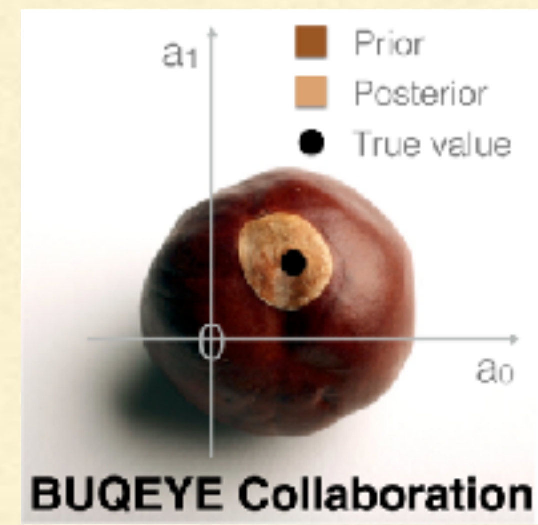
Improving nuclear force inference with correlated models of truncation errors

Daniel Phillips



Research supported by the US DoE, the NSF Office of Advanced Cyberinfrastructre, and the Swedish Research Council

Falsifying claims about an EFT expansion for observables

Consider χEFT, where we have two light scales, p and m_π

- General χEFT series for observable to order k: $y = y_{\rm ref} \sum_{n=0}^{k} c_n(p_{\rm typ}/m_\pi) Q^n$:

$$Q = \frac{(p_{\rm typ}, m_\pi)}{\Lambda_b}; \quad \Lambda_b \approx 600 \text{ MeV}$$

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I will argue that we can:

- Make a probabilistic definition of "The c_n are order 1"
- Falsify claims that all orders behave in the same way
- Estimate the Λ_b that makes the c_n as similar in size as possible
- Check what c_n is a function of (E? p? E and theta? p and q?)
- Test different assumptions for how soft scales appear in Q

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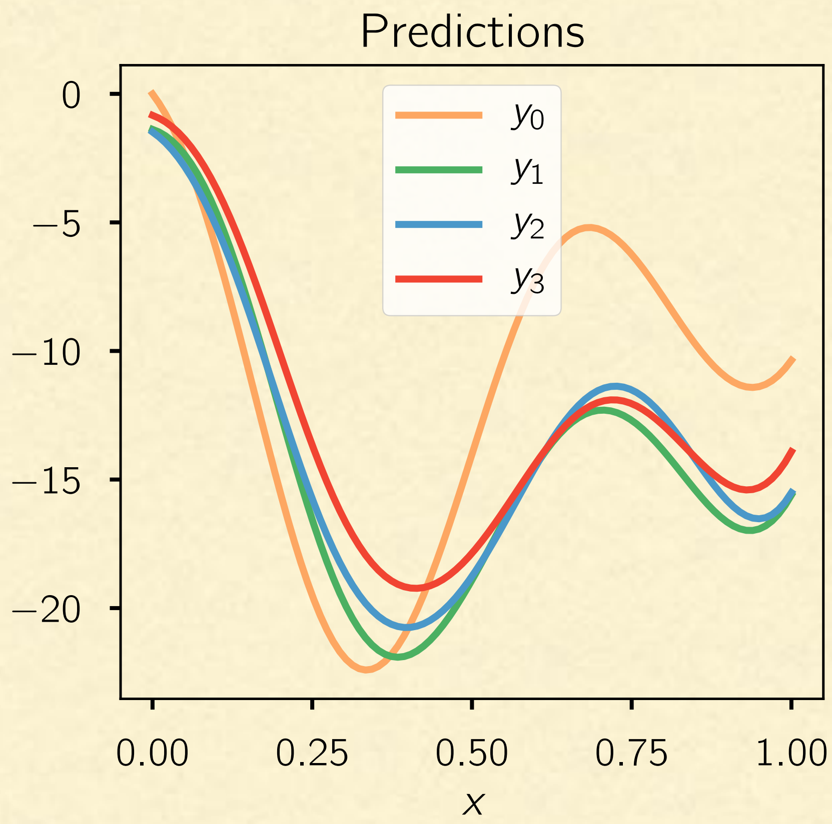

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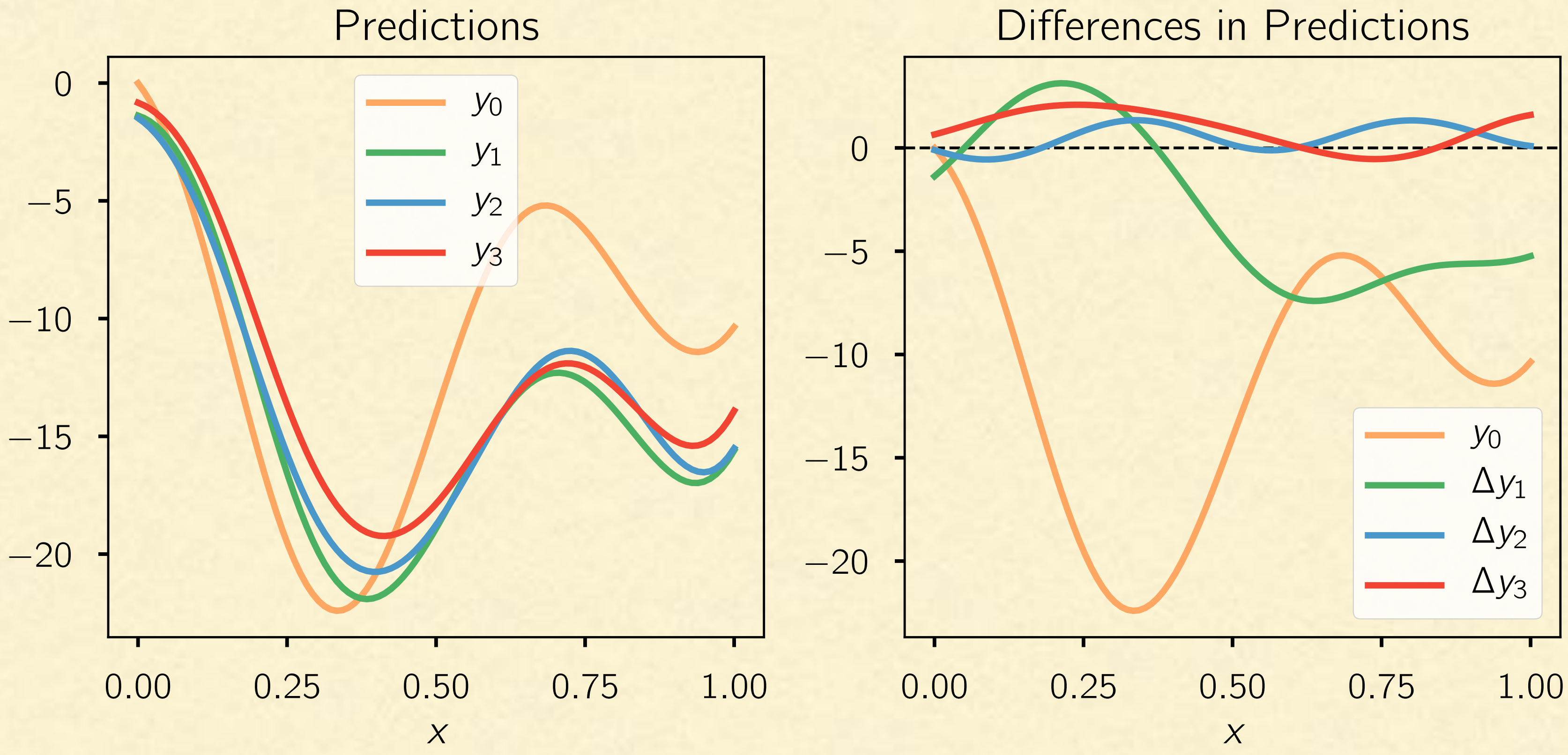

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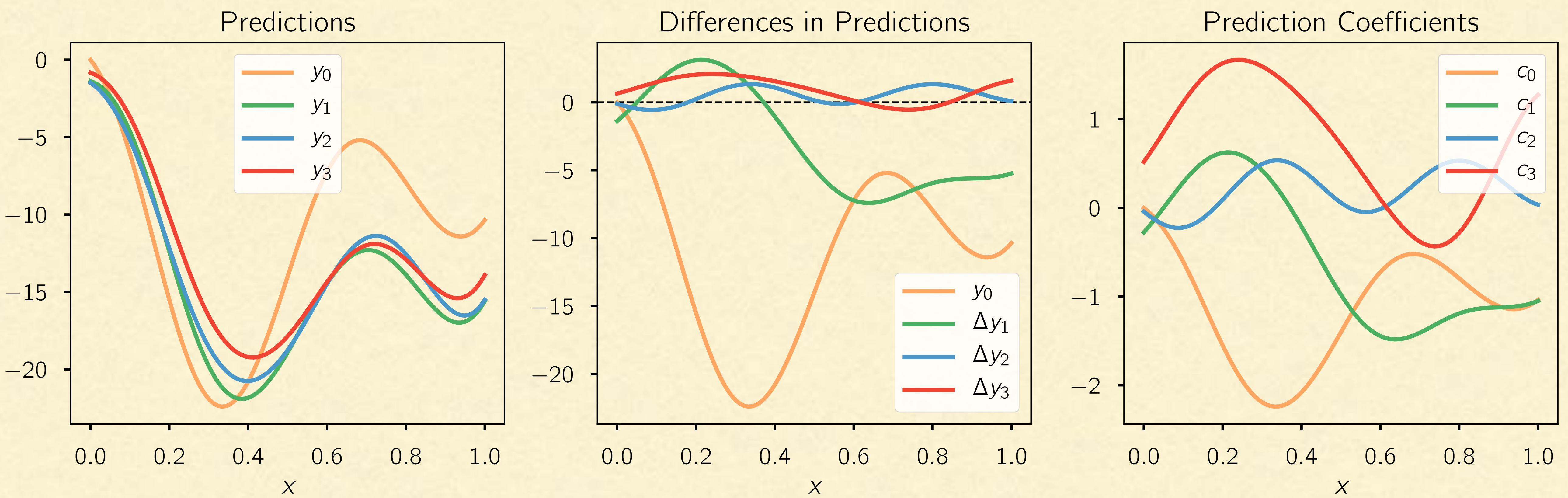

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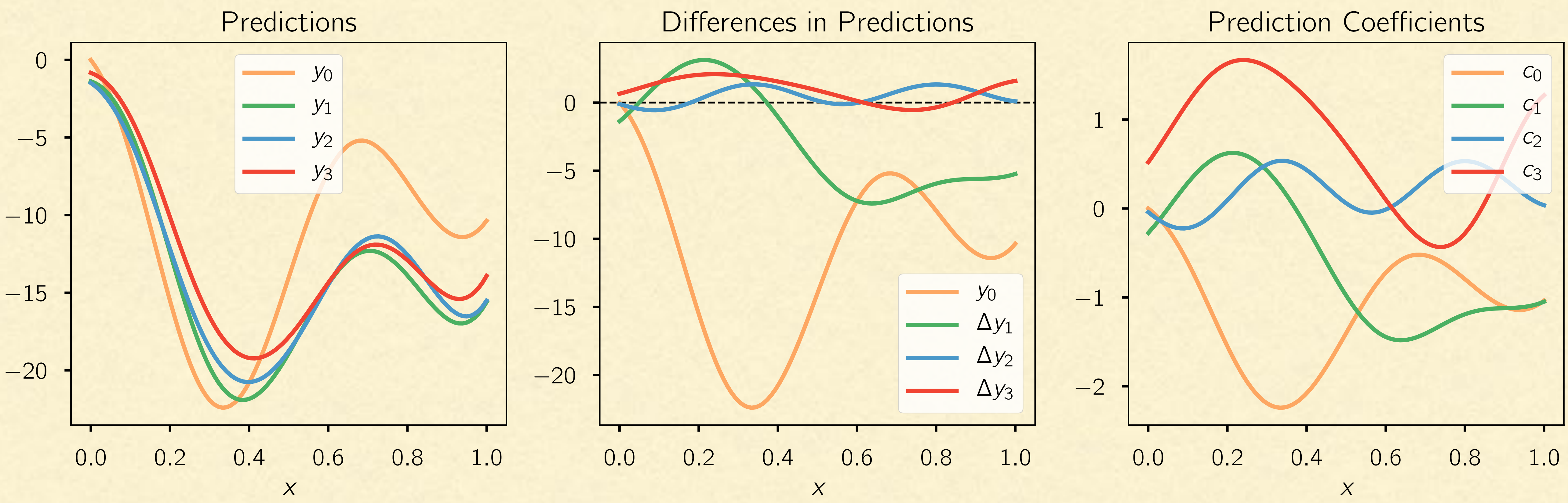

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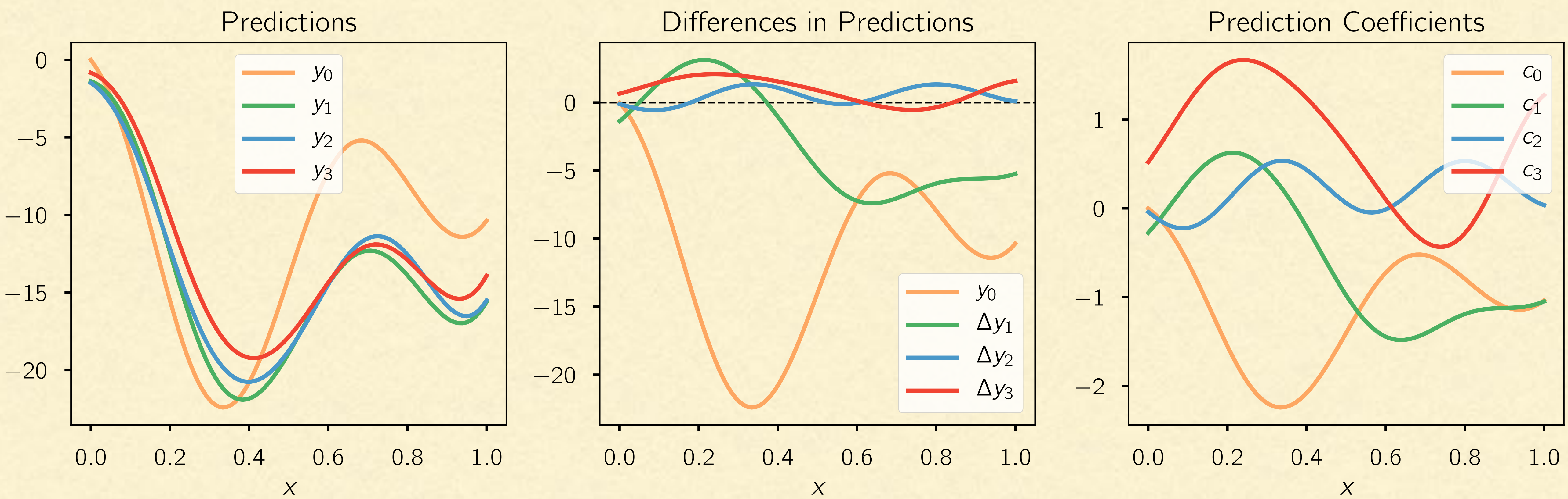

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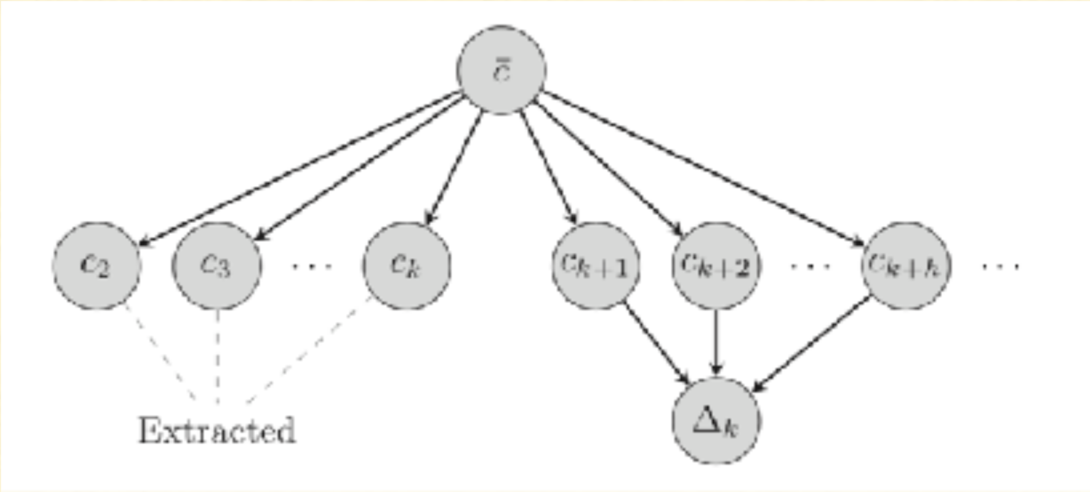
Data for analysis: EFT predictions at different orders across “input space”

A statistical model for EFT coefficients

Furnstahl, Klco, DP, Wesolowski, PRC (2015); after Cacciari & Houdeau, JHEP (2011)

$$y = y_{\rm ref} \sum_{n=0}^{k} c_n Q^n:$$



If we know the distribution from which c_0, c_1, …, c_k are drawn we can predict size of truncation error

Parameter $\bar{c}$ sets size of all dimensionless coefficients

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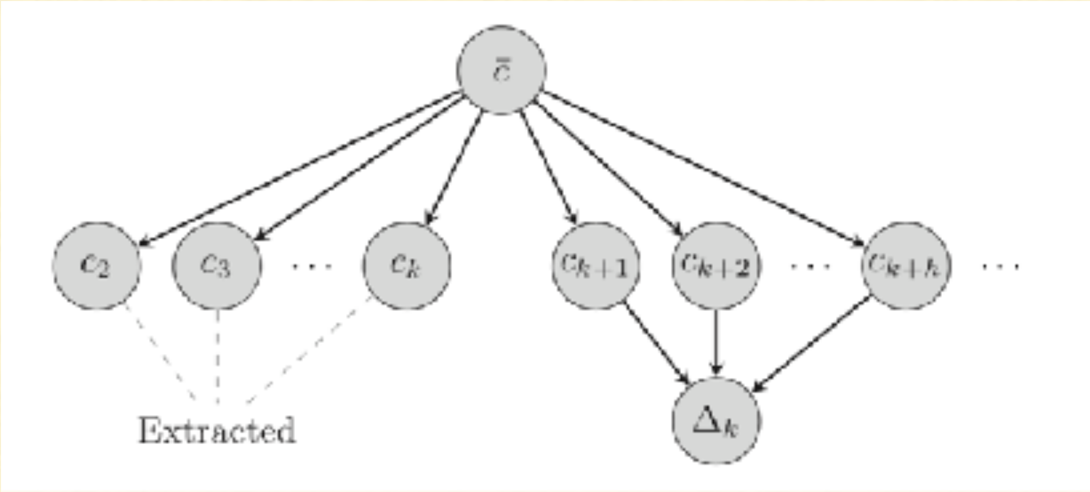

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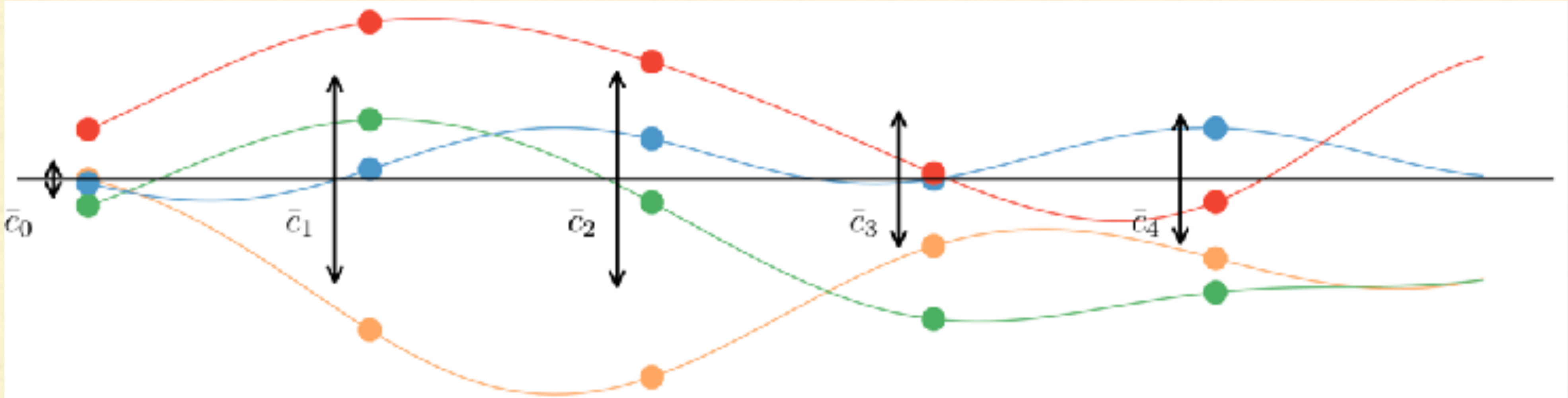
"Pointwise"



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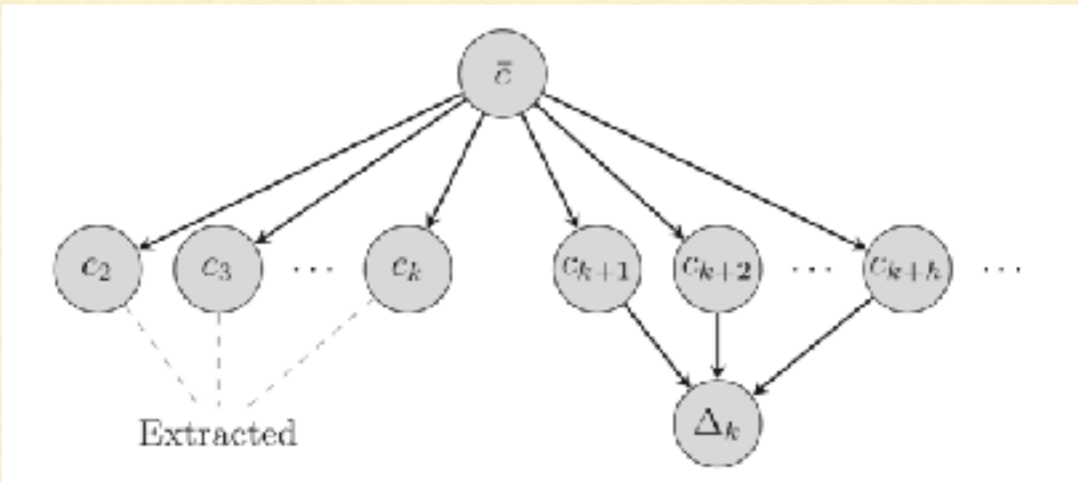

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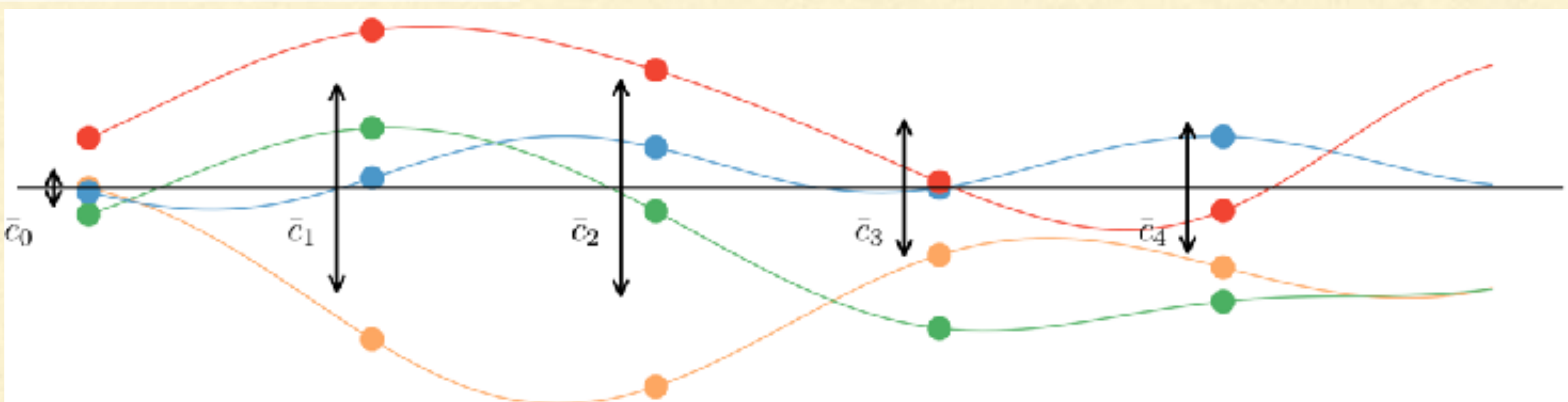
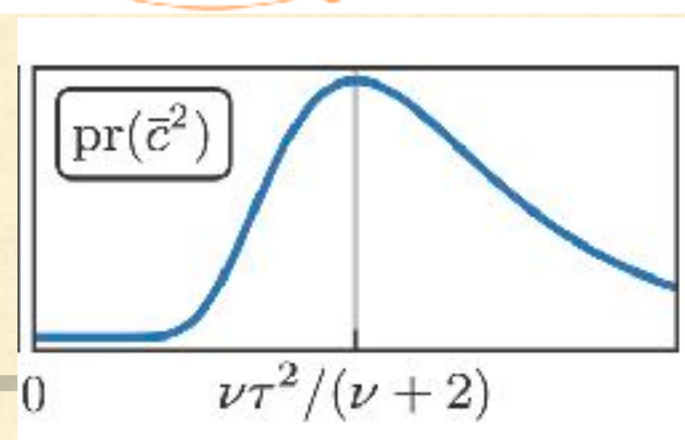


For this talk $c_n \sim \mathcal{N}(0, \bar{c}^2)$

$$\bar{c}^2 \sim \chi^{-2}(\nu, \tau^2)$$



$$\nu = \nu_0 + n_c;$$

$$\nu\tau^2 = \nu_0\tau_0^2 + \vec{c}_k^{\,2}$$

From pointwise to curvewise

Melendez, Wesolowski, Furnstahl, DP, Pratola, PRC (2019)

$$y = y_{\rm ref} \sum_{n=0}^{k} c_n(p/m_\pi) Q^n$$

Function c_n is not a constant.
But the c_n's at different values of p aren't independent random variables either

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EFT coefficients at different orders can be modeled as independent draws from a *Gaussian Process* with a stationary kernel

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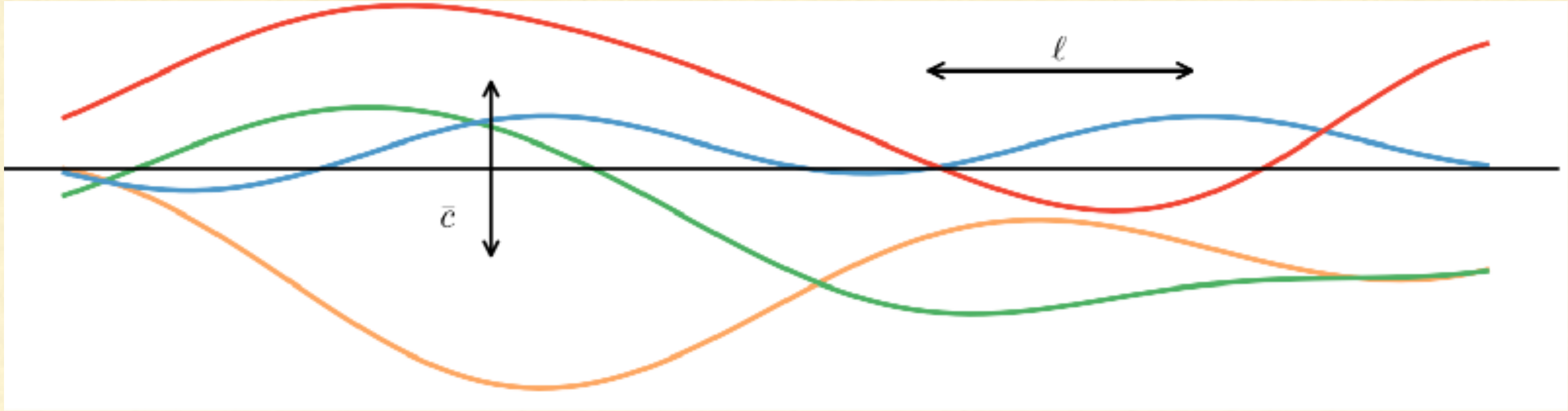

- Gaussian distribution at each point
- With correlation structure parameterized by a single $\bar{c}^2$ and ℓ at all orders

A bit more on Gaussian Processes

- Non-parametric, probabilistic model for a function
- Suppose we already know f at $x_1, x_2, x_3, \ldots, x_n$.
- Specify how f(y) is correlated with $f(x_1)$, $f(x_2)$, …..; don't specify underlying functional form.
- But value of f(y) is not deterministic: it's given by a (Gaussian) probability distribution.
- Correlation decreases as points get further away from each other.
- Specify correlation matrix of f at x and y, e.g.:

$$k(f(x), f(y)) = \bar{c}^2 \exp\left(-\frac{(x-y)^2}{2\ell^2}\right)$$

- Two parameters $\bar{c}^2$ and ℓ: $\mathrm{pr}(\bar{c}^2 \,|\, I) \sim \chi^{-2}(\nu_0, \tau_0^2)$; $\mathrm{pr}(\ell \,|\, I)$ uniform

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Statistical model choices

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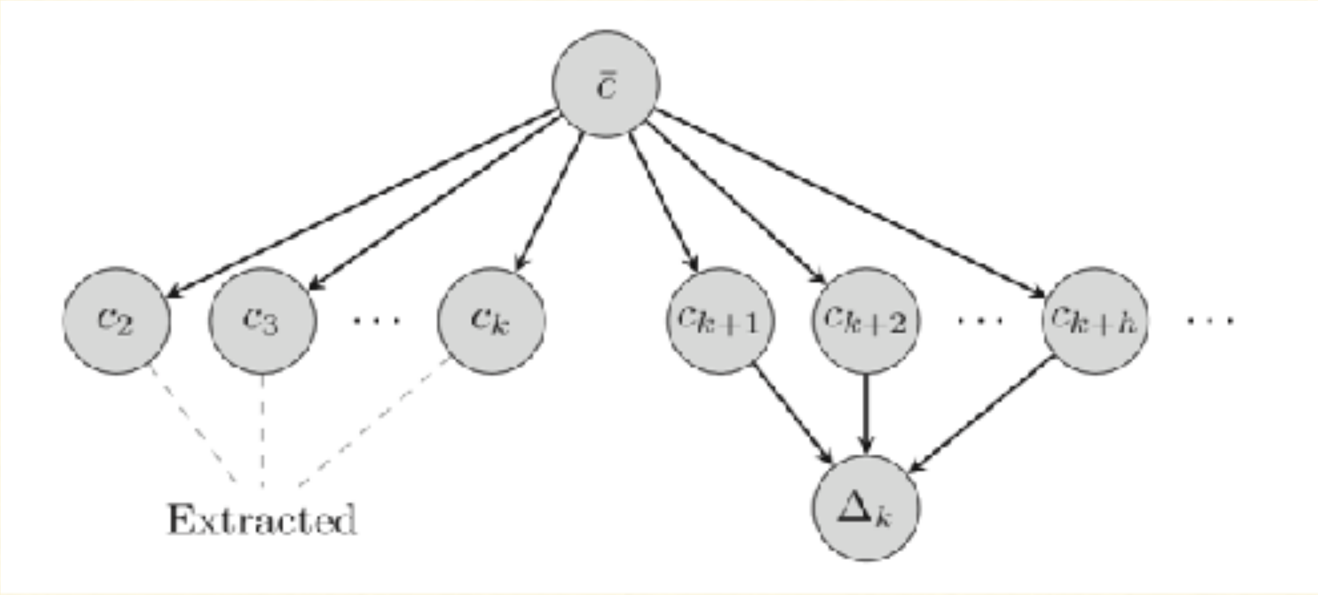
Inferring the next coefficient(s)



Gaussian process "model" for χEFT coefficients, trained on c_2 -c_5, can be used to predict distribution of N^5LO corrections

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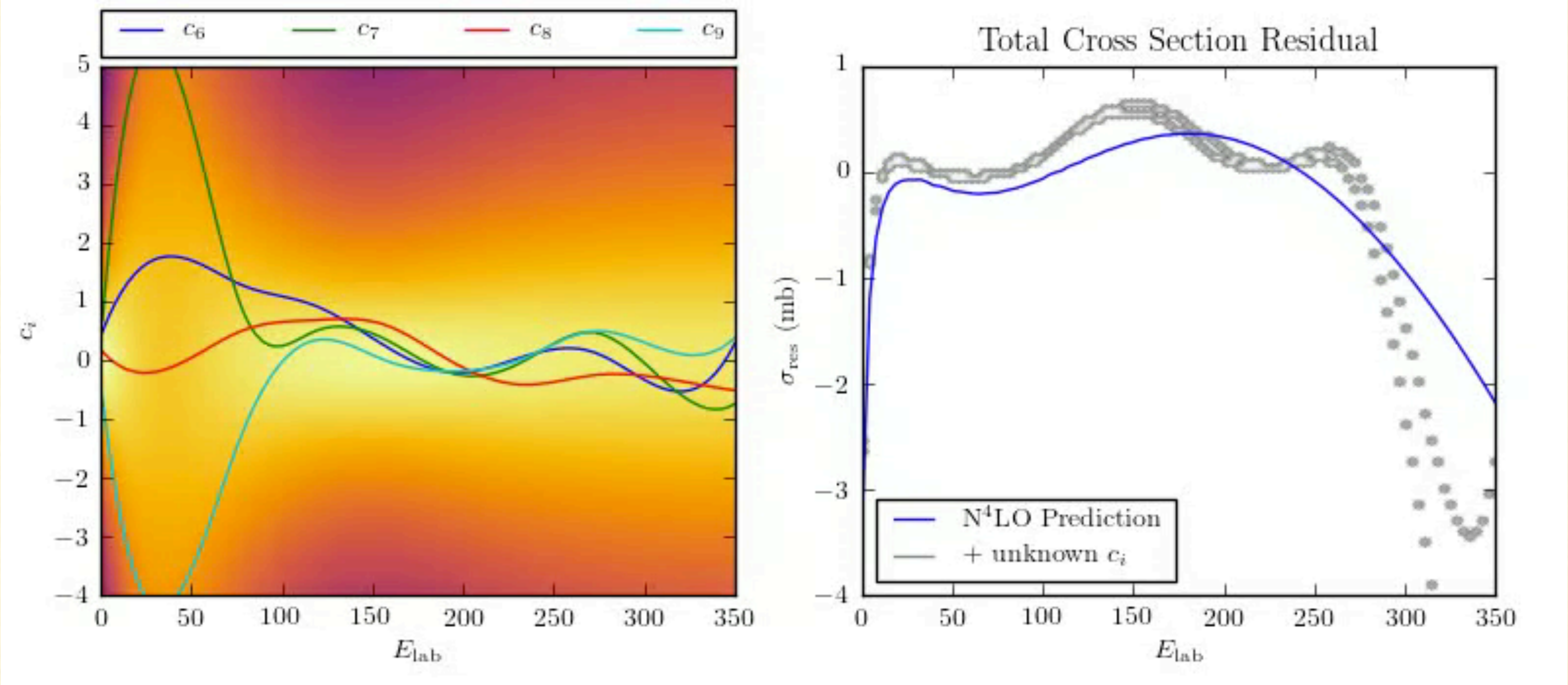


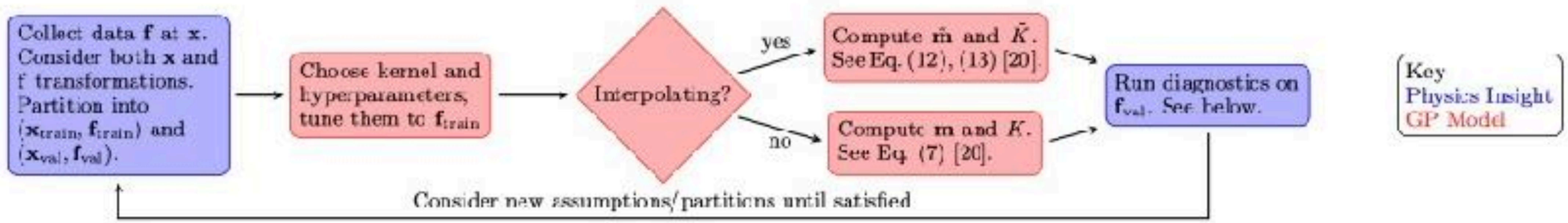
Gaussian process "model" for χEFT coefficients, trained on c_2 -c_5, can be used to predict distribution of N^{5}LO corrections

$$\Delta\sigma(E) = \sigma_{\rm ref}[c_6(E)Q^6 + c_7(E)Q^7 + c_8(E)Q^8 + c_9(E)Q^9 + c_{10}(E)Q^{10}]$$

Model checking

https://github.com/buqeye/gsum

Melendez et al. (2019), Millican et al. (2024), Bastos & O'Hagan (2009)

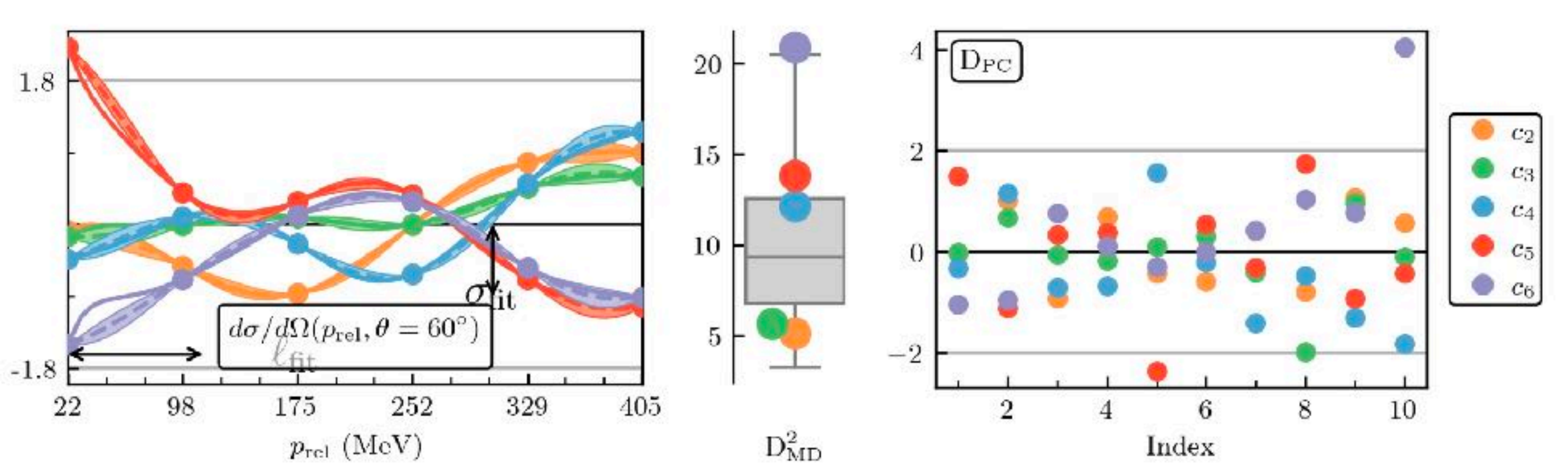


Diagnostic	Formula	Motivation	Success	Failure
Visualize the function	—	Does $\mathbf{f}_{\rm val}$ look like a draw from a GP? What kind of GP?	$\mathbf{f}_{\rm val}$ "looks similar" to draws from a GP	$\mathbf{f}_{\rm val}$ "stands out" compared to GP draws
Mahalanobis Distance $D^2_{\rm MD}$	$(\mathbf{f}_{\rm val}-\mathbf{m})^\intercal K^{-1}(\mathbf{f}_{\rm val}-\mathbf{m})$	Can we *quantify* how much the $\mathbf{f}_{\rm val}$ looks like a GP?	$D^2_{\rm MD}$ follows its theoretical distribution (χ^2_M)	$D^2_{\rm MD}$ lies too far away from the expected value of M
Pivoted Cholesky $D_{\rm PC}$	$G^{-1}(\mathbf{f}_{\rm val}-\mathbf{m})$	Can we understand why $D^2_{\rm MD}$ is failing?	At each index, points follow standard Gaussian	Many cases (see below)
Credible Interval $D_{\rm CI}(P)$ for $P\in[0,1]$	$\frac{1}{M}\sum_{i=1}^{M}\mathbf{1}[\mathbf{f}_{{\rm val},i}\in {\rm CI}_i(P)]$	Do $100P\%$ credible intervals capture data roughly $100P\%$ of the time?	Plot $D_{\rm CI}(P)$ for $P\in[0,1]$; the curve should be within errors of $D_{\rm CI}(P)=P$.	$D_{\rm CI}(P)$ is far from $100P\%$, particularly for large $100P\%$ (e.g., 68% and 95%).

Variance	Length Scale	Observed Pattern in $D_{\rm PC}$
$\sigma_{\rm est}=\sigma_{\rm true}$	$\ell_{\rm est}=\ell_{\rm true}$	Points are distributed as a standard Gaussian, with no pattern across index (e.g., only $\approx 5\%$ of points outside 2σ lines).
$\sigma_{\rm est}=\sigma_{\rm true}$	$\ell_{\rm est}>\ell_{\rm true}$	Points look well distributed at small index but expand to a too-large range at high index.
$\sigma_{\rm est}=\sigma_{\rm true}$	$\ell_{\rm est}<\ell_{\rm true}$	Points look well distributed at small index but shrink to a too-small range at high index.
$\sigma_{\rm est}>\sigma_{\rm true}$	$\ell_{\rm est}=\ell_{\rm true}$	Points are distributed in a too-small range at all indices.
$\sigma_{\rm est}<\sigma_{\rm true}$	$\ell_{\rm est}=\ell_{\rm true}$	Points are distributed in a too-large range at all indices.

What does success look like?

Millican, Furnstahl, Melendez, DP, Pratola (2024)



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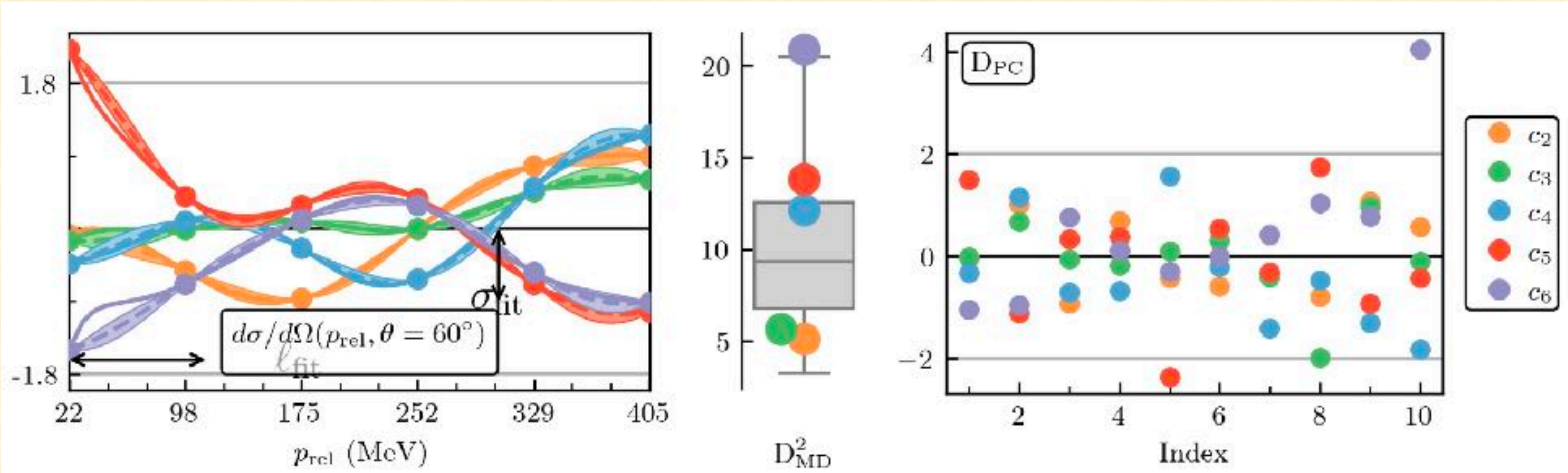
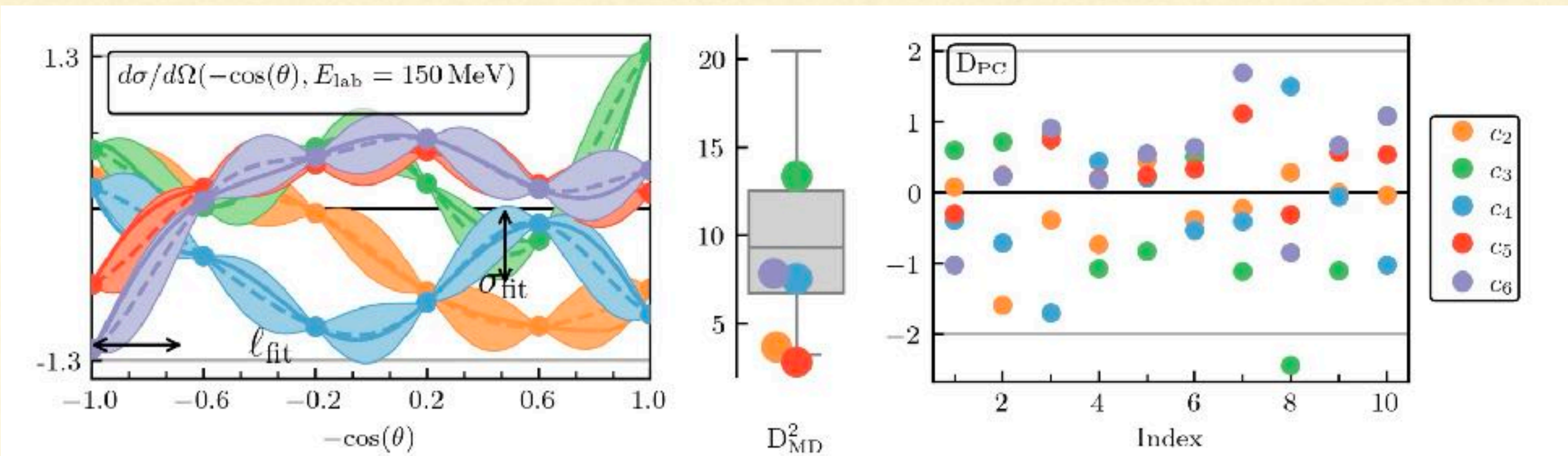
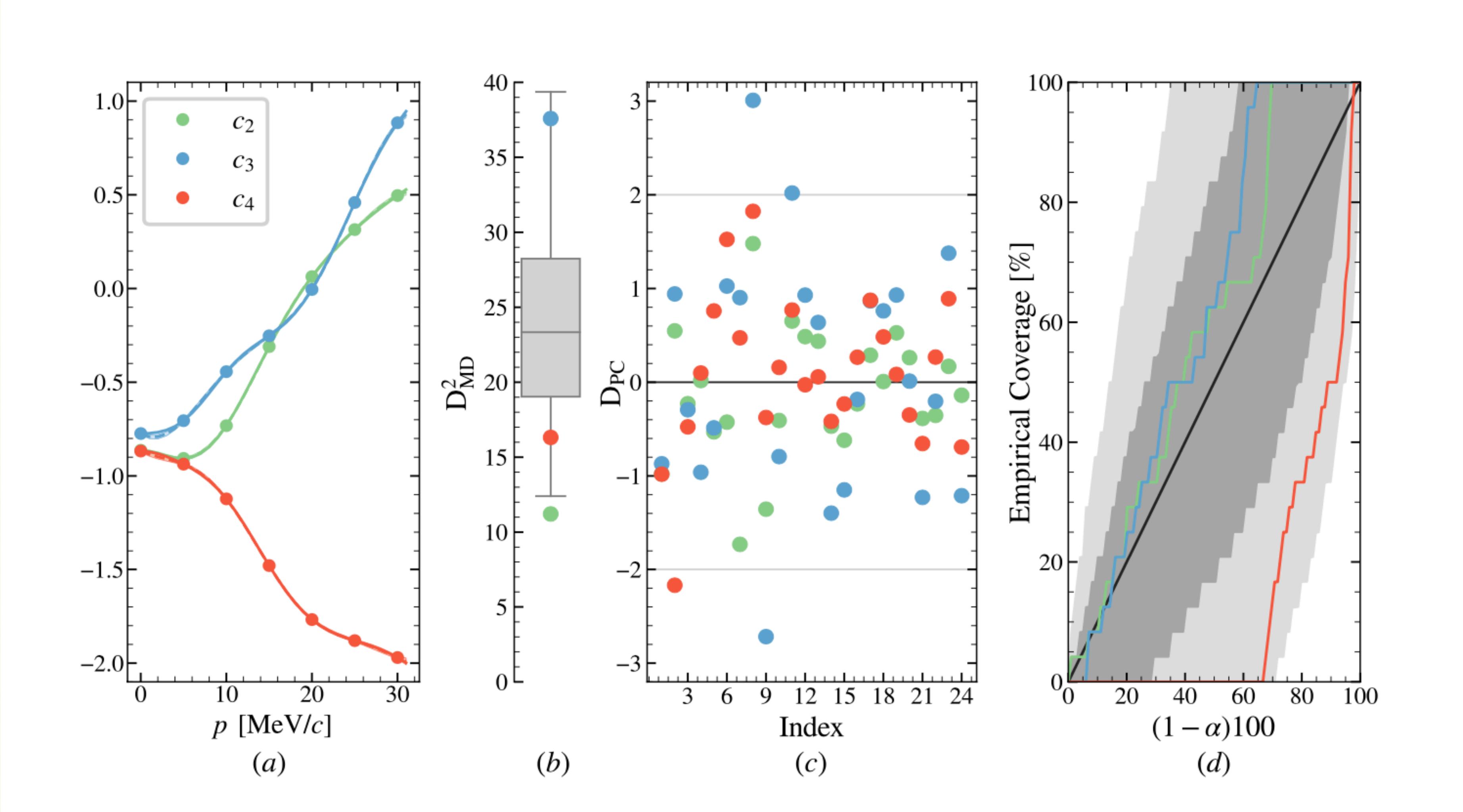




Application to np→dγ

Acharya, Bacca, PLB 2022



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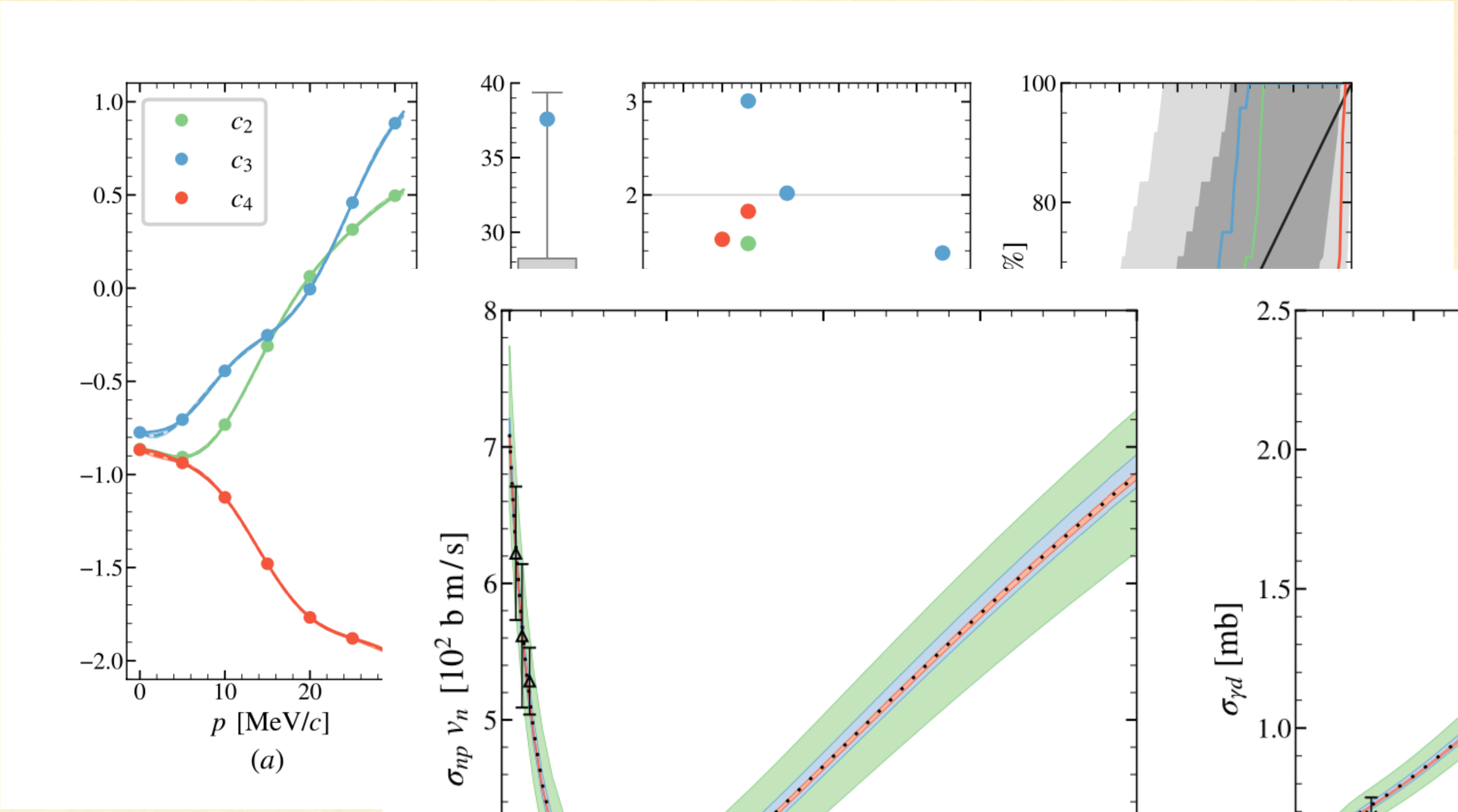

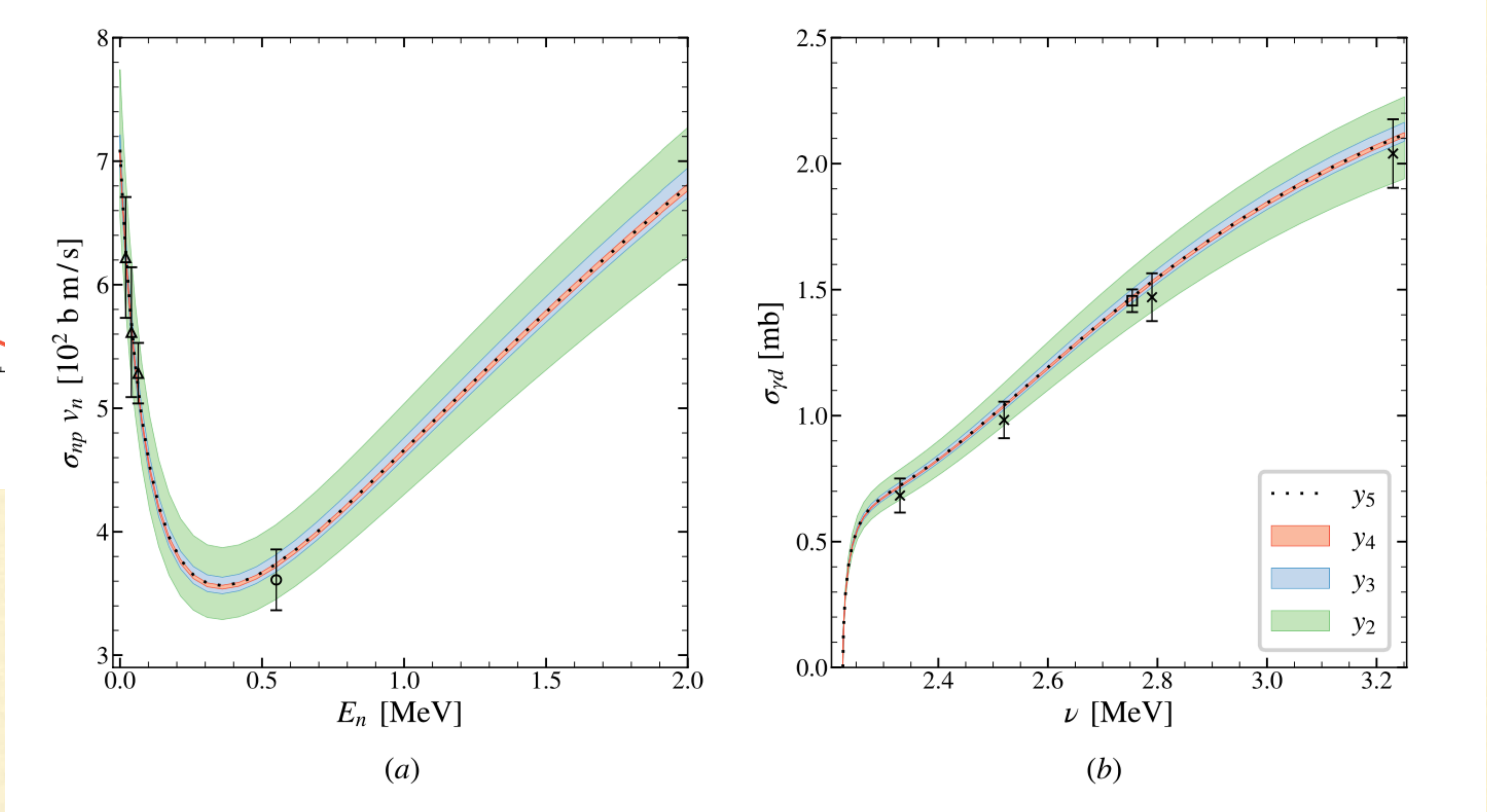

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Application to $\mu^- d \rightarrow nn\nu_\mu$

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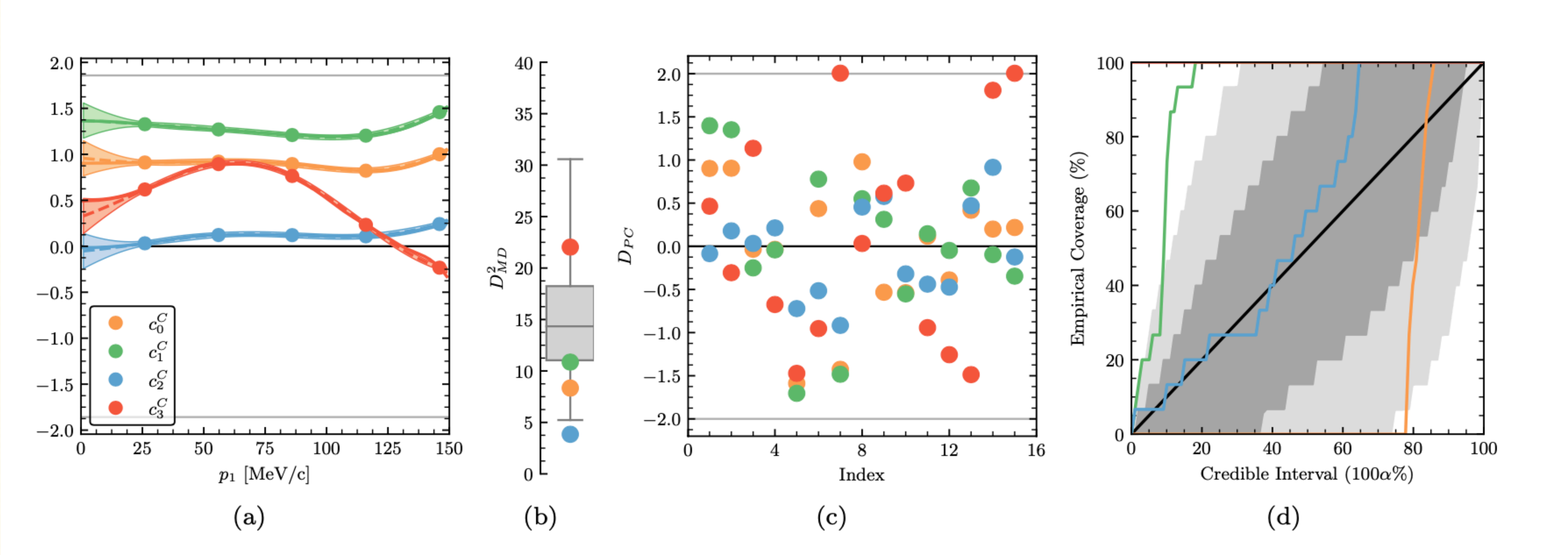

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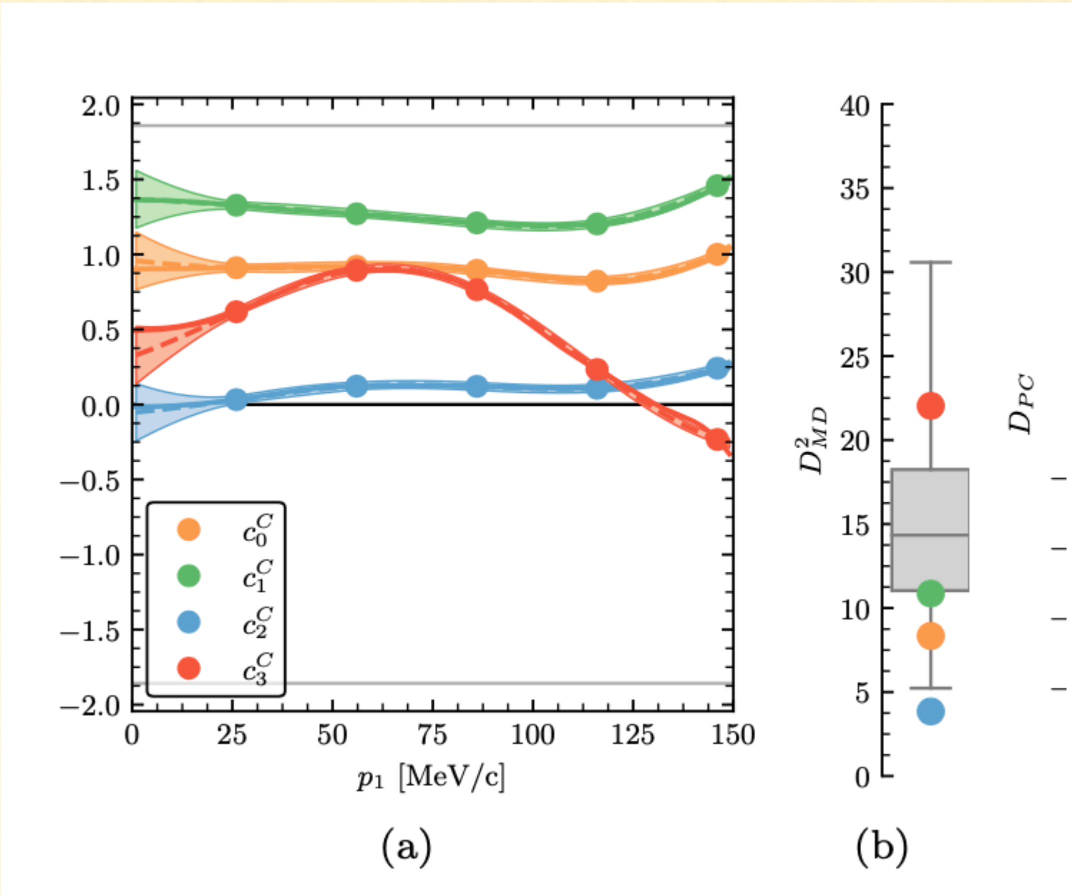
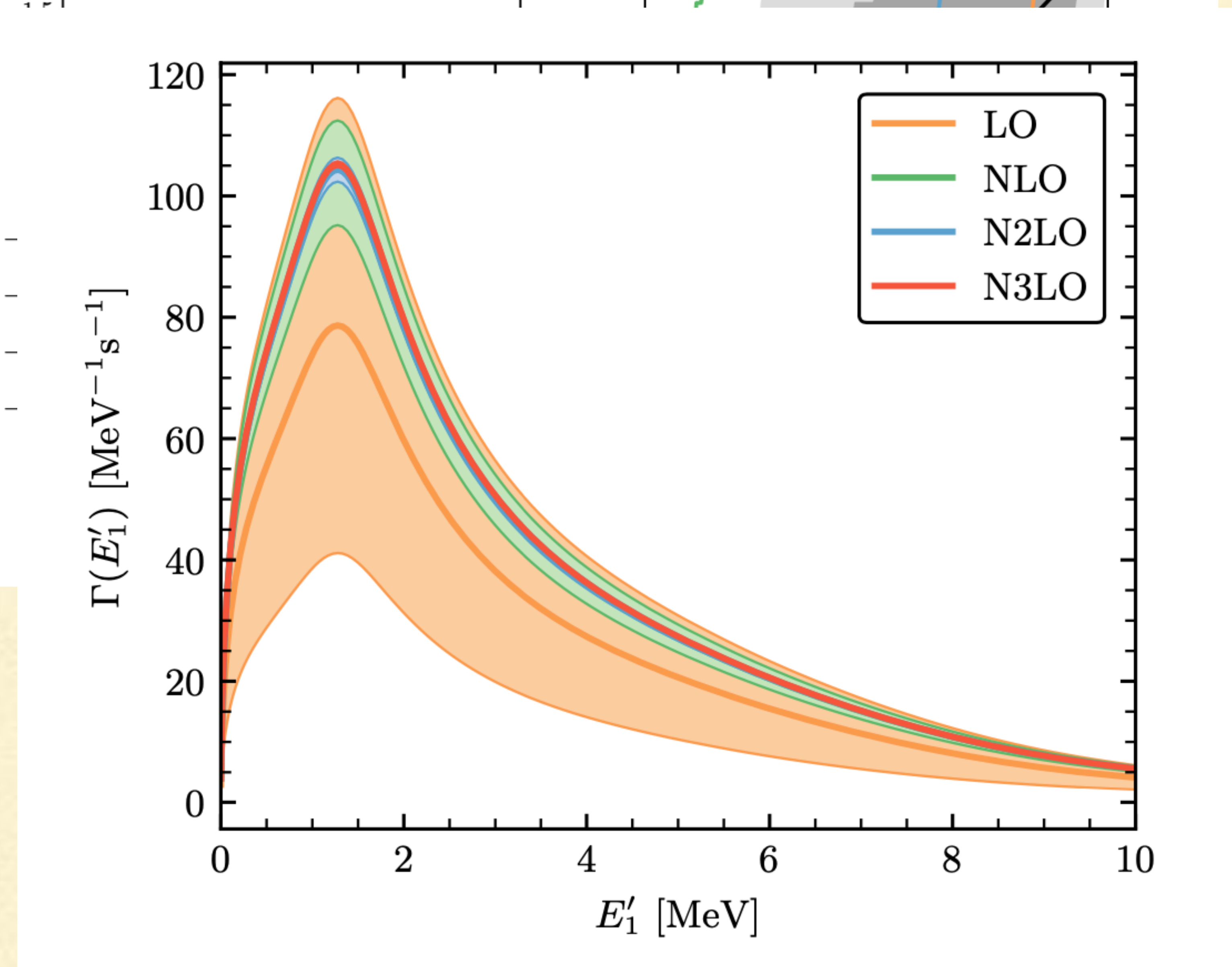
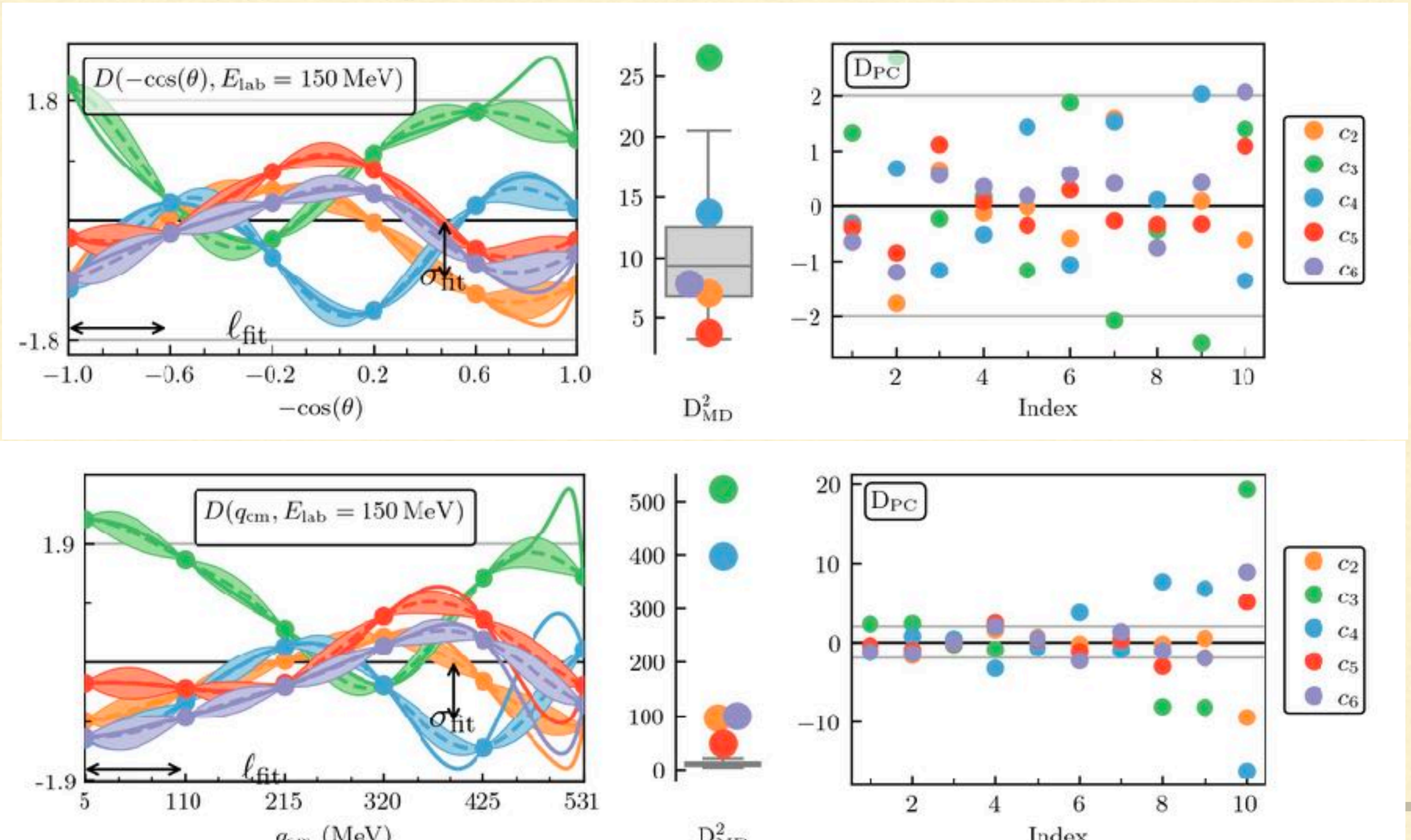




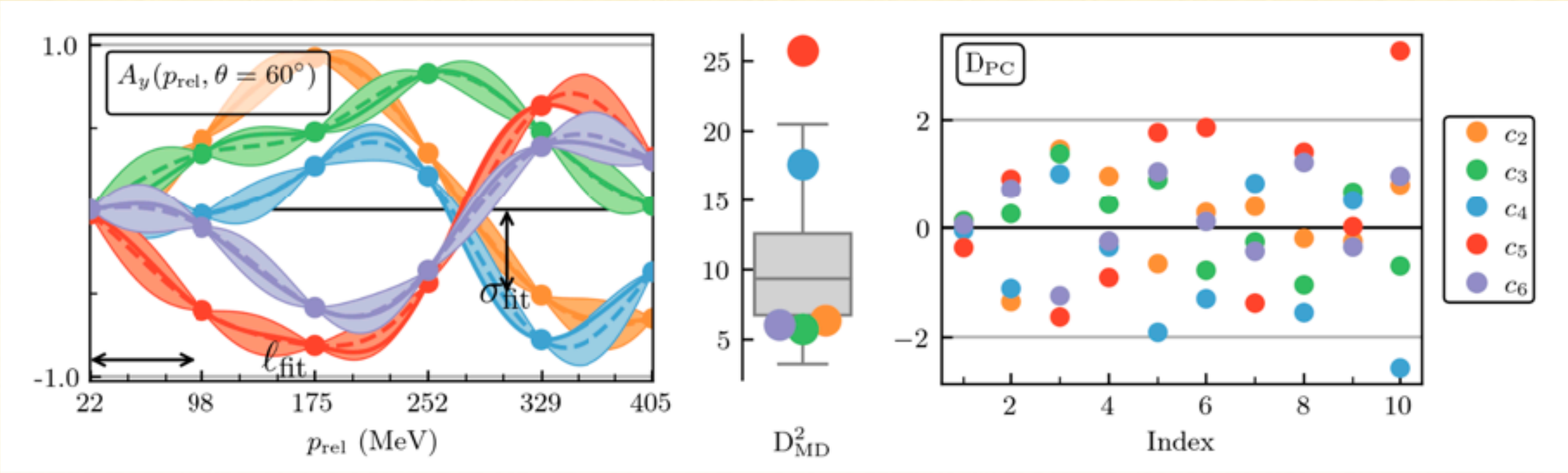
NN physics choices 0 & I: y_{ref} & angular space

- $y_{ref}=y_0$ or y_k. Don't choose something that goes to zero
- Is it $c_n(\cos(\theta))$ or $c_n(q)$ that has a single length scale?



NN physics choice II: energy/momentum?

- Is it $c_n(p)$ or $c_n(E)$ that has a single length scale?



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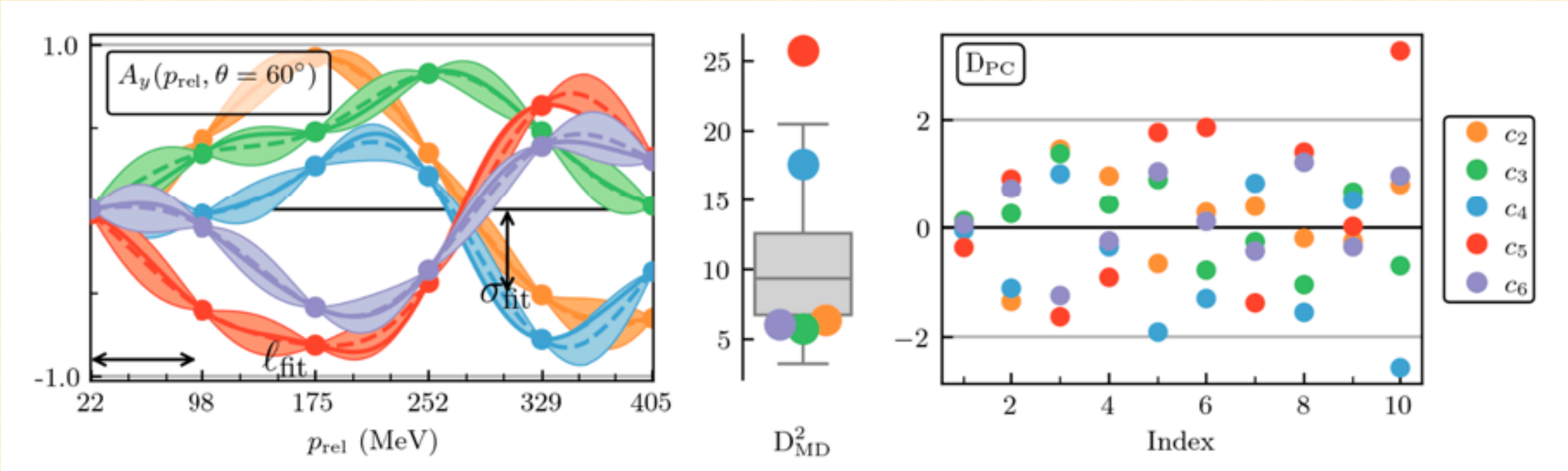
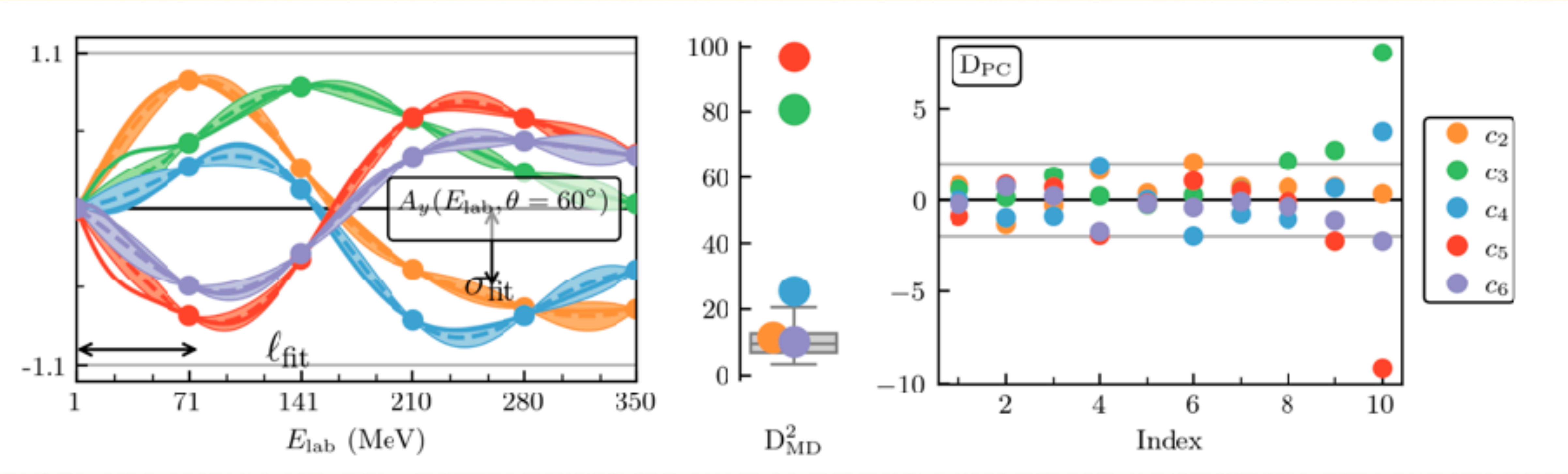
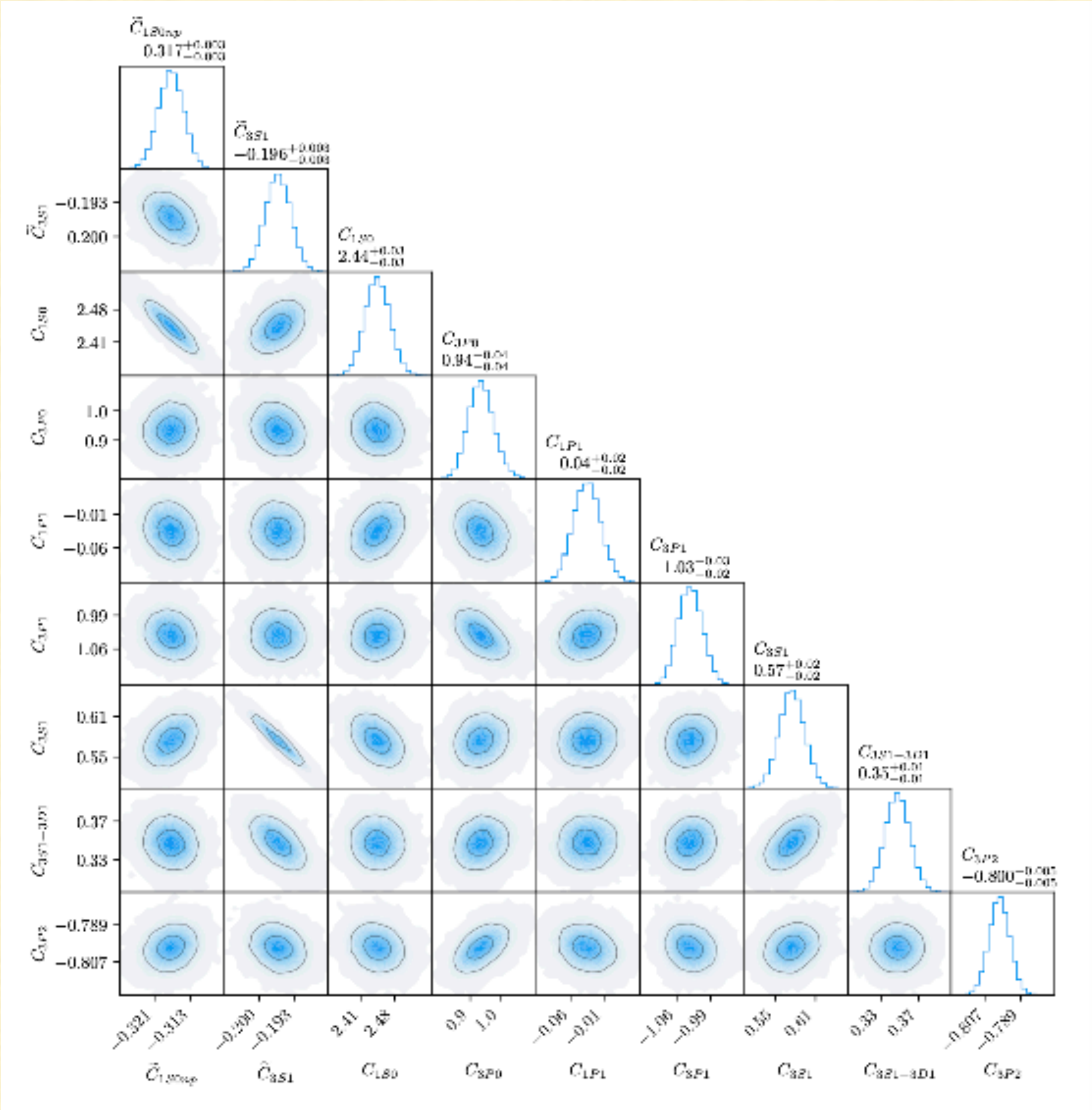




Calibrating NN LECs with a GP error model

Svennson, Ekstróm, Forssén, PRC (2024)



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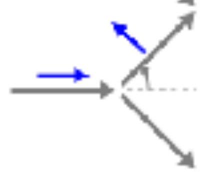
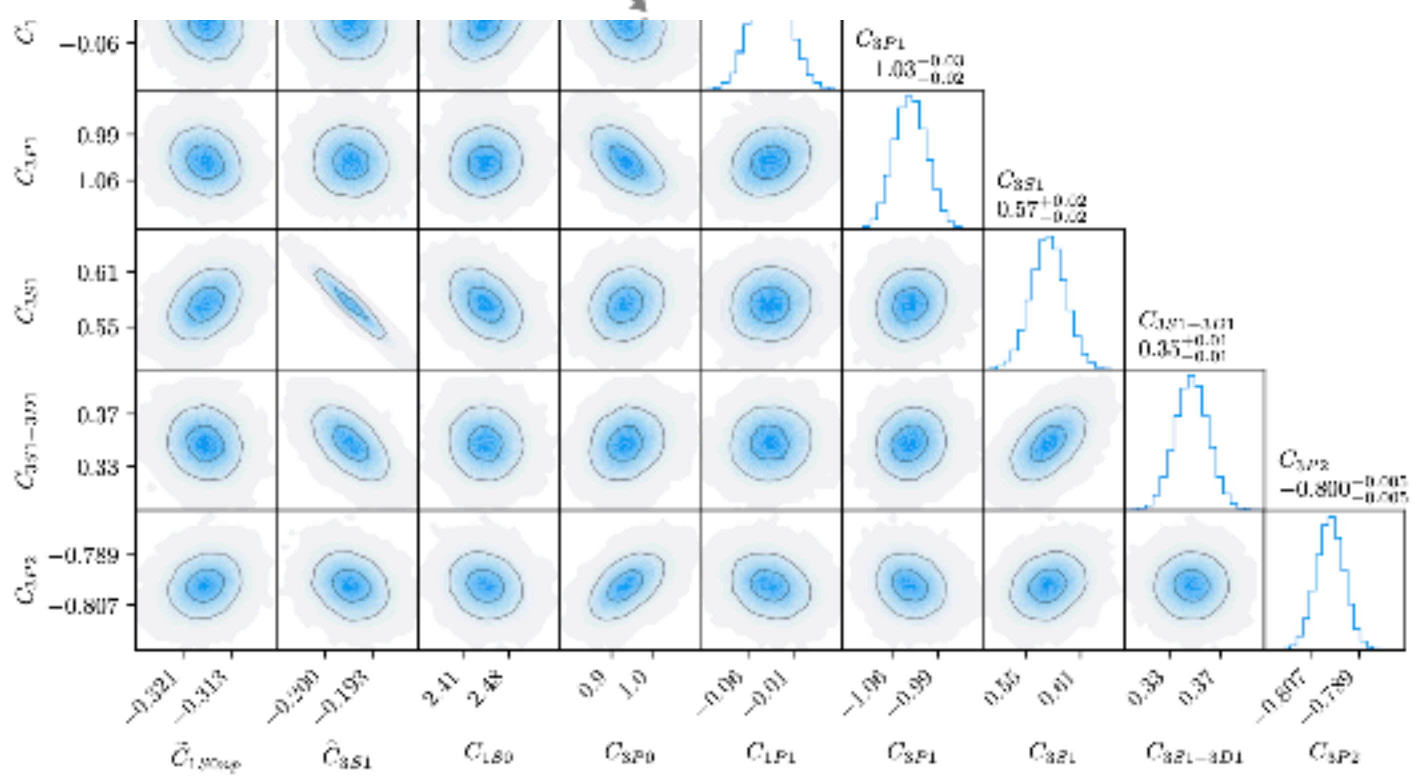
Notation	Definition	Acronym	$N_{d,y}$	$N_{T_{\text{lab}},y}$	n_{eff}	$\widehat{\ell_{T_{\text{lab}}}}$ (MeV)		$\widehat{\bar{c}^2}$
σ_{tot}	total cross section	SGT	119	113	30.8	49		0.56^2
σ_T	$\sigma_{\text{tot}}(\uparrow\downarrow) - \sigma_{\text{tot}}(\uparrow\uparrow)$	SGTT	3	3	—	—		—
σ_L	$\sigma_{\text{tot}}(\leftrightarrows) - \sigma_{\text{tot}}(\rightrightarrows)$	SGTL	4	4	3.8	47		1.95^2

Notation	Tensor	Illustration	Acronym	$N_{d,y}$	$N_{T_{\text{lab}},y}$	n_{eff}	$\widehat{\ell_{T_{\text{lab}}}}$ (MeV)	$\widehat{\ell_\theta}$ (deg)	$\widehat{\bar{c}^2}$
$\sigma(\theta)$	I_{0000}		DSG	1207	68	352.9	73	39	0.61^2
$A(\theta)$	D_{s0k0}		A	5	1	5.0	68	37	0.65^2



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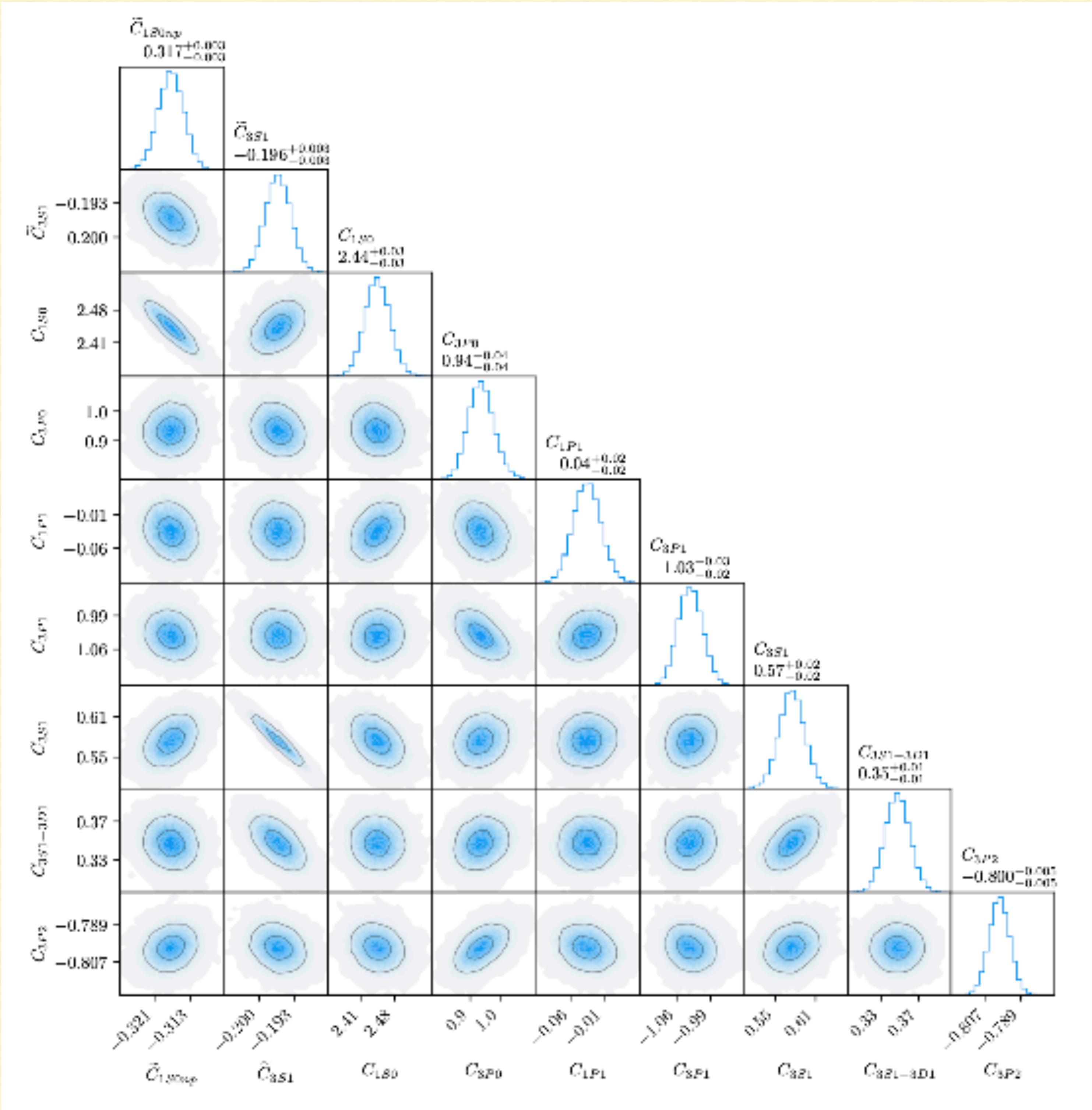

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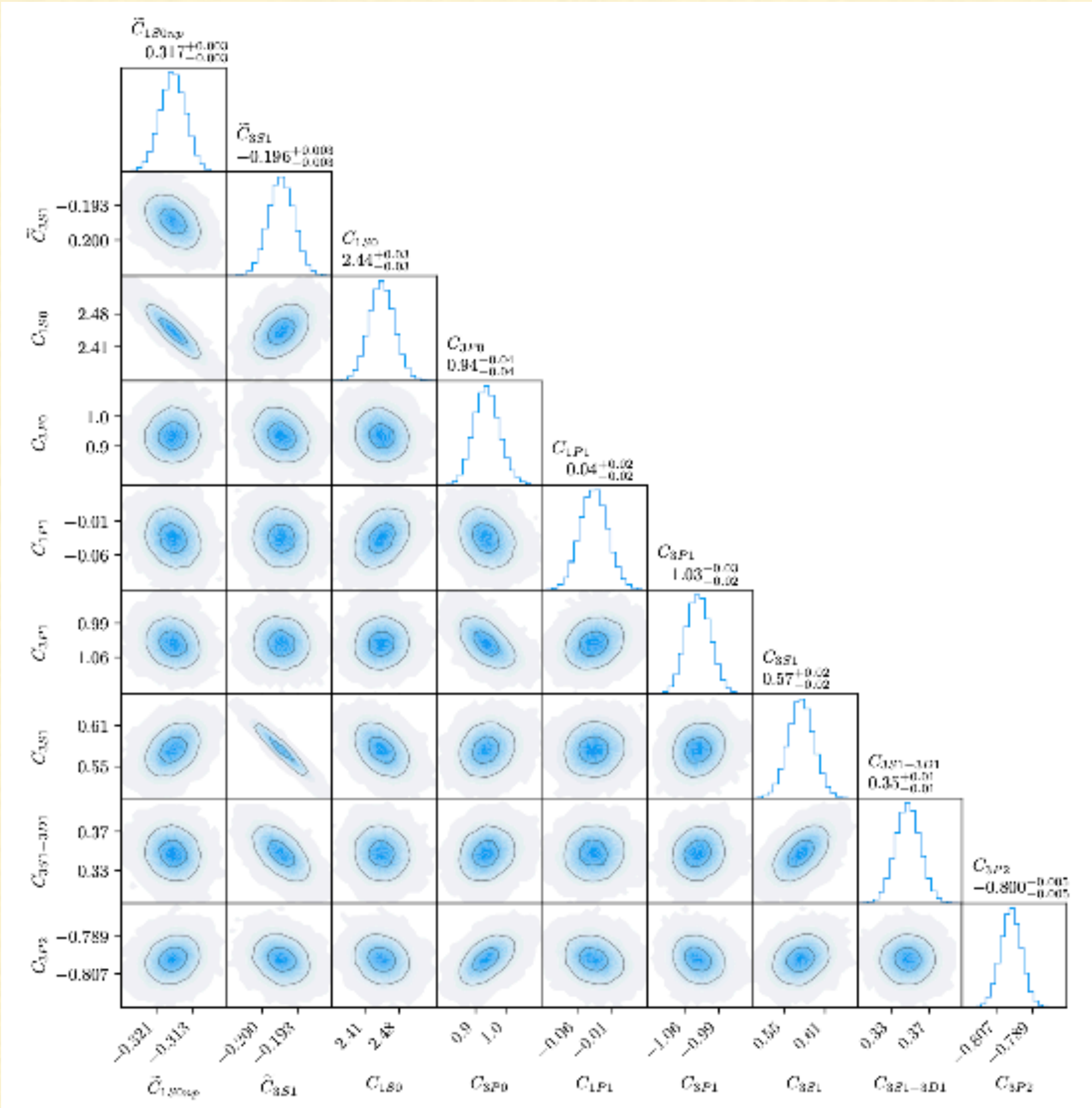
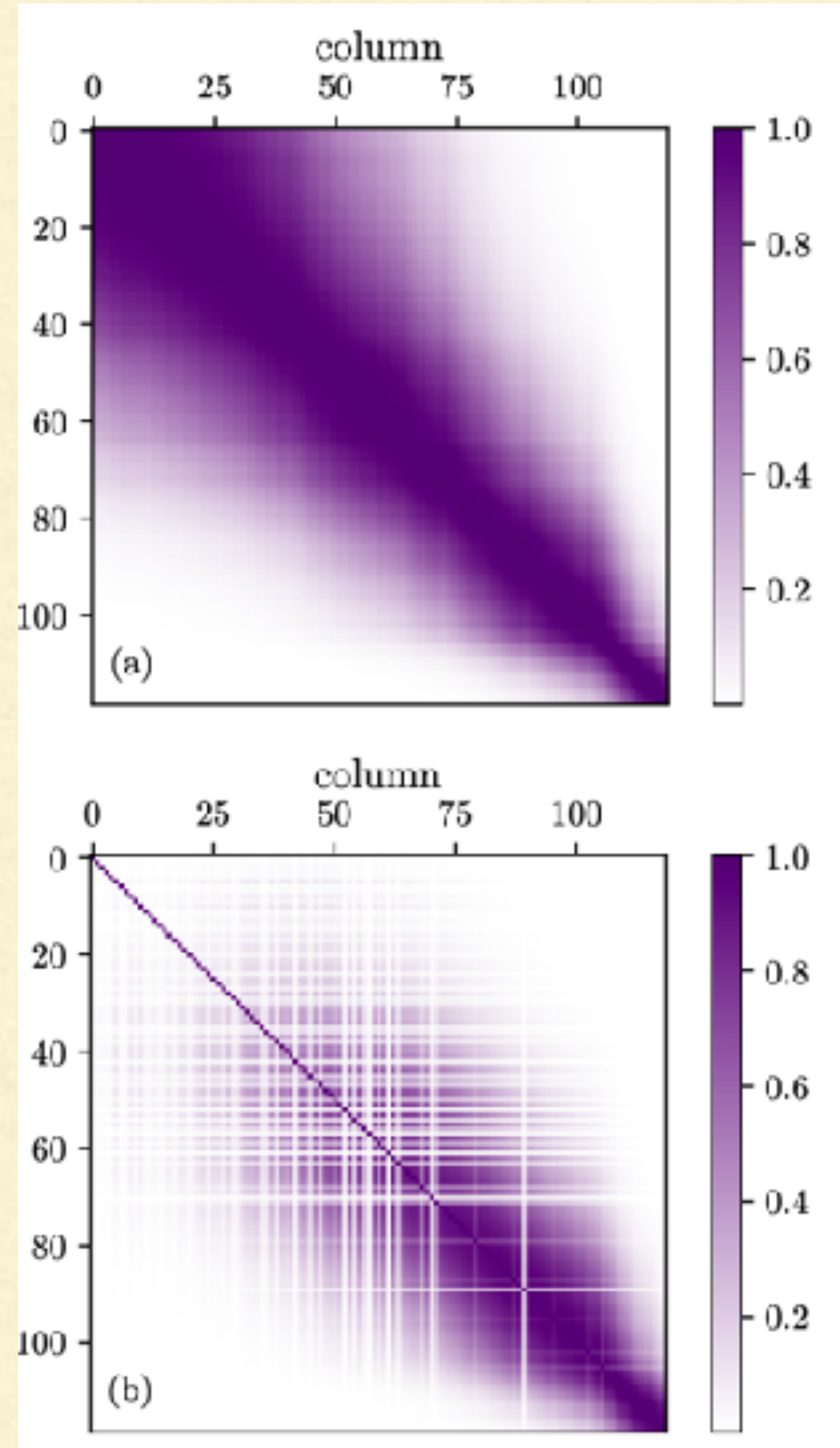
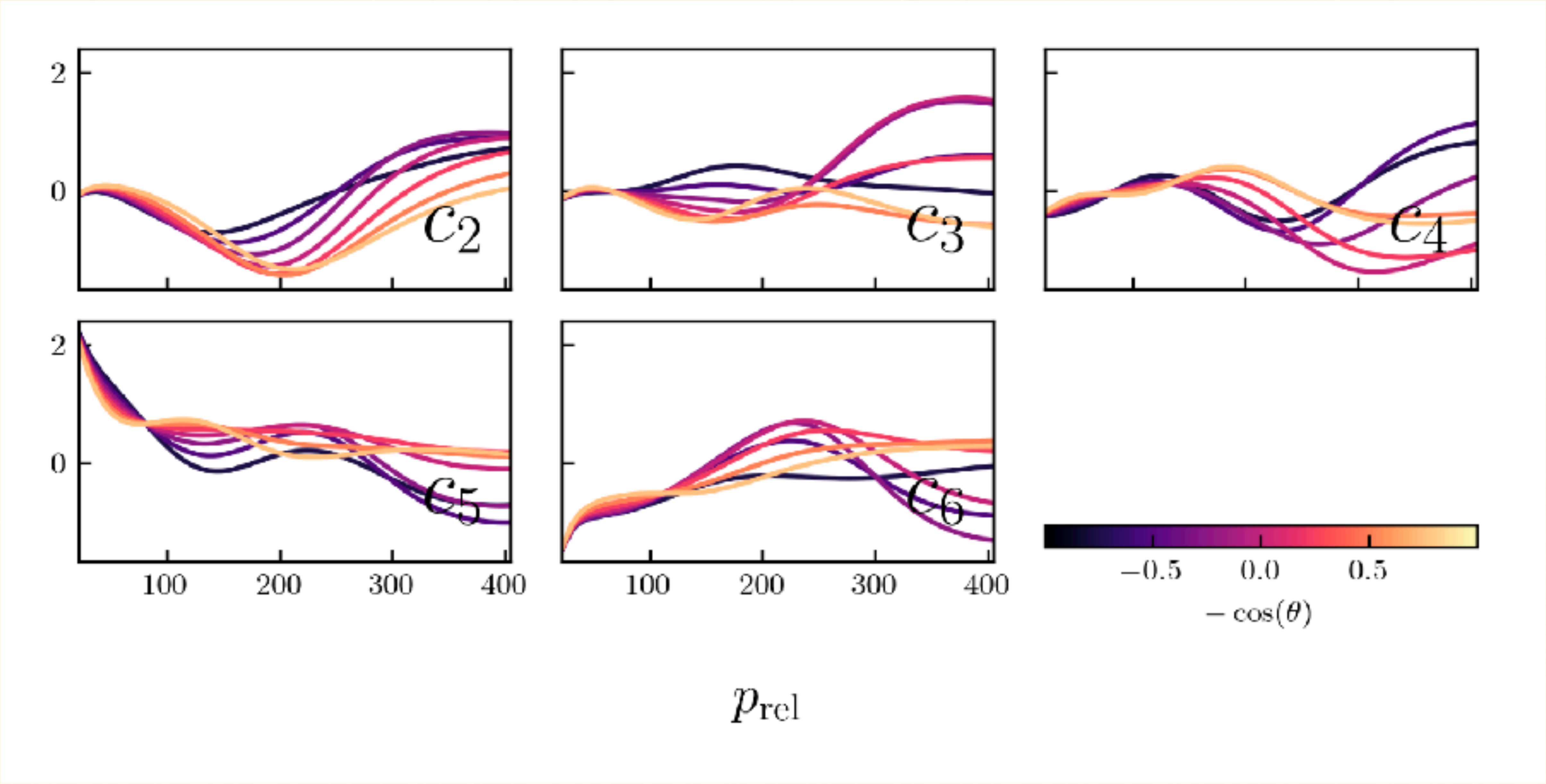




σ_{tot} correlations by data index

So what do coefficients look like for SMS potentials?

Millican et al. (2024B)



SMS 500 MeV

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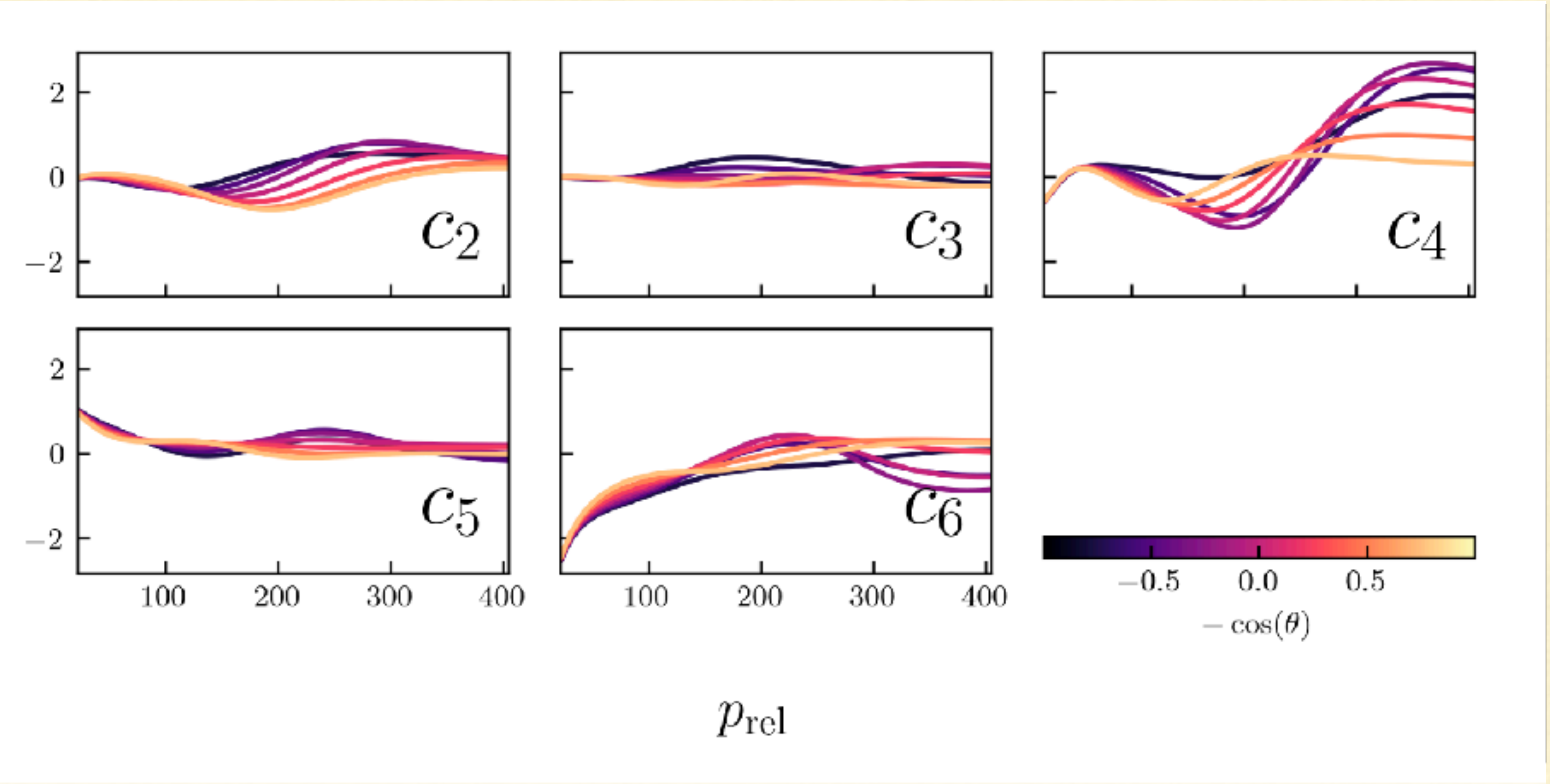
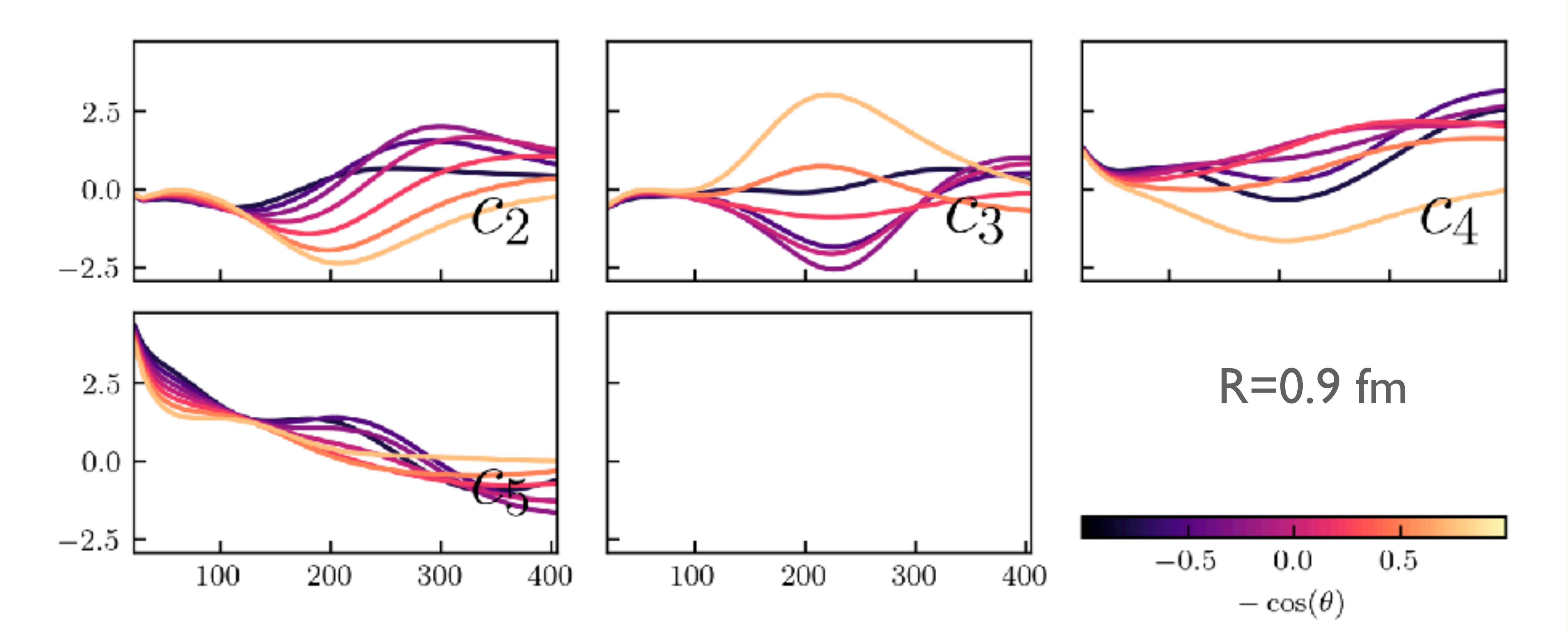


SMS 400 MeV

Similar pattern for SCS potentials



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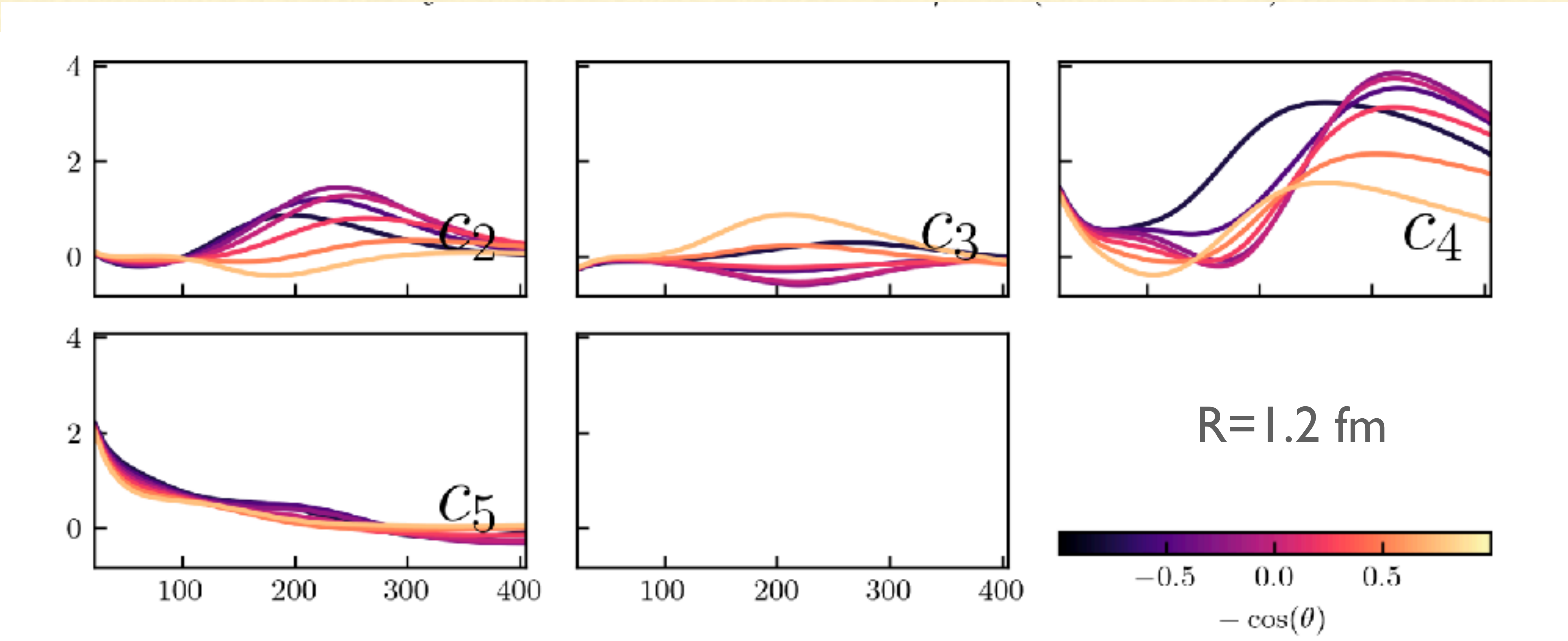

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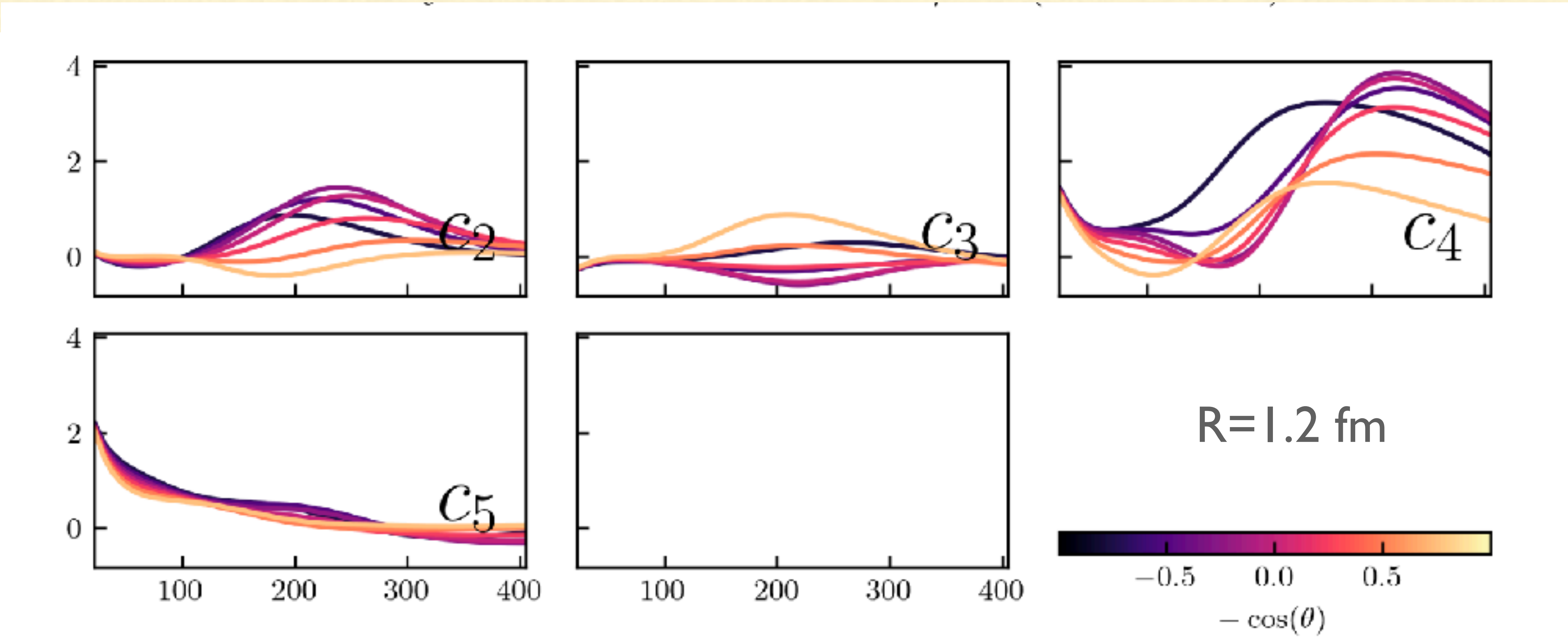
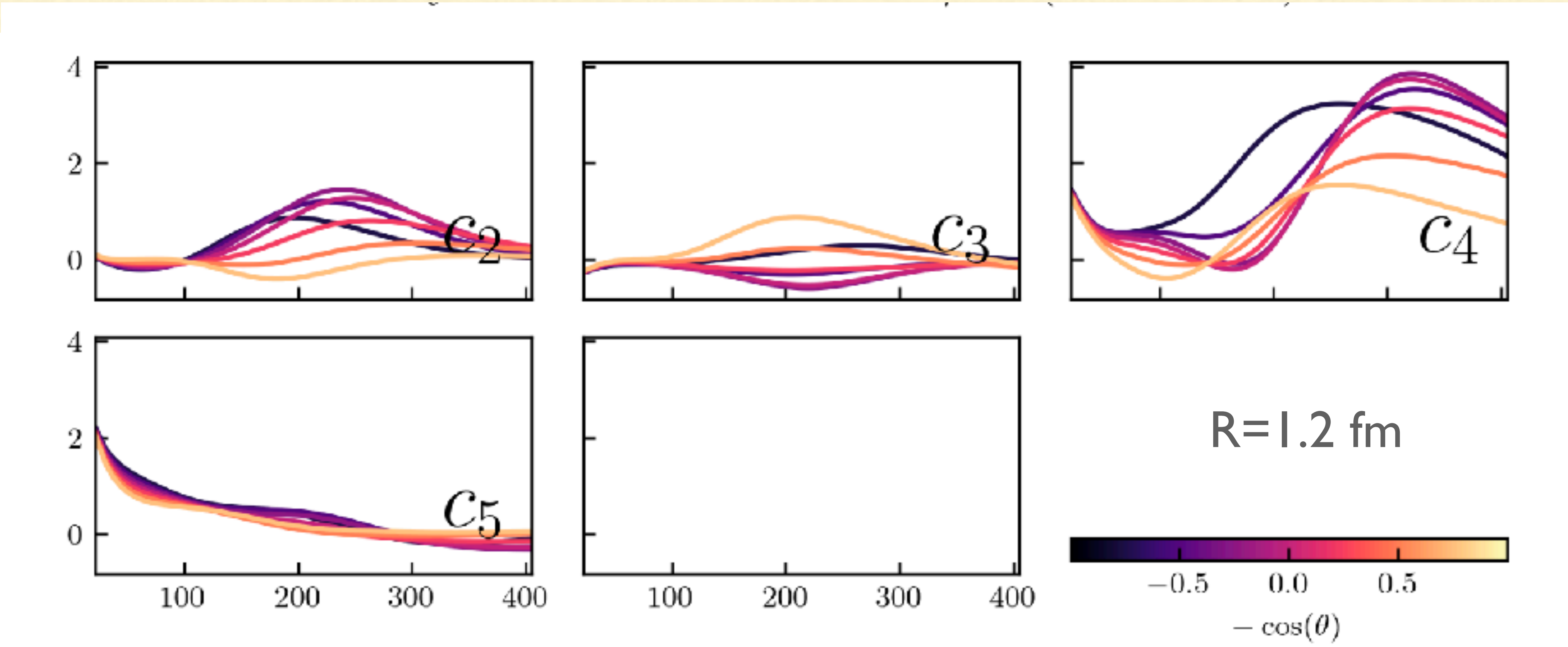


Soft potentials reshuffle contributions across orders, leaving large even orders (especially c_4) and small odd orders

Similar pattern for SCS potentials



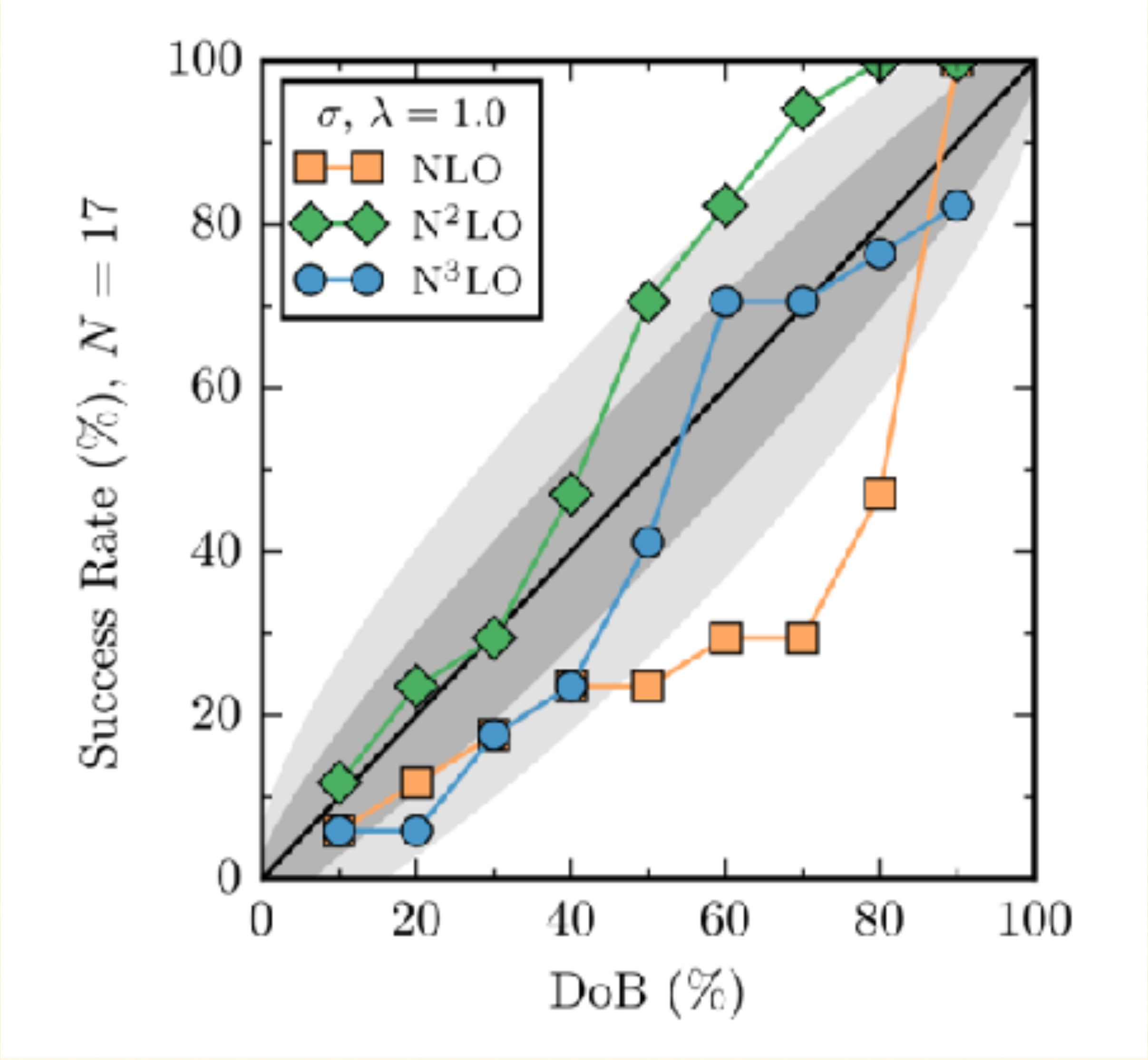
Soft potentials reshuffle contributions across orders, leaving large even orders (especially c_4) and small odd orders

~~The BUQEYE~~ Any statistical model assuming regular convergence fails for them

Impact on error bars

Melendez , Furnstahl, Wesolowski (2017)

R=0.9 fm

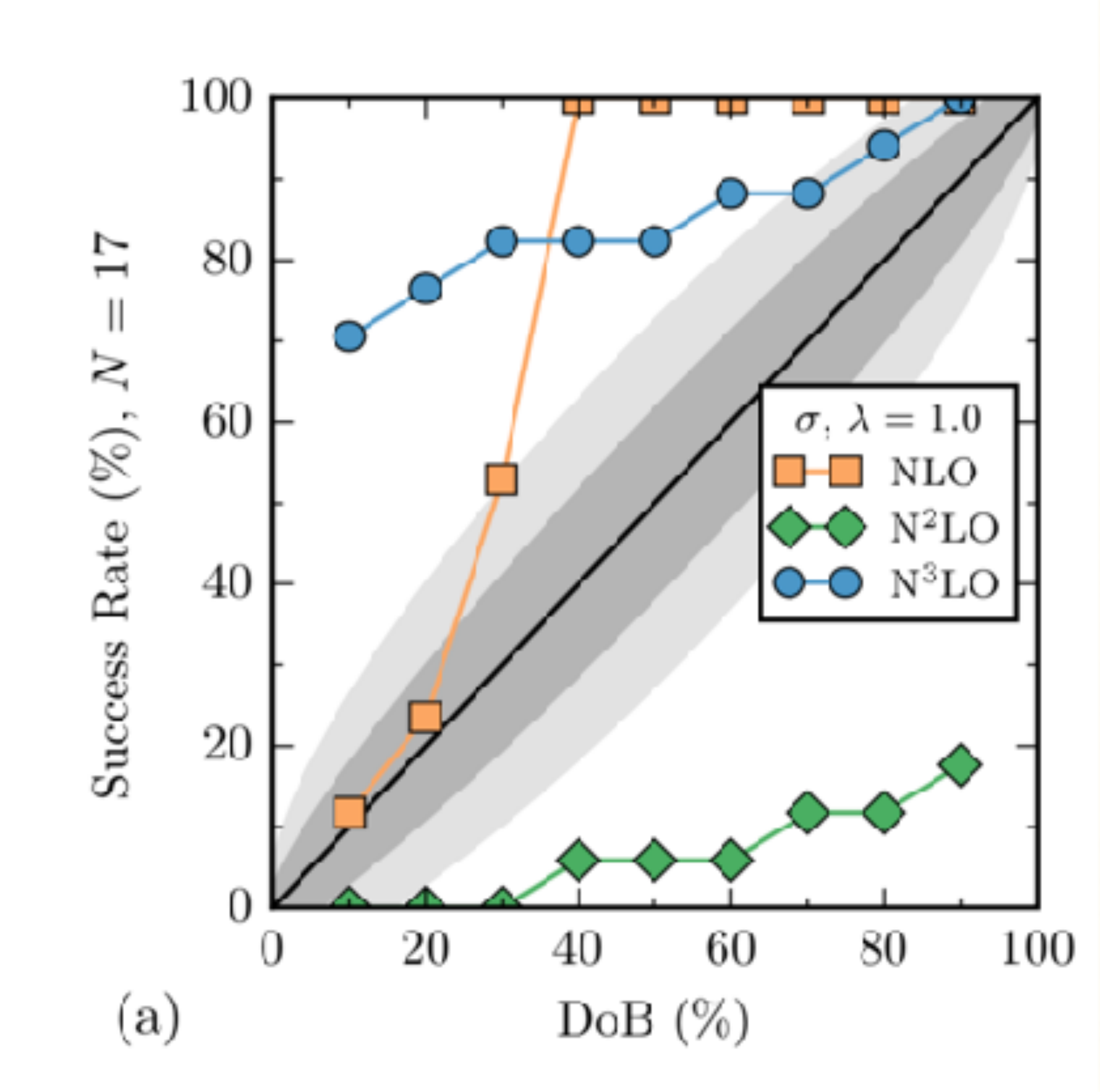


- Now we consider predictions at each order, with their error bars, as data and test them to see if the procedure is consistent
- Fix a given DOB interval, compute actual success ratio and compare
- Look at this over EKM predictions at four different orders and four different energies
- “Pointwise” analysis

Impact on error bars

Melendez , Furnstahl, Wesolowski (2017)

R=1.2 fm



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Inferring Q

If Q too big then c_n will shrink with n
(and so will error bars)

If Q too small then c_n will grow with n
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Once we have $\mathrm{pr}(\vec{\mathbf{c}}_k \,|\, \ell, I)$ we derive

$$\mathrm{pr}(\mathbf{Q} \,|\, \vec{\mathbf{y}}_k, \ell, I) \propto \frac{\mathrm{pr}(Q \,|\, I)}{\tau^{\nu} \prod_{i,n} |\, Q^n(x_i) \,|}$$

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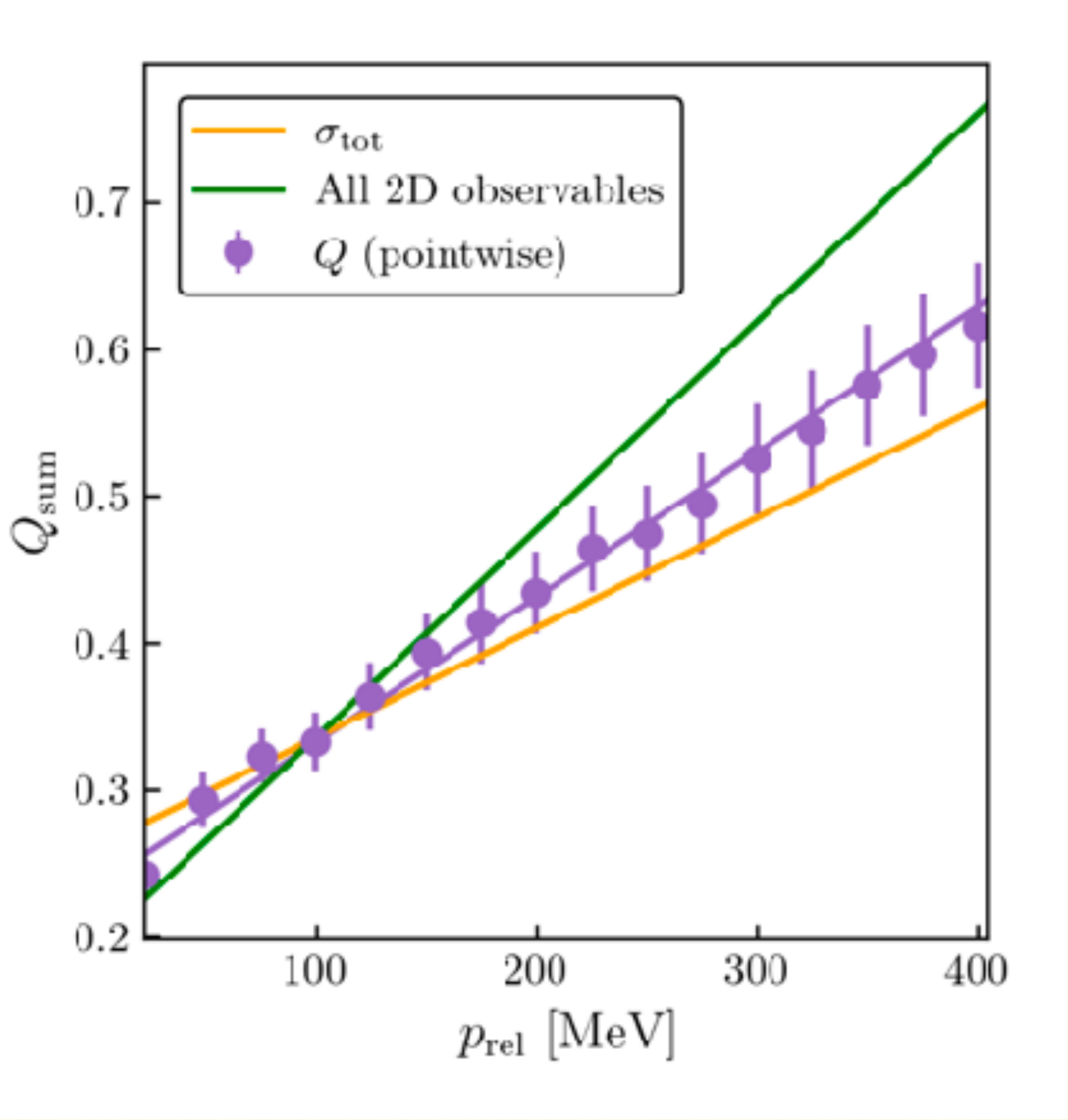
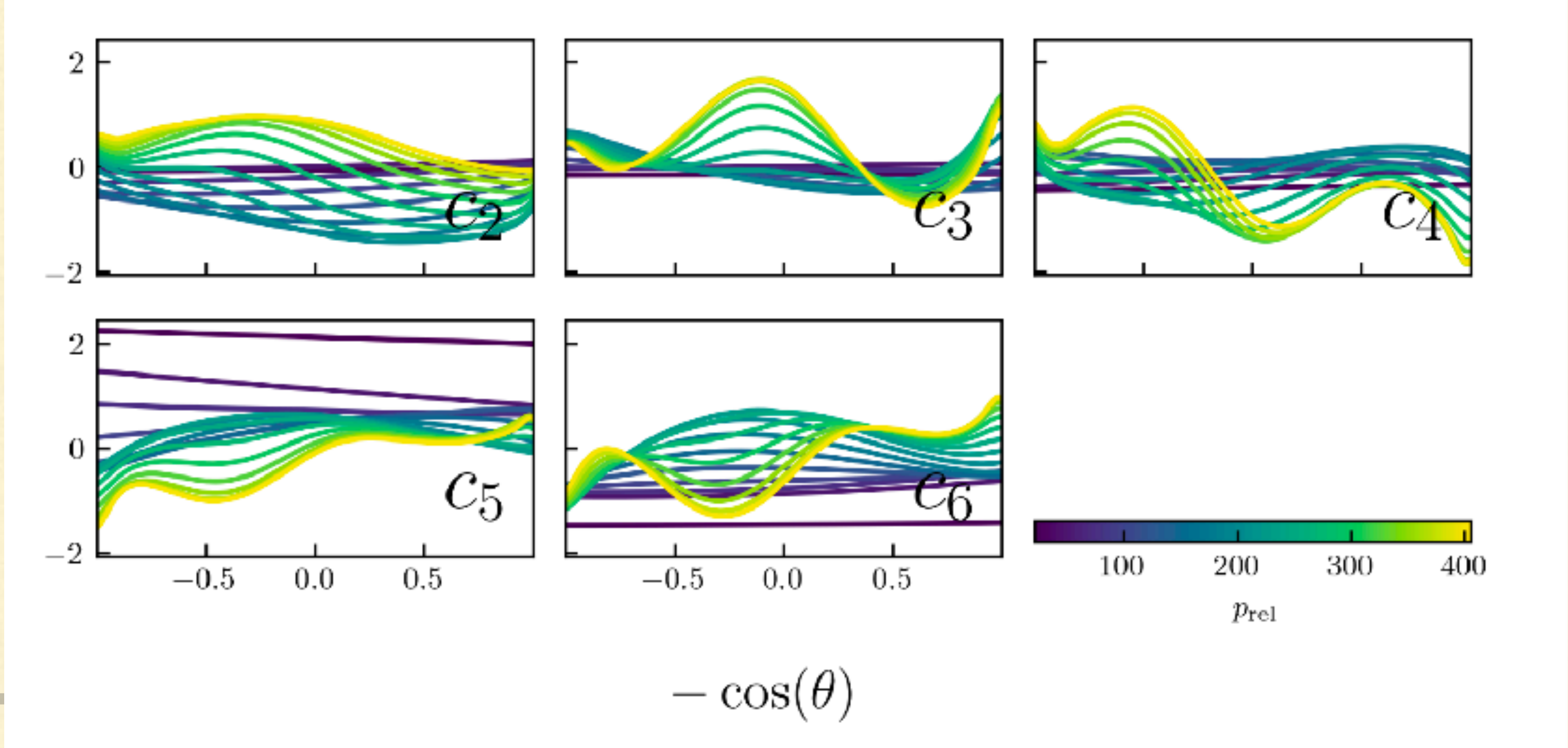


$Q_{max} = \frac{(\mathrm{p}, \mathrm{m}_\pi)}{\Lambda_b}$ or $Q_{sum} = \frac{p + m_\pi}{\Lambda_b + m_\pi}$?

Preliminary conclusions

- Input space (p, -cos(θ))
- Expansion parameter $Q_{\text{sum}} = \dfrac{p + m_\pi}{\Lambda_b + m_\pi}$
- No evidence that N4LO+ is pathological cf. other coefficients

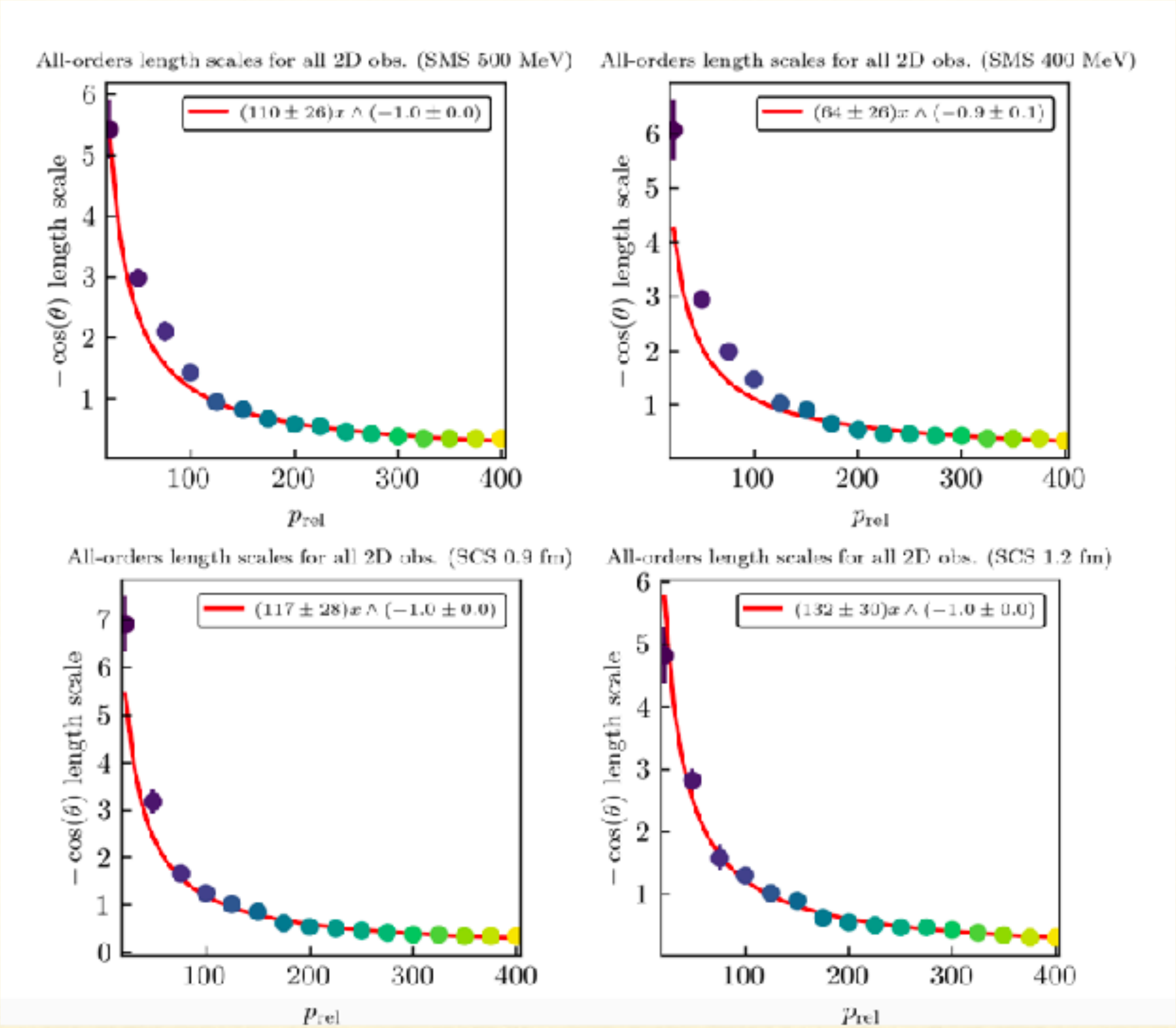
BUT



The GP is not **actually** 2D stationary

Millican et al. (2024B)

PRELIMINARY



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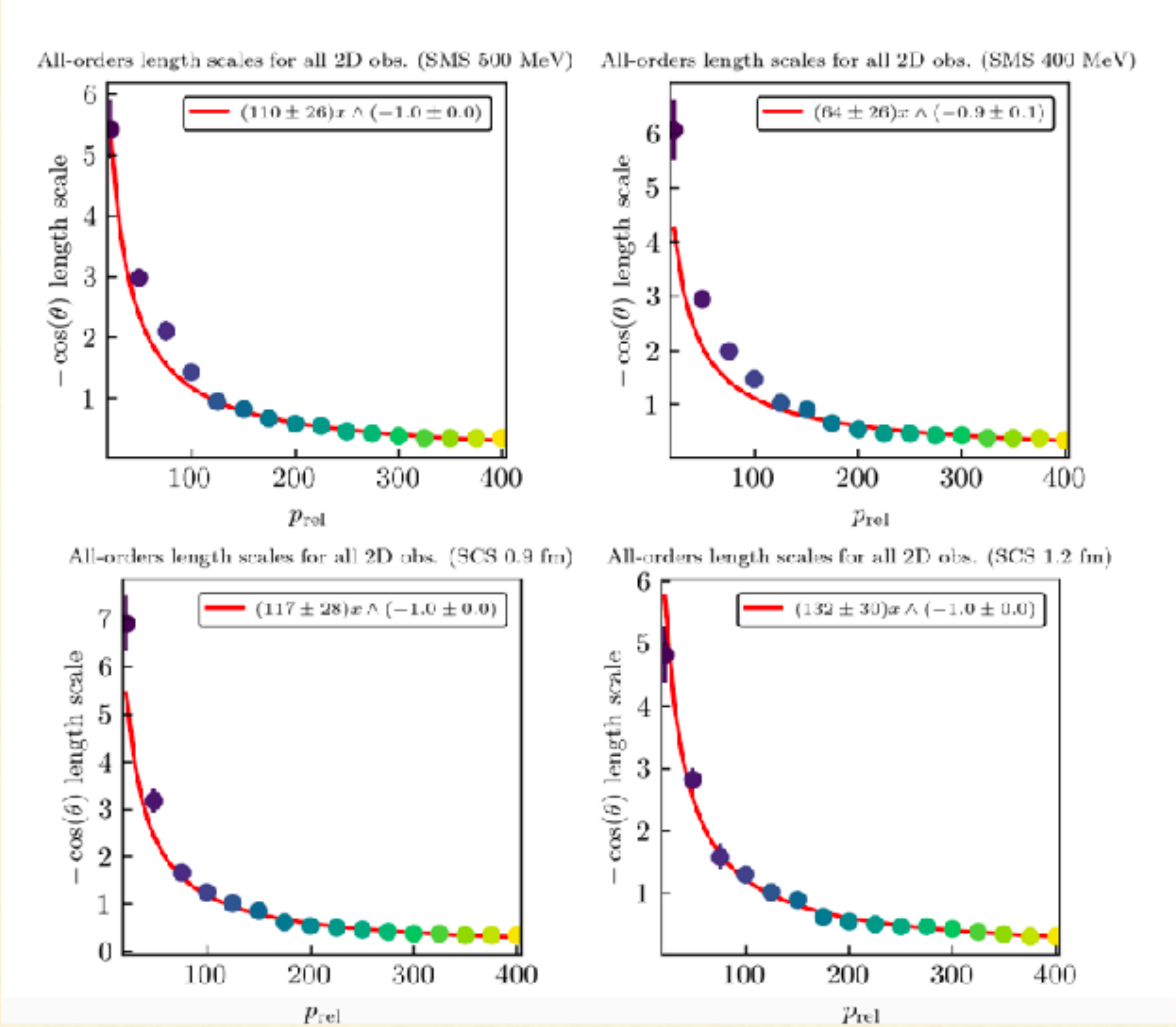
- $\ell_\theta \sim 1/p$



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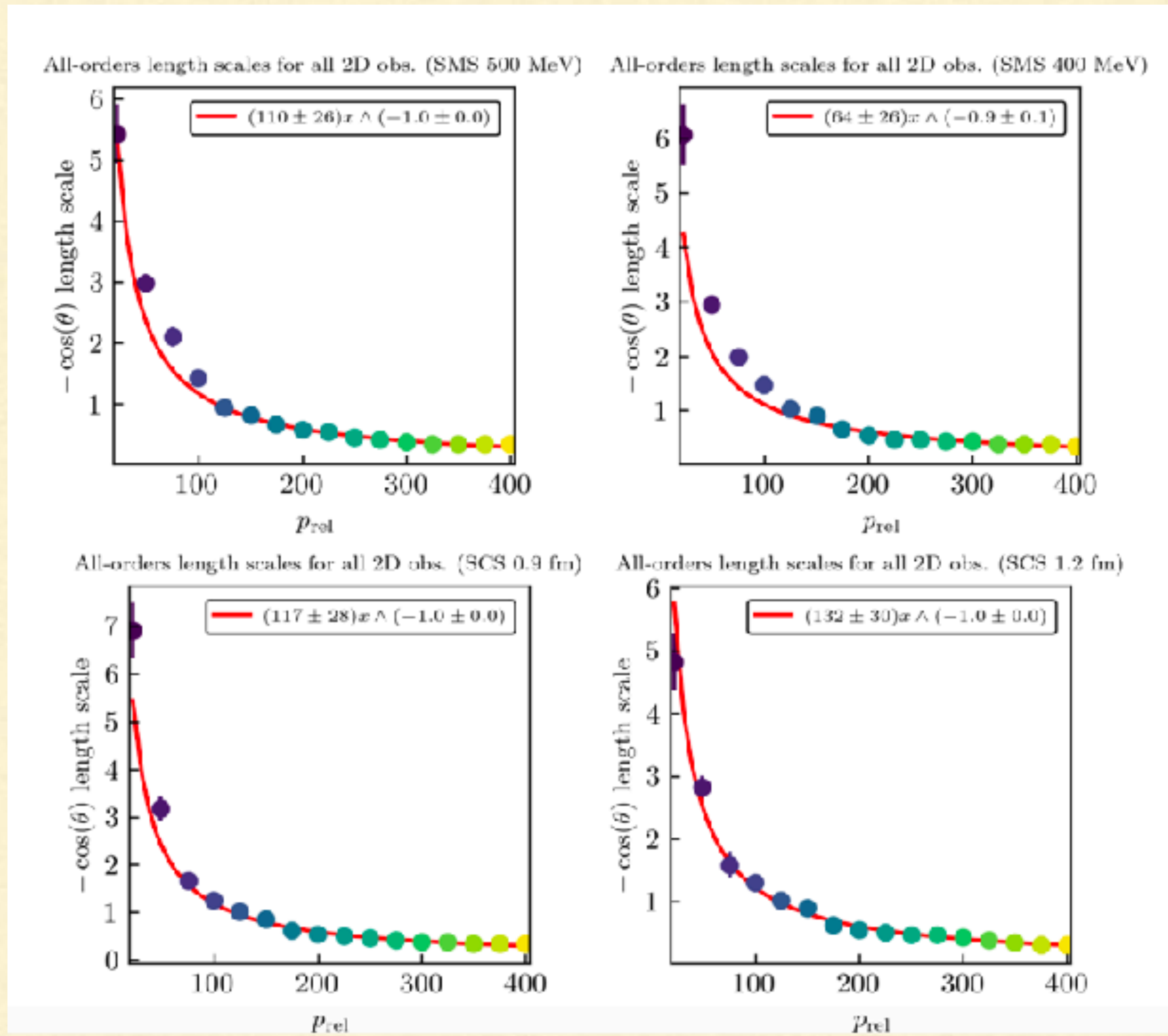

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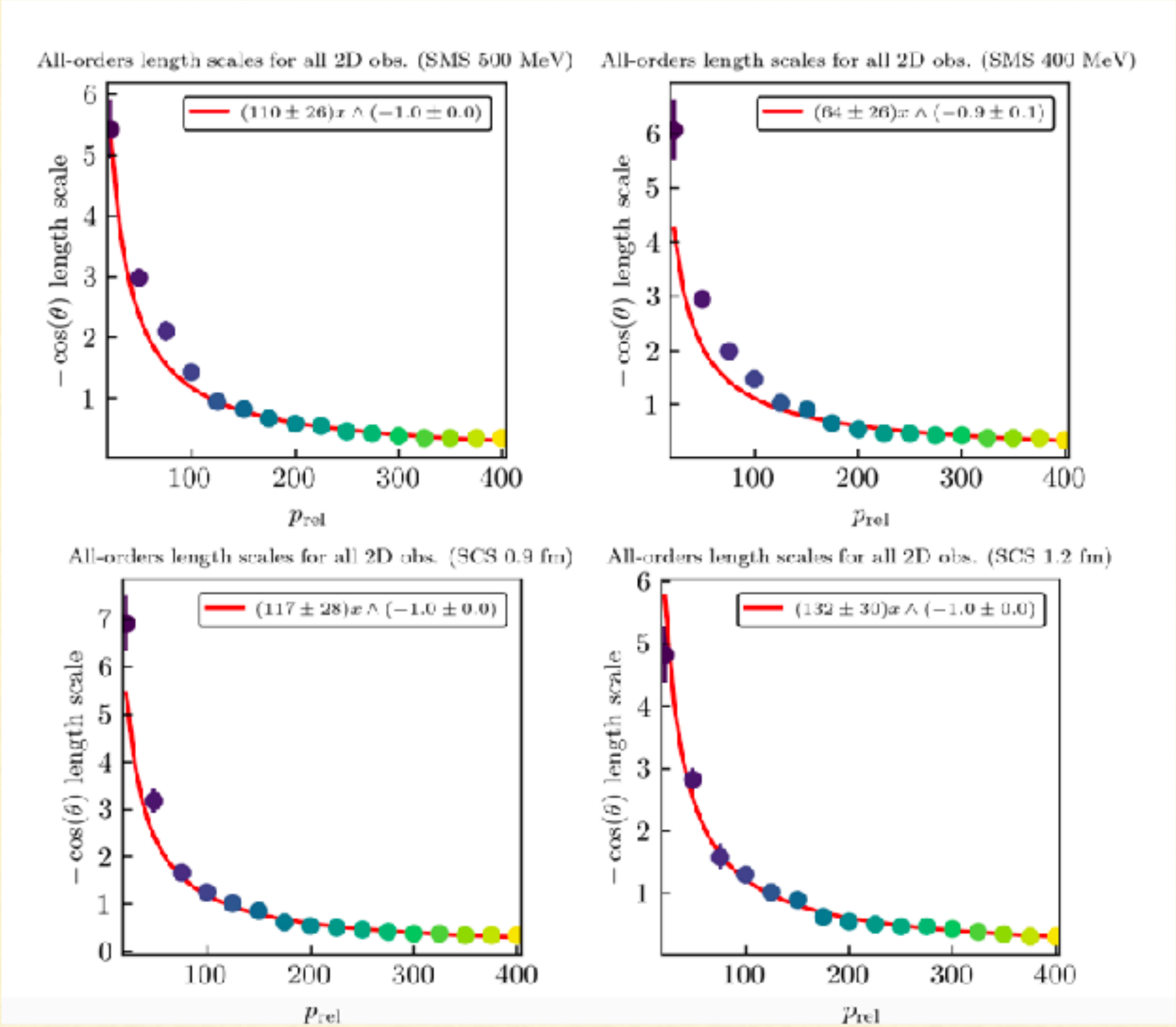

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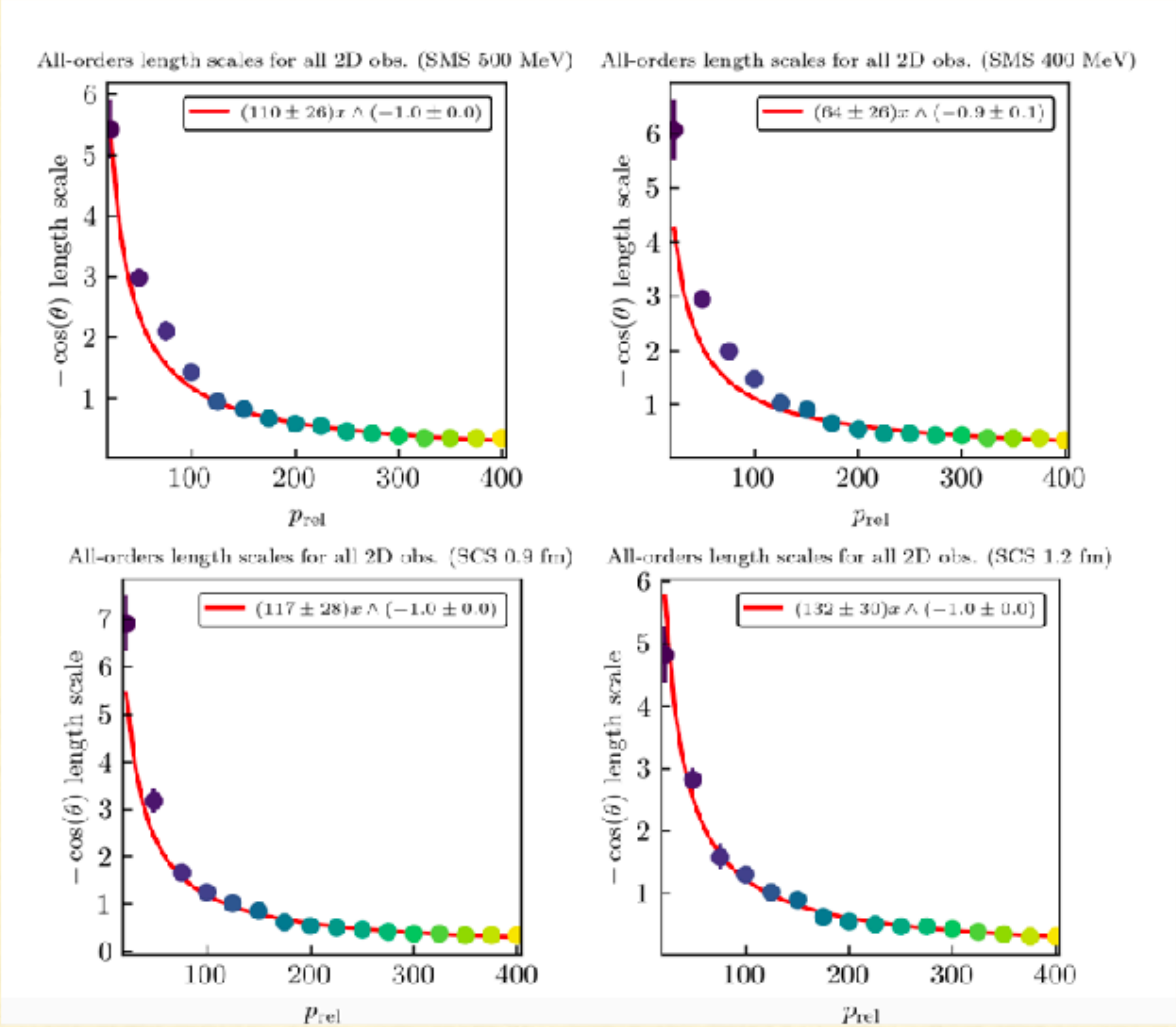

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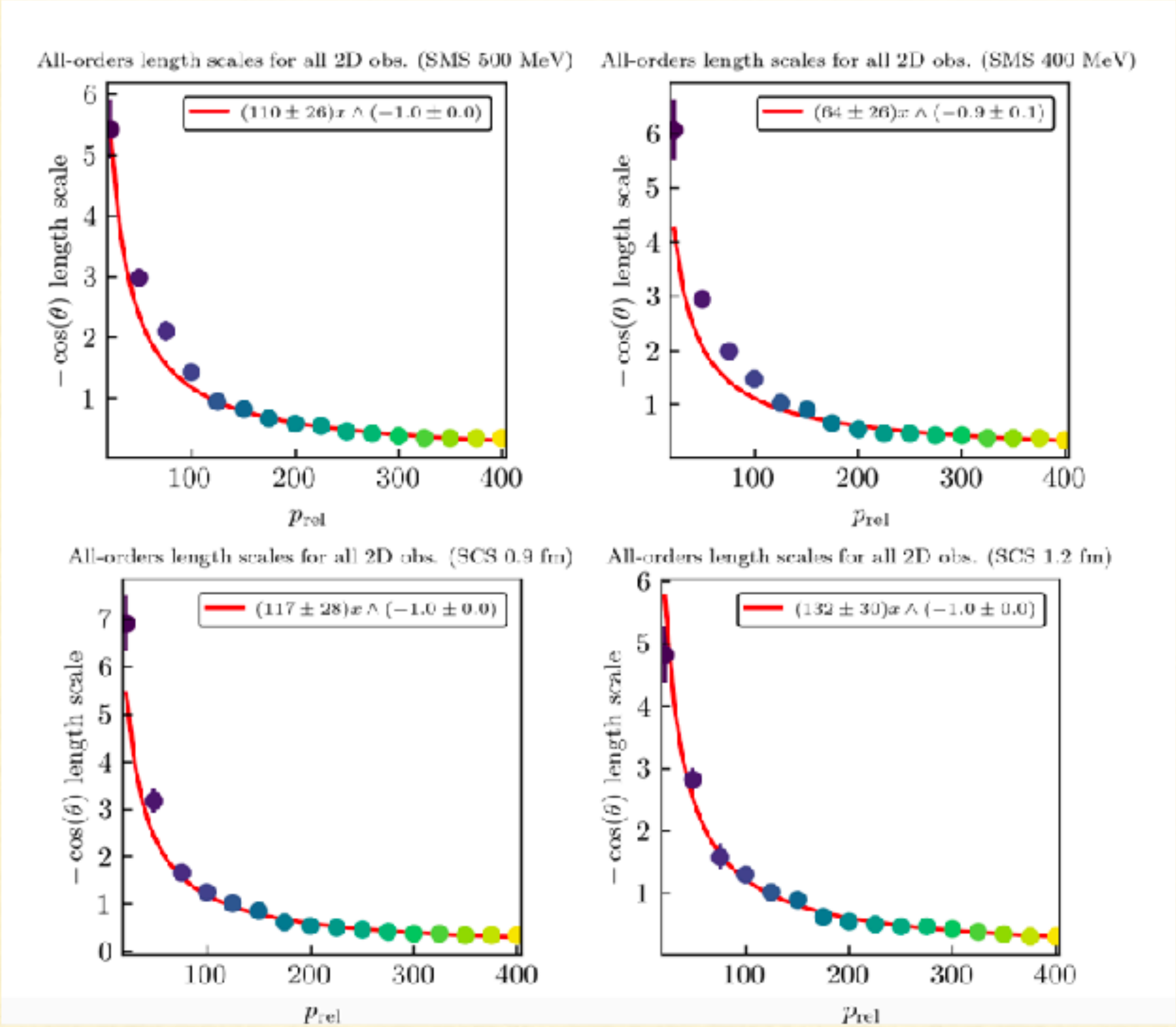
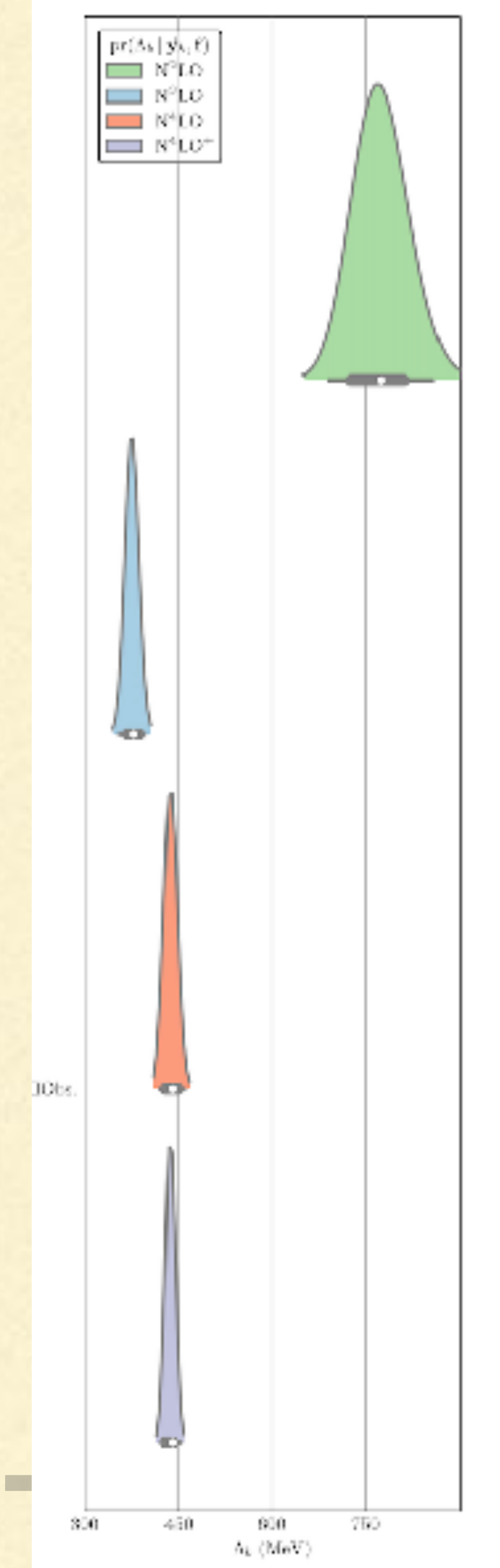
- Duh
- And $\bar{c}_\theta^2$ has mild p dependence
- "Warp" input space to account for 1/p effect
- Fit Lorentzian parameters

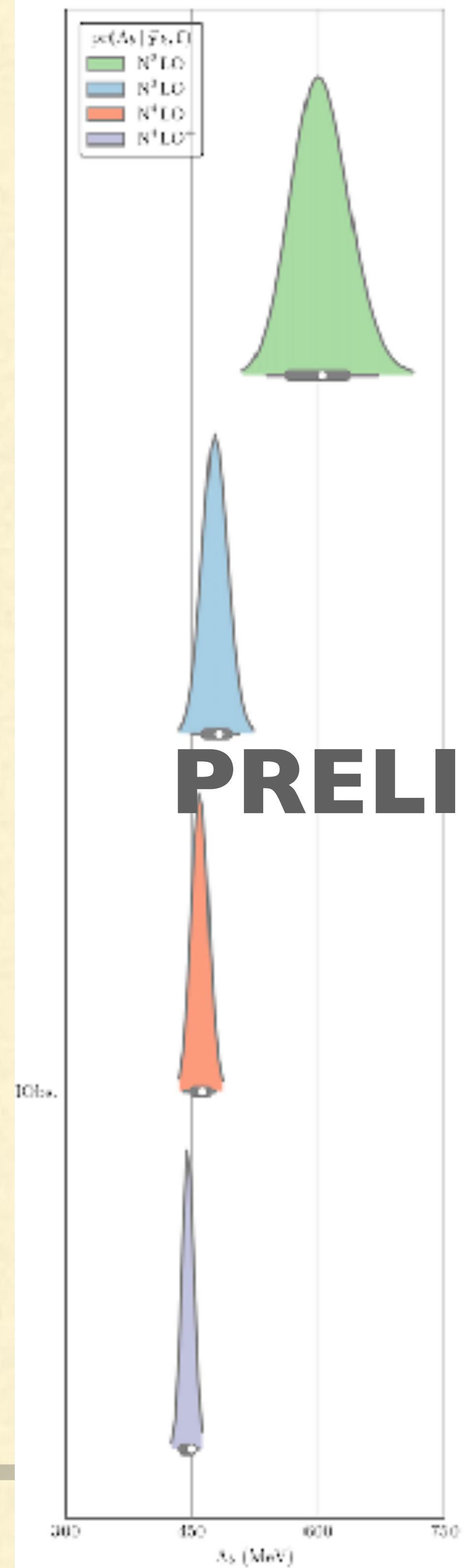


Results for Λ_b for SMS potentials

400 MeV

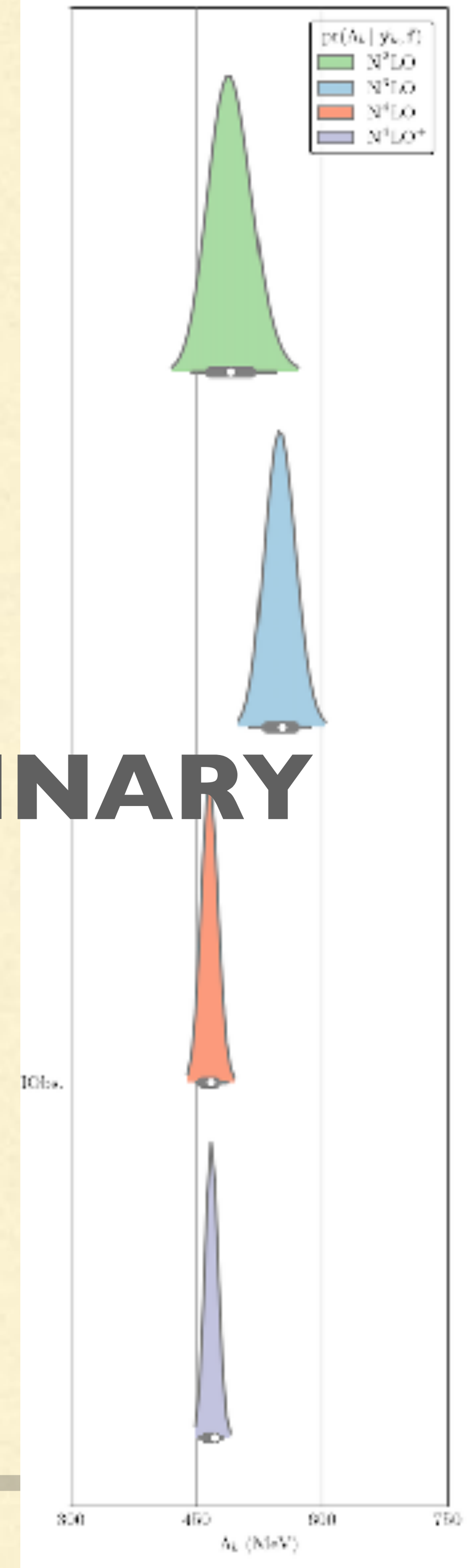


450 MeV

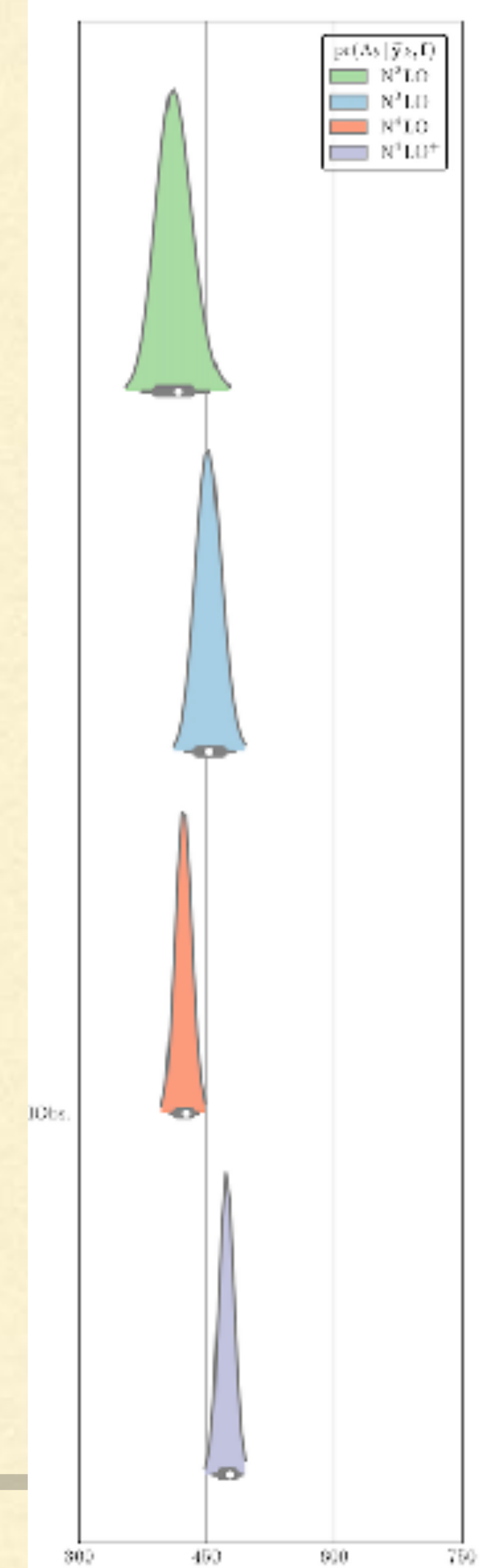


PRELIMINARY

500 MeV



550 MeV



Are observables the right place to look?

McClung, Elster, DP (in progress)

Wolfenstein amplitudes

Wolfenstein & Ashkin (1952)

$$\begin{aligned}\overline{M}(q,\theta) =& A(q,\theta)\mathbb{1} \\ &+ iC(q,\theta)(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\hat{\mathbf{n}} \\ &+ M(q,\theta)(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{n}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}}) \\ &+ [G(q,\theta) - H(q,\theta)](\boldsymbol{\sigma}_1\cdot\hat{\mathbf{q}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{q}}) \\ &+ [G(q,\theta) + H(q,\theta)](\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{\mathcal{K}}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{\mathcal{K}}}).\end{aligned}$$

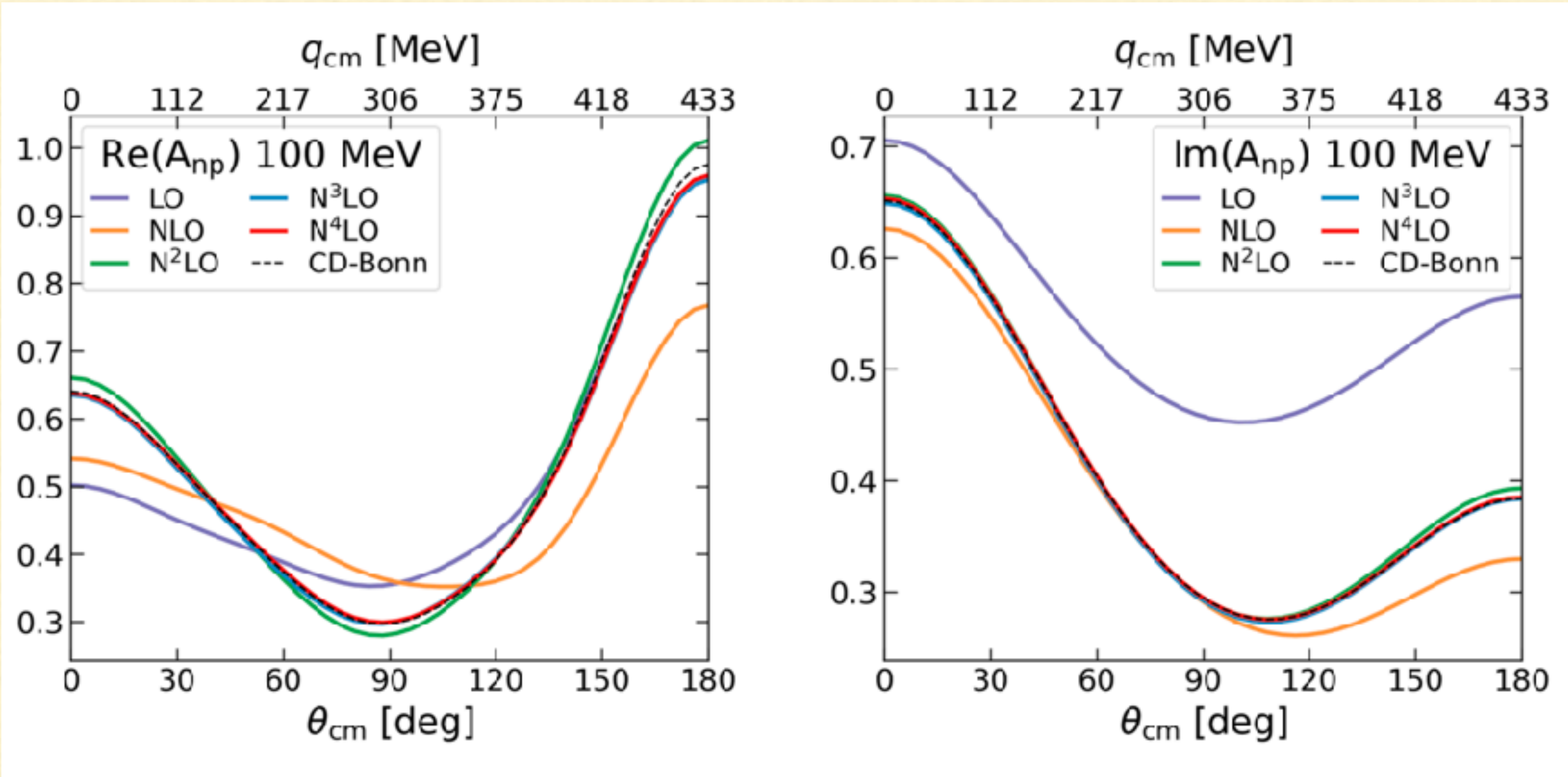
$$\mathbf{K} = \frac{1}{2}(\mathbf{p}+\mathbf{p}'); \mathbf{q} = \mathbf{p}' - \mathbf{p}; \mathbf{n} = \mathbf{p}\times\mathbf{p}'$$

A: central part

C: spin-orbit

M, G, and H: tensor effects

Why not decompose these order by order?



SCS 0.9 fm

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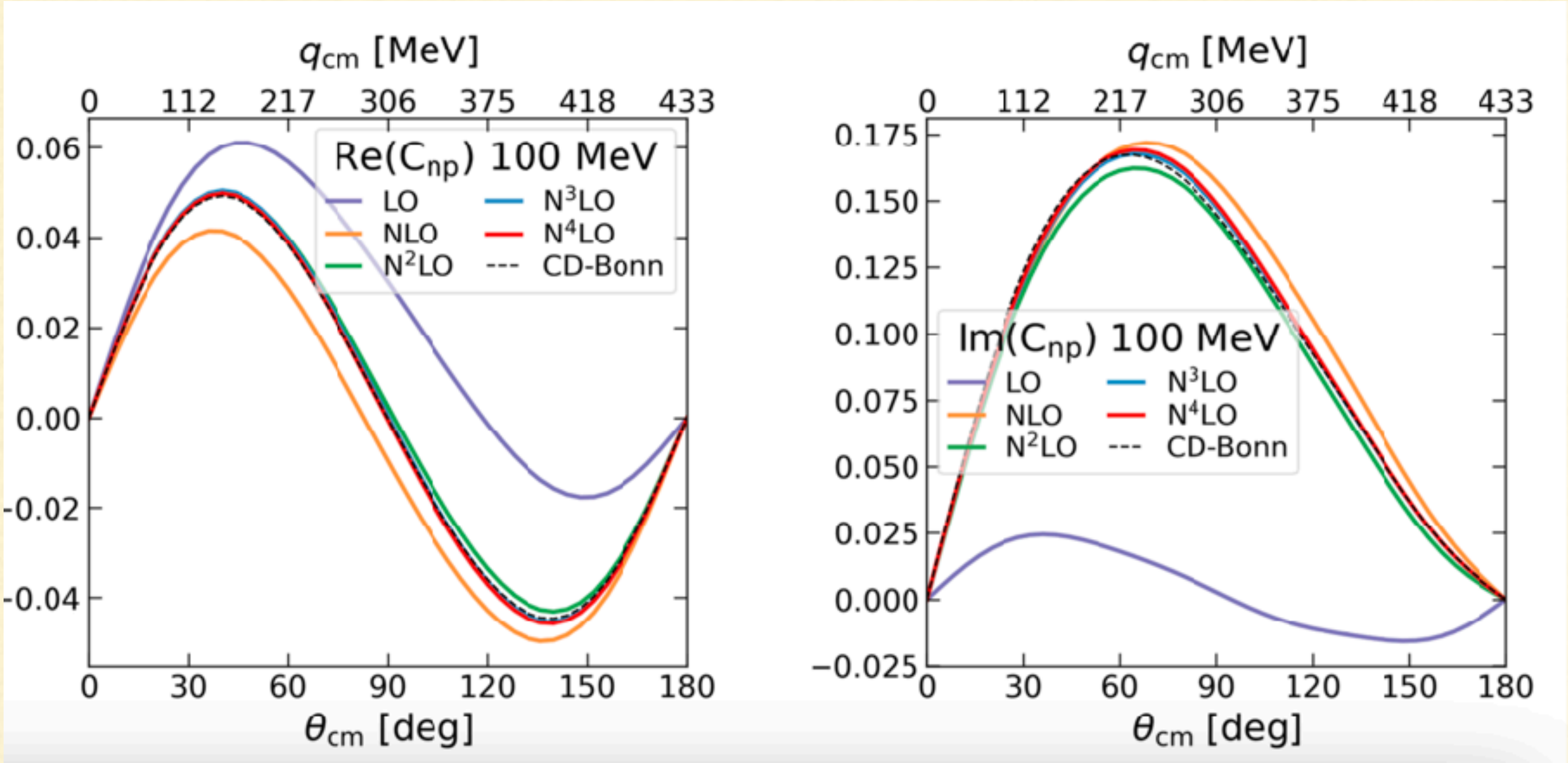
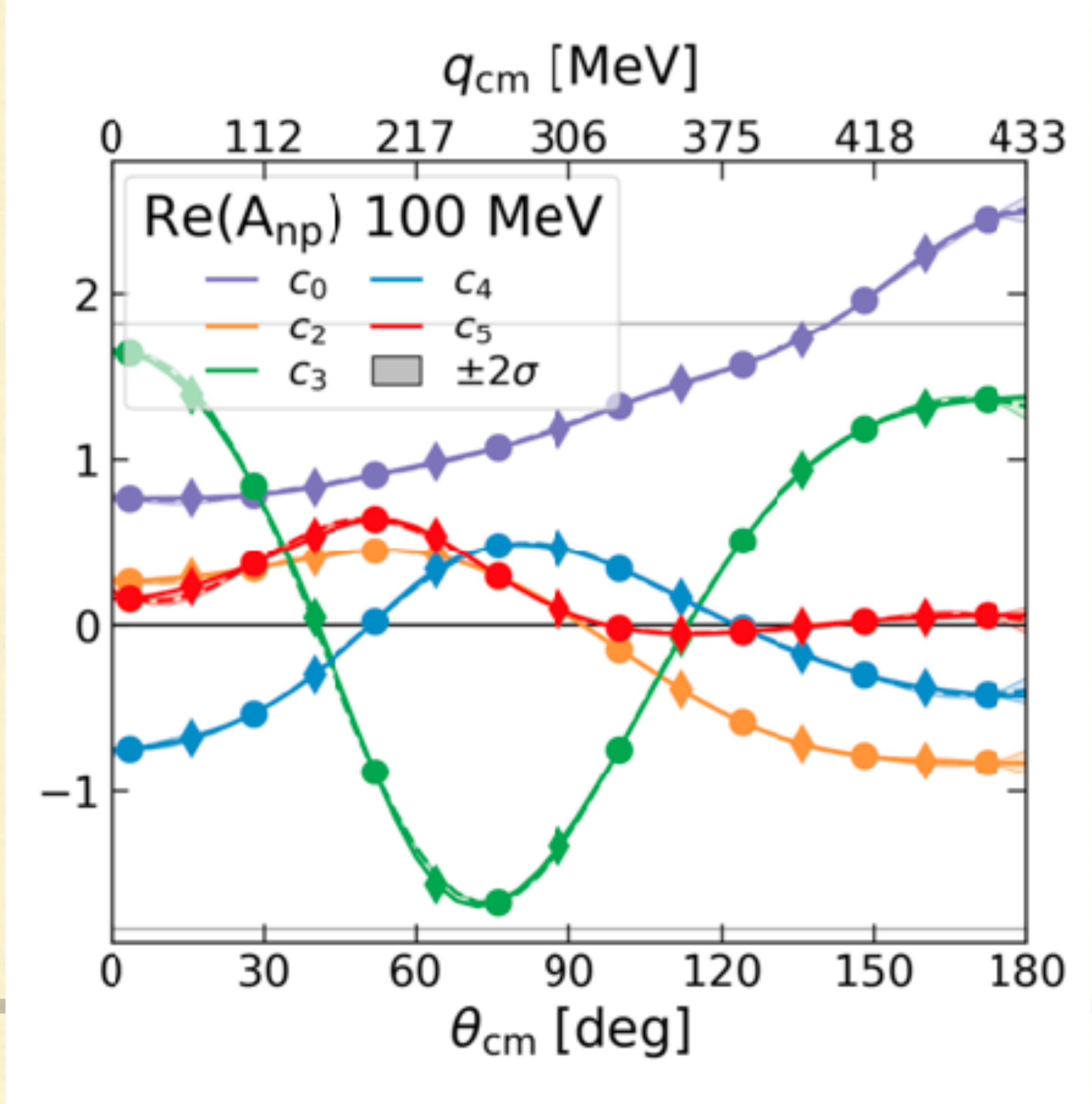


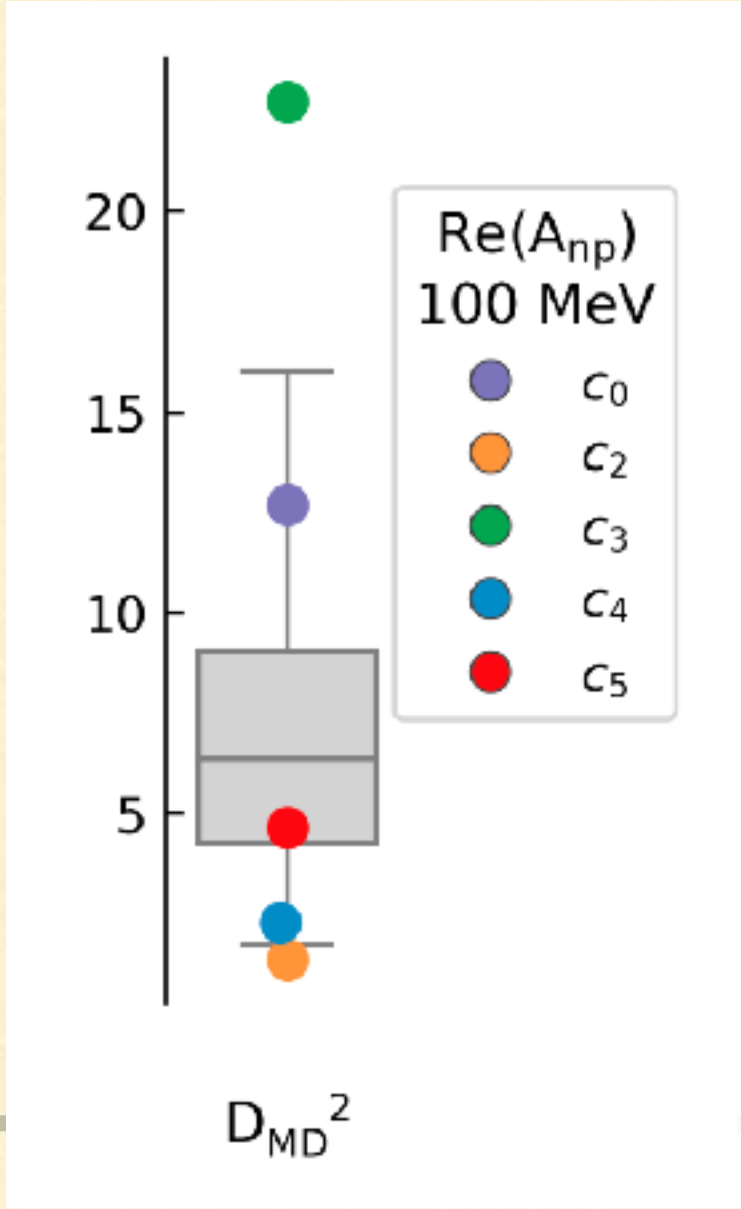
SCS 0.9 fm

Works well for amplitudes at 100 MeV

PRELIMINARY

- y_{ref}=Im(A)
- $$Q = \frac{\max(p, q) + m_\pi}{\Lambda_b + m_\pi}$$





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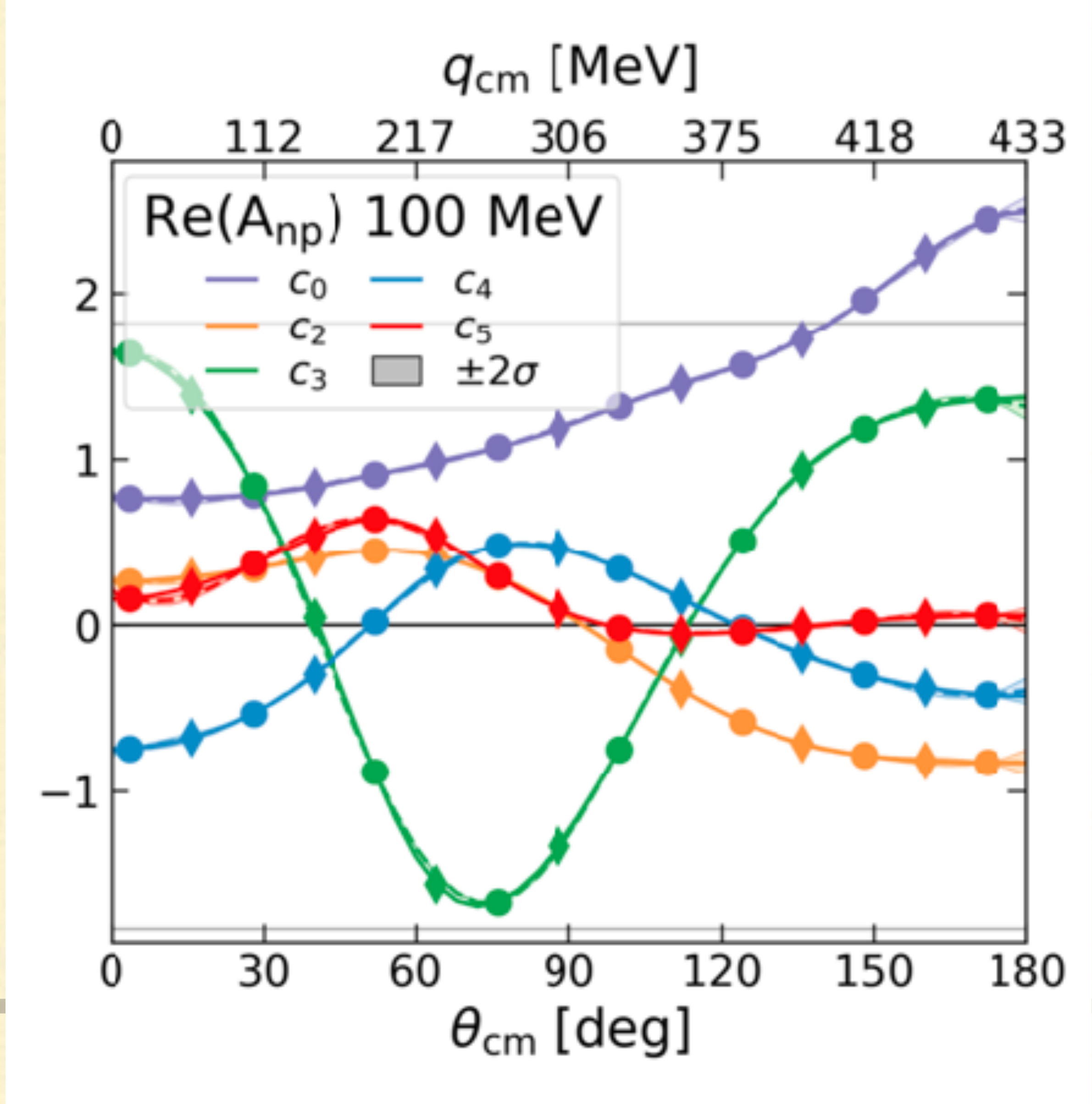
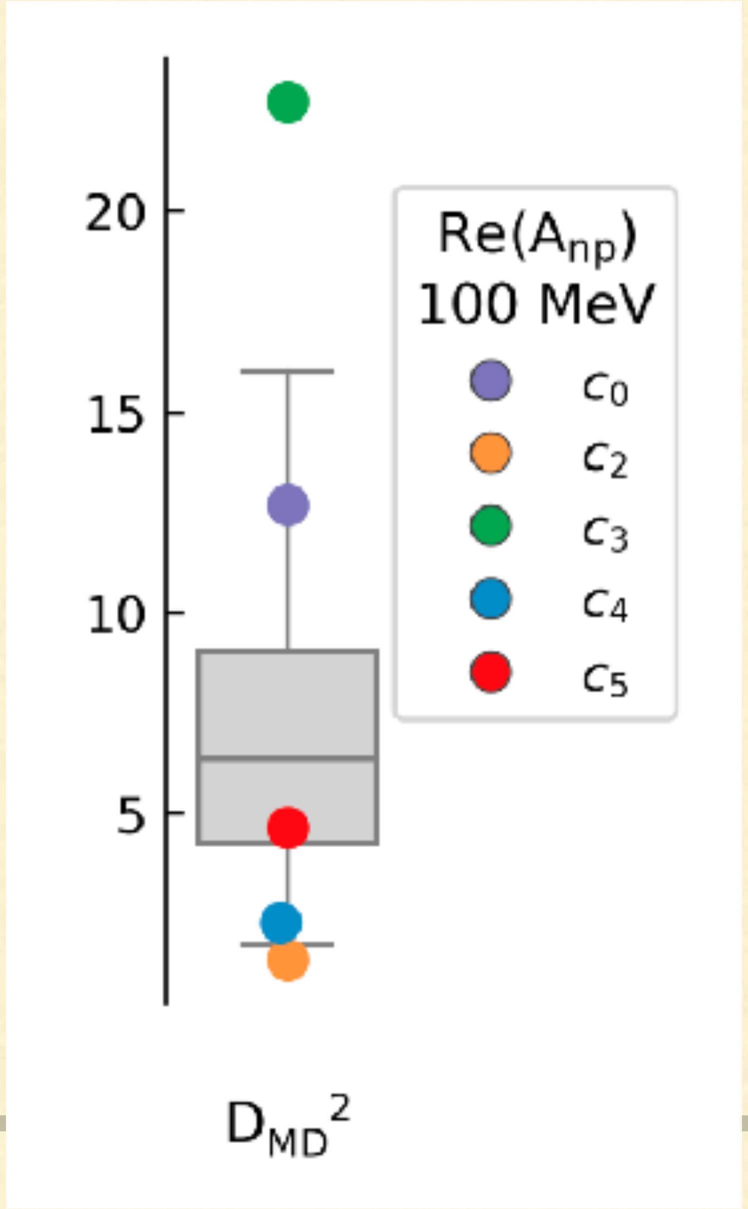
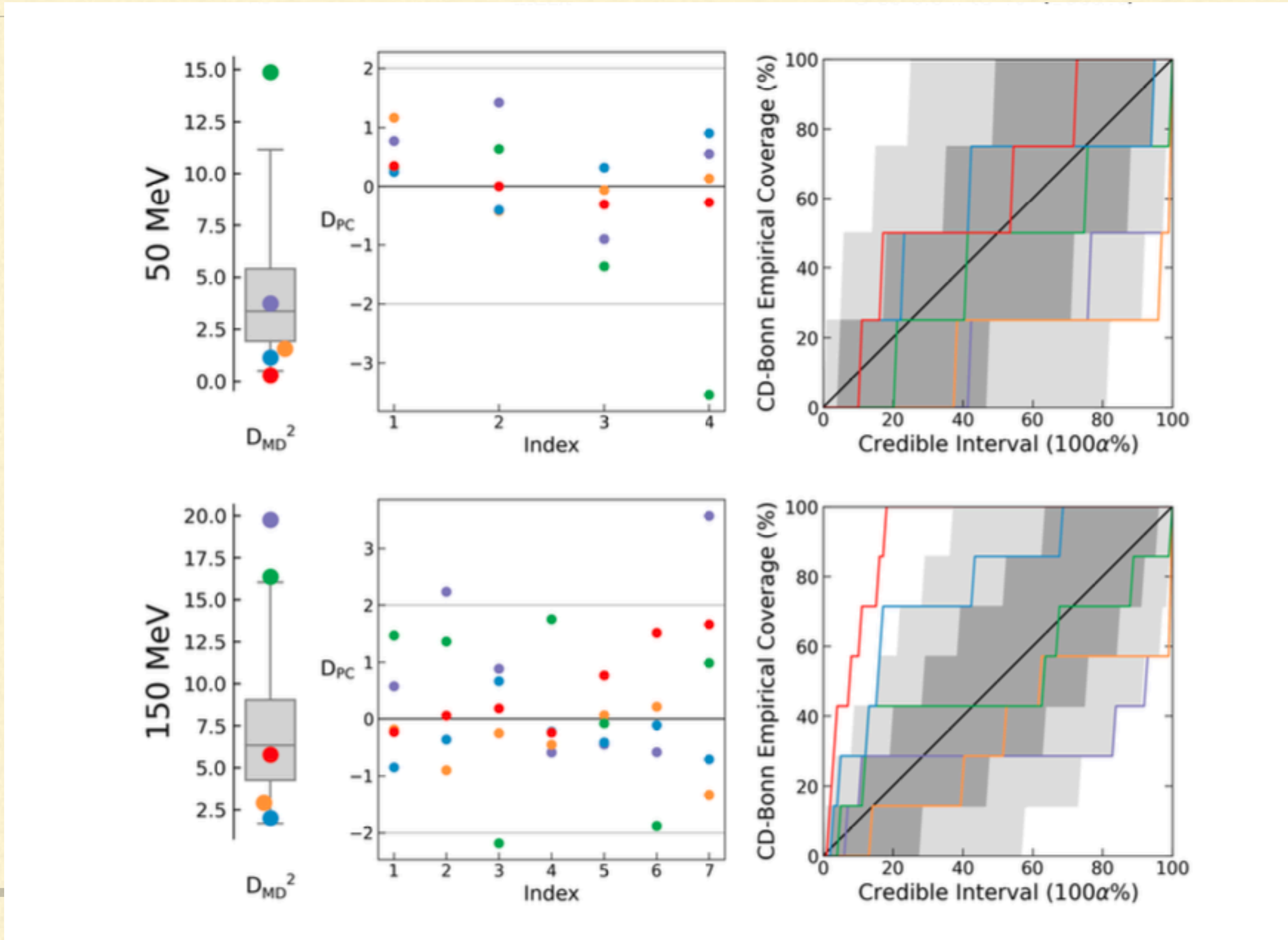
See 1/p dependence of ℓ_θ in this analysis too





And at other energies too



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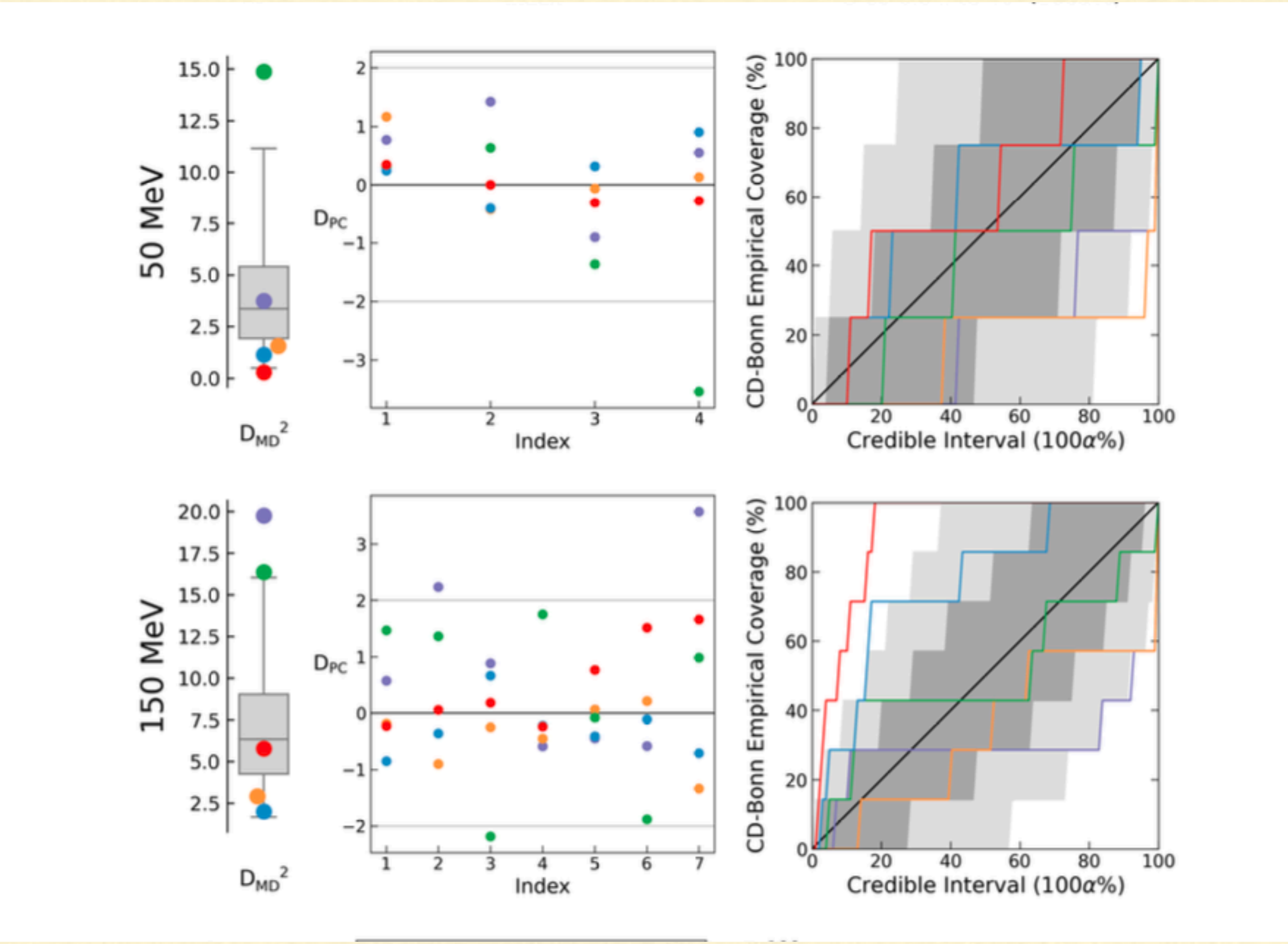

Summary, omissions, and future work

- Statistical modeling for EFT coefficients means quantifying what we mean by EFTheological statements
- Since probability theory is "the logic of science" (Jaynes) once we do this we can then check if EFTs are behaving in accord with the statistical model.
- Building & checking such models doesn't *only* get us truncation error estimates
- Do all orders *really* have the same size? What range can Λ_b be in?
- Could also make "pion mass" a parameter of the statistical model
- Forthcoming: analyses for many potentials. Amplitudes or observables?
- Future: incorporate correlated truncation error in fit of LECs to NN data
  Svennson, Ekström, Forssén, PRC (2024)
- Apply technology to other observables, other nuclear physics EFTs
- How to combine information across different observables ?