

Investigating gluonic saturation through DVMP at the twist-3

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in collaboration with

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based on

[R. Boussarie, M. F., L. Szymanowski, S. Wallon (to appear)]

Diffraction and gluon saturation at the LHC and the
EIC, Trento, 10-14 June 2024



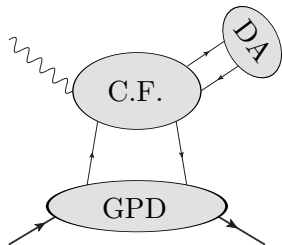
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Deeply virtual meson production (DVMP)

- Exclusive ρ -meson leptonproduction

$$\gamma^{(*)}(p_\gamma) + P(p_0) \rightarrow \rho(p_\rho) + P(p'_0)$$

- Extensively studied at HERA



- NLO corrections to the production of a longitudinally polarized ρ -meson at small- x

[Ivanov, Kotsky, Papa (2004)]

[Boussarie, Grabovsky, Ivanov, Szymanowski, Wallon (2017)]

[Mäntysaari, Pentalla (2022)]

- Transversally polarized ρ -meson production start at the **twist-3**

[Diehl, Gousset, Pire (1999)] [Collins, Diehl (2020)]

- Collinear treatment at the twist-3 leads to **end point singularities**

[Mankiewicz, Piller (2000)] [Anikin, Teryaev (2002)]

Transversely polarized ρ -meson production

- Momentum space impact factor for the exclusive ρ -meson production at the twist-3 in the dilute limit (BFKL scheme) and forward case

[Anikin, Ivanov, Pire, Szymanowski, Wallon (2009)]

- Phenomenological studies at small- x

[Besse, Szymanowski, Wallon (2013)]

[Bolognino, Celiberto, Ivanov, Papa (2018)]

[Bolognino, Szczurek, Schäfer (2019)]

[Mäntysaari, Pentalla (2022)]

i. Restricted to s -channel helicity conserving (SCHC) amplitudes

- Exclusive light vector meson production at the **twist-3** within the Shockwave approach

[Boussarie, M.F. , Szymanowski, Wallon (to appear)]

i. DVMP in the **non-linear** regime in the transversely polarized case

ii. Both **forward** and **non-forward** results

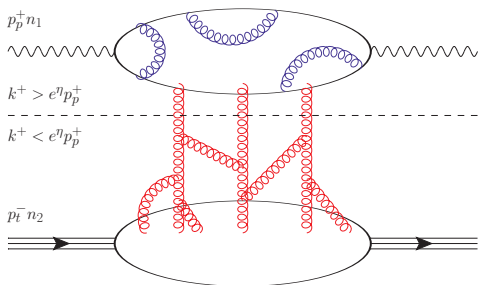
iii. **Beyond SCHC** amplitudes

iiii. **Coordinate** and **momentum space** representations

iiii. Linearization [Caron-Huot (2013)] \Rightarrow **BFKL results**

Shockwave approach

- High-energy approximation $s = (p_p + p_t)^2 \gg \{Q^2\}$



$$p_p = p_p^+ n_1 - \frac{Q^2}{2p_p^+} n_2$$

$$p_t = \frac{m_t^2}{2p_t^-} n_1 + p_t^- n_2$$

$$p_p^+ \sim p_t^- \sim \sqrt{\frac{s}{2}}$$

$$n_1^2 = n_2^2 = 0 \quad n_1 \cdot n_2 = 1$$

- Separation of the gluonic field into “fast” (quantum) part and “slow” (classical) part through a rapidity parameter $\eta < 0$

[I. Balitsky (1996-2001)]

