Wilsonian RG and Halo EFT

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Outline

- RG in few-body sector of chiral EFT;
- "RG invariant" approach or "peratization"?
- Lorentz invariance and low energy EFT;
- Chiral EFT for P-wave halo states and RG;
  - Subtractive renormalization;
  - Wilsonian RG with two cutoffs;
- Questions for discussion

Disclaimer:
It is by no means my intention to offend anybody ...
not even Weinberg!

Just:
Amicus Plato, sed magis amica veritas!

Imagine a hypothetic situation:

In 1991: S. Weinberg states:

"Capital of Georgia is Atlanta!"

In 1998 X, Y., Z. and other experts of Geography argue giving perturbative as well as non-perturbative water-proof arguments that

"Capital of Georgia is Tbilisi!

... in 1991 one (including Weinberg) did not have good understanding of Geography of former Sovjet Union, but now we know it for sure ... ”

And next 26 years we hear again and again ...

"Everybody knows that Weinberg was wrong !!! "

"I do not know that ... let us have a closer look ..."

"You are NOT Everybody!!!"

"Sorry mite, ... even prophets can be wrong ..."

"There is nothing to discuss ... !"

"Maybe we first should agree on terminology ... "

2. Power counting and its problems

We shall deal here with reactions involving arbitrary numbers of low-energy pions and nucleons, all with three-momenta (in the rest frame of any one of the initial nucleons) less than an amount Q, of the order of one or two hundred MeV. To calculate the matrix elements for such reactions, we shall use an effective lagrangian in which we integrate out all other particle types, including heavy mesons and nucleon isobars. We also integrate out those nucleon loops that are connected to the rest of the diagram only by pion lines, burying their contributions in the constants of the effective low-energy pion interactions. The ultraviolet divergences that arise in calculations using this effective theory are absorbed into a renormalization of the parameters of this lagrangian, using renormalization points also characterized by momenta of order Q or less. After renormalization, the effective cut-off is Q, not only on virtual pion four-momenta but also on the four-momenta transferred to or from the remaining internal nucleon lines. Since Q is taken small compared with typical QCD scales such as the nucleon mass, we may treat the nucleons (but not the pions) non-relativistically, with corrections to the non-relativistic limit regarded as additional interactions with extra derivatives, and we may order terms in perturbation theory according to the number ν of powers of Q that they contain. In I this number was calculated using the methods of

S. Weinberg, *Effective chiral Lagrangian for nucleon-pion interactions and nuclear forces*, Nucl. Phys. **B363** 3, (1991).

RG in few-body sector of chiral EFT

Short reminder about renormalization:

In QFT bare couplings of the Lagrangian c_i are represented as

$$c_i(\Lambda) = c_i^R(\mu) + \delta c_i(\Lambda, c_j^R(\mu), \mu),$$

$c_i^R(\mu)$ - renormalized couplings, μ - renormalization point, δc_i - counter terms.

By differentiating with μ we obtain Gell-Mann-Low RG equations

$$\mu \frac{\partial}{\partial \mu} c_i^R = -\mu \frac{\partial}{\partial \mu} \delta c_i(\Lambda, c_j^R(\mu), \mu).$$

By differentiating with Λ we obtain Wilsonian RG equations

$$\Lambda \frac{\partial}{\partial \Lambda} c_i(\Lambda) = \Lambda \frac{\partial}{\partial \Lambda} \delta c_i(\Lambda, c^R(\mu), \mu).$$

Wilsonian RG controls cutoff-dependence of bare couplings!

Gell-Mann-Low RG controls renormalization point-dependence of renormalized couplings!

While the full physical quantities do not depend on μ, in any perturbative expansion the relative size of different contributions (i.e., power counting) essentially depends on the choice of the renormalization scheme!

Best-known example: Perturbative QCD ...

Demonstrating the usage of RG in PT

Let a physical quantity be given by exact expression

$$f(g) = \frac{g}{1 - \hbar g} \,,$$

where g is a parameter and $\hbar$ controls the quantum corrections.

Suppose for whatever reason we calculate this quantity in PT.

