pi} d\mu_1 \frac{1}{\mu_1(2k-\mu_1)} \\ &\quad \theta(\mu_1-3m_\pi)\int_{2m_\pi}^{\mu_1-m_\pi} d\mu_2 \frac{1}{\mu_2(2k-\mu_2)} \\ &\quad \theta(\mu_2-2m_\pi)\int_{m_\pi}^{\mu_2-m_\pi} d\mu_3 \frac{1}{\mu_3(2k-\mu_3)}.\end{aligned}$$

Iterative diagrams

- **nπ exchange (with $\mu_0 \equiv 2k$)**

$$\Delta_{n\pi}(A) = -\theta(n^2L - A)\left(\frac{g_A^2 m_\pi^2}{4f_\pi^2}\right)^n \left(\frac{M_N}{4\pi}\right)^{n-1} \frac{\pi}{4k^2} \prod_{j=1}^{n-1} \theta(\mu_{j-1} - (n+1-j)m_\pi) \int_{(n-j)m_\pi}^{\mu_{j-1}-m_\pi} d\mu_j \frac{1}{\mu_j(2k-\mu_j)}$$

- **This is the formal solution of the integral equation (with $\Delta(A) = \tilde{\Delta}(A, 2k)$)**

$$\tilde{\Delta}(A, \bar{\mu}) = \Delta_{1\pi}(A) - \left(\frac{M_N k^2}{\pi^2}\right) \theta(\bar{\mu} - 2m_\pi) \int_{m_\pi}^{\bar{\mu}-m_\pi} d\mu \frac{\Delta_{1\pi}(A)\tilde{\Delta}(A, \mu)}{\mu(2k-\mu)}$$

- **When $k \gg m_\pi$**

$$\Delta(A) = -\frac{\lambda\pi^2}{M_N k^2} e^{\frac{2\lambda}{k}\operatorname{arctanh}\left(1-\frac{m_\pi}{k}\right)} = -\frac{\lambda\pi^2}{M_N k^2}\left(\frac{m_\pi}{2k - m_\pi}\right)^{-\lambda/k}$$

$$\lambda = \frac{g_A^2 m_\pi^2}{4f_\pi^2}\frac{M_N}{4\pi}$$

General Case

- **Use the analytical properties of** T
- **Use the analytical properties of** $V \rightarrow$ **Dynamical cuts**
- **Dynamical cut for** T **are the same as for** V
- **Make the analytical extension of the Lippmann-Schwinger Eq. to the LHC and look for** $\Delta(A)$**.**

The general Equation for S **waves is**

$$\begin{aligned}
\Delta T(\nu,k) &= \Delta V(\nu,k) + \theta(k-m_\pi)\theta(k-\nu-2m_\pi)\frac{M_N}{\pi^2}\int_{\nu+m_\pi}^{k-m_\pi} d\nu_1 \frac{\nu_1^2}{k^2-\nu_1^2} \\
&\quad \Delta V(\nu,\nu_1)\Delta T(\nu_1,k) \\
\Delta(A) &= -\Delta T(-k,k)
\end{aligned}$$

with ($\epsilon > \delta > 0$)

$$\begin{aligned}
2\Delta T(\nu,k) &\equiv \lim_{\epsilon\to 0}\lim_{\delta\to 0} \mathrm{Im}T(i\nu+\epsilon-\delta, ik+\epsilon) - \mathrm{Im}T(i\nu+\epsilon+\delta, ik+\epsilon) \\
2\Delta V(\nu,\nu_1) &\equiv \lim_{\epsilon\to 0}\lim_{\delta\to 0} \{\mathrm{Im}V(i\nu+\epsilon-\delta, i\nu_1+\epsilon) - \mathrm{Im}V(i\nu+\epsilon+\delta, i\nu_1+\epsilon)\}
\end{aligned}$$

General Case

For OPE in the $^1S_0$ partial wave

$$\Delta V(\nu_1,\nu) = -\pi \frac{g_A^2 m_\pi^2}{16 f_\pi^2} \frac{1}{\nu_1 \nu} \mathrm{sign}(\nu_1-\nu)\theta[|\nu_1-\nu|-m_\pi])$$

We get the same Equation with

- **For OPE in the $^1S_0$ partial wave** $\Delta \hat{V}(\nu_1,\nu) = k^2 \Delta_{1\pi}(A)$
- **Making the change** $\mu = k - \nu$
- **Identifying** $\tilde{\Delta}(A,\mu) = k^2 \Delta \hat{T}(\nu = k-\mu, k)$

General Case

For $l > 0$ we define

$$\begin{aligned} \Delta\hat{T}(\nu_1, \nu_2) &= \nu_1^{1+l}\nu_2^{1+l}\Delta T(\nu_1, \nu_2) \\ \Delta\hat{V}(\nu_1, \nu_2) &= \nu_1^{1+l}\nu_2^{1+l}\Delta V(\nu_1, \nu_2) \end{aligned}$$

we get

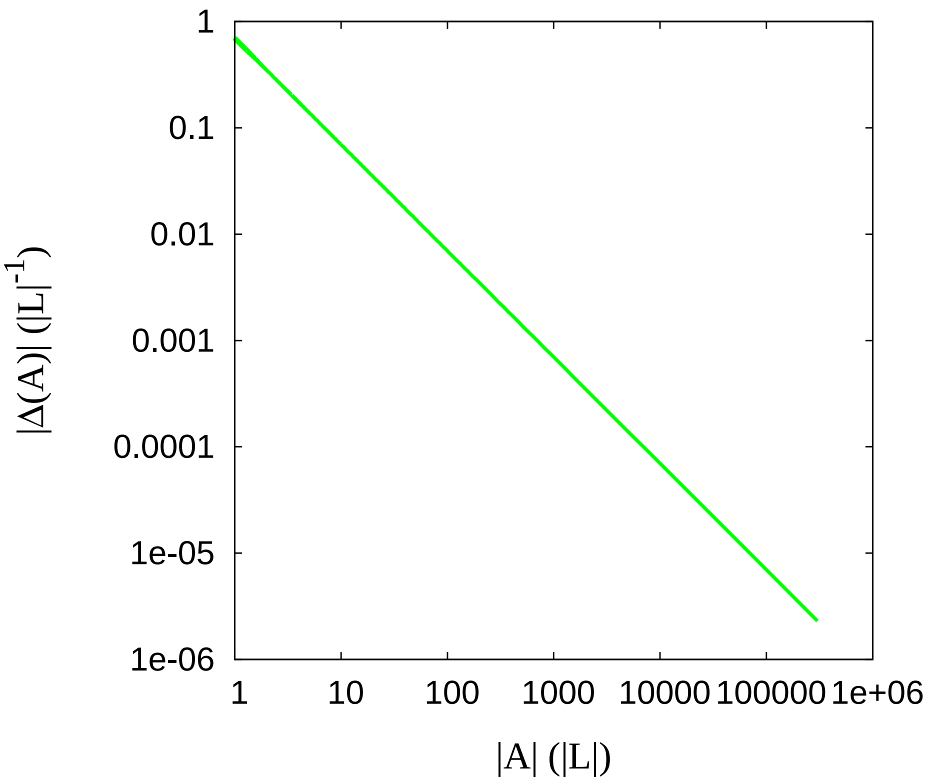
$$\begin{aligned} \Delta\hat{T}(\nu, k) &= \Delta\hat{V}(\nu, k) + \frac{M_N\theta(k-m_\pi)\theta(k-\nu-2m_\pi)}{\pi^2} \\ & \int_{\nu+m_\pi}^{k-m_\pi} d\nu_1 \frac{\nu_1^2}{k^2-\nu_1^2}\left(\frac{1}{2(\nu_1+i\epsilon)^{2l+1}}+\frac{1}{2(\nu_1-i\epsilon)^{2l+1}}\right)\Delta\hat{V}(\nu,\nu_1)\Delta\hat{T}(\nu_1,k) \end{aligned}$$

- **The input is $\Delta\hat{V}(\nu, \nu_1)$**
- **The integral equation has finite integration limits (but depending on ν)**
- **The result $\Delta\hat{T}(\nu, \nu_1)$ is always finite even for singular interactions**
- $\Delta\hat{T}(-k, k) = -k^{2l+2}\Delta(A)$

Regular case

At LO only OPE is present which is a non-singular interaction

Results for the physical g_A



$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ • $\Delta(A)$ **(for $k \gg m_\pi$)**

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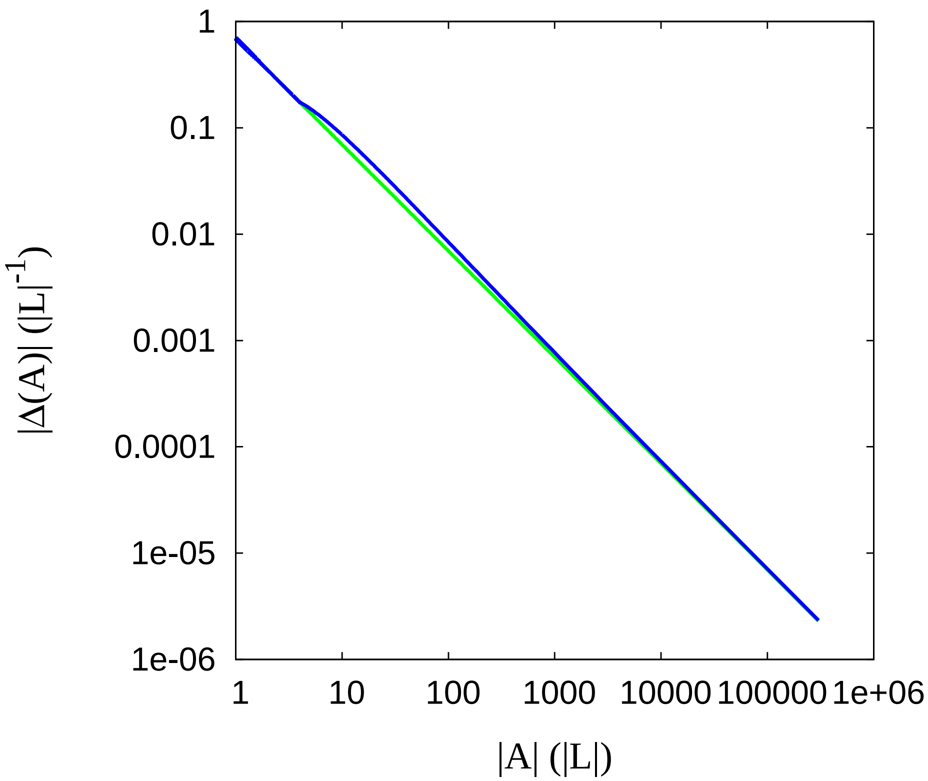

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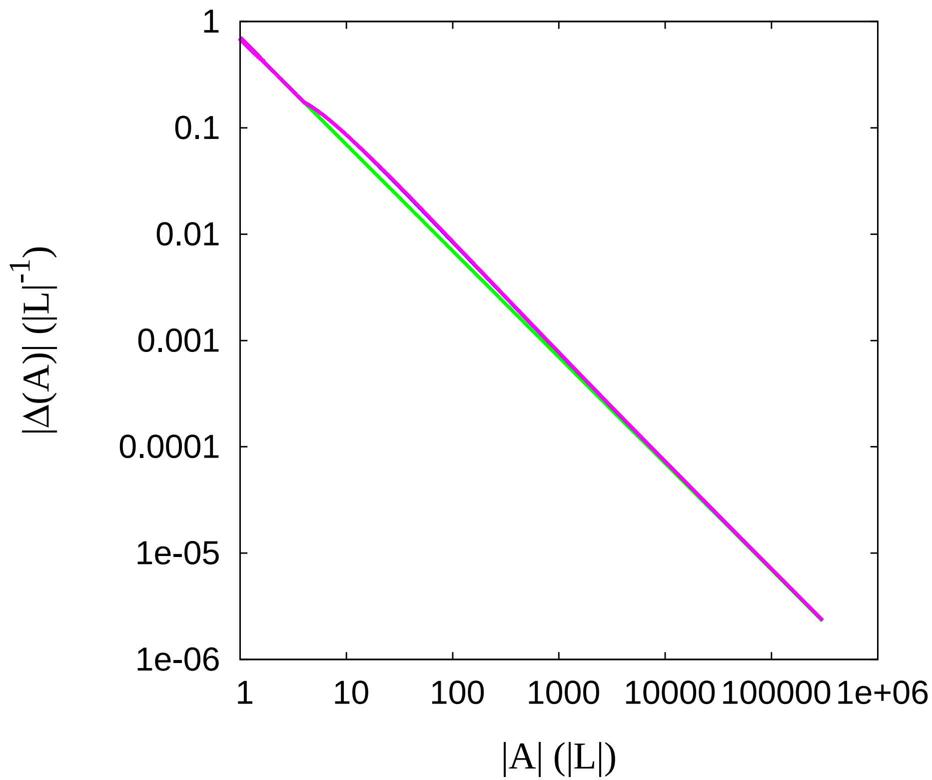

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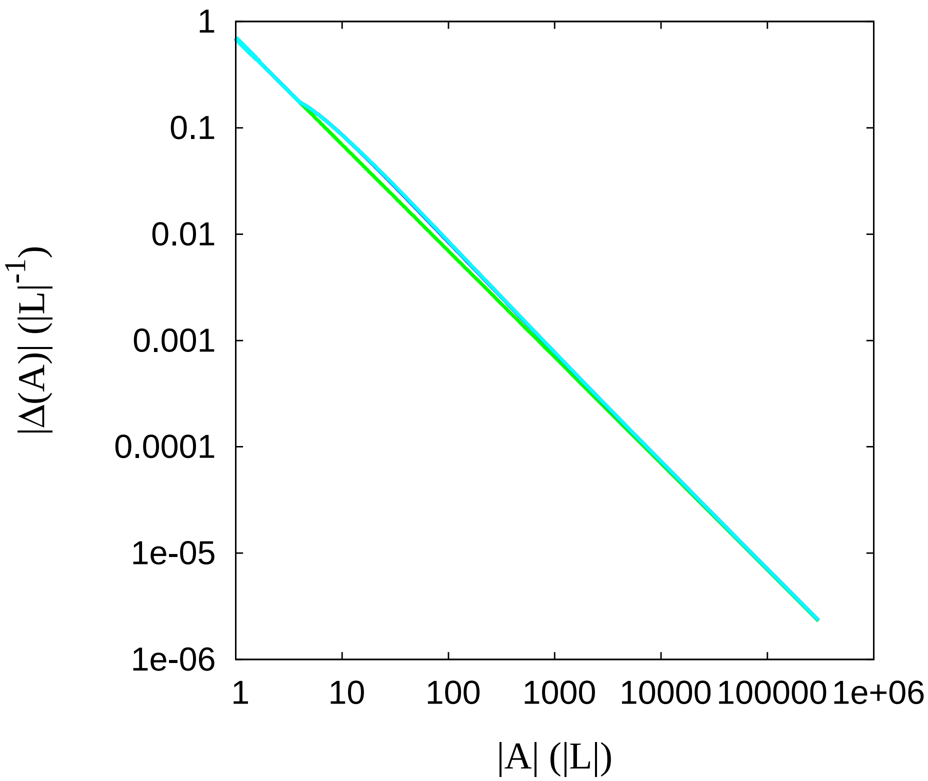

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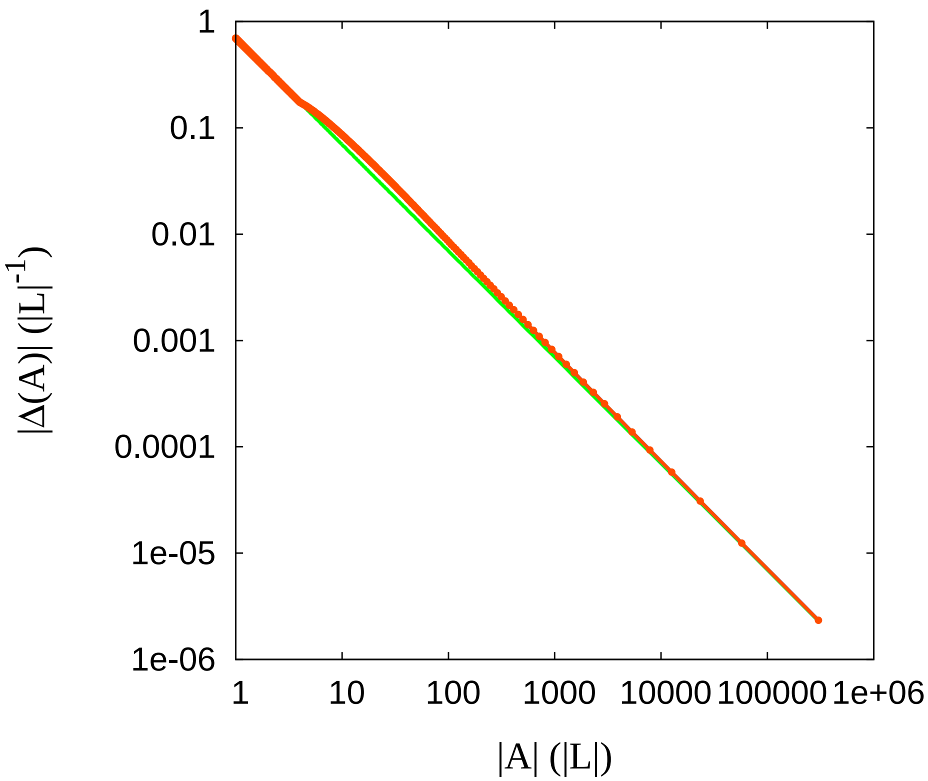

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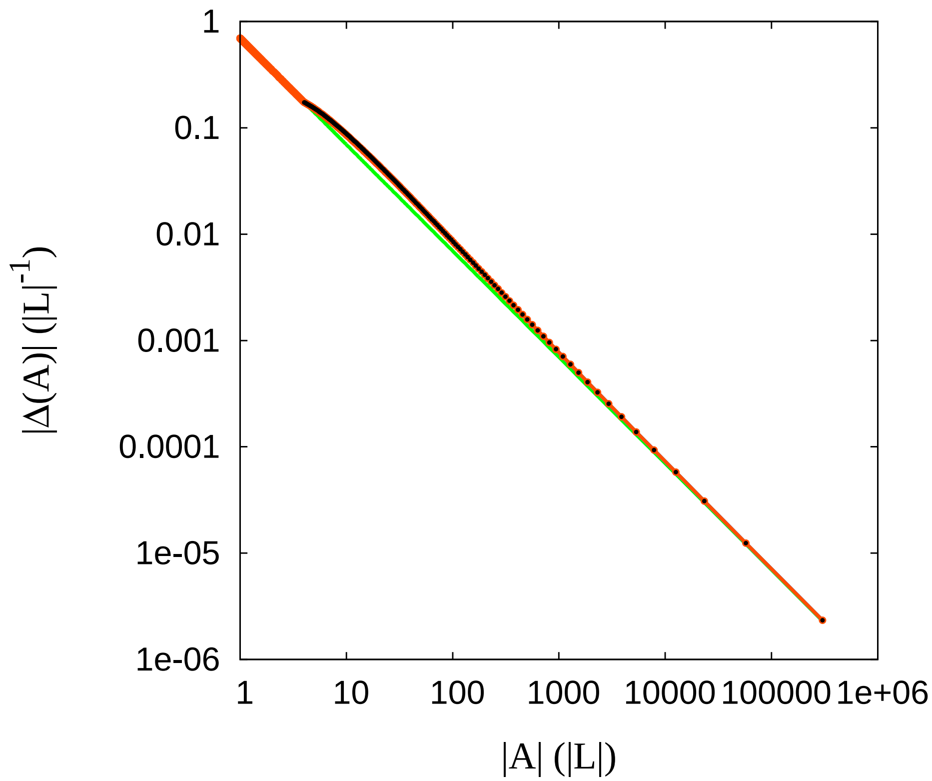

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Regular case N/D$_{01}$

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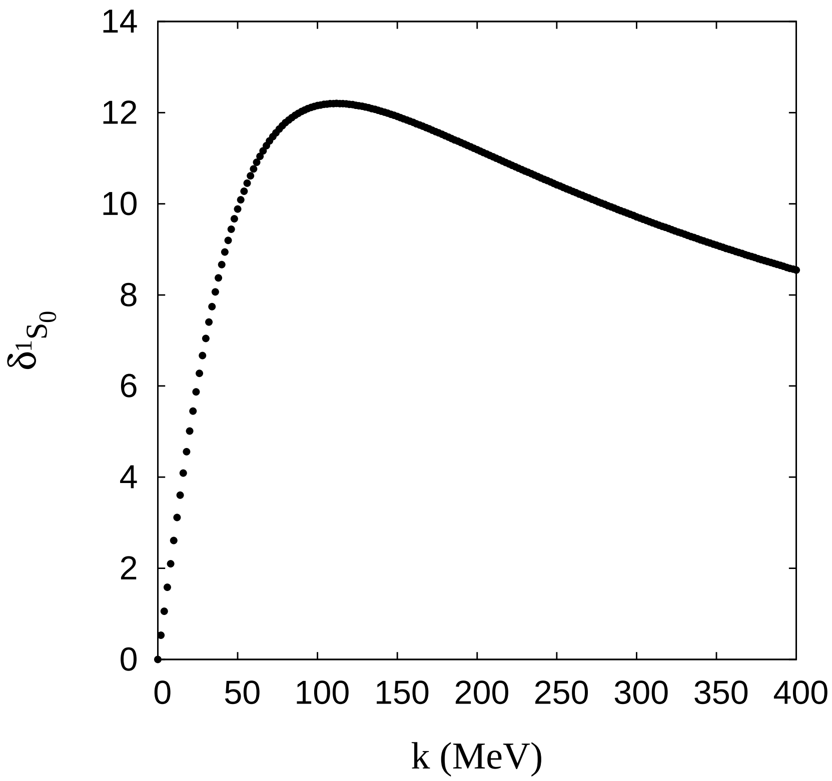

Solution of N/D$_{01}$

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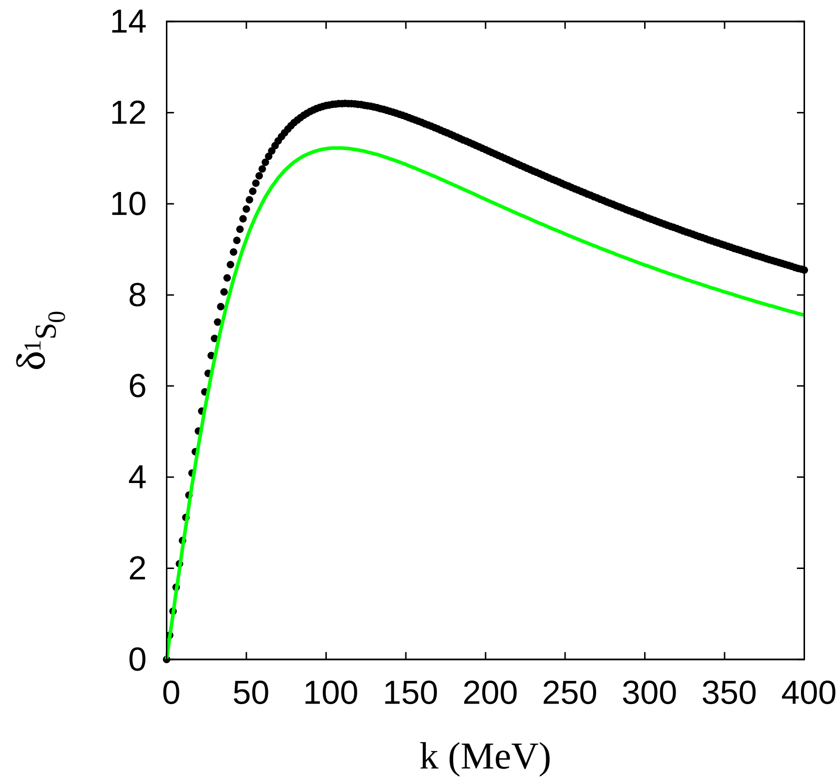

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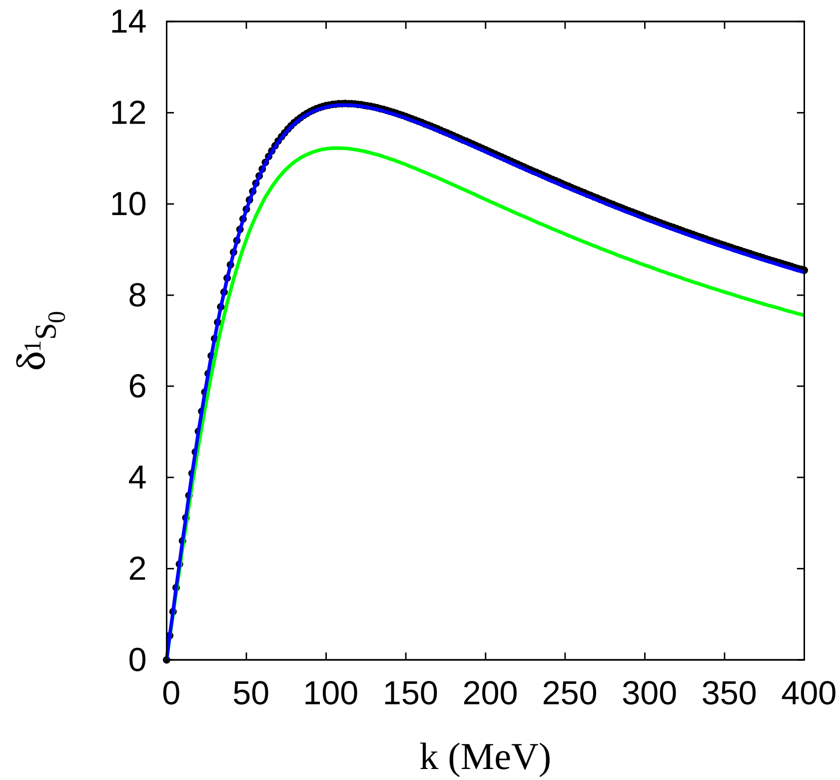

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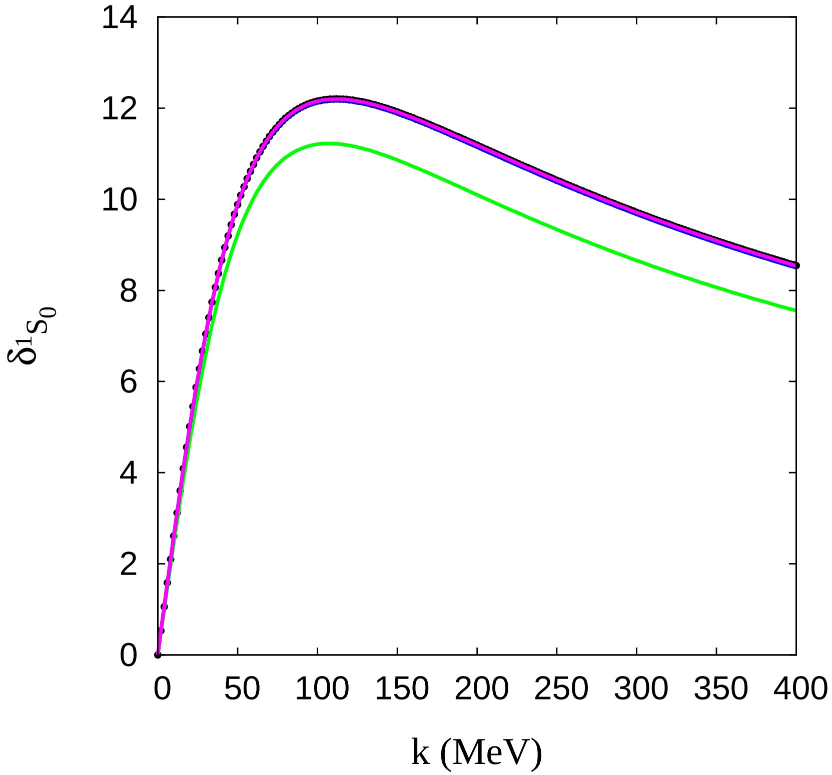

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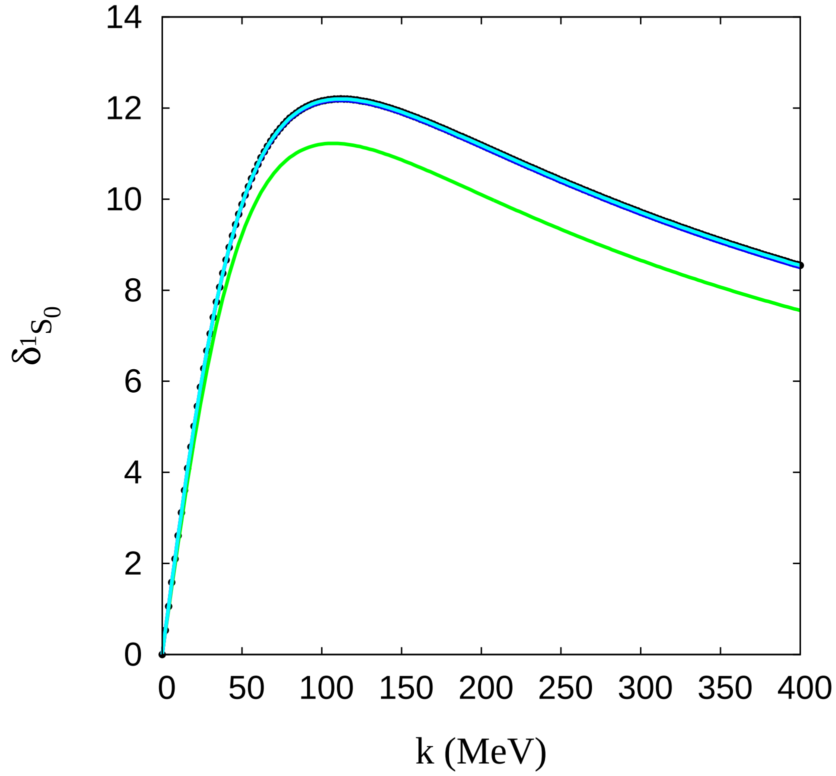

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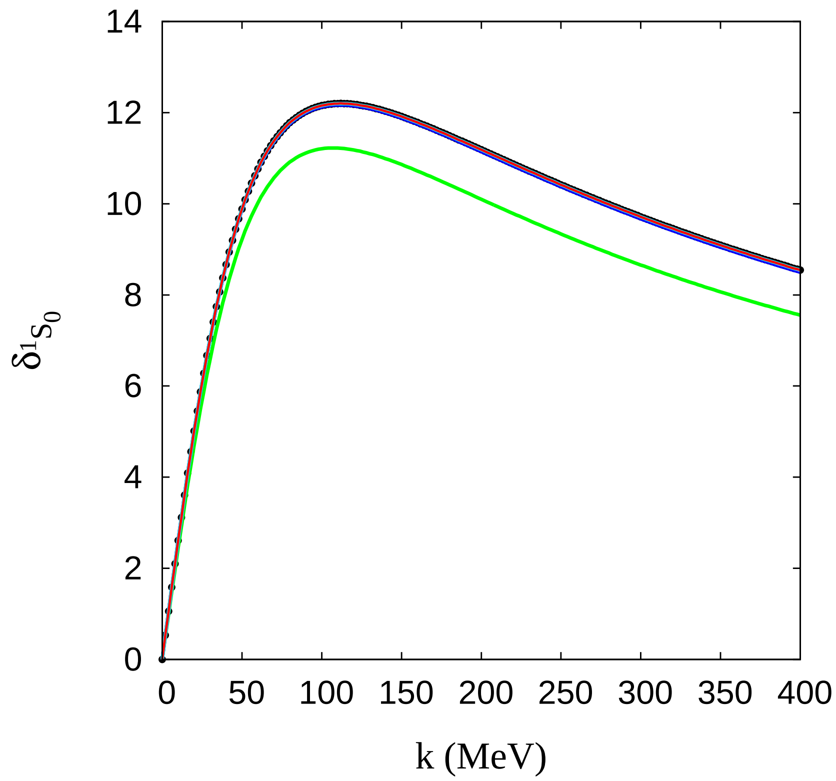

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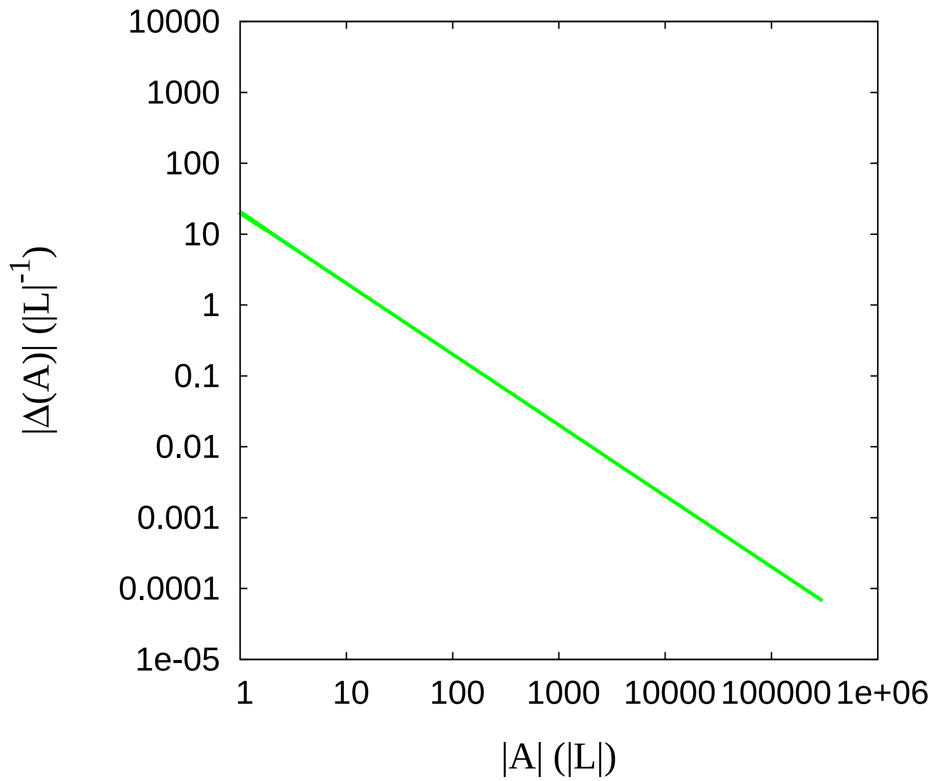
Regular case

Results for the unphysical $g_A = 6{,}80$

Now the problem is non-perturbative



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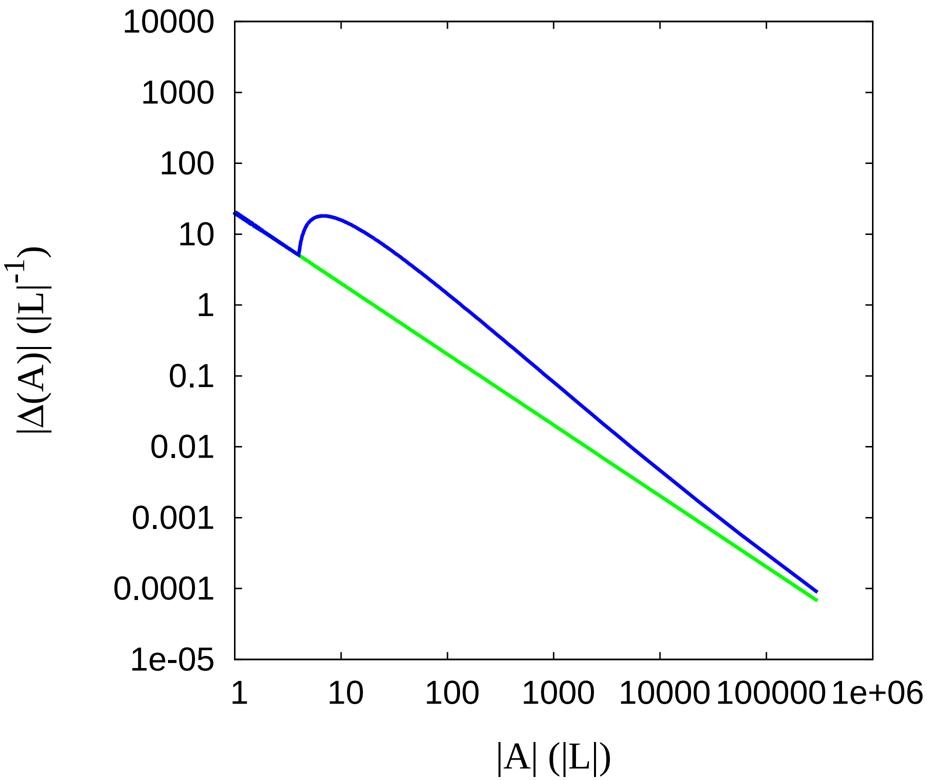

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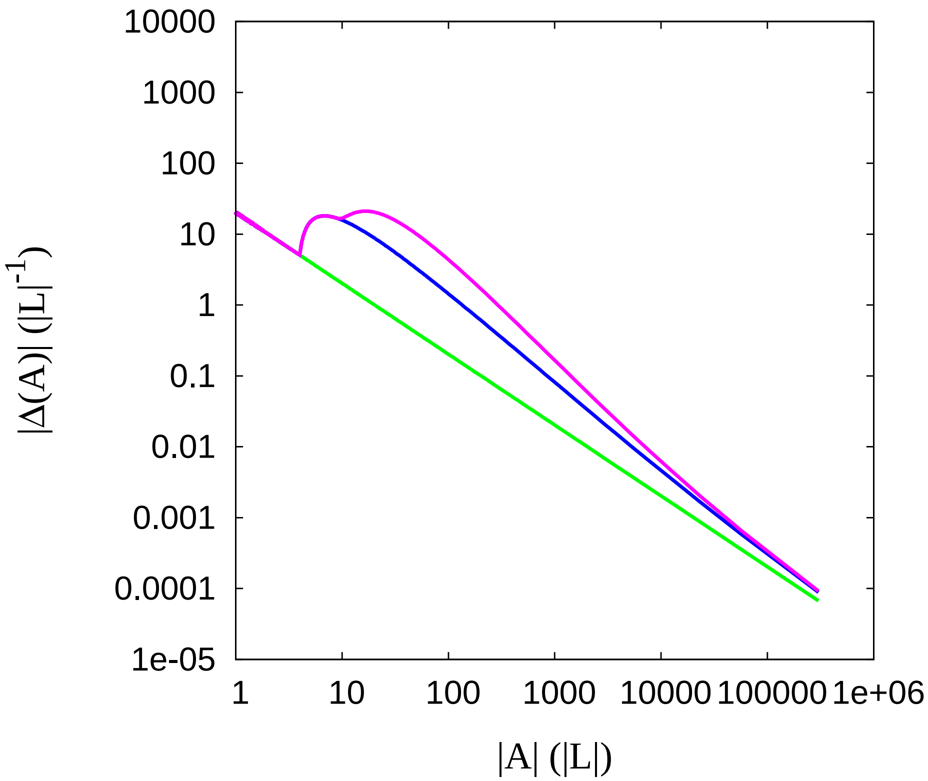

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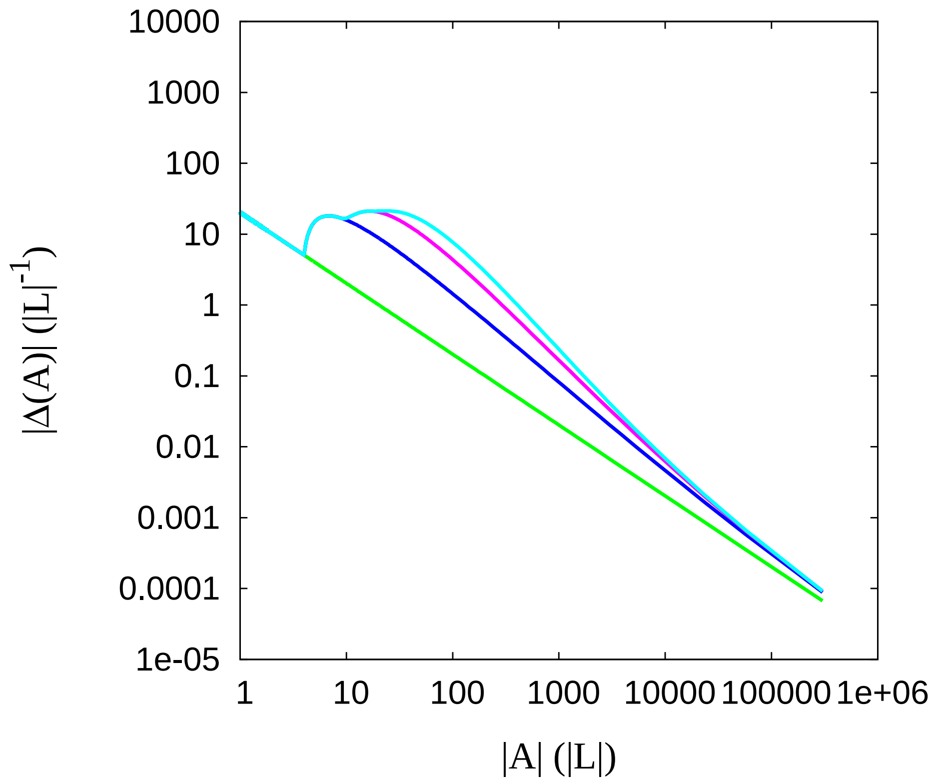

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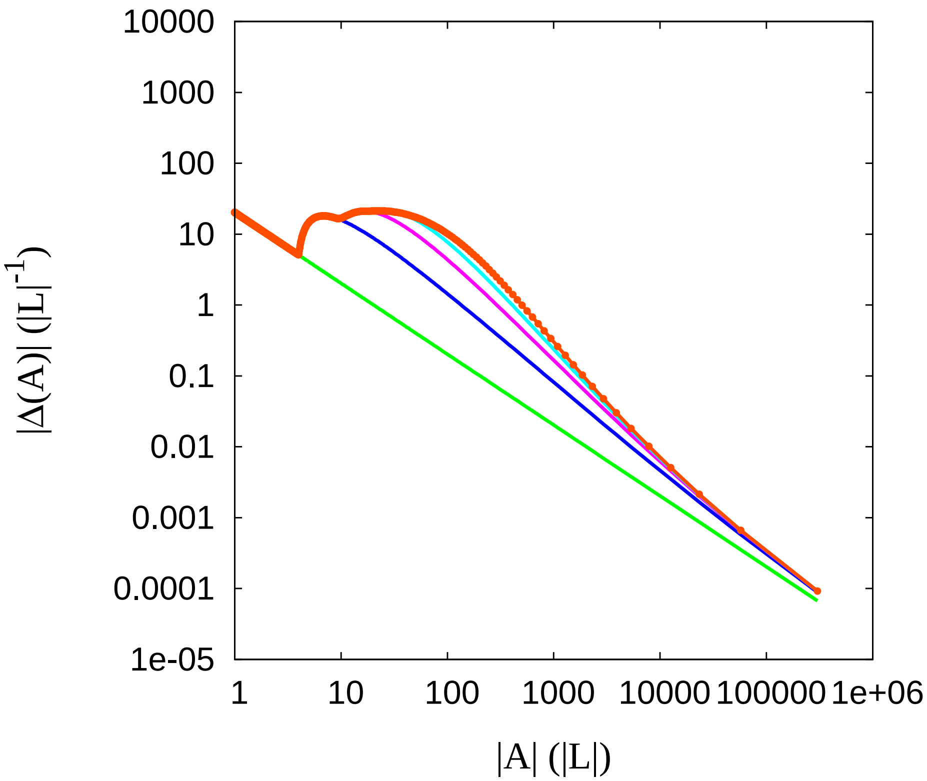

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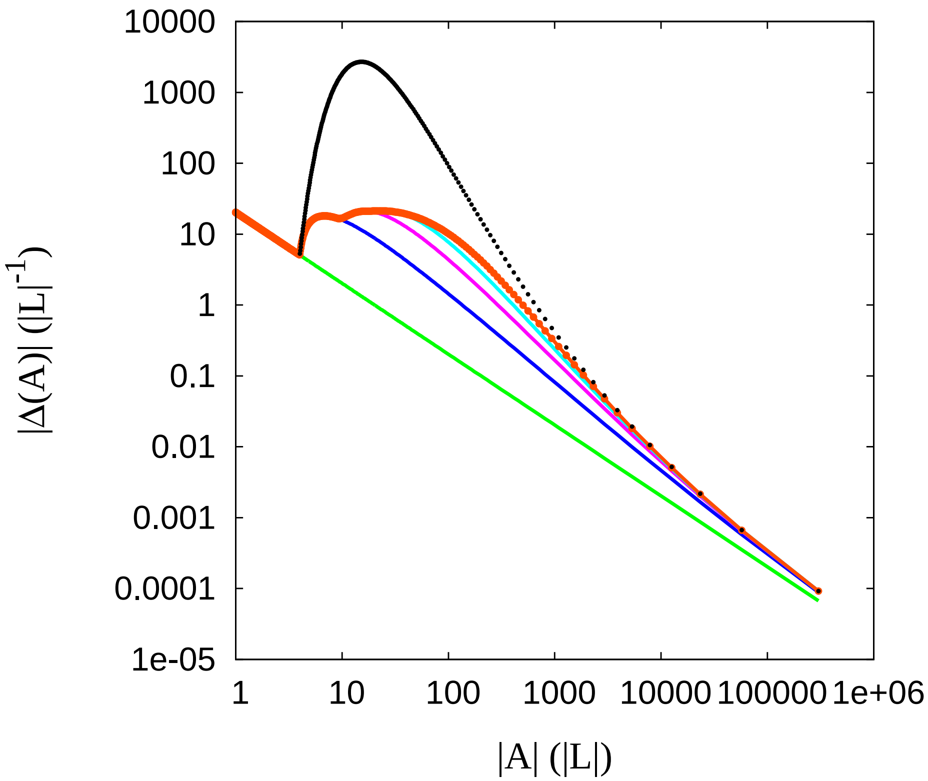
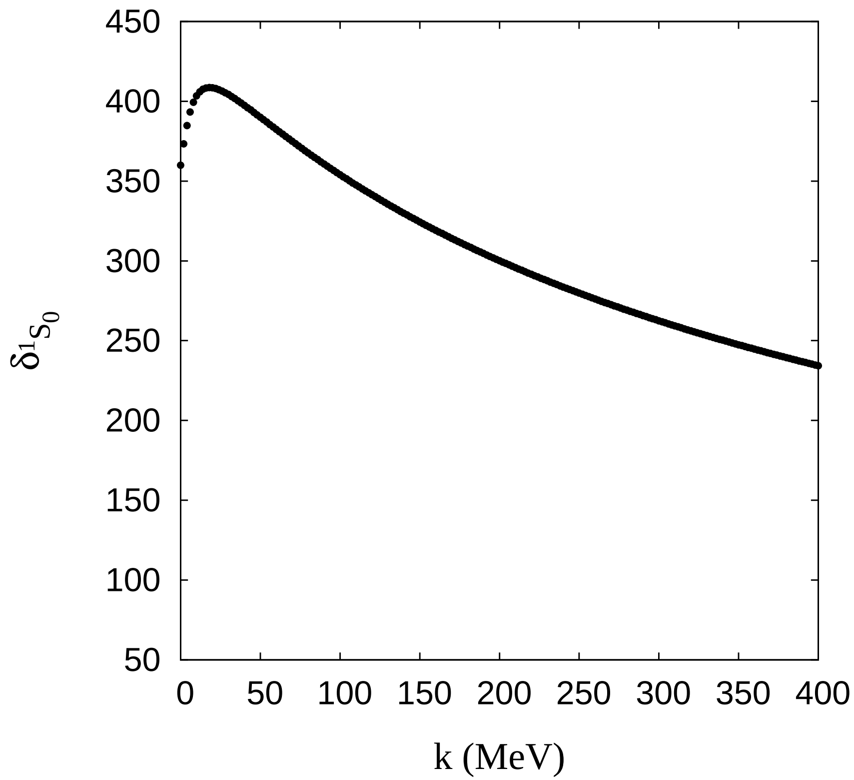


$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ $\bullet\ \Delta(A)$ **(for** $\sqrt{-A} \gg m_\pi$

Regular case N/D$_{01}$

Results for the unphysical $g_A = 6{,}80$

Now the problem is non-perturbative



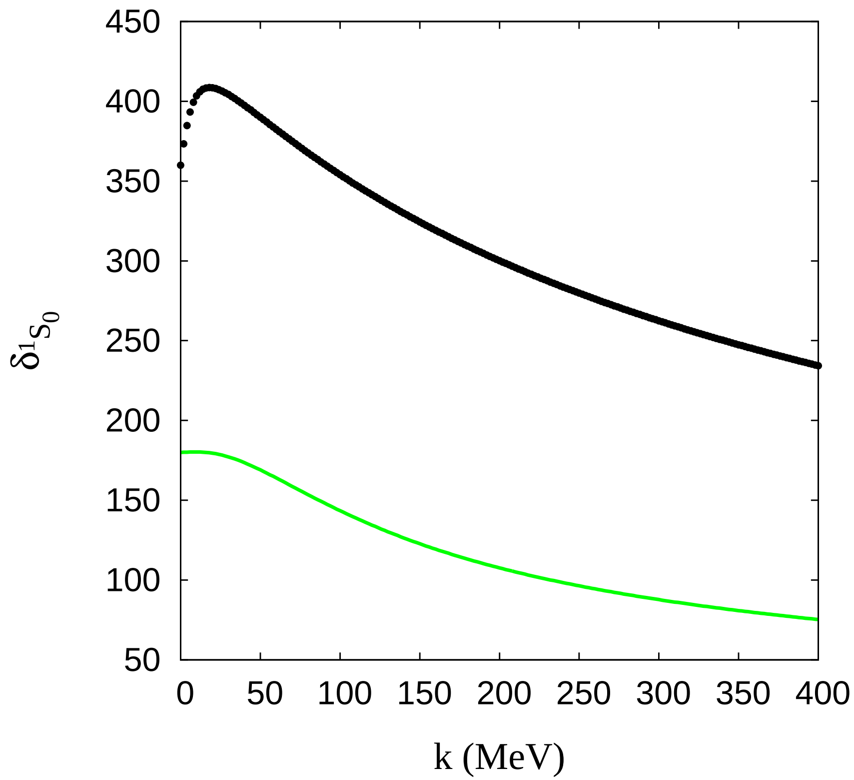
Solution of N/D$_{01}$

$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ • **LS**

Regular case N/D$_{01}$

Results for the unphysical $g_A = 6{,}80$

Now the problem is non-perturbative



Solution of N/D$_{01}$

$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ • **LS**

Regular case N/D$_{01}$

Results for the unphysical $g_A = 6{,}80$

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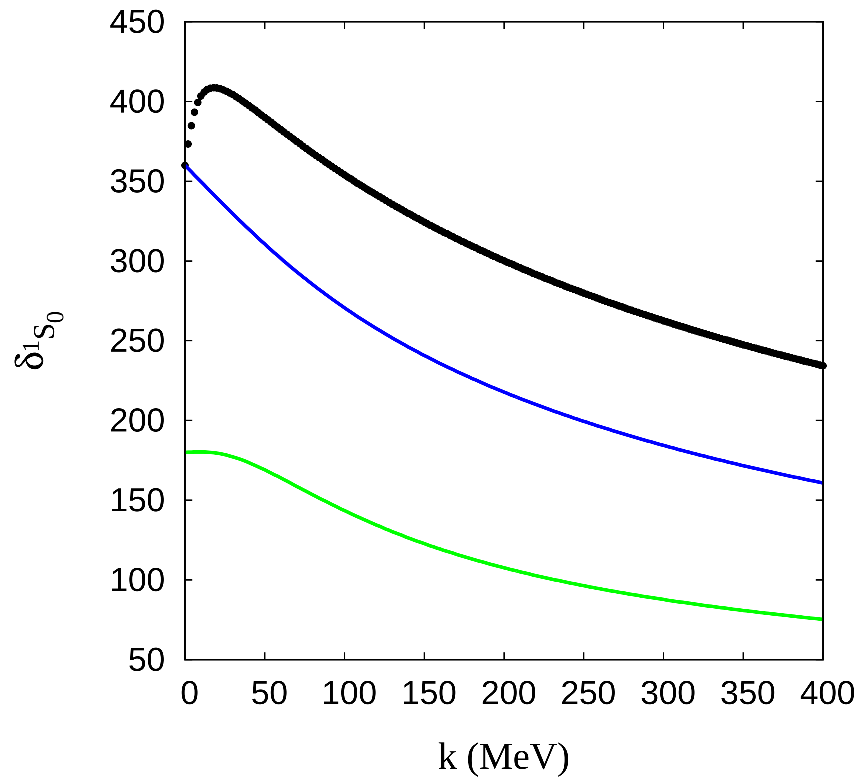


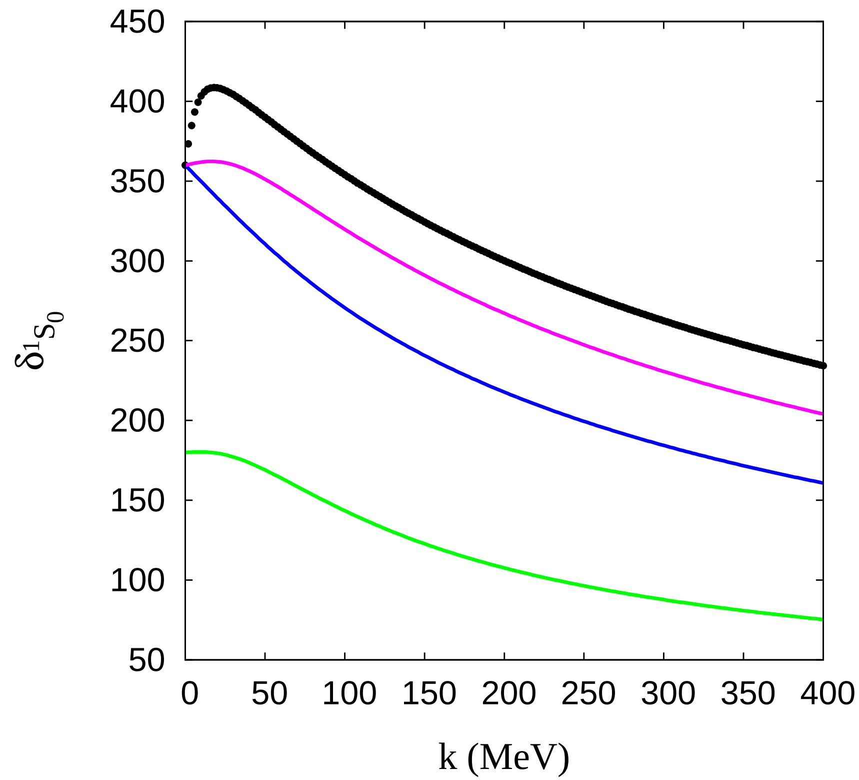
Solution of N/D$_{01}$

$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ • **LS**

Regular case N/D_{01}

Results for the unphysical $g_A = 6{,}80$

Now the problem is non-perturbative



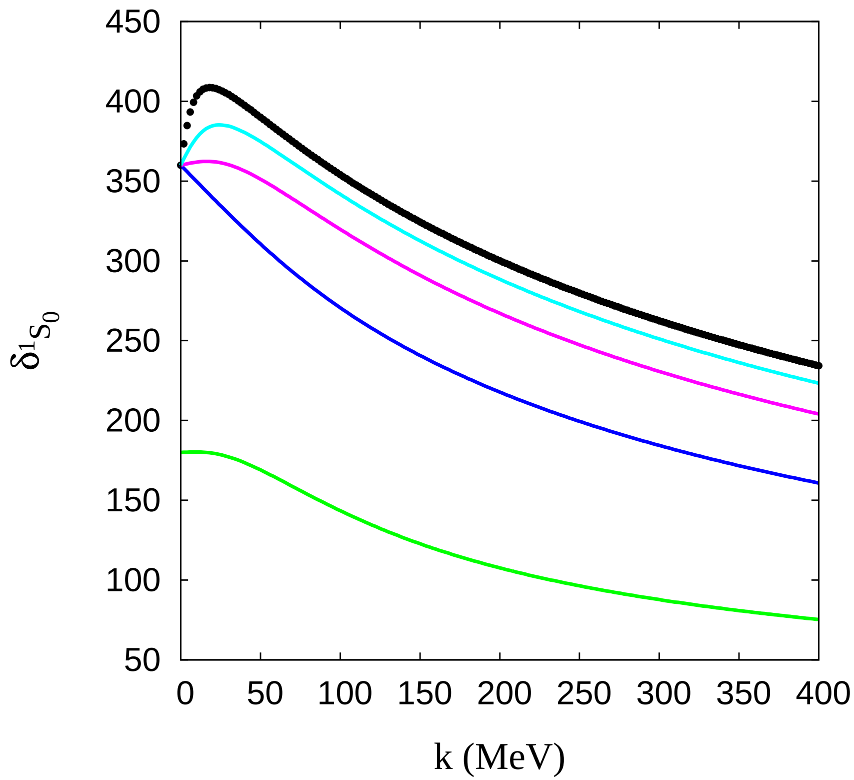
Solution of N/D_{01}

$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ • **LS**

Regular case N/D$_{01}$

Results for the unphysical $g_A = 6{,}80$

Now the problem is non-perturbative



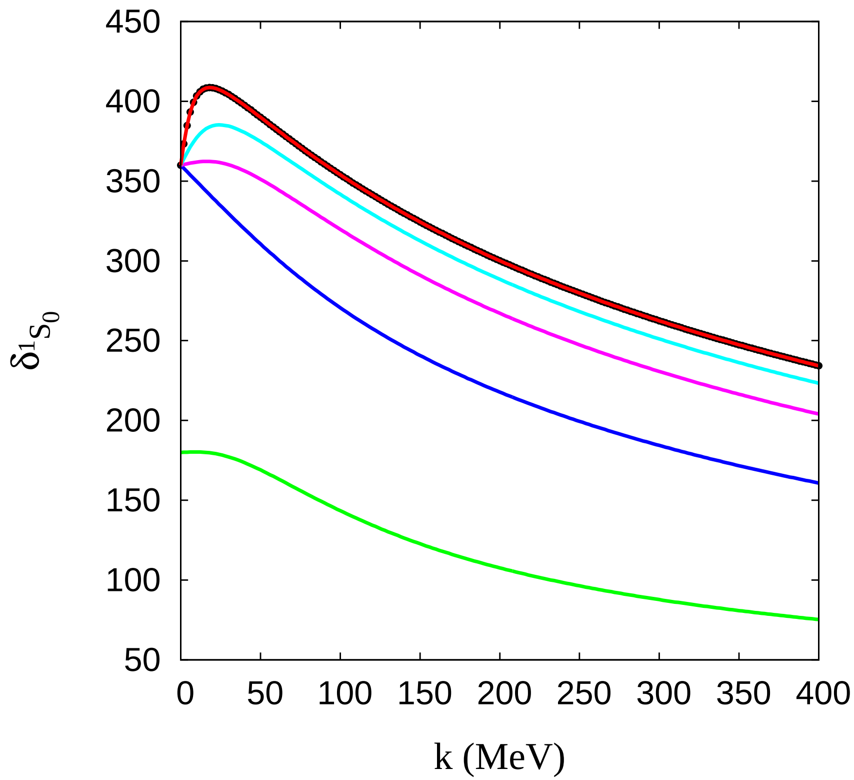
Solution of N/D$_{01}$

$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ • **LS**

Regular case N/D_{01}

Results for the unphysical $g_A = 6{,}80$

Now the problem is non-perturbative



Solution of N/D_{01}

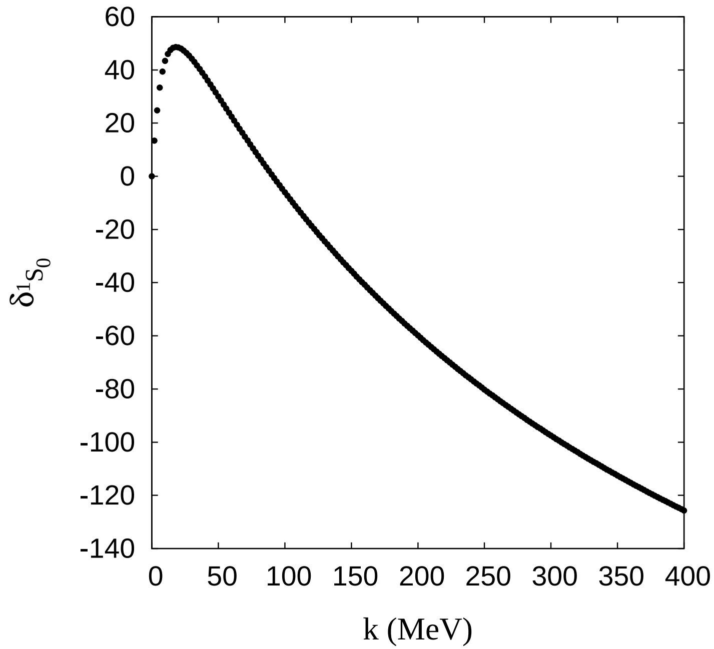
$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ • **LS**

Regular case N/D$_{11}$

Results for the unphysical $g_A = 6{,}80$

Now the problem is non-perturbative but it looks perturbative

a fitted to the value given by LS



Solution of N/D$_{11}$

$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ • **LS**

Regular case N/D$_{11}$

Results for the unphysical $g_A = 6{,}80$

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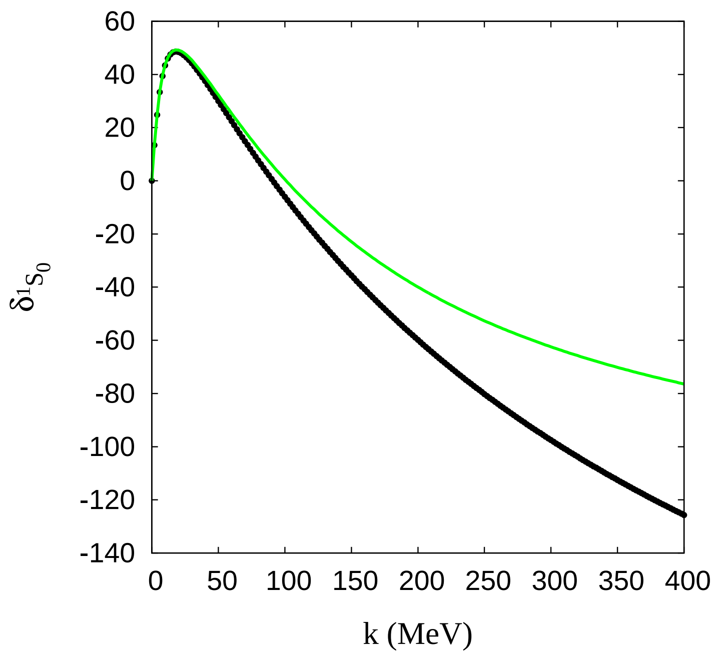

Solution of N/D$_{11}$

$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ • **LS**

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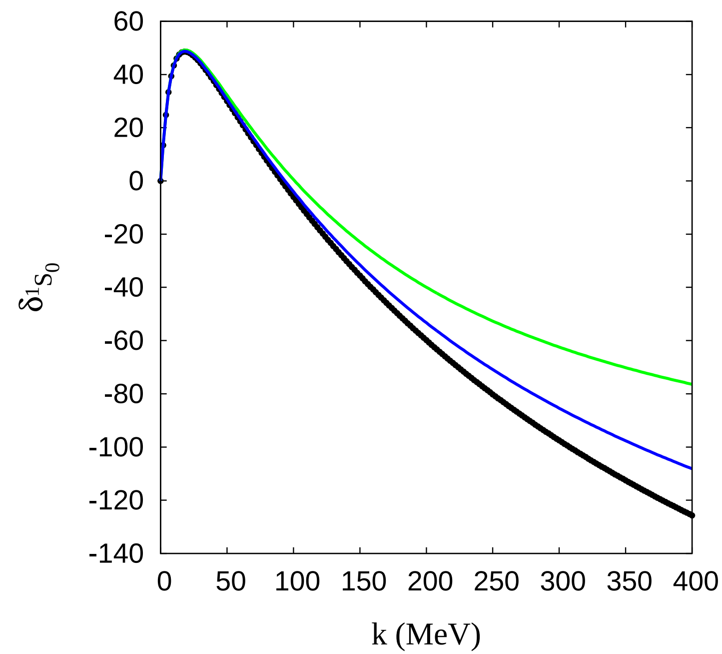

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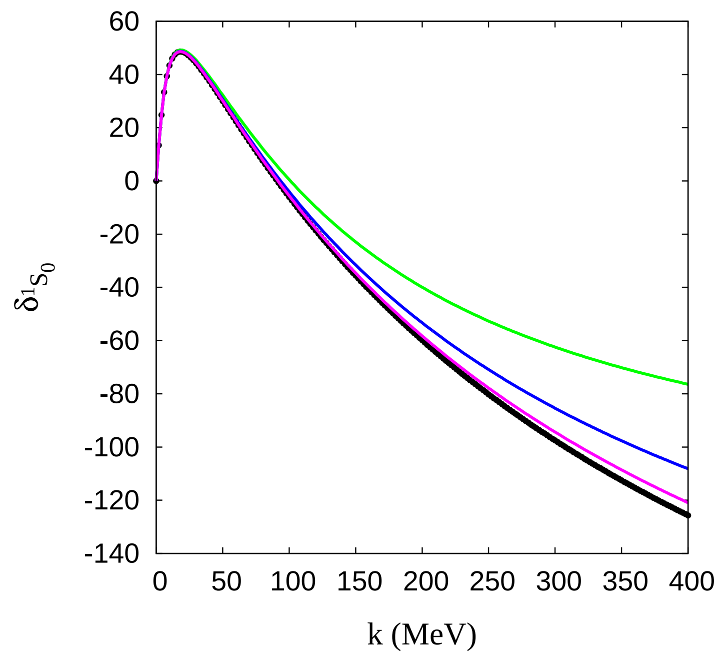

Solution of N/D$_{11}$

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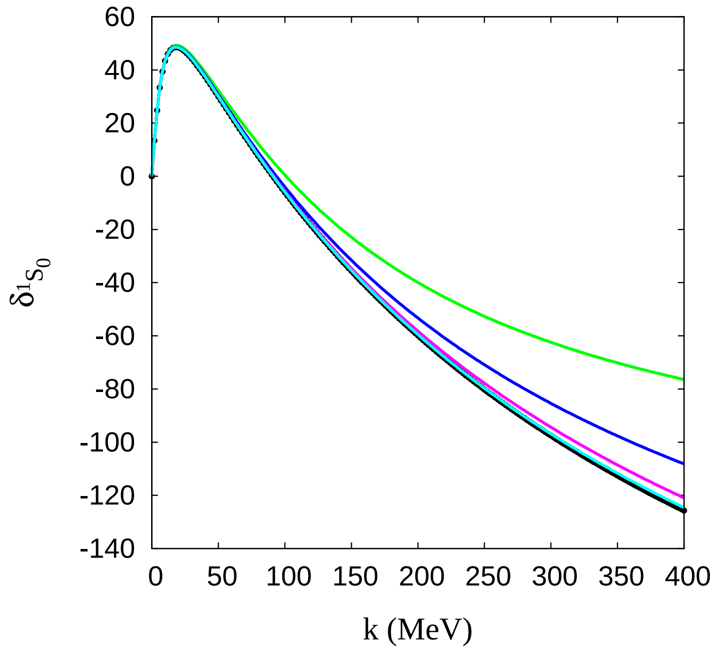

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$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ • **LS**

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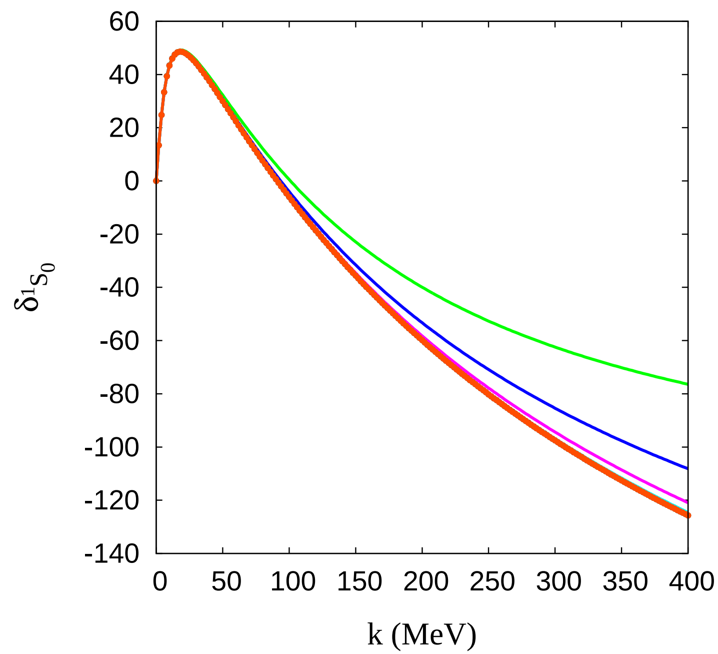


Solution of N/D$_{11}$

$\Delta_{1\pi}(A)$ $\Delta_{2\pi}(A)$ $\Delta_{3\pi}(A)$ $\Delta_{4\pi}(A)$ $\Delta(A)$ • **LS**

Regular case

Results for the unphysical $g_A = 6{,}80$

	a_s (fm)	r (fm)	v_2 (fm^3)	v_3 (fm^5)	v_4 (fm^7)	v_5 (fm^9)	v_6 (fm^{11})
N/D$_{01}$							
$\Delta_{1\pi}$	1.66	0.714	-0.168	0.847	-5.35	35.5	-241
$\Delta_{2\pi}$	3.53	2.03	-5.70 10^{-2}	3.38	-27.4	234	-1.99 10^3
$\Delta_{3\pi}$	1.80	1.15	-8.71 10^{-2}	0.924	-5.69	36.9	-247
$\Delta_{4\pi}$	-6.89	13.7	47.5	356	2.63 10^3	1.97 10^4	1.46 10^5
Non Perturbative	-23.75	8.90	18.7	89.8	411	2.00 10^3	8.98 10^3
N/D$_{11}$							
$\Delta_{1\pi}$	-23.75	8.56	15.3	60.5	221	906	3.08 10^3
$\Delta_{2\pi}$	-23.75	8.80	17.7	80.4	346	1.60 10^3	6.71 10^3
$\Delta_{3\pi}$	-23.75	8.88	18.4	87.5	395	1.90 10^3	8.40 10^3
$\Delta_{4\pi}$	-23.75	8.90	18.6	89.3	408	1.98 10^3	8.87 10^3
Non-perturbative	-23.75	8.90	18.7	89.8	411	2.00 10^3	8.98 10^3

Regular case

We encrease $g_A = 7{,}45$ to have a near threshold bound state

	N/D$_{01}$	N/D$_{11}$	Schrödinger
1π		**2.02**	
2π		**2.18**	
3π		**2.21**	
4π	**0.89**	**2.22**	
Non-perturbative	**2.22**	**2.22**	**2.22**

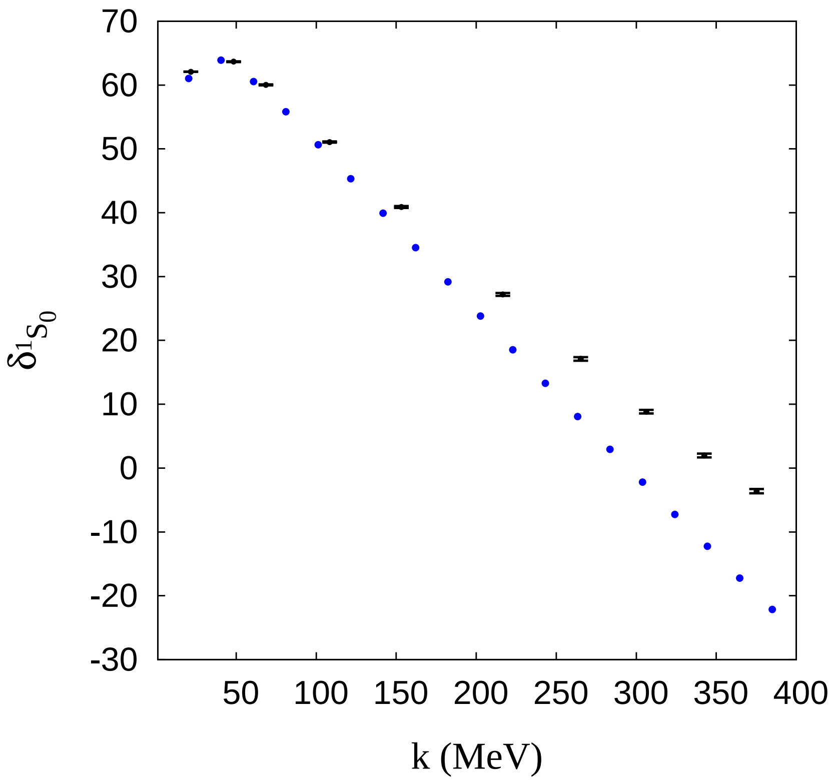
Renormalization procedures

- Three equivalent methods for $\Lambda \to \infty$
  - Renormalization with boundary conditions
  - Renormalization with counter terms
  - Substractive renormalization

  The methods gives the same result
  - Only one renormalization condition for singular attractive case
  - No renormalization condition for singular repulsive case
- The N/D method is equivalent
  - N/D_{11} for singular attractive case
  - N/D_{01} for singular repulsive case
- The N/D method with more substractions can go further

Singular atractive case - NNLO

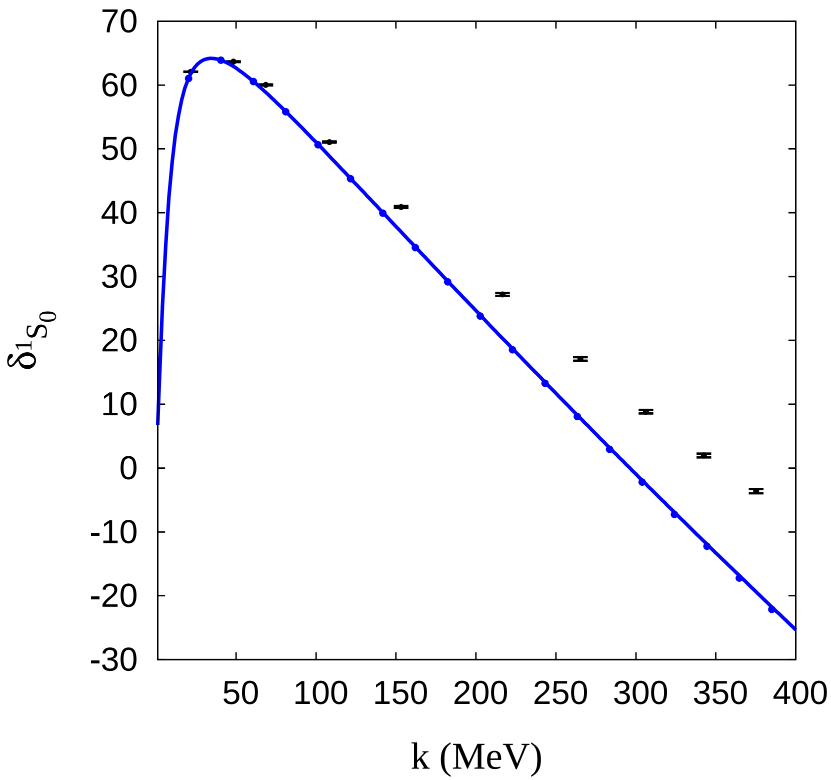
N/D$_{12}$ does not converge



N/D$_{11}$ N/D$_{22}$ • Granada • Substractive renormalization

Singular atractive case - NNLO

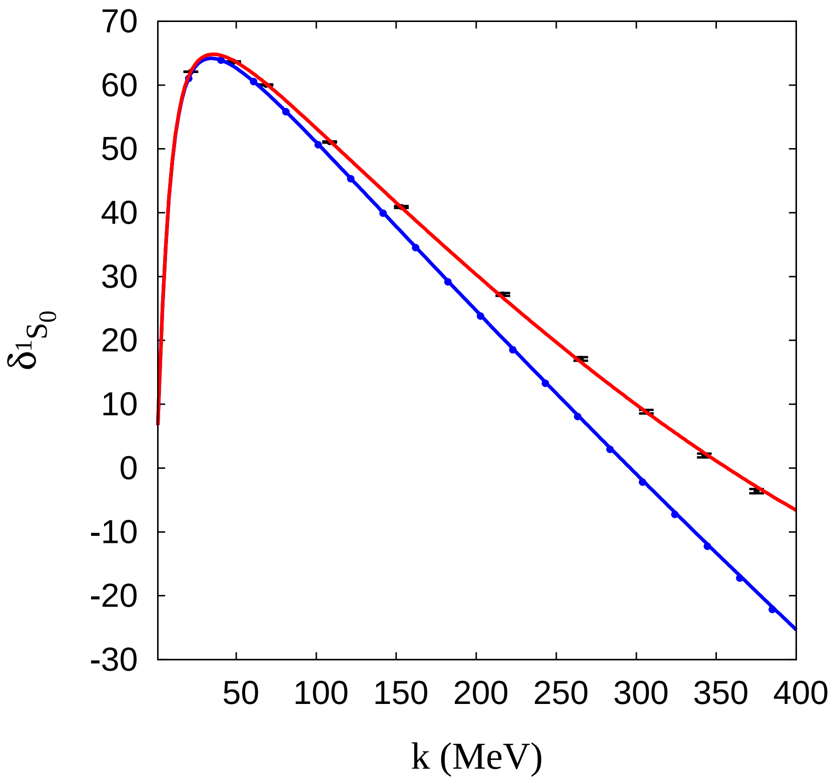
N/D$_{12}$ does not converge



N/D$_{11}$ **N/D$_{22}$** **• Granada** **• Substractive renormalization**

Singular atractive case - NNLO

N/D_{12} does not converge



N/D_{11} N/D_{22} • Granada • Substractive renormalization

Toy model

$$
\begin{aligned}
V(r) &= V_1(r) + V_2(r) + V_3(r) + V_4(r) \\
V_1(r) &= -\alpha \frac{e^{-m_\pi r}}{r} \\
V_2(r) &= \alpha_1 \frac{e^{-2m_\pi r}}{r^3} \\
V_3(r) &= -\alpha_1 (m_2 - 2m_\pi) \frac{e^{-m_1 r}}{r^2} \\
V_4(r) &= -\alpha_1 \frac{e^{-m_2 r}}{r^3}
\end{aligned}
$$

- $V_1(r)$ **long range regular interaction**
- $V_2(r)$ **middle range singular interaction (**$\alpha_1 > 0$ **repulsive,** $\alpha_1 < 0$ **atractive)**
- $V_3(r)$ **and** $V_4(r)$ **short range singular interaction (**$m_1 = 1$ **GeV** $m_2 = 1{,}2$ **GeV)**
- $V(r)$ **regular interaction**

$$
\lim_{r\to 0} V(r) \quad \to \quad \frac{1}{2} \frac{-2\alpha + \alpha_1 (2m_\pi - m_2)(2m_\pi - 2m_1 + m_2)}{r}
$$

Toy model

Consider we only know $V_1(r)$ and $V_2(r)$, Can we describe low energy results of the full $V(r)$?

With renormalization with one counter term (or substractive renormalization) and renormalization with boundary conditions (which are equivalent):

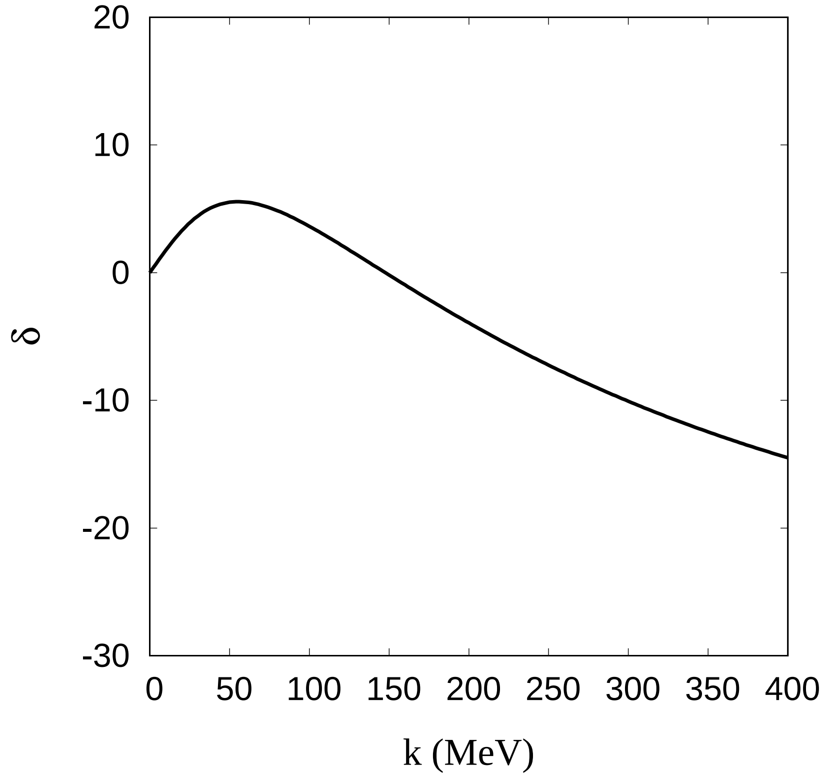
- **If we only consider $V_1(r)$ the interaction is regular and can be renormalized**
- **If we consider $V_2(r)$ the interaction is singular**
  - $\alpha_1 < 0$ **we can fix only one low energy data (scattering length a)**
  - $\alpha_1 > 0$ **we can not fix any low energy data**

This is the main limitation of these renormalization procedures, main criticism of this way to renormalize

Can the exact N/D method with multiple substractions do it better?

The N/D$_{22}$ converges but, has it a physical meaning?

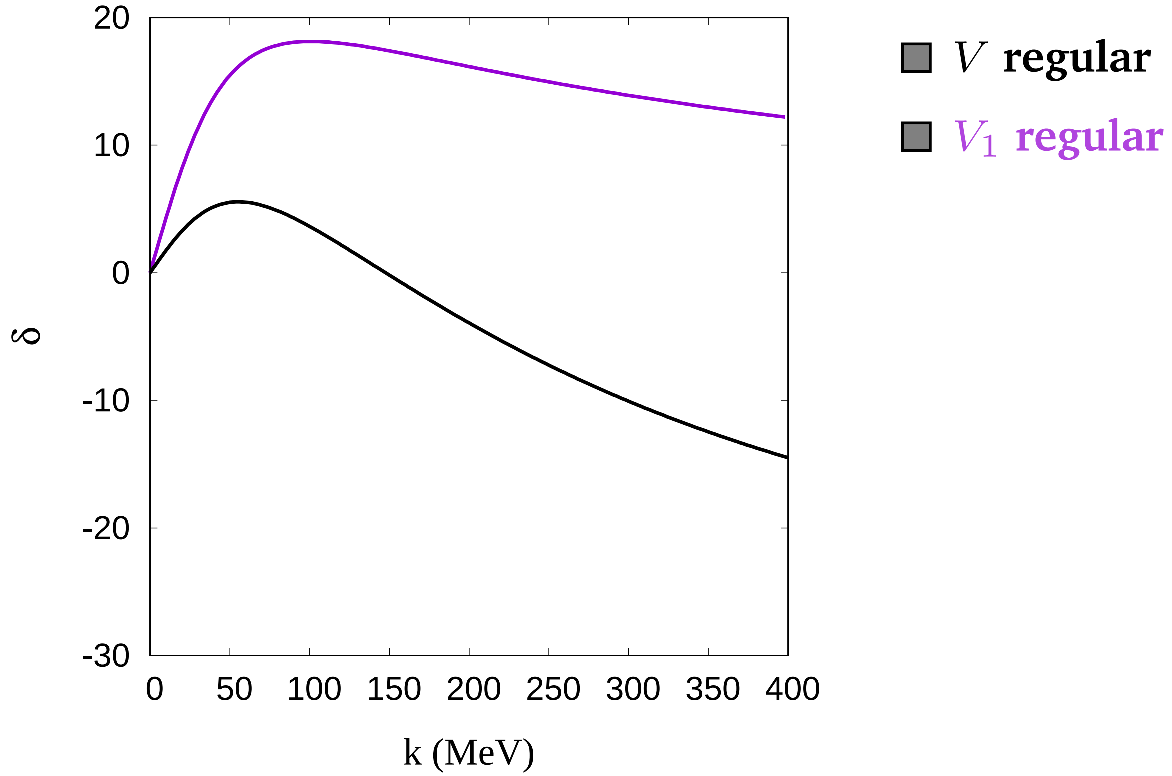
Problem of Renormalization



- V regular

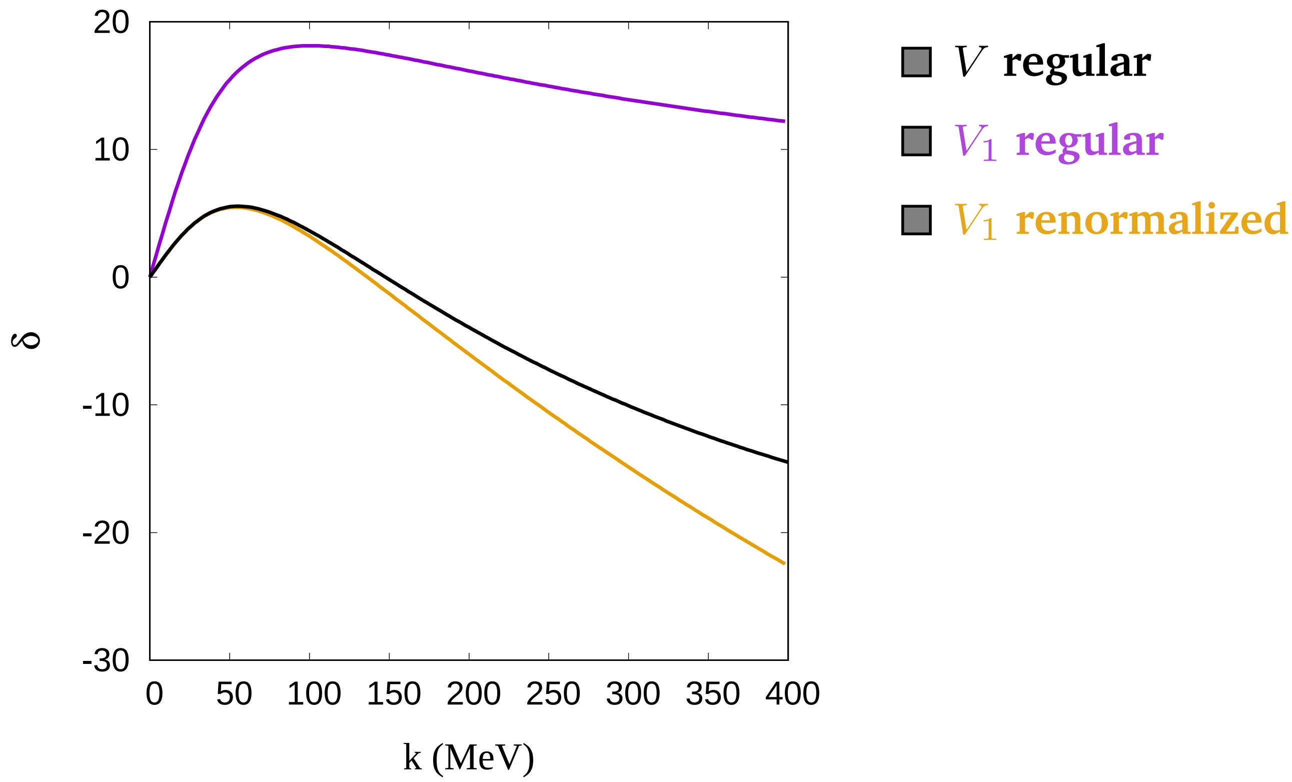
Only one or none renormalization conditions

Problem of Renormalization



Only one or none renormalization conditions

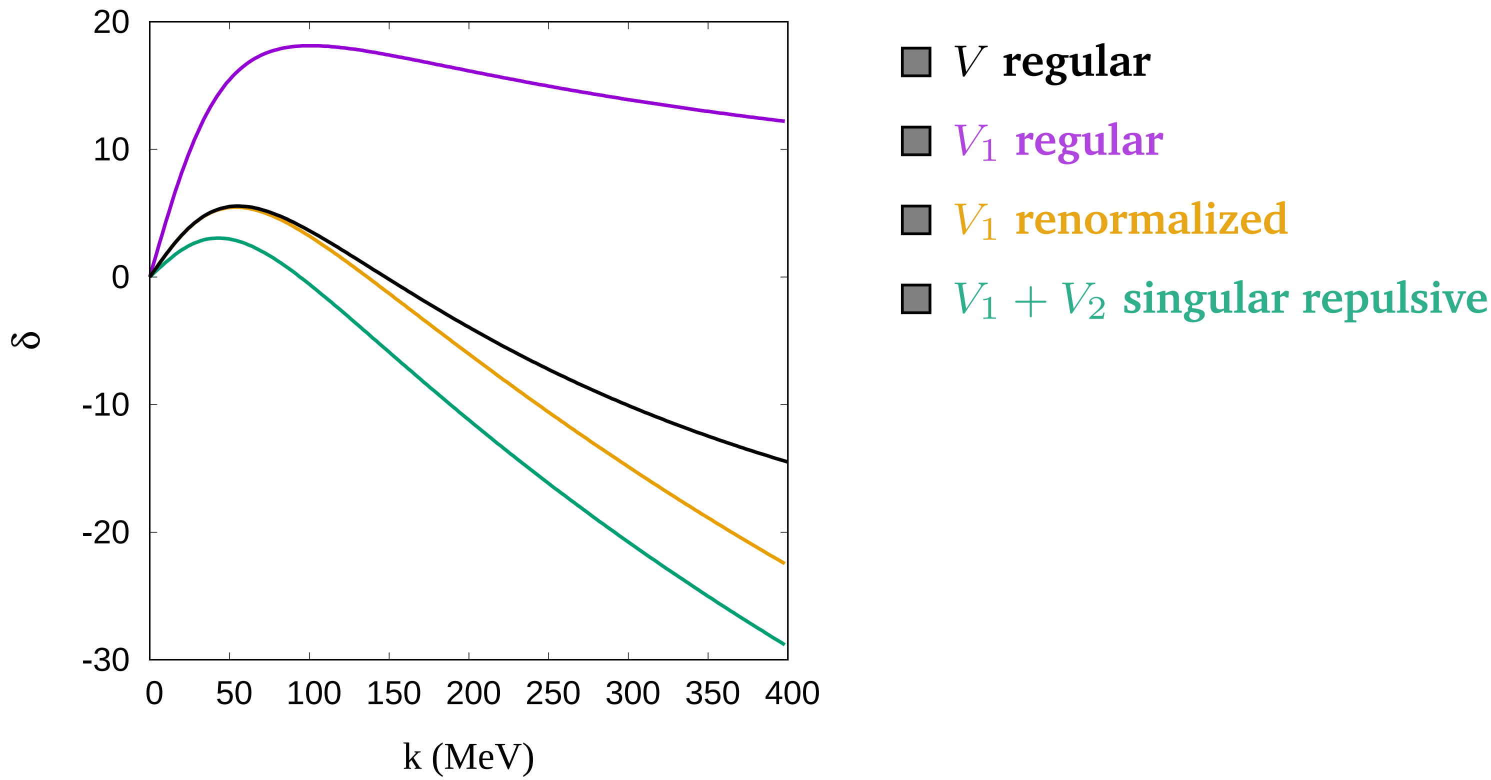
Problem of Renormalization



Only one or none renormalization conditions

Renormalization allow to get the correct low energy behavior

Problem of Renormalization

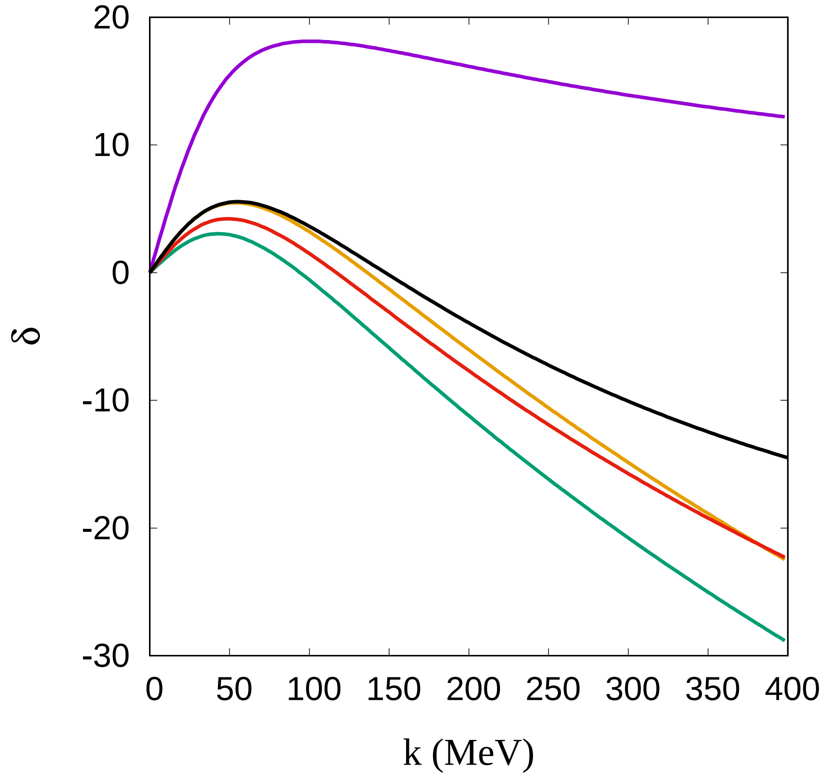


Only one or none renormalization conditions

Renormalization allow to get the correct low energy behavior

Including singular repulsive interactions does not allow to renormalize and even the low energy behavior is wrong

Problem of Renormalization



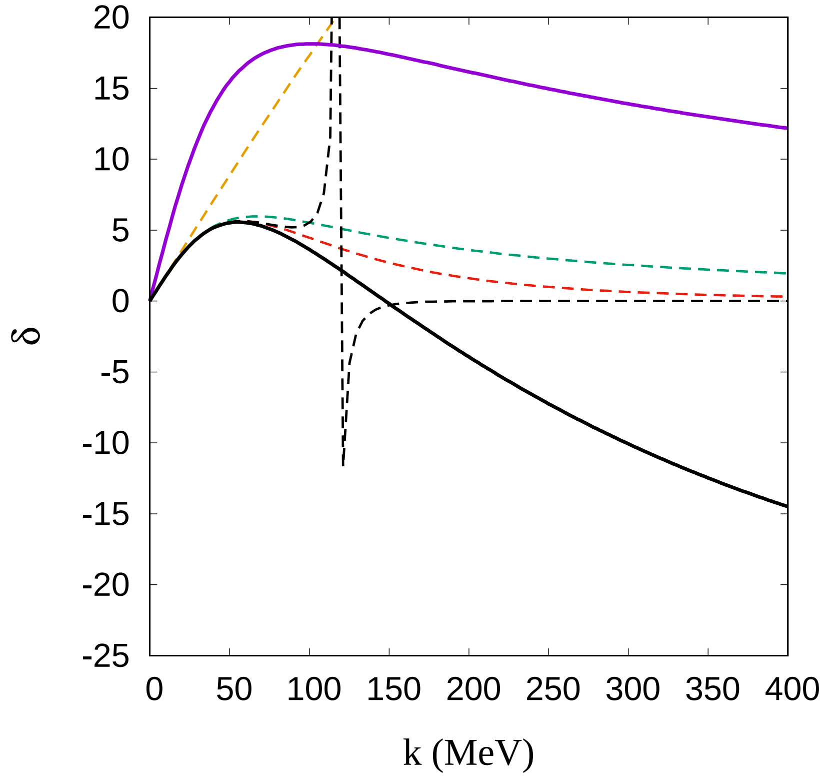
- V regular
- V_1 regular
- V_1 renormalized
- $V_1 + V_2$ singular repulsive
- $V_1 + V_2 + V_3$ singular repulsive

Only one or none renormalization conditions

Renormalization allow to get the correct low energy behavior

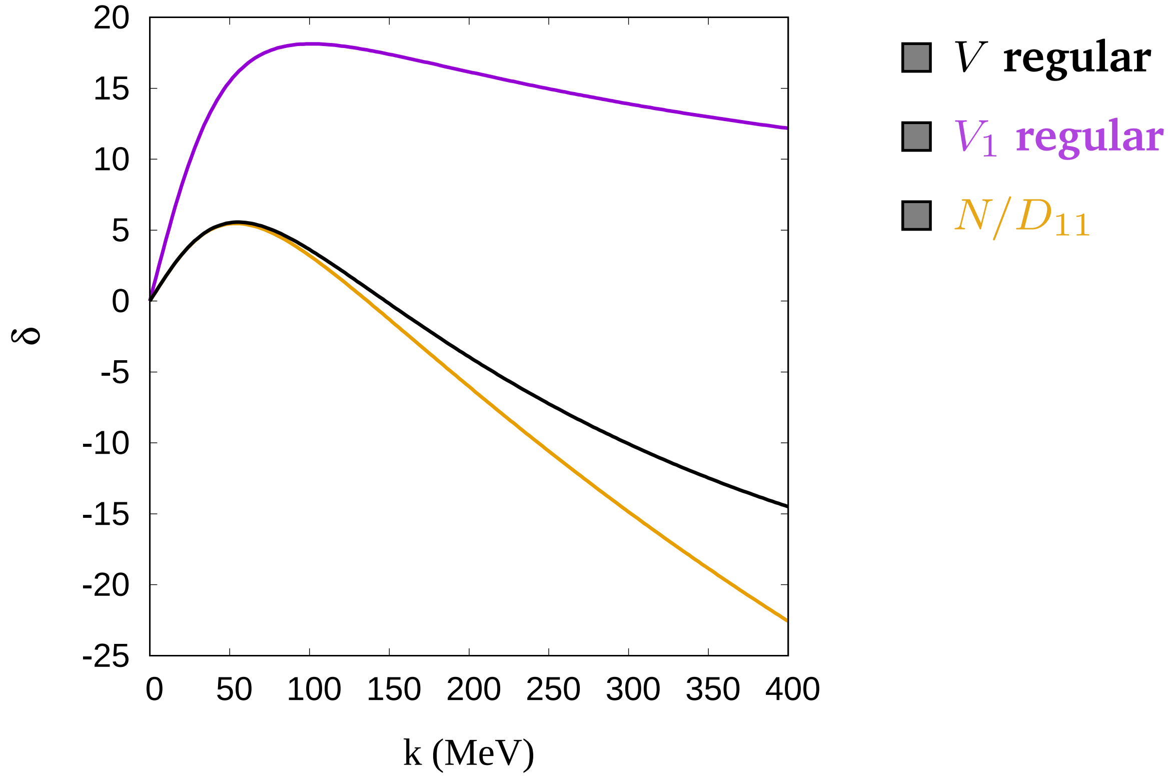
Including singular repulsive interactions does not allow to renormalize and even the low energy behavior is wrong

Only V_1 (non-singular)

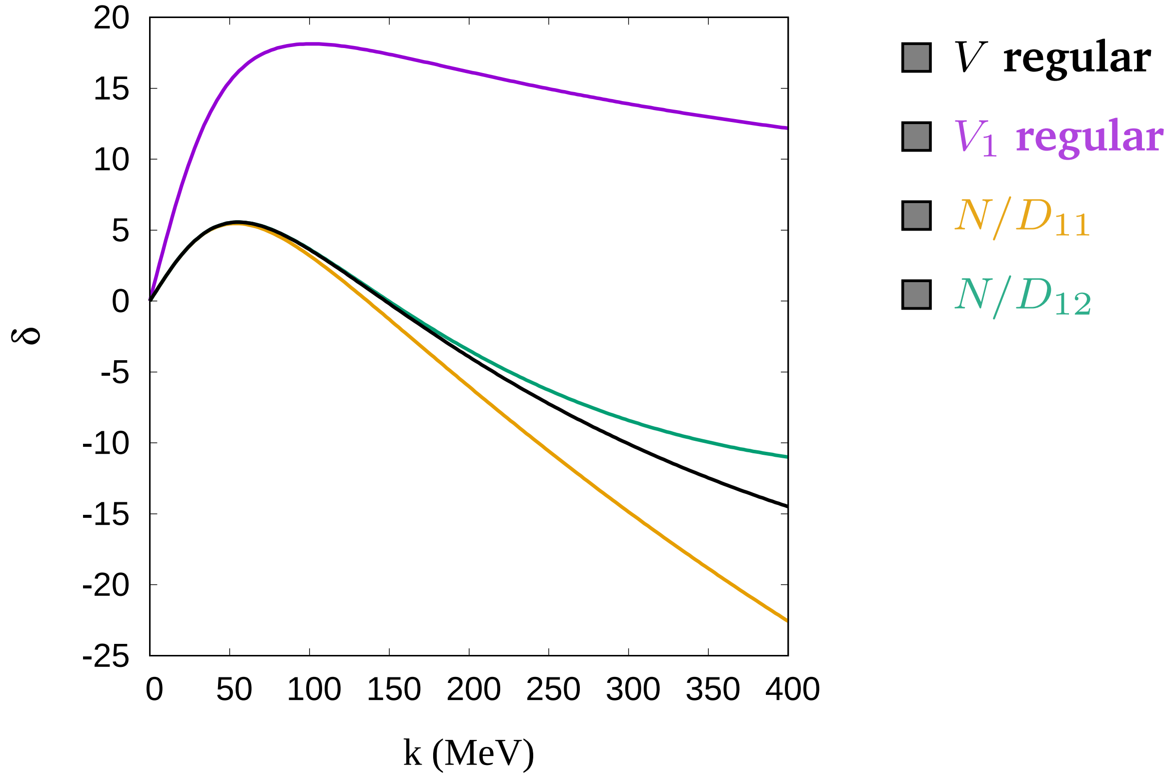


- V **regular**
- V_1 **regular**
- **Dash ERE at** k^{2n}, $n = 0$, $n = 1$, $n = 2$, $n = 6$

Only V_1 (non-singular)

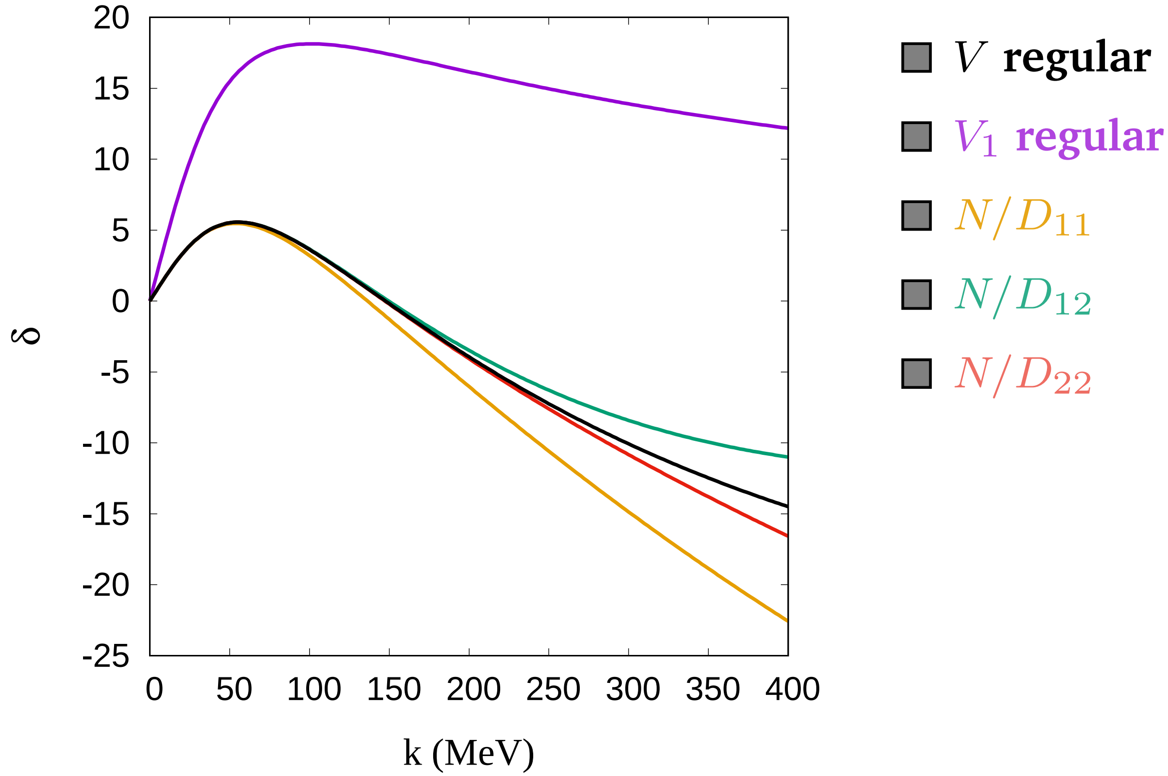


Only V_1 (non-singular)



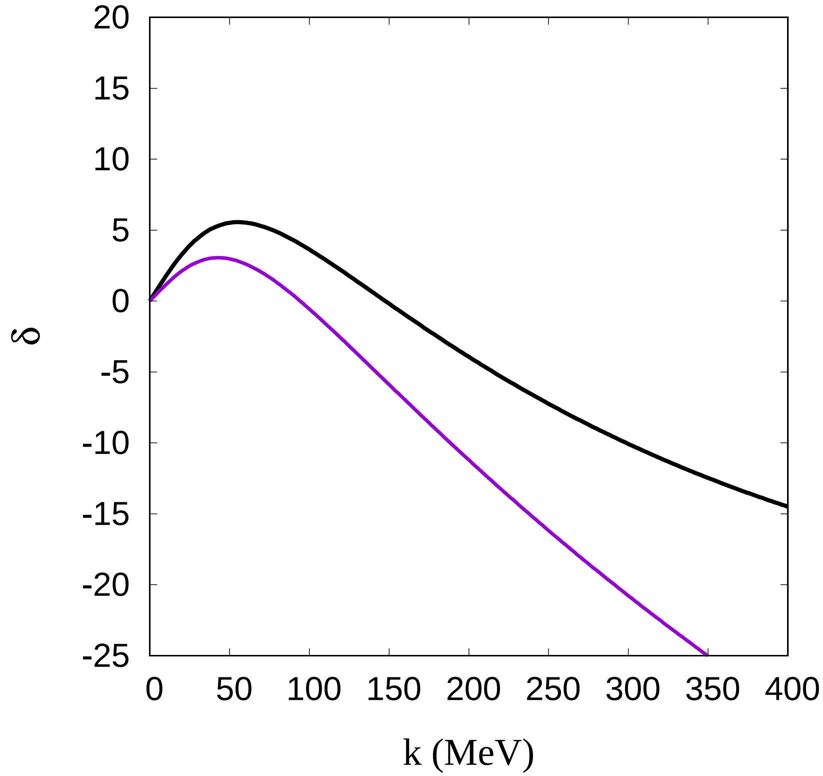
Including more substractions allow to go to higher energies

Only V_1 (non-singular)



Including more substractions allow to go to higher energies

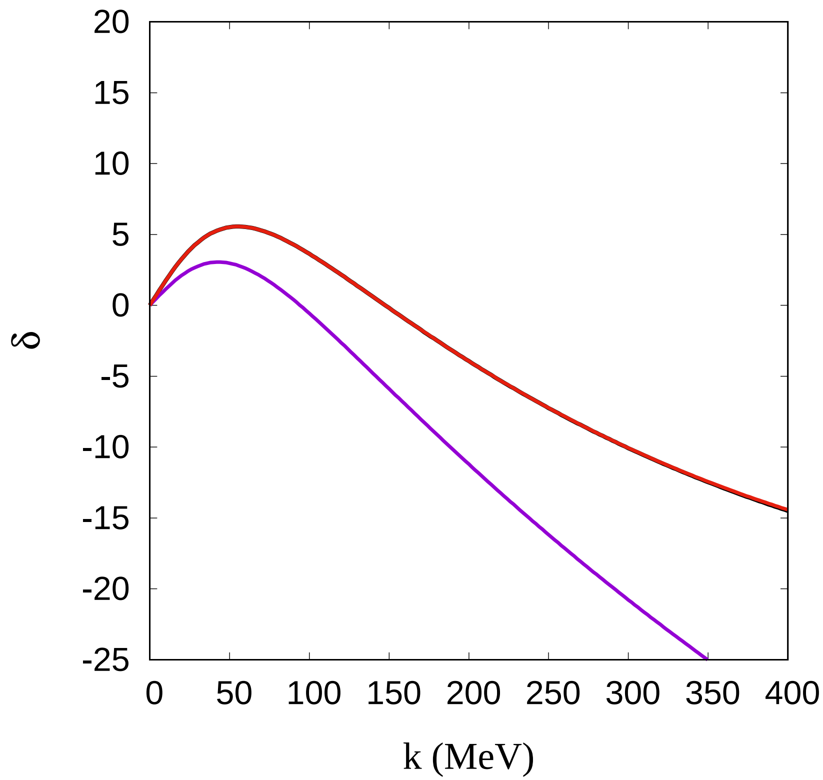
$V_1 + V_2$ (singular repulsive)



- V **regular**
- $V_1 + V_2\ N/D_{01}$
- N/D_{11} **and** N/D_{12} **does not converge**

For singular repulsive case we can not fix only a or a and r

$V_1 + V_2$ (singular repulsive)

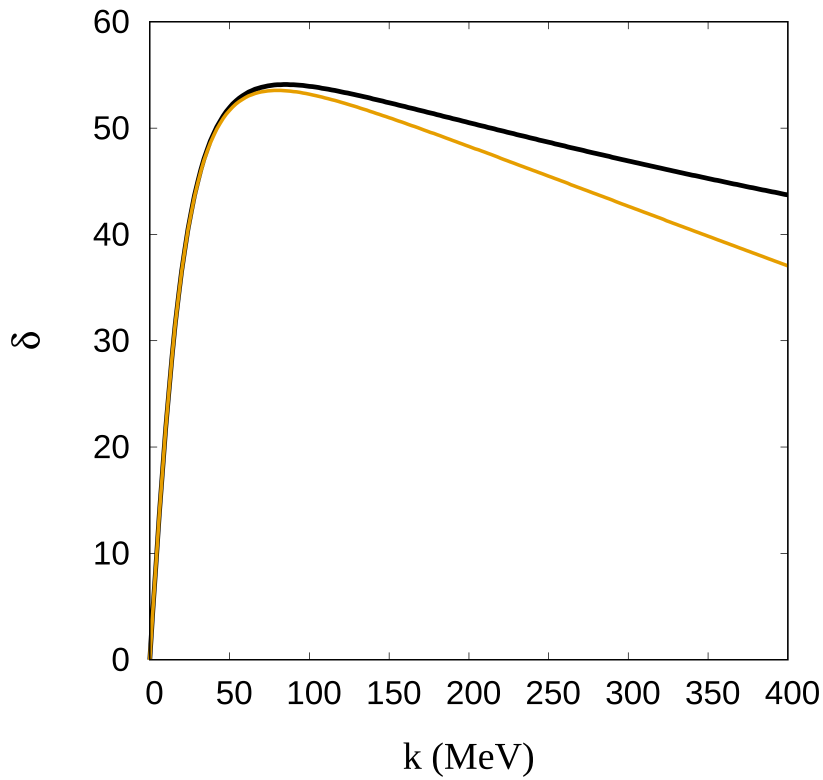


- V regular
- $V_1 + V_2$ N/D_{01}
- N/D_{11} and N/D_{12} does not converge
- N/D_{22}

For singular repulsive case we can not fix only a or a and r

But we can fix a, r and v_2 and the result agrees with the full theory upto 400 MeV

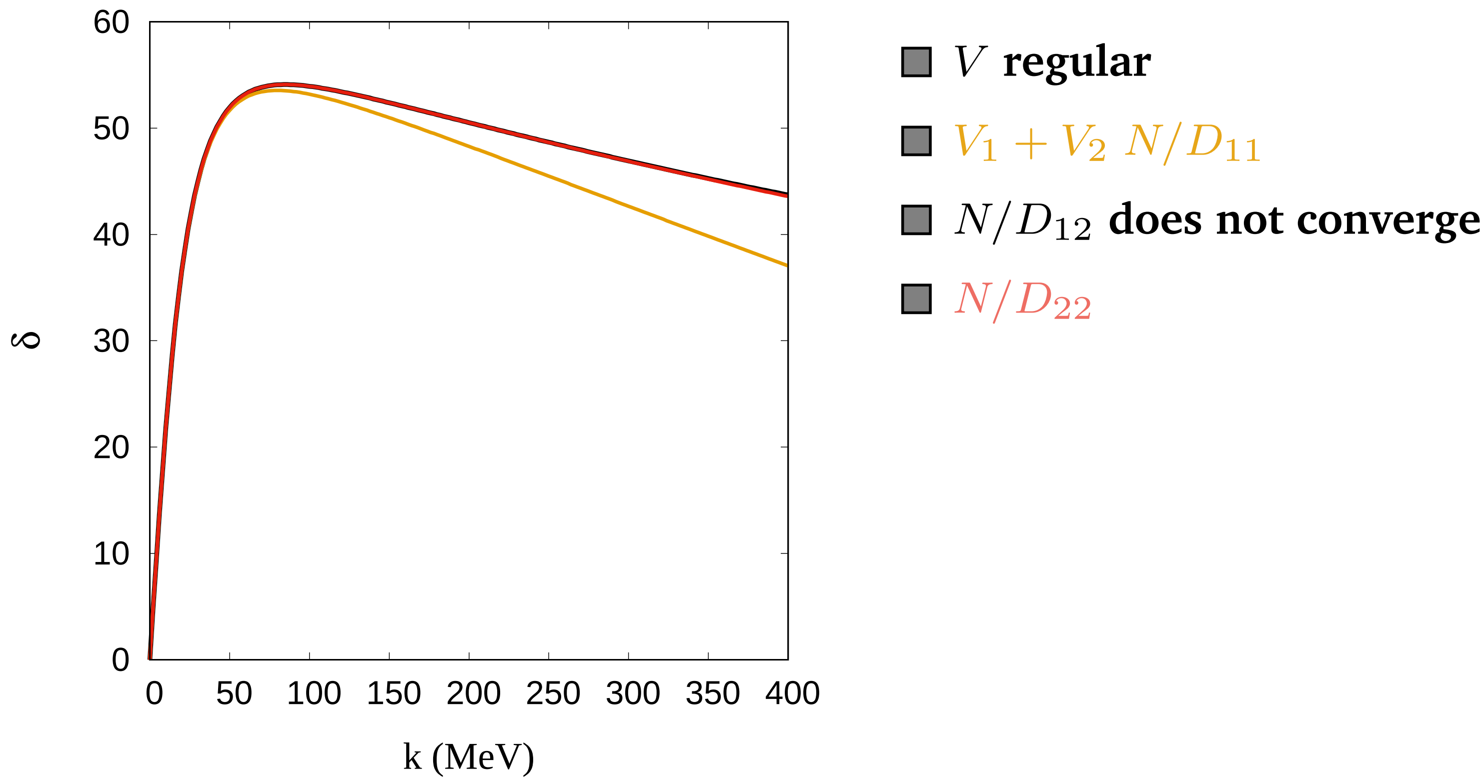
$V_1 + V_2$ (singular attractive)



- V **regular**
- $V_1 + V_2\ N/D_{11}$
- N/D_{12} **does not converge**

For singular atractive case we can fix a but not a and r

$V_1 + V_2$ (singular attractive)



For singular atractive case we can fix a but not a and r

But we can fix a, r and v_2 and the result agrees with the full theory upto 400 MeV

LS with dimensional Reg.

- We follow J. Nieves, Physics Letters B 568, 109 (2003)
- Distorted Wave Theory from T. Barford and M.C. Birse, Phys. Rev. C 67, 064006 (2003) is used
- We consider the finite range regular OPE in the $^1S_0$ partial wave

$$V_\pi(p',p) = -\alpha_\pi \frac{M_N}{2\pi}\frac{1}{p'p}\log\left(\frac{(p'+p)^2+m_\pi^2}{(p'-p)^2+m_\pi^2}\right)$$

- We add zero range contact interactions in the $^1S_0$ partial wave

$$\begin{aligned} V_s(p',p) &= \sum_{s=0}^{n}\sum_{m=0}^{s} g_{m,s-m}\, p'^{2m} p^{2(s-m)} \\ &= g_{00} + g_{01}(p'^2+p^2) + \dots \end{aligned}$$

- The full potential is

$$V(p',p) = V_\pi(p',p) + V_s(p',p)$$

LS with dimensional Reg.

- **The solution from DWT (which is exact) is**

$$T = T_\pi + (I + T_\pi G_0) T_s (I + G_0 T_\pi)$$

- **The solution for the regular OPE potential is needed**

$$T_\pi = V_\pi + V_\pi G_0 T_\pi$$

- **And T_s is given by**

$$T_s = V_s + V_s G_\pi T_s$$

with $G_\pi = G_0 + G_0 T_\pi G_0$

- **Since V_s is separable, the equation for T_s can be solved algebraically with the ansatz**

$$\begin{aligned} T_s(p', p; k) &= \sum_{s=0}^{n} \sum_{m=0}^{3} \alpha_{sm} p'^{2s} p^{2m} \\ &= \alpha_{00} + \alpha_{01}(p'^2 + p^2) + \alpha_{11} p'^2 p^2 + \ldots \end{aligned}$$

LS with dimensional Reg.

- **Since V_s is singular, divergent integrals appear and they are treated using dimensional regularization**
- **All divergent integrals are written in terms of two (using dimensional regularization)**
  - **One convergent**

$$L_0(k) \quad \equiv \quad \int_0^\infty dq \frac{q^2 T_\pi(k,q;k)}{k^2 - q^2 + i\epsilon}$$

  - **One logarithmically divergent**

$$J_0(k) \quad \equiv \quad \int_0^\infty dq' \int_0^\infty dq \frac{q'^2 q^2 T_\pi(q',q;k)}{(k^2 - q'^2 + i\epsilon)(k^2 - q^2 + i\epsilon)}$$

  - **We evaluate $\bar{J}_0(k) \equiv J_0(k) - J_0(0)$ which is convergent**

LS with dimensional Reg.

- **The solution is**

$$T(k) = T_\pi(k) + \frac{(1+L_0(k))^2}{V_s^{-1} + i\frac{\pi k}{2} - \bar{J}_0(k) - J_0^R}$$

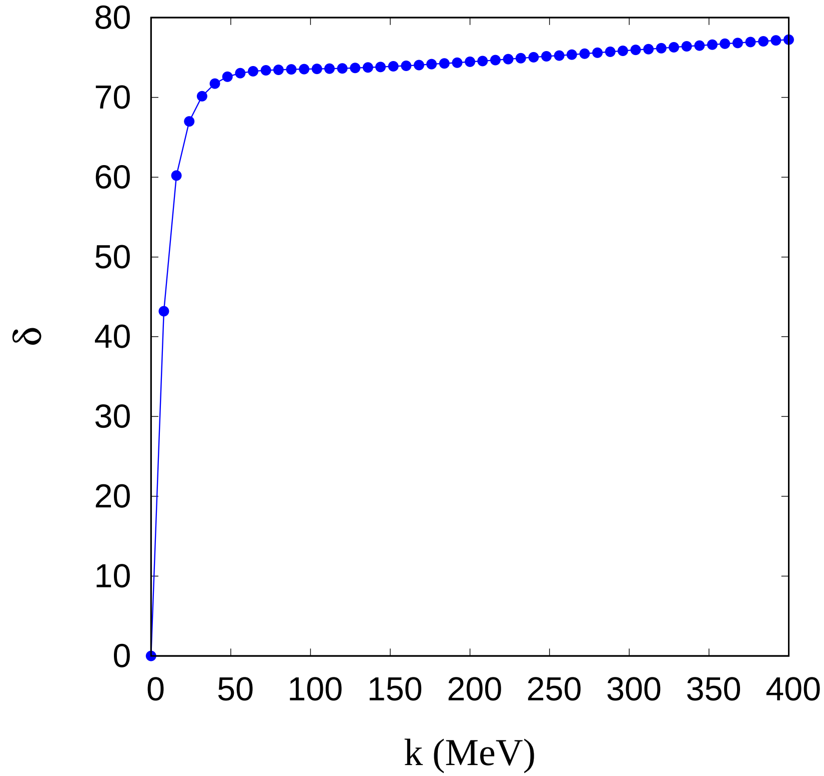
$$V_s^{-1} \equiv \left[g_{00} + 2g_{01}(k^2 - \eta) + \dots\right]^{-1}$$

  with $\eta \equiv M_N m_\pi \alpha_\pi$ **and** $J_0^R \equiv J(0)$

- **For** $|k| < \frac{m_\pi}{2}$ **it is straight forward to do the analytical continuation to complex** k
- **Original work** $g_{00} = g_0$ **and** $g_{01} = g_1$ **and the case with only one contact is included**
- **We have generalized upto** $n = 3$ **(contacts**$\sim Q^6$**)**

Only one contact

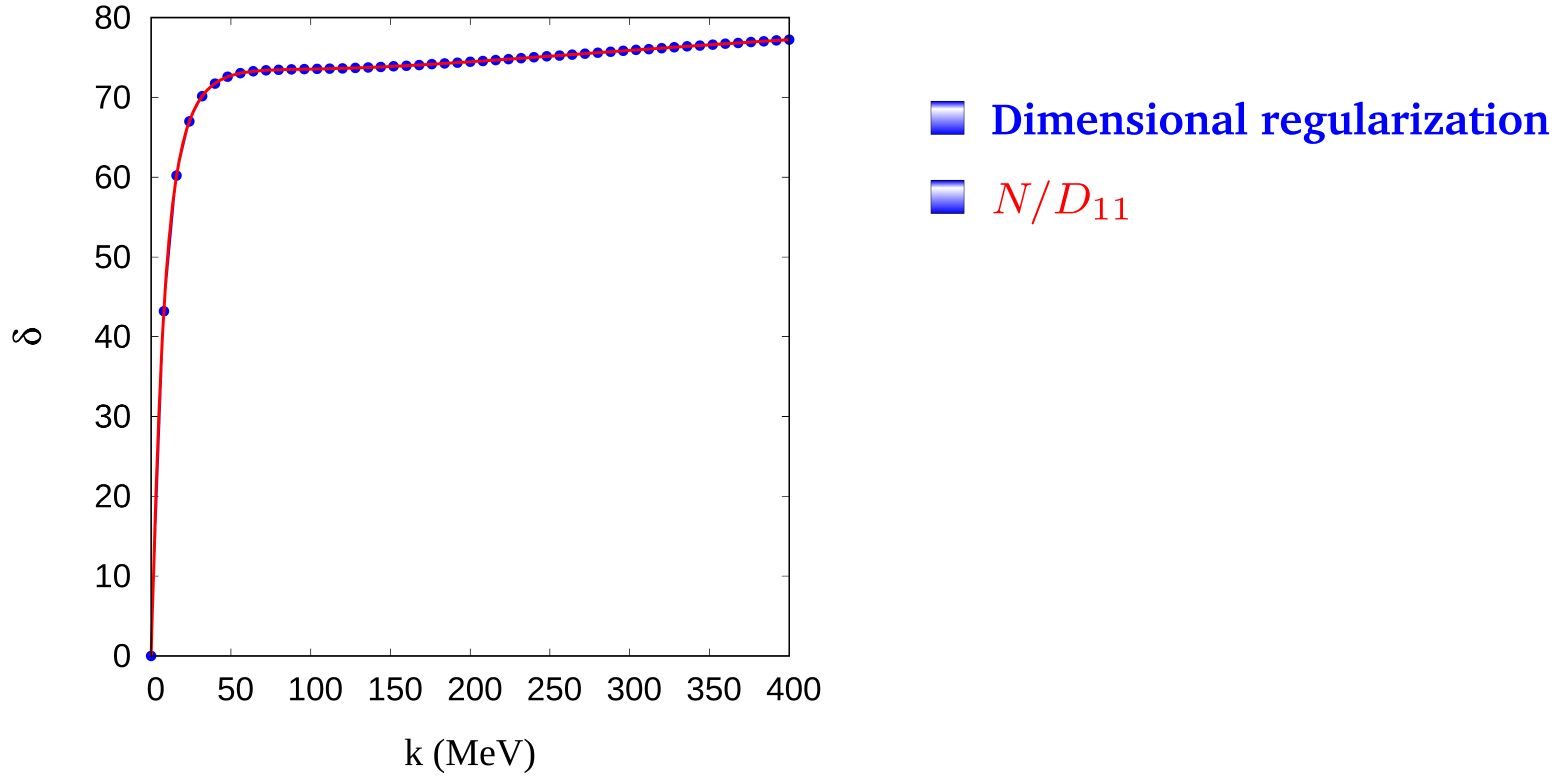
- We only have one free parameter $\delta_0 \equiv \frac{1}{g_{00}} - J_0^R$
- We can fit δ_0 to the scattering length a and we get



- Dimensional regularization

Only one contact

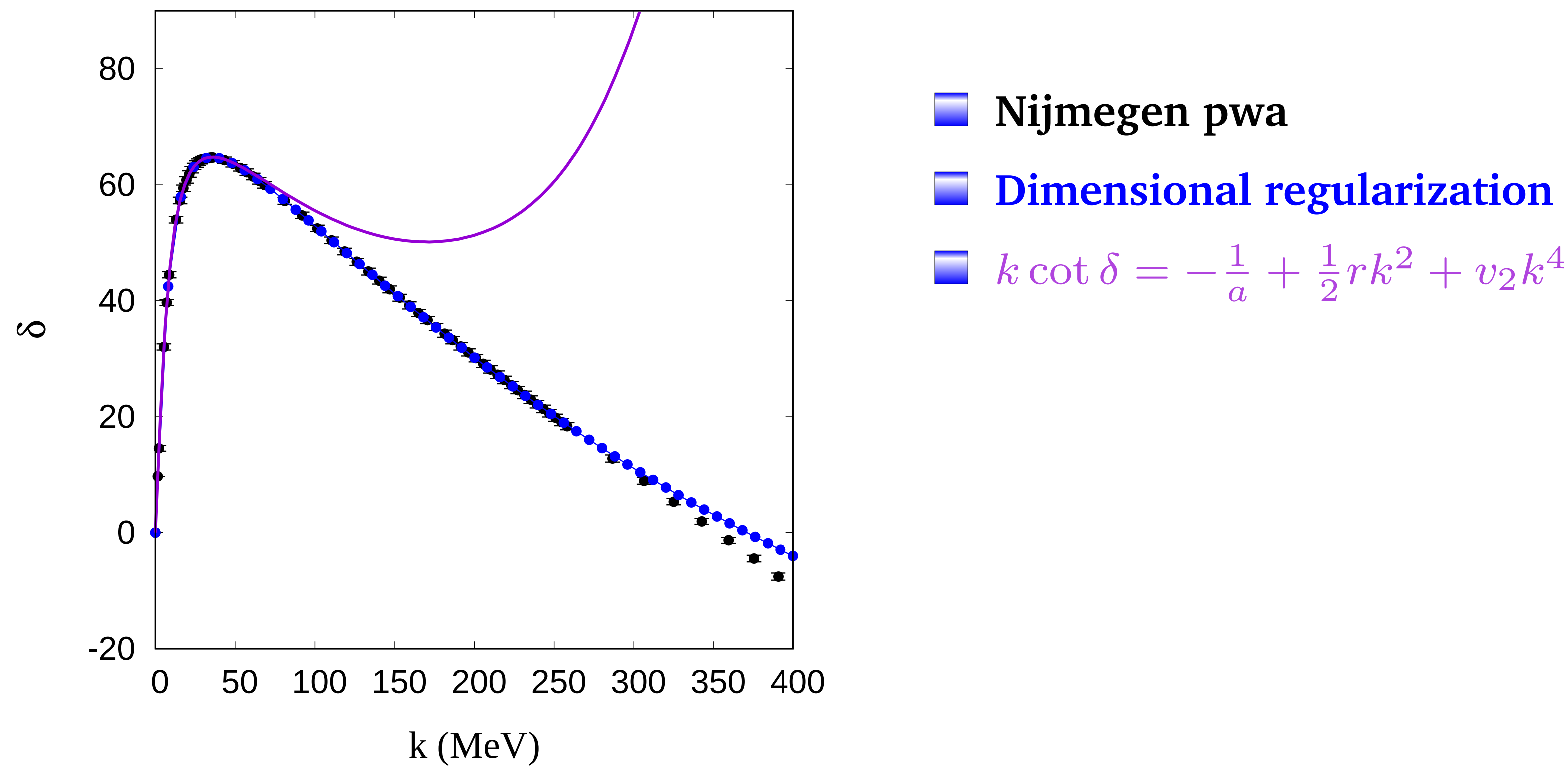
- **We only have one free parameter** $\delta_0 \equiv \frac{1}{g_{00}} - J_0^R$
- **We can fit** δ_0 **to the scattering length** a **and we get**



- **Previous works (Pavón-Valderrama and Ruiz-Arriola in coordinate space and Nogga, Timmermans and van Kolck in momentum space) given by** N/D_{11} **agree**

Two contacts

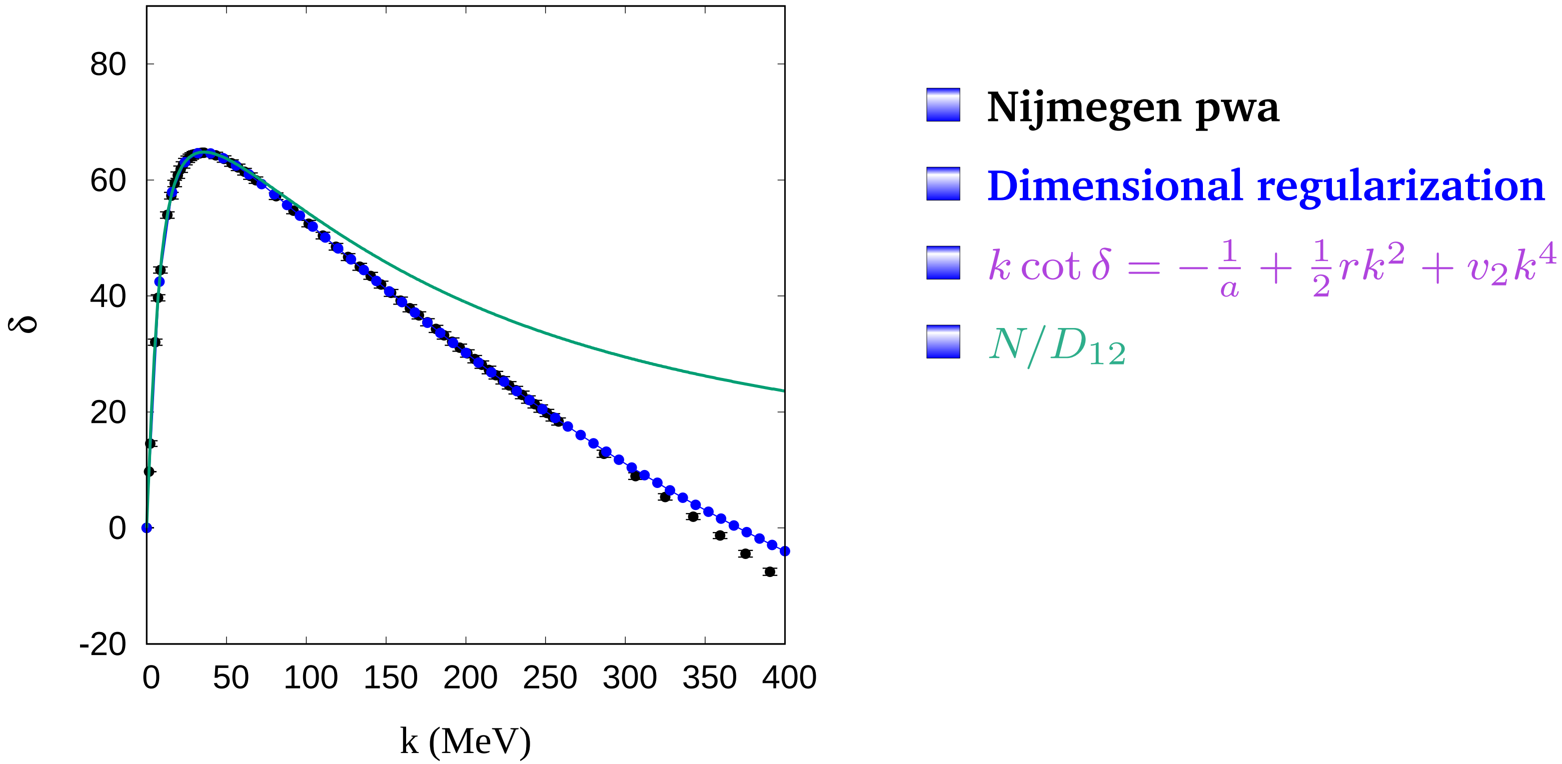
- **Three parameters** $g_0 m = -0{,}2762$, $g_1 m^3 = 0{,}3470$ **and** $J_0^R/m = -3{,}207$ **from original calculation**



	a (fm)	r (fm)	v_2 (fm^3)	v_3 (fm^5)	v_4 (fm^7)	v_5 (fm^9)	v_6 (fm^{11})
DR	-23.7588	2.67286	-0.571399	5.00026	-29.2871	185.602	-1224.71

Two contacts

- **Three parameters $g_0 m = -0{,}2762$, $g_1 m^3 = 0{,}3470$ and $J_0^R/m = -3{,}207$ from original calculation**



	a (fm)	r (fm)	v_2 (fm^3)	v_3 (fm^5)	v_4 (fm^7)	v_5 (fm^9)	v_6 (fm^{11})
DR	**-23.7588**	**2.67286**	**-0.571399**	**5.00026**	**-29.2871**	**185.602**	**-1224.71**
N/D_{12}	**-23.7588**	**2.67286**	**-0.838494**	**4.57628**	**-27.7419**	**177.298**	**-1176.81**

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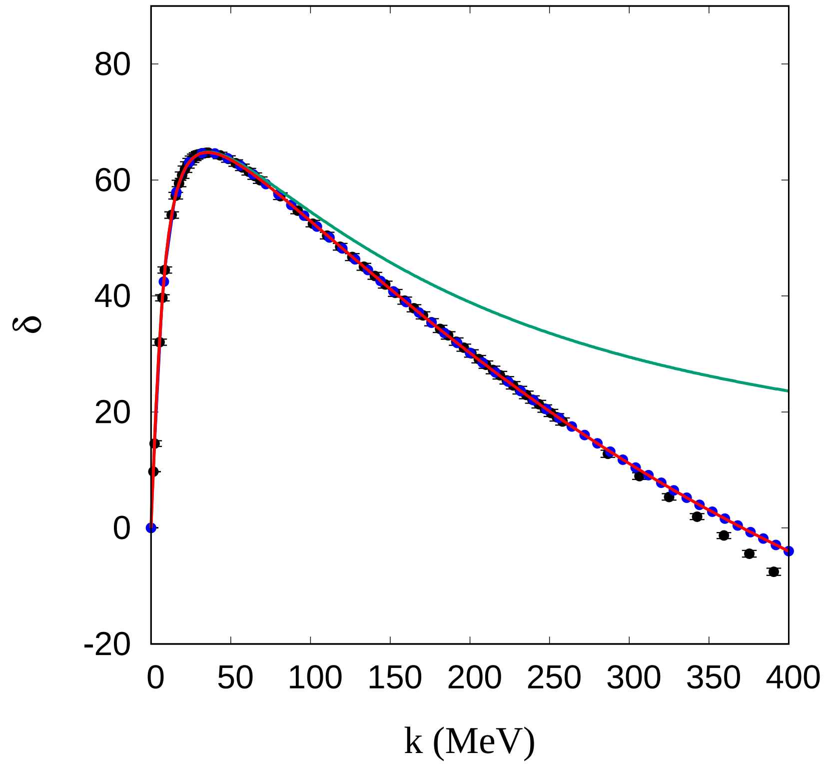

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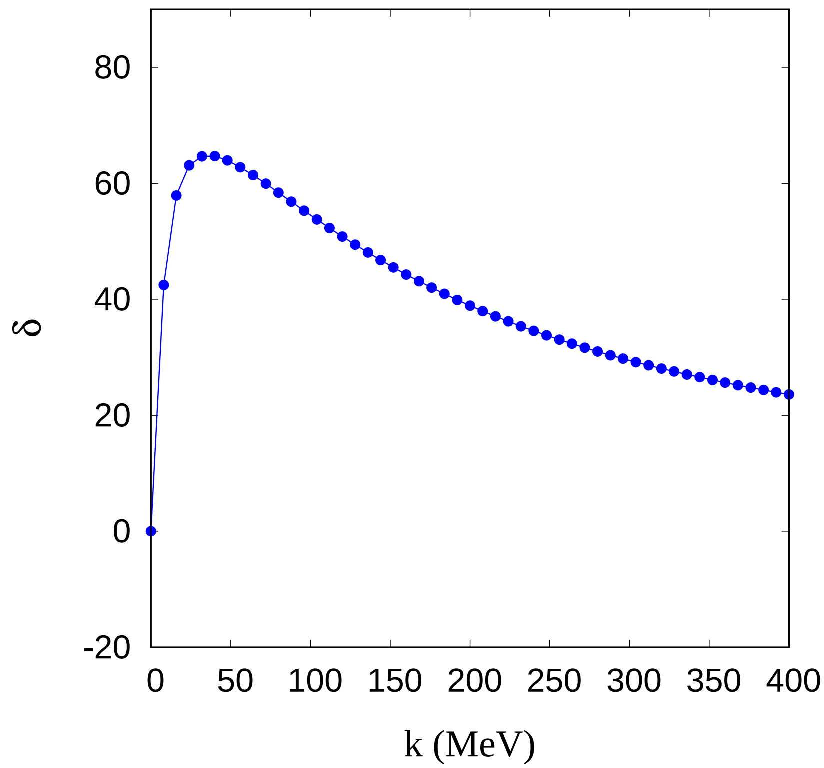
N/D_{22}	-23.7588	2.67286	-0.571399	5.00026	-29.2871	185.602	-1224.71

Two contacts

- If we consider $J_0^R \to \infty$ we can fix a and r with g_0 and g_1



- **Dimensional regularization fixing a and r and $J_0^R \to \infty$**

Two contacts

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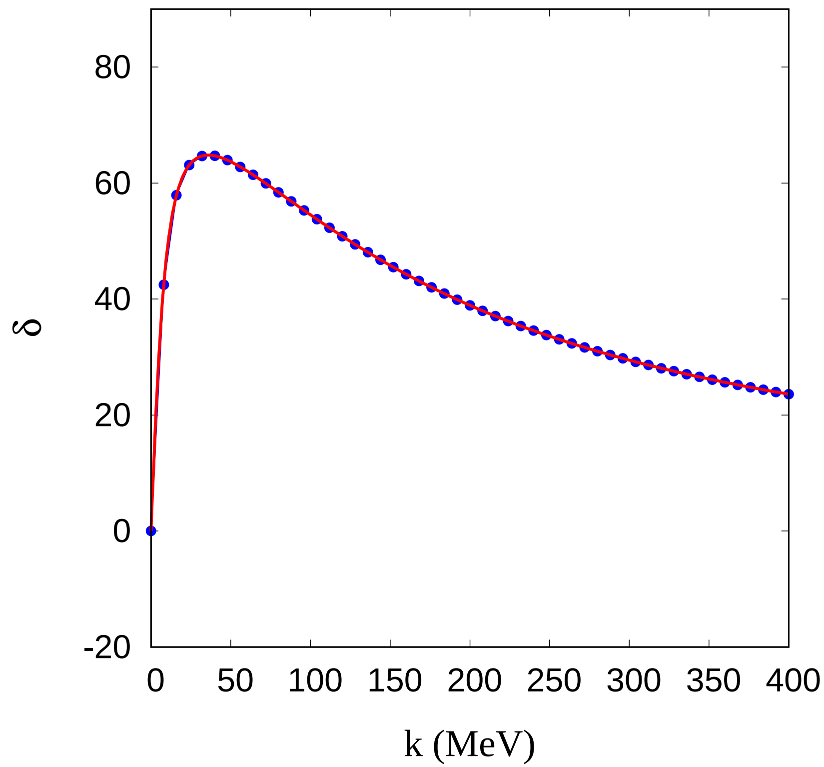
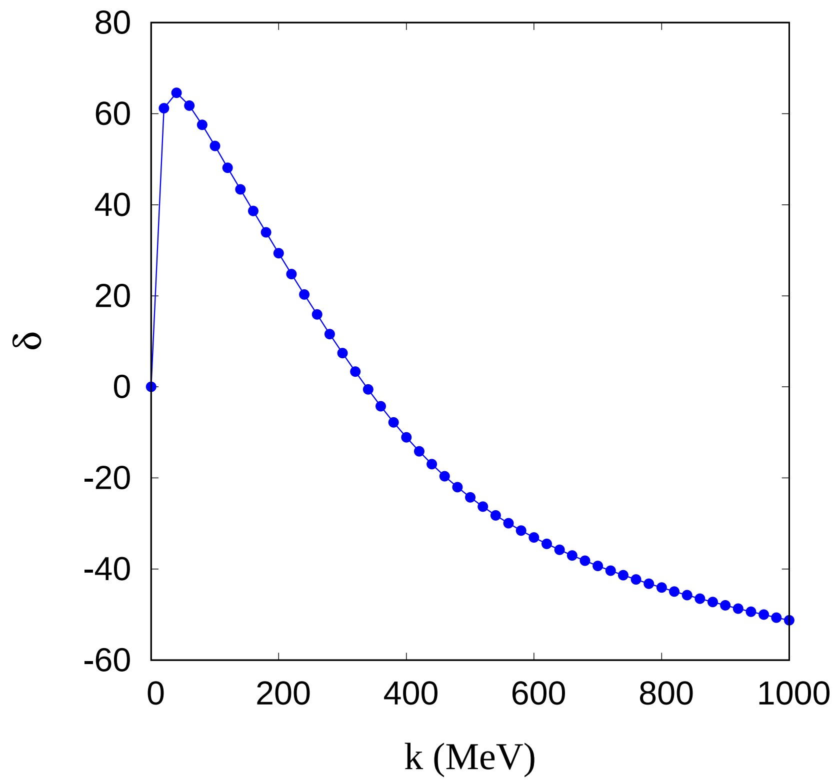


- **Dimensional regularization fixing a and r and $J_0^R \to \infty$**
- N/D_{12}

The low energy constants in both calculations perfectly agree

$n = 3$

- **We use** $J_0^R/m = -3{,}207$ **and fix** a**,** r**,** v_2 **and** v_3

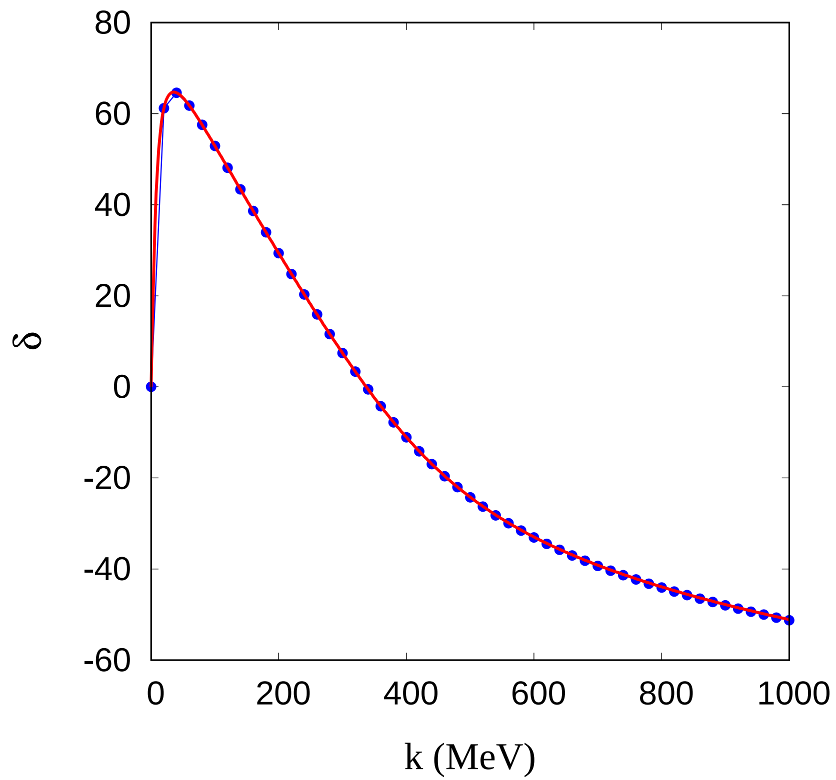


- **Dimensional regularization**

$$
\begin{aligned}
a &= -23{,}7588\,\mathrm{fm} \\
r &= 2{,}67286\,\mathrm{fm} \\
v_2 &= -0{,}571399\,\mathrm{fm}^3 \\
v_3 &= 5{,}02179\,\mathrm{fm}^5 \\
v_4 &= -29{,}2442\,\mathrm{fm}^7 \\
v_5 &= 185{,}489\,\mathrm{fm}^9 \\
v_6 &= -1224{,}08\,\mathrm{fm}^{11} \\
v_7 &= 8328{,}23\,\mathrm{fm}^{13} \\
v_8 &= -57993{,}3\,\mathrm{fm}^{15} \\
v_9 &= 411258\,\mathrm{fm}^{17}
\end{aligned}
$$

$n = 3$

- **We use $J_0^R/m = -3{,}207$ and fix a, r, v_2 and v_3**



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\end{aligned}
$$

- N/D_{44}

The low energy constants in both calculations perfectly agree

Summary

- Renormalization methods for singular interactions were limited to one or none renormalization conditions
- The exact N/D method
  - The N/D_{01} method with an exact $\Delta(A)$ is equivalent to the Lippmann-Schwinger Eq. for regular interactions. For singualar interactions
    - Singular attractive N/D_{11}
    - Singular repulsive N/D_{01}
  - $\Delta(A)$ has only contributions from finite range interactions and the effect of contact terms is taken through substractions in the N/D equations
  - The IE for $\Delta(A)$ is always finite and no regularization is needed even for singular interactions
  - For singular interactions the N/D method with more substractions improves the low energy behavior for the toy model
  - The integral equation for D does not fulfill the Fredholm conditions and we don't know a priori if a solution exist.

Questions

- In the toy model a, r and v_2 were known
- In the real case these are not known
- Let's say that we generate pseudo-data with the full-model
- The question is: What should be the priori for the fit? considering that V_1 and V_2 are exactly known and we want to extract a, r and v_2 from the data

- For the regular OPE potential the N/D method is equivalent to the LS equation with singular contact terms treated with DR
- Can we obtain similar relations for the singular case?