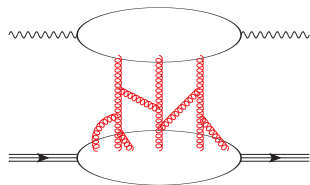
$$A^\mu(k^+, k^-, \vec{k}) = A^\mu(k^+ > e^\eta p_p^+, k^-, \vec{k}) + b^\mu(k^+ < e^\eta p_p^+, k^-, \vec{k})$$

$$e^\eta \ll 1$$

Shockwave approach

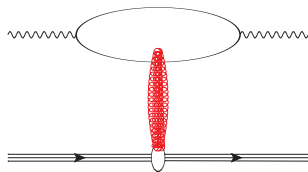
- Large longitudinal Boost: $\Lambda = \sqrt{\frac{1+\beta}{1-\beta}} \sim \frac{\sqrt{s}}{m_t}$

$$\begin{cases} b^+(x^+, x^-, \vec{x}) &= \Lambda^{-1} b_0^+(\Lambda x^+, \Lambda^{-1} x^-, \vec{x}) \\ b^-(x^+, x^-, \vec{x}) &= \Lambda b_0^-(\Lambda x^+, \Lambda^{-1} x^-, \vec{x}) \\ b^i(x^+, x^-, \vec{x}) &= b_0^i(\Lambda x^+, \Lambda^{-1} x^-, \vec{x}) \end{cases}$$



$$b_0^\mu(x)$$

boost \rightarrow



$$b^\mu(x^+, x^-, \vec{x}) = \delta(x^+) \mathbf{B}(\vec{x}) n_2^\mu + \mathcal{O}(\Lambda^{-1})$$

Shockwave approximation

- Light-cone gauge $A \cdot n_2 = 0$

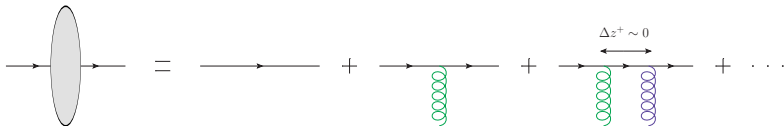
$A \cdot b = 0 \implies$ *Simple effective Lagrangian*

Shockwave approach

- Interactions with the simple shockwave field

- Independence on $x^- \implies$ conservation of p^+ (**eikonal** approximation)
- $\delta(x^+) \implies$ interactions at a **single transverse coordinate**.

- Quark line through the shockwave



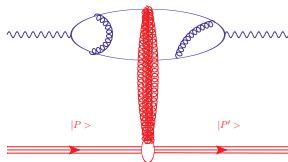
$$V_{\vec{z}_i} = 1 + ig \int_{-\infty}^{+\infty} dz_i^+ b_{\eta}^- (z_i^+, \vec{z}_i) + (ig)^2 \int_{-\infty}^{+\infty} dz_i^+ dz_j^+ b_{\eta}^- (z_i^+, \vec{z}_i) b_{\eta}^- (z_j^+, \vec{z}_i) \theta(z_{ij}^+) + \dots$$

- Multiple interactions with the target \rightarrow **path-ordered Wilson lines**

$$V_{\vec{z}}^{\eta} = \mathcal{P} \exp \left[ig \int_{-\infty}^{+\infty} dz_i^+ b_{\eta}^- (z_i^+, \vec{z}) \right]$$

Shockwave approach

- Factorization in the Shockwave approximation



$$\mathcal{M}^\eta = N_c \int d^d z_1 d^d z_2 \Phi^\eta(z_1, z_2) \langle P' | \mathcal{U}_{12}^\eta(z_1, z_2) | P \rangle$$

- Dipole operator

$$\mathcal{U}_{ij}^\eta = 1 - \frac{1}{N_c} \text{Tr} \left(V_{\vec{z}_i}^\eta V_{\vec{z}_j}^{\eta\dagger} \right)$$

- Evolution equations

- Balitsky-JIMWLK** evolution equations

[Balitsky (1995)]

[Jalilian-Marian, Iancu, McLerran, Weigert, Kovner, Leonidov]

- Large $N_c \rightarrow$ **Balitsky-Kovchegov (BK)** non-linear equation

[Balitsky (1995)] [Kovchegov (1999)]