For $|g| > 1$ our expansion in g leads to a divergent series.
In this case we can expand in a different way:

$$\begin{aligned} f(g) &\equiv \frac{g}{1 - \hbar\mu\, g - \hbar\, g(1-\mu)} \\ &= \frac{g_\mu}{1-\mu}\left(1 + \hbar g_\mu + \hbar^2 g_\mu^2 + \cdots\right), \end{aligned}$$

where $g_\mu = g(1-\mu)/(1 - \hbar\mu\, g)$.

The exact $f(g)$ is μ-independent, however the sum of the first N terms depends on μ.

While formally this dependence is of higher order $\sim \hbar^{N+1}$, the convergence of the series crucially depends on the choice of μ.

For example, for $g = 2$ this series converges only if $\mu > 3/4$, the convergence being best close to $\mu = 1$.

Wilsonian RG in few-body sector of chiral EFT:

For definiteness we concentrate on the NN scattering.

We assume that QCD is the correct theory of the strong interaction.

Given the NN scattering amplitude of QCD we could obtain a corresponding NN potential.

In particular, we would extend the physical amplitude $T_{\rm on}(E,\Theta) = T_{\rm on}(\vec{p}\,',\vec{p})$ to an off-shell function $T(\vec{p}\,',\vec{p},E)$ and obtain the potential by solving the equation:

$$V\left(\vec{p}\,',\vec{p}\,,E\right) = T\left(\vec{p}\,',\vec{p}\,,E\right) - \int d^3k \; V\left(\vec{p}\,',\vec{k}\,,E\right) G(E,k) \; T\left(\vec{k},\vec{p}\,,E\right)$$

Considering all possible off-shell extensions of the amplitude one covers all potentials which lead to the same physical amplitude.

Wilsonian RG:

We regularize the integral equation and modify the potential so that the scattering amplitude does not change:

$$T\left(\vec{p}\,',\vec{p}\right)=V\left(\Lambda,\vec{p}\,',\vec{p}\right)+\int d^3k\ V\left(\Lambda,\vec{p}\,',\vec{k}\right)G(\Lambda,E,k)\,T\left(\vec{k},\vec{p}\right)\,,$$

where

$$G(\Lambda,E,k)=F(\Lambda,k)G(\Lambda,E,k)\,,$$

and $F(\Lambda,k)$ is a regulator function.

The off-shell amplitude coincides with the original one if the potential satisfies RG equation:

$$\begin{aligned}V\left(\Lambda,\vec{p}\,',\vec{p}\,,E\right)&=V\left(\vec{p}\,',\vec{p}\,,E\right)\\&+\int d^3k\ V\left(\Lambda,\vec{p}\,',\vec{k}\,,E\right)G(E,k)(1-F(\Lambda,k))\ V\left(\vec{k},\vec{p}\,,E\right)\,.\end{aligned}$$

The solution to this equation defines the RG flow of the potential.

M. C. Birse, J. A. McGovern, K. G. Richardson, Phys. Lett. B **464**, 169 (1999).

Any value of Λ leads to the same amplitude if the full cutoff-dependent potential $V\left(\Lambda, \vec{p}\,', \vec{p}\,, E\right)$ is substituted.

If one expands $V\left(\Lambda, \vec{p}\,', \vec{p}\,, E\right)$ by applying some method of approximations, the convergence of the corresponding perturbative series for the amplitude, in general, strongly depends on the choice of the value of Λ.

One chooses such values of Λ which lead to optimal convergence of perturbative series for physical quantities.

Does there exist any regulated potential following from QCD which admits a systematic expansion in some region of Λ, which can be reproduced order-by-order using an effective Lagrangian?

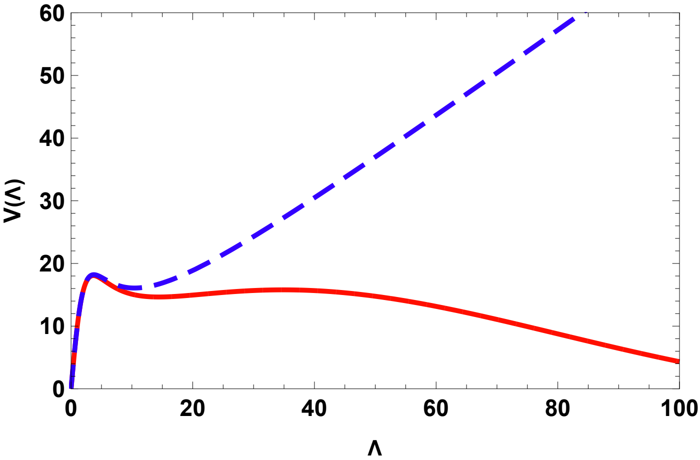
If such an expansion does not exist in the RG space of potentials then the Wilsonian RG approach to chiral EFT is guaranteed to fail.

If it does, one still needs to identify the region of values of Λ and the expansion of the potential in terms of chiral EFT, i.e., the corresponding power counting for various terms of the Lagrangian.

For a system with scale separation $M_{low} \ll M_{high}$
for energies $E \sim M_{low}^2/m$

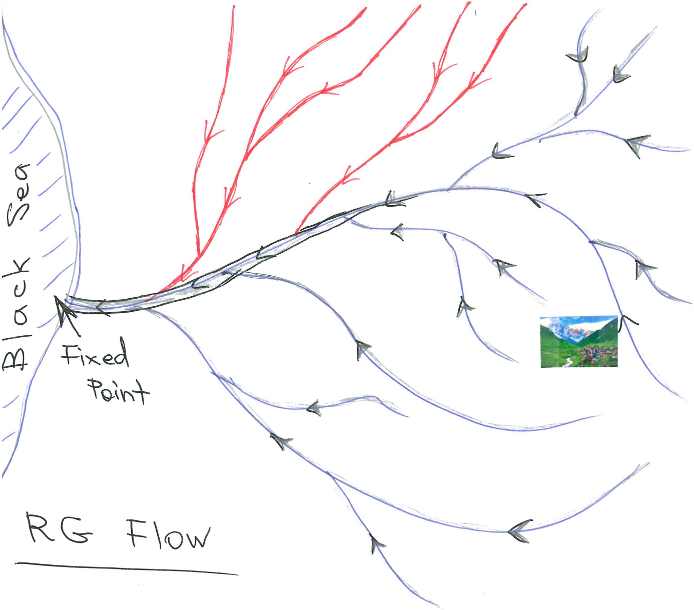
by choosing $M_{low} \ll \Lambda \ll M_{high}$

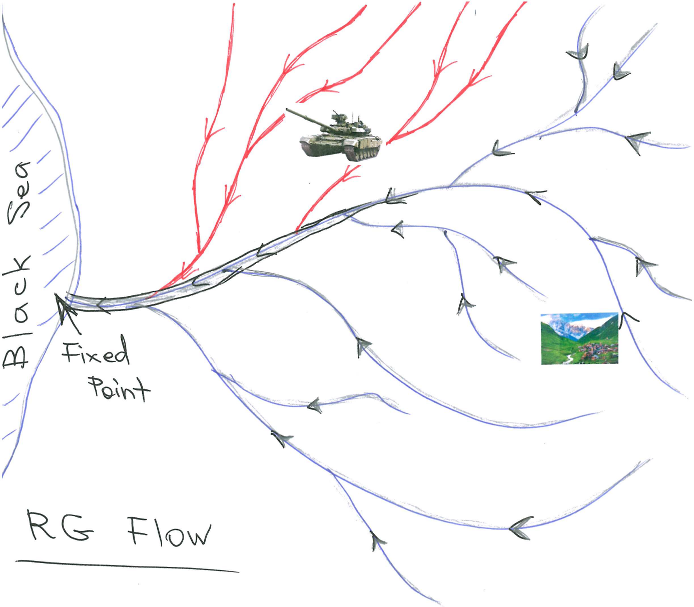
we can expand the regulated potential in powers of
$\epsilon \sim M_{low}/\Lambda \sim \Lambda/M_{high}$:



RG flow, red line - exact potential, blue line - approximate potential.

Georgian mountains ... "Svaneti"


Black Sea
Fixed Point
RG Flow


Black Sea
Fixed Point
RG Flow

"RG invariant" approach or "peratization"?

Following
J. B. Habashi, S. Sen, S. Fleming and U. van Kolck, Annals Phys. **422**, 168283 (2020),
U. van Kolck, Symmetry **14**, 1884 (2022),
H. W. Griesshammer and U. van Kolck, Eur. Phys. J. A **59**, 289 (2023)
we revisit the contact interaction potential with first two terms for $a \sim r_e \sim 1/Q \gg 1/\tilde{Q}$, where Q and $\tilde{Q}$ are the soft and hard scales of the problem.

To avoid the complications caused by the Wigner bound (D. R. Phillips and T. D. Cohen, Phys. Lett. B **390**, 7-12 (1997)) encountered in "RG invariant" approach it was assumed that the effective range is negative.

Following

D. R. Phillips, S. R. Beane and T. D. Cohen, Nucl. Phys. A **631**, 447C-451C (1998)

one can obtain the corresponding amplitude in closed analytic form.

Using sharp cutoff and fitting C and C_2 to a and r_e one obtains:

$$
\begin{aligned}
T &= \frac{N}{mD}\,, \\
N &= 4i\pi^2 a\left(\pi a\Lambda q^2 r_e - 4a\hbar\left(\Lambda^2+q^2\right)+2\pi\Lambda\right) \\
D &= 2i\pi^2\Lambda - 2a\pi(\pi q + 2i\Lambda)\Lambda\hbar \\
&- iaq\hbar\left(-2\pi\Lambda + 4a\left(q^2+\Lambda^2\right)\hbar - a\pi q^2\Lambda r_e\right)\ln\frac{\Lambda-q}{\Lambda+q} \\
&+ a^2 q\hbar\left(-8iq\Lambda\hbar + 4\pi\left(q^2+\Lambda^2\right)\hbar + \pi q(-\pi q + 2i\Lambda)\Lambda r_e\right)\,.
\end{aligned}
$$

By taking $\Lambda \sim \tilde{Q}$ or larger the above amplitude can be written as

$$T = -\frac{4\pi}{m}\frac{1}{-1/a + q^2 r_e/2 - iq\hbar} + \frac{\pi^2 q^4 r_e^2}{2\Lambda m\hbar\left(-1/a + q^2 r_e/2 - iq\hbar\right)^2} + \mathcal{O}\left(\frac{1}{\Lambda^2}\right)$$

This expression satisfies conditions of "RG invariant" approach.

However it has to be applicable also for momenta $q \ll 1/a$, where the perturbative loop expansion converges.

That is, for such values of q the re-summation of diagrams is not necessary because renormalized perturbative series converges.

The problem with the non-perturbatively "renormalized" amplitude is manifested in its perturbative expansion in $\hbar$:

$$T = \frac{4\pi a}{m} + \frac{2\pi a^2 q^2 r_e}{m} - \frac{i a^2 q \hbar}{m}\left[-2iq\left(a^2 \Lambda q^2 r_e^2 + \frac{4}{\Lambda}\right) + 2i\tanh^{-1}\left(\frac{q}{\Lambda}\right)\left(a q^2 r_e + 2\right)^2 + \pi\left(a q^2 r_e + 2\right)^2\right] + \mathcal{O}(\hbar^2)\,.$$

Starting at one-loop order it contains positive powers of Λ.

The term linear in Λ is suppressed relative to LO contribution only if $\frac{a^3 \Lambda q^4 \hbar r_e^2}{2\pi} \ll 1$ which implies that Λ cannot be taken much larger than $Q \sim 1/a \sim 1/r_e$.

The above "RG Invariant" treatment leads to differing results for the dimensional and the cutoff regularizations
D. R. Phillips, S. R. Beane and T. D. Cohen, Nucl. Phys. A **631**, 447C-451C (1998).
In particular, using dimensional regularization one obtains

$$T = -\frac{4\pi}{m}\frac{1}{-1/(a+q^2a^2r_e/2)-iq\hbar}+\mathcal{O}(d-4)\,,$$

where d is the space-time dimension.
Notice that the difference between two expressions is *not* of higher order for $a \sim r_e \sim 1/Q \gg 1/\tilde{Q}$.

Different regularizations leading to different results is a clear indication that the applied "renormalization" procedure is not consistent with the standard QFT formalism.

In fact this "non-perturbative renormalization" is actually "peratization" of

G. Feinberg and A. Pais, Phys. Rev. 131, 2724 (1963).

G. Feinberg and A. Pais, Phys. Rev. 133, B477 (1964).

Peratization:

$\lim_{\Lambda\to\infty} \frac{A+B\Lambda}{C+D\Lambda} = \frac{B}{D}$.

Renormalization:

$\lim_{\Lambda\to\infty} \frac{A+B\Lambda}{C+D\Lambda} = \frac{A^R}{C^R} \neq \frac{B}{D}$.

The same difficulty can be seen in a different way if we define the renormalized value of C as the value of T at a variable renormalization point, $\vec{k}^2 = -\mu^2$. The solution of the renormalization group equation here is obtained by simply substituting $-\mu^2$ for $\vec{k}^2$ in eq. (19),

$$C(\mu) = \frac{C_R}{1 + m_N C_R \mu / 4\pi} . \tag{22}$$

Note that for $\mu \gg 4\pi / m_N C_R$, the dimensionless quantity $C(\mu)\mu m_N$ approaches the fixed point 4π. It is not surprising that the renormalized value of C scales as μ^{-1}. At the renormalization scale μ the delta-function potential (14) is effectively smeared over a radius $b \simeq \mu^{-1}$, so in order to have a volume integral $C(\mu)$, the depth V_0 of the potential must be of order $C(\mu)\mu^3$; the familiar condition $V_0 \sim b^{-2}$ for a short-ranged potential to have a finite effect thus requires that $C(\mu) \propto \mu^{-1}$.

S. Weinberg, *Effective chiral Lagrangian for nucleon-pion interactions and nuclear forces*, Nucl. Phys. **B363** 3, (1991).

Lorentz invariance and low energy EFT

Lorentz invariance is a fundamental symmetry of QCD!

Is it important at low energies?

Apparent answer: No! ...

Does that apply to real world, or to its mathematical image (... called theoretical Physics)?

If to both, then how is it realized in mathematical image?

Does Lorentz-invariant formulation offer any advantages in few-body sector? ...

INT Program 13-1a

Computational and Theoretical Advances ...

March 25 - April 19, 2013

available on INT homepage

van Kolck:

Root of the problem:
pion exchanges (long-ranged, contribute to waves higher than S)
are singular (sensitive to short-range physics, require counterterms)
This has Nothing to do with relativity...
(For the opposite opinion, see Epelbaum + Gegelia ’12)

E. Epelbaum and J. Gegelia, Phys. Lett. B **716**, 338 (2012).

- From the long-range behavior of the potential follows **Nothing** about its short-range behavior!
- Chiral EFT potential is valid only for small momenta!
- Singular behavior of chiral potentials comes from large momenta - it is solely artefact of extrapolation!
- QCD does not fix the NN potential uniquely, however the spectrum of QCD fixes the class of possible potentials: **NON-SINGULAR!**
- Any short-range extrapolation
  ... in the class of non-singular potentials is going to be wrong!

  ... in the class of singular potentials is not going to be even wrong!
- What does that to do with Lorentz invariance?

Lorentz-invariant formulation versus NR approach

Is Lorentz invariance important at low energies?

No, provided that one takes proper care!

Non-relativistic theory should be adequate at low energies, however ...

Per definition, non-relativistic expansion means:

1. Lorentz invariant effective Lagrangian – Lorentz invariance is a fundamental symmetry!
2. Quantum corrections.
3. Regularization (Λ) and renormalization.
4. Non-relativistic expansion (expansion in $1/m$) of renormalized quantities.

On the other hand, non-relativistic EFT:

1. Lorentz-invariant EFT Lagrangian – expanded in $1/m \Rightarrow$ non-relativistic EFT Lagrangian.
2. Quantum corrections.
3. Regularization (Λ), Renormalization.
4. Renormalized quantities are given as non-relativistic series.

- Proper non-relativistic expansion:
  first – calculation of quantum corrections,
  after – $1/m$ expansion.
- Non-relativistic EFT:
  first – expansion in $1/m$,
  after – calculation of quantum corrections.
- Expansions in $1/m$ and calculation of quantum corrections are not commutative!
- Difference ("error") can be compensated by adding terms in non-relativistic EFT Lagrangian. A finite number of terms needed at any fixed order in one nucleon sector.

