DWBA, again:

Sticking with a modernist view of nuclear forces

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A Stuckist Manifesto

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Effective field theories

Ideal goal (unachievable?)

- convergent expansion of observables in powers of Q/Λ_0
  low-energy scales, Q: momenta, m_π ($\lesssim$ 200 MeV)
  scales of underlying physics, Λ_0: $4\pi F_\pi$, M_N, m_ρ ($\gtrsim$ 800 MeV)
- requires Q/Λ_0 small enough (good separation of scales)

Ingredients

- effective Lagrangian or Hamiltonian with contact interactions (describing unresolved physics), respecting symmetries of underlying theory
- the renormalisation group (tool for analysing scale dependence)
- list of low-energy scales

Starting point

- scale-free system*
- fixed point of the RG**

Observables expanded around this

- in powers of Q/Λ_0
- according to the “power counting” for the fixed point
  (anomalous dimensions)

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Classic example

- chiral perturbation theory [Weinberg (1979)]
- hidden chiral symmetry of QCD
  $\rightarrow$ pions as Goldstone bosons interact weakly at low energies
- trivial fixed point (no interactions)
- expansion in powers of momenta and m_π
- power counting: naive dimensional analysis (perturbative)

* or one with only low-energy scales

** and any marginal (logarithmic) terms or limit cycles

Nuclear EFTs

More complicated

- some interactions too strong for perturbative treatment
  (bound states!)
- basic nonrelativistic loop diagram of order Q [Weinberg (1991)]

$$\frac{M}{(2\pi)^3}\int\frac{\mathrm{d}^3 q}{p^2-q^2+\mathrm{i}\varepsilon}=-\mathrm{i}\,\frac{M\,p}{4\pi}+\text{analytic in } p^2$$

- any interaction enhanced to order Q^{-1} cancels Q from loop
  iterations not suppressed $\to$ nonperturbative
  part of a nontrivial fixed point

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Classic example

- “pionless” EFT [Weinberg (1991, again)]
- contact interactions only
- “unitary” fixed point: zero-energy point state, $1/a=0$

$\rightarrow$ effective-range expansion [Schwinger (1947)]

Low-energy scales in nuclear physics

- momenta, $m_\pi \simeq 140$ MeV
- S-wave scattering lengths $1/a \lesssim 40$ MeV
- OPE strength?

$$\lambda_{NN} = \frac{16\pi F_\pi^2}{g_A^2 M_N} \simeq 290 \text{ MeV}$$

  built out of high-energy scales ($4\pi F_\pi$, M_N) but $\sim 2m_\pi$
- N-Δ splitting?

$$M_\Delta - M_\mathrm{N} \simeq 300 \text{ MeV}$$

Adding λ_{NN} to list of low-energy scales

- promotes OPE to order Q^{-1} ("leading order")

→ new fixed point(s)

Distorted waves

To analyse systems with strong long-range forces:

- easiest to work in basis of DWs
- DW scattering amplitude or K matrix

$$\widetilde{K}(p) = -\frac{4\pi}{Mp}\tan\Big(\delta_{\mathrm{PWA}}(p) - \delta_{\mathrm{OPE}}(p)\Big)$$

  removes rapid energy dependence on scales of long-range potential
- remainder can be expanded using DWBA
  or DW (modified) effective-range expansion [Bethe *et al* (1949)]

Spin-triplet NN scattering

Wave functions dominated by $1/r^3$ tensor OPE

- for small r solutions satisfy (uncoupled waves)

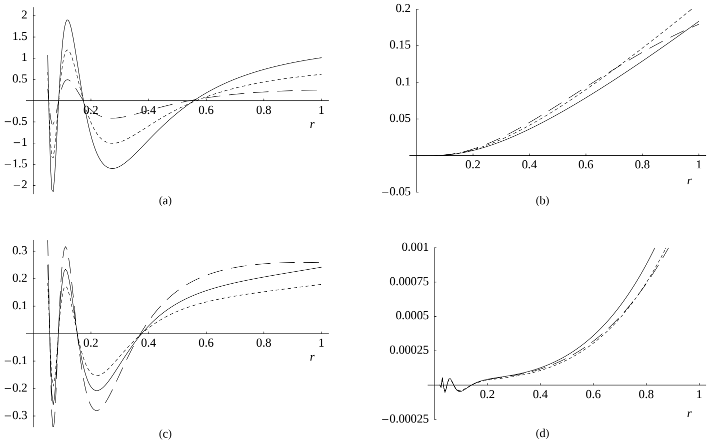
$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{L(L+1)}{r^2} - \frac{\beta_{LJ}}{r^3}\right]\psi_0(r) = 0 \qquad \left(\beta \propto \frac{1}{\lambda_\pi}\right)$$

$\rightarrow$ Bessel functions for attractive channels, $\beta < 0$ (cf WKB)

$$\psi_0(r) \propto r^{-1/2}\left[\sin\alpha\, J_{2L+1}\left(2\sqrt{\frac{|\beta_{LJ}|}{r}}\right) + \cos\alpha\, Y_{2L+1}\left(2\sqrt{\frac{|\beta_{LJ}|}{r}}\right)\right]$$

α: fixes phase of short-distance oscillations

(self-adjoint extension or leading short-distance parameter)



Wave functions $\psi(r)/p^L$ for (a) $^3P_0$, (b) $^3P_1$, (c) $^3D_2$, (d) $^3G_4$.
Short-dashed lines: $T = 5$ MeV; long-dashed lines: $T = 300$ MeV;
solid lines: energy-independent asymptotic form

Critical relative momenta

From Bessel-function expansion [Birse (2005)]
but standard perturbation theory gives very similar results
(7-loop order!) [Kaplan (2019)]

Channel	p_c
$^3S_1$–$^3D_1$	66 MeV
$^3P_0$	182 MeV
$^3P_1$	365 MeV
other P, D waves	$\sim$ 400 MeV
F waves and above	$\gtrsim$ 2000 MeV

S, P, D waves: PT applicable only for momenta below $\sim 2m_\pi$
(much lower for $^3S_1$, $^3P_0$)

Power counting

Easiest using RG analysis with running cutoff applied to DWs

For $L \leq 2$ partial waves $\psi_0(r) \sim r^{-1/4}\cos\left(2\sqrt{\beta/r}\right)$

$\rightarrow$ new power counting

- leading contact interaction promoted to order $Q^{-1/2}$
  very weakly irrelevant $\rightarrow$ treat nonperturbatively
  [Nogga *et al* (2005); developments by: Pavon Valderrama; Long and Yang; Gasparyan and Epelbaum (2021)]
- higher energy-dependent terms at orders $Q^{3/2}$, $Q^{7/2}$, ...
- corresponds to DWBA amplitude expanded in powers of energy

Long-range potentials not renormalised:

- leading 2π exchange: Q^2
- subleading: Q^3

DWBA for effective potential

Regulate using energy-dependent δ-shell form

$$V_S(p,r) = \frac{1}{4\pi R^2 |\psi_0(R)|^2}\, \widetilde{V}(p)\, \delta(r-R)$$

- cutoff $\Lambda = \dfrac{1}{R}$ arbitrary but $\gg p,\ m_\pi,\ \lambda_{NN}$
- divide by $|\psi_0(R)|^2$ to remove dependence on R (as $R \to 0$)

Strength determined directly from residual K matrix using DWBA

$$\widetilde{V}(p) = \frac{|\psi_0(R)|^2}{|\psi(p,R)|^2}\, \widetilde{K}(p)$$

$\psi(p,r)$: full energy-dependent distorted wave

Note: zeros of full $\psi(p, r)$ and energy-independent form $\psi_0(r)$ coincide only in limit $r \to 0$

- generates poles in $\widetilde{V}(p)$ for finite values of R
- but poles get more widely spaced and strengths tend to zero (not a limit cycle!)

$\rightarrow$ well-defined large-Λ limit exists
[contrast: Gasparyan and Epelbaum (2022)]

- in practice take Λ in flat "plateau" between two poles (large enough that good plateaux exist – provided we work perturbatively following our power counting, counterterms exist to cancel all divergences)

Results for $^3P_0$

Presented as a Lepage-style plot of $\ln \widetilde{V}(p)$ against $\ln(T_{\rm lab})$

- starting from Nijmegen PWA93 and two Nijmegen potentials (newer Granada 2013 PWA gives similar behaviours)
- cutoff $R = 0.3$ fm
- short-distance counterterms from polynomials fitted to range $T_{\rm lab} = 40 - 100$ MeV

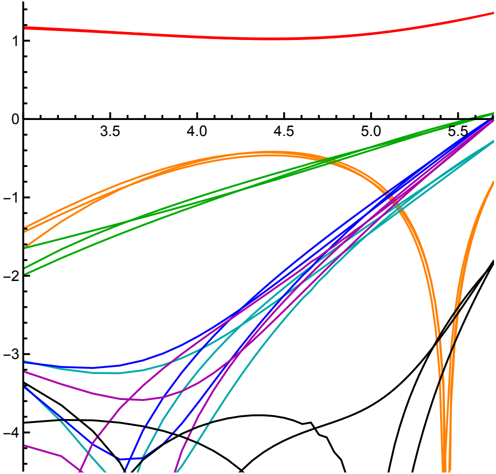
To avoid the Reign of Terror [© Timmermans] we need…

A Napoleonic Code

"Deconstruct" by subtracting order by order in our counting:

Q^{-1} iterated leading OPE with $\alpha = 0$ (DWBA)
$Q^{-1/2}$ and a constant
Q^{1} iterated leading OPE with $\alpha = 0.53$
(fit to very-low-energy scattering)
$Q^{3/2}$ and a term linear in energy $\propto p^2$
Q^{2} and leading TPE plus subleading OPE
Q^{3} and next-to-leading TPE
$Q^{7/2}$ and a term quadratic in energy $\propto p^4$

Removed:
$\mathcal{O}(Q^{-1})$
$\mathcal{O}(Q^{-1/2})$
$\mathcal{O}(Q^{1})$
$\mathcal{O}(Q^{3/2})$
$\mathcal{O}(Q^{2})$
$\mathcal{O}(Q^{3})$
$\mathcal{O}(Q^{7/2})$



- breakdown scale $\Lambda_0 \sim 550$ MeV
- next-to-leading TPE (Q^3) larger than expected

$\rightarrow$? need to include Δ (signs hidden)

Observations and questions

Analysis uses large values of the cutoff

- not a problem provided we have identified a fixed point
- and we treat everything else as perturbations around it
- then the counting for that fixed point will ensure we have counterterms needed to cancel all divergences
- here $\mathcal{O}(Q^3)$ TPE has $1/r^6$ singularity $\rightarrow$ divergences linear and quadratic in energy:
  renormalised by $\mathcal{O}(Q^{-1/2,3/2})$ contact interactions

Can we extend this to understand the power counting for three-body forces?

- first solve with tensor OPE plus contact term treated nonperturbatively, with large cutoffs
- then add a three-body contact term as a perturbation
- renormalise to keep some low-energy three-body observable fixed as Λ varied

Can we understand the transition from perturbative to nonperturbative regimes?

- iterating tensor OPE generates divergences of ever higher orders
- infinite number of these...
- but can we sum at least the leading ones for each loop:

$$\propto \frac{1}{\lambda_{NN}}\left(\frac{\Lambda}{\lambda_{NN}}\right)^n$$

(Kaplan reached $n = 7$ but need to get to ∞ ... how?
N/D method [Oller and Entem]?)

Does including explicit Δ improve convergence in this channel?

- should reduce $\mathcal{O}(Q^3)$ TPE
- does it also reduce residual p^4 term?
  (surprisingly large and cutoff dependent)

Modified/DW effective range expansion

Schematic form

$$k \cot[\delta(k) - \delta_L(k)] = |\psi_L^I(k,R)|^2 F(k^2) - \mathrm{Re}[G_L(R,R;k)]$$

- $F(k^2)$ effective-range function (meromorphic in k^2)
- $\delta_L(k)$ phase shift for long-range V_L
- $\psi_L^I(k,R)$ irregular DW solution for V_L (dressed vertex)
- $G_L(R,R;k)$ DW Green's function (loop integral dressed with V_L)
- waves evaluated at nonzero R if V_L singular
  (powers of R, k, numerical factors omitted)

Contribution of $F(k^2)$ to observables enhanced by DWs at small r

Expansion of $F(k^2)$ (short-range physics) not tied to expansion of G_L etc (long-range forces)

Griesshammer plot

Differences between $\widetilde{V}(p)$ for $R = 0.1$ fm and 0.3 fm

Removed:
$\mathcal{O}(Q^{-1})$
$\mathcal{O}(Q^{-1/2})$
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