- Evolution at the NLO

[Balitsky, Chirilli (2007)] [Kovner, Lublinsky, Mulian (2013)]

Theoretical framework

- **Effective background field operator** formalism of small- x physics

$$[\psi_{\text{eff}}(z_0)]_{z_0^+ < 0} = \psi(z_0) - \int d^D z_2 G_0(z_{02}) \left(V_{z_2}^\dagger - 1 \right) \gamma^+ \psi(z_2) \delta(z_2^+)$$

$$[\bar{\psi}_{\text{eff}}(z_0)]_{z_0^+ < 0} = \bar{\psi}(z_0) + \int d^D z_1 \bar{\psi}(z_1) \gamma^+ (V_{z_1} - 1) G_0(z_{10}) \delta(z_1^+)$$

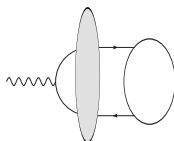
$$[A_{\text{eff}}^{\mu a}(z_0)]_{z_0^+ < 0} = A^{\mu a}(z_0) + 2i \int d^D z_3 \delta(z_3^+) F_{-\sigma}^b(z_3) G^{\mu\sigma\perp}(z_{30}) \left(U_{z_3}^{ab} - \delta^{ab} \right)$$

- Shockwave effective Feynman rules are easily reproduced

$$[v_{\alpha}^{ij}(p_{\bar{q}}, z_0)]_{z_0^+ < 0} \equiv [\psi_{\text{eff}, \alpha}^j(z_0)]_{z_0^+ < 0} |i, p_{\bar{q}}\rangle = -\frac{(-i)^{d/2}}{2(2\pi)^{d/2}} \left(\frac{p_{\bar{q}}^+}{-z_0^+} \right)^{d/2} \theta(p_{\bar{q}}^+) \theta(-z_0^+) \\ \times \int d^d z_2 V_{z_2}^{ij\dagger} \frac{-z_0^+ \gamma^- + \hat{z}_{20\perp}}{-z_0^+} \gamma^+ \frac{v(p_{\bar{q}})}{\sqrt{2p_{\bar{q}}^+}} \exp \left\{ i p_{\bar{q}}^+ \left(z_0^- - \frac{\vec{z}_{20}^2}{2z_0^+} + i0 \right) - i \vec{p}_{\bar{q}} \cdot \vec{z}_{20} \right\} \\ G_{ij}(z_2, z_0)|_{z_2^+ > 0 > z_0^+} \equiv \overbrace{\psi_i(z_2)} \left[\bar{\psi}_{\text{eff}, j}(z_0) \right]_{z_0^+ < 0} \\ = \frac{i\Gamma(d+1)}{4(2\pi)^{d+1}} \int d^d \vec{z}_1 V_{ij}(\vec{z}_1) \frac{\left(z_2^+ \gamma^- + \hat{z}_{21\perp} \right) \gamma^+ \left(-z_0^+ \gamma^- + \hat{z}_{10\perp} \right)}{\left(-z_0^+ z_2^+ \right)^{\frac{D}{2}} \left(-z_{20}^- + \frac{\vec{z}_{21}^2}{2z_2^+} - \frac{\vec{z}_{10}^2}{2z_0^+} + i\varepsilon \right)^{d+1}} \theta(z_2^+) \theta(-z_0^+)$$

ρ -meson production: diagrams

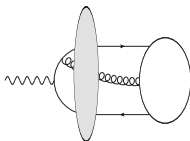
- **Two-body contribution**



- i. Dependence of the leading Fock state wave function – with a minimal number of (valence) partons – on **transverse momentum**

$$\mathcal{A}_2 = -ie_f \int d^D z_0 \theta(-z_0^+) \langle P(p') M(p_M) | \bar{\psi}_{\text{eff}}(z_0) \hat{\varepsilon}_q e^{-i(q \cdot z_0)} \psi_{\text{eff}}(z_0) | P(p) \rangle$$