  J. G. and G. Japaridze, Phys. Rev. D **60**, 114038 (1999).
- Due to solving an integral equations, an infinite number of compensating terms needed in few-body sector already at LO.

Solutions:

- Keep $\Lambda \lesssim m$ – successfully implemented by various groups:

  C. Ordonez, L. Ray and U. van Kolck, ...
  E. Epelbaum, W. Gloeckle and U. -G. Meißner, ...
  D. R. Entem and R. Machleidt ...

- Take into account compensating terms of non-relativistic EFT Lagrangian:

  - Realized in KSW approach (perturbative pions)
  D. B. Kaplan, M. J. Savage and M. B. Wise, Phys. Lett. B **424**, 390 (1998) ...

  - Problematic if pions are included non-perturbatively!

- Prove that an infinite number of compensating terms can be safely dropped – has (can)not been done!

- Use the original Lorentz invariant Lagrangian without $1/m$ expansion ... Resulting OPE $\sim 1/r^2$.
  E. Epelbaum and J. G., Phys. Lett. B **716**, 338 (2012).

Chiral EFT for P-wave halo states and RG

C.A. Bertulani, H.-W. Hammer, U. Van Kolck, Nucl.Phys. A712, 37, (2002) .

Consider two non-relativistic particles with range of interaction $R \sim 1/M_{\rm hi}$.

ERE for the orbital angular momentum l:

$$T(k) \propto \frac{1}{k\cot\delta - ik} \simeq \frac{k^{2l}}{\left(-1/a + 1/2\, r\, k^2 + v_2 k^4 + \ldots\right) - ik^{2l+1}}\,,$$

a, r, v_i - scattering length, effective range and the shape parameters.

Assume that the first two terms in the ERE are fine tuned as

$$1/a \sim M_{\rm lo}^3\,, \qquad r \sim M_{\rm lo}\,, \qquad v_n \sim M_{\rm hi}^{3-2n}\,. \tag{1}$$

In low-energy EFT with contact interactions only the two lowest-order contact interactions in the effective potential

$$V = C_2\, p' p + C_4\, p' p \left(p'^2 + p^2 \right) + \dots\,,$$

need to be iterated in the LS equation to all orders.

We solve

$$T(p',p) = V(p',p) + m\int_0^\Lambda \frac{l^2 dl}{2\pi^2}\,\frac{V(p,l)\,T(l,p')}{k^2 - l^2 + i\,\epsilon}\,,$$

and obtain for the on-shell amplitude $T(k) \equiv T(k,k)$:

$$\frac{k^2}{T(k)} = -I(k)\,k^2 - I_3 + \frac{(C_4 I_5 - 1)^2}{C_4\left(k^2(2 - C_4 I_5) + C_4 I_7\right) + C_2}\,,$$

where the integrals I_n and $I(k)$ are defined via

$$I_n = -m\int_0^\Lambda \frac{l^2 dl}{2\pi^2}\, l^{n-3}, \quad n = 1,3,5,\ldots\,,$$

$$I(k) = \int_0^\Lambda \frac{l^2 dl}{2\pi^2}\,\frac{m}{k^2 - l^2 + i\epsilon}\,.$$

Renormalization and RG can be implemented *a la* Gell-Mann and Low or *a la* Wilson.

Subtractive renormalization

We renormalize the amplitude by applying BPHZ procedure, i.e. subtracting *all* UV divergences prior to taking the limit $\Lambda \to \infty$.

Specifically, we first separate out power-like UV divergences in the appearing integrals in the most general way via

$$
\begin{aligned}
I_n &= -m\int_0^{\mu_n} \frac{l^2 dl}{2\pi^2} l^{n-3} - m\int_{\mu_n}^{\Lambda} \frac{l^2 dl}{2\pi^2} l^{n-3} \equiv I_n^R(\mu_n) + \Delta_n(\mu_n)\,, \\
& \quad \text{with} \quad n = 1, 3, 5, \ldots\,, \\
I(k) &\equiv I^R(k, \mu_1) - \Delta_1(\mu_1)\,,
\end{aligned}
$$

where μ_n denote the corresponding renormalization scales.

We renormalize the amplitude by replacing the integrals I_n and $I(k)$ with $I_n^R(\mu_n)$ and $I^R(k, \mu_1)$ and the bare couplings C_2 and C_4 by renormalized couplings C_2^R and C_4^R.

Since the renormalized amplitude depends only on UV-convergent integrals, we can now safely take the limit $\Lambda \to \infty$.

Fitting the renormalized LECs to a and r leads to:

$$k^3 \cot\delta = -\frac{1}{a} + \frac{1}{2} r k^2 - \frac{3k^4}{2\pi} \frac{(4\mu_1 + \pi r)^2}{6\pi a^{-1} - 4\mu_3^3 + 3k^2(4\mu_1 + \pi r)} .$$

The renormalized scattering amplitude depends on μ_1 and μ_3.

The choice of μ_i plays a key role in setting up a self-consistent power counting.

For the resonant P -wave scattering the choice of renormalization conditions is rather delicate due to the strong fine tuning.

Indeed, one *must* choose $\mu_3 \sim M_{\rm hi}$ since setting $\mu_3 \sim M_{\rm lo}$ would lead to poles in the effective range function located at $k \sim M_{\rm lo}$, thereby resulting in enhanced values of the coefficients in the ERE.

Consequently, no KSW-like scheme is possible for resonant P-wave systems under consideration.

A self-consistent Weinberg-like scheme with all LECs scaling according to NDA emerges if we set $\mu_5 \sim \mu_7 \sim \mu_9 \sim \ldots \sim M_{\rm lo}$. The scale μ_1 can be chosen either as $\mu_1 \sim M_{\rm hi}$ or $\mu_1 \sim M_{\rm lo}$.

Wilsonian RG with two cutoffs

Using the approach of
E. Epelbaum, J. G. and U. -G. Meißner, Commun. Theor. Phys. **69**, no.3, 303 (2018)
we rewrite the potential as

$$V = (C_2 + 2C_4k^2)pp' + C_4pp'(p^2 - k^2 + p'^2 - k^2)\,,$$

and introduce two cutoffs via

$$\begin{aligned} V &= (C_2 + 2C_4k^2)pp'\theta(\Lambda_1 - p)\theta(\Lambda_1 - p') \\ &+ C_4pp'\theta(\Lambda_1 - p)\theta(\Lambda_1 - p') \\ &\times \left[(p^2 - k^2)\theta(\Lambda_2 - p) + (p'^2 - k^2)\theta(\Lambda_2 - p')\right]\,, \end{aligned}$$

where it is implied that $\Lambda_1 \geq \Lambda_2$.

This potential can be represented in a separable form:

$$
\begin{aligned}
V &= \left(\; p'\theta(\Lambda_1 - p'), \quad p'(p'^2 - k^2)\theta(\Lambda_2 - p') \; \right) \\
&\times \left(\begin{array}{cc} C_2 + 2C_4k^2, & C_4 \\ C_4, & 0 \end{array} \right) \left(\begin{array}{c} p\theta(\Lambda_1 - p) \\ p(p^2 - k^2)\theta(\Lambda_2 - p) \end{array} \right),
\end{aligned}
$$

and therefore the corresponding LS equation for the scattering amplitude can be straightforwardly solved analytically.

Matching the solution to the ERE we fix the LECs C_2 und C_4:

$$C_2 = \frac{1}{350 m\pi^2(3\pi - 2a\Lambda_1^3)}\Big\{75 C_4^2 m^2 \pi \Lambda_2^7 + a\big[4200\pi^4 + 840 C_4 m\pi^2 \Lambda_2^5 + 2C_4^2 m^2 \Lambda_2^7 (21\Lambda_2^3 - 25\Lambda_1^3)\big],$$

$$C_4 = \frac{10\sqrt{5}\pi^2(3\pi - 2a\Lambda_1^3)^2}{m\Lambda_2^5\sqrt{(3\pi - 2a\Lambda_1^3)^2\,\alpha(\Lambda_1,\Lambda_2)}} - \frac{10\pi^2}{m\Lambda_2^5}\Big\}.$$

The LECs C_2 and C_4 must be real, therefore the argument of the square root has to be non-negative.

This leads to the condition

$$\alpha(\Lambda_1,\Lambda_2) \equiv 45\pi^2 + 4a^2\Lambda_1(5\Lambda_1^5 - 9\Lambda_2^5) - 3a\pi(20\Lambda_1^3 + 3ar\Lambda_2^5) \geq 0\,.$$

For two independent cutoffs Λ_1 and Λ_2, the condition that $\alpha(\Lambda_1, \Lambda_2)$ has to be non-negative can be satisfied for any values of a and r.

To check the convergence of the ERE we subtract $-\frac{1}{a} + \frac{1}{2} r k^2$ from the calculated expression of $k \cot \delta$ and obtain the remainder:

$$S_{Rest} = \frac{k^3}{2\pi} \left(-\frac{3ak(\pi r + 4\Lambda_1)^2}{3\pi(2 + ak^2 r) - 4a\Lambda_1(\Lambda_1^2 - 3k^2)} - 2 \ln \frac{\Lambda_1 - k}{\Lambda_1 + k} \right) .$$

The second term in the bracket has a convergent expansion in k^2 for $\Lambda_1 \gg k$ and the expansion of the first term converges if

$$-1 < \frac{3 (\pi r + 4 \Lambda_1)}{6 \pi / a - 4 \Lambda_1^3} k^2 < 1 .$$

By taking sufficiently large Λ_1 this condition can always be fulfilled. For considered system this amounts to taking $\Lambda_1 \sim M_{\rm hi}$ or larger.

By taking $\Lambda_1 \sim M_{\rm hi}$ and $\Lambda_2 \sim M_{\rm lo}$, we find that $C_2 \sim 1/(m M_{\rm hi}^3)$ and $C_4 \sim 1/(m M_{\rm hi}^5)$, i.e. both are of natural size.

For $\Lambda_1 = \Lambda_2 = \Lambda$ we have

$$\alpha(\Lambda, \Lambda) = 45\pi^2 - 16a^2\Lambda^6 - 9\pi a^2 r\Lambda^5 - 60\pi a\Lambda^3 \,,$$

which turns negative for sufficiently large Λ values and, therefore, the LECs C_2 and C_4 become complex.

For our system the cutoff Λ cannot be taken larger than $\sim M_{\rm lo}$. This observation is in line with the causality bound $r \leq -2/R\,(1 + \mathcal{O}(R^3/a))$ obtained in

H. W. Hammer and D. Lee, Annals Phys. **325**, 2212-2233 (2010).

if the range of the interaction R is identified with $1/\Lambda$.

EFT & renormalization of singular potentials

A clever person solves the problem.
A wise person avoids it.
— Albert Einstein (?)

- What does „RGI EFT" mean? Which criteria should the amplitude fulfill to qualify for RGI? If residual Λ-dependence allowed, why not $\Lambda \sim \Lambda_b$? Is it essential to allow for $\Lambda \gg \Lambda_b$? If so, what about the examples with a wrong $\lim_{\Lambda\to\infty} T$, e.g. 0906.3822, 2104.01823, 2202.01105 ?
  + talk by Ashot

- Can renormalization by itself impose constraints on physical quantities? E.g., in π-less EFT, on relative sizes and signs of a, r, … as claimed by Habashi et al. '20,'21 ? Has χEFT for NN any predictive power beyond χ symmetry (= long-range tail of the interaction)?

- PC in explicitly renorm. EFTs (ChPT, KSW, …) $\overset{?}{\longleftrightarrow}$ PC in π-full nuclear cutoff EFTs
  powers of Q, not unique (renorm. cond.) $\overset{?}{\longleftrightarrow}$ large-Λ behavior, powers of what?

- How have simpler EFTs, such as pionless and halo/cluster EFTs, influenced Chiral EFT?
  → test/benchmark different approaches! For resonant P-wave systems, see 2104.01823

  dimer EFT ↔ subtractively renormalized NN EFT (W. and KSW) ↔
  ↔ implicitly renormalized cutoff EFT ↔ RG analysis

  Perhaps, we should (try to) agree on the analytically solvable pionless/halo EFT first…