- **Three-body contribution**

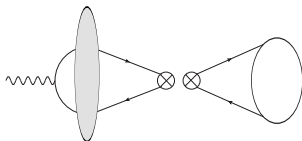


- i. Distribution with a **non-minimal parton configuration**

$$\begin{aligned} \mathcal{A}_{3,q} &= (-ie_q) (ig) \int d^D z_4 d^D z_0 \theta(-z_4^+) \theta(-z_0^+) \\ &\times \langle P(p') M(p_M) | \bar{\psi}_{\text{eff}}(z_4) \gamma_\mu A_{\text{eff}}^{\mu a}(z_4) t^a G(z_4 0) \hat{\varepsilon}_q e^{-i(q \cdot z_0)} \psi_{\text{eff}}(z_0) | P(p) \rangle \end{aligned}$$

ρ -meson production: factorization

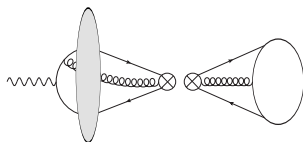
- Two-body contribution**



$$\mathcal{A}_2 = ie_f \int d^D z_0 \int d^D z_1 \int d^D z_2 \theta(-z_0^+) \delta(z_1^+) \delta(z_2^+) \langle M(p_M) | \bar{\psi}(z_1) \Gamma^\lambda \psi(z_2) | 0 \rangle$$

$$\times \langle P(p') | 1 - \frac{1}{N_c} \text{tr} (V_{z_1} V_{z_2}^\dagger) | P(p) \rangle \frac{1}{4} \text{tr}_D [\gamma^+ G_0(z_{10}) \hat{\varepsilon}_q e^{-i(q \cdot z_0)} G_0(z_{02}) \gamma^+ \Gamma_\lambda]$$

- Three-body contribution**



$$\mathcal{A}_{q3} = -ie_q \int d^D z_4 d^D z_3 d^D z_2 d^D z_1 d^D z_0 \theta(-z_4^+) \delta(z_3^+) \delta(z_2^+) \delta(z_1^+) \theta(-z_0^+) e^{-i(q \cdot z_0)}$$

$$\times \langle P(p') | \text{tr} (V_{z_1} t^a V_{z_2}^\dagger t^b U_{z_3}^{ab}) | P(p) \rangle \langle M(p_M) | \bar{\psi}(z_1) \Gamma^\lambda g F_{-\sigma}(z_3) \psi(z_2) | 0 \rangle$$

$$\times \frac{1}{N_c^2 - 1} \text{tr}_D [\gamma^+ G_0(z_{14}) \gamma_\mu G^{\mu\sigma\perp}(z_{34}) G_0(z_{40}) \hat{\varepsilon}_q G_0(z_{02}) \gamma^+ \Gamma_\lambda] - \text{n.i.}$$

Results: two-body contribution

- **Dipole amplitude**

$$A_2 = \int_0^1 dx \int d^2\mathbf{r} \Psi(x, \mathbf{r}) \int d^d\mathbf{b} e^{i(\mathbf{q}-\mathbf{p}_M)\cdot\mathbf{b}} \left\langle P(p') \left| 1 - \frac{1}{N_c} \text{tr} \left(V_{\mathbf{b}+\bar{x}\mathbf{r}} V_{\mathbf{b}-x\mathbf{r}}^\dagger \right) \right| P(p) \right\rangle$$

- **Coordinate space impact factor**

$$\Psi_2(x, \mathbf{r}) = e_q \delta \left(1 - \frac{p_M^+}{q^+} \right) \left(\varepsilon_{q\mu} - \frac{\varepsilon_q^+}{q^+} q_\mu \right) \\ \times \left[\phi_{\gamma^+}(x, \mathbf{r}) \left(2x\bar{x}q^\mu - i(x-\bar{x}) \frac{\partial}{\partial r_{\perp\mu}} \right) + \epsilon^{\mu\nu+-} \phi_{\gamma^+\gamma^5}(x, \mathbf{r}) \frac{\partial}{\partial r_{\perp\nu}} \right] K_0 \left(\sqrt{x\bar{x}Q^2\mathbf{r}^2} \right)$$

- **Two-body vacuum to meson matrix elements**

$$\phi_{\gamma^+}(x, \mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dr^- e^{ixp_M^+ r^-} \left\langle M(p_M) \left| \bar{\psi}(r) \gamma^+ \psi(0) \right| 0 \right\rangle_{r^+=0}$$

$$\phi_{\gamma^+\gamma^5}(x, \mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dr^- e^{ixp_M^+ r^-} \left\langle M(p_M) \left| \bar{\psi}(r) \gamma^+ \gamma^5 \psi(0) \right| 0 \right\rangle_{r^+=0}$$

Results: three-body contribution

- Three-body amplitude

$$\begin{aligned} \mathcal{A}_3 &= \left(\prod_{i=1}^3 \int dx_i \theta(x_i) \right) \delta(1 - x_1 - x_2 - x_3) \int d^2 \mathbf{z}_1 d^2 \mathbf{z}_2 d^2 \mathbf{z}_3 e^{iq(x_1 \mathbf{z}_1 + x_2 \mathbf{z}_2 + x_3 \mathbf{z}_3)} \\ &\times \Psi_3(x_1, x_2, x_3, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) \left\langle P(p') \left| \mathcal{U}_{\mathbf{z}_1 \mathbf{z}_3} \mathcal{U}_{\mathbf{z}_3 \mathbf{z}_2} - \mathcal{U}_{\mathbf{z}_1 \mathbf{z}_3} - \mathcal{U}_{\mathbf{z}_3 \mathbf{z}_2} + \frac{1}{N_c^2} \mathcal{U}_{\mathbf{z}_1 \mathbf{z}_2} \right| P(p) \right\rangle \end{aligned}$$

- Coordinate space impact factor

$$\begin{aligned} \Psi_3(x_1, x_2, x_3, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) &= \frac{e_q q^+}{2(4\pi)} \frac{N_c^2}{N_c^2 - 1} \left(\varepsilon_{q\rho} - \frac{\varepsilon_q^+}{q^+} q_\rho \right) \\ &\times \left\{ \chi_{\gamma^+ \sigma} \left[\left(4ig_{\perp\perp}^{\rho\sigma} \frac{x_1 x_2}{1 - x_2} \frac{Q}{Z} K_1(QZ) + T_1^{\sigma\rho\nu}(x_1, x_2, x_3) \frac{z_{23\perp\nu}}{z_{23}^2} K_0(QZ) \right) - (1 \leftrightarrow 2) \right] \right. \\ &\left. - \chi_{\gamma^+ \gamma^5 \sigma} \left[\left(4\epsilon^{\sigma\rho\mu\nu} \frac{x_1 x_2}{1 - x_2} \frac{Q}{Z} K_1(QZ) + T_2^{\sigma\rho\nu}(x_1, x_2, x_3) \frac{z_{23\perp\nu}}{z_{23}^2} K_0(QZ) \right) + (1 \leftrightarrow 2) \right] \right\} \end{aligned}$$

- Three-body vacuum to meson matrix elements

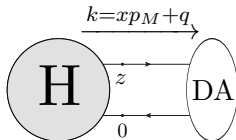
$$\chi_{\Gamma^\lambda, \sigma} \equiv \chi_{\Gamma^\lambda, \sigma}(x_1, x_2, x_3, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) =$$

$$\int_{-\infty}^{\infty} \frac{dz_1^-}{2\pi} \frac{dz_2^-}{2\pi} \frac{dz_3^-}{2\pi} e^{-ix_1 q^+ z_1^- - ix_2 q^+ z_2^- - ix_3 q^+ z_3^-} \left\langle M(p_M) \left| \bar{\psi}(z_1) \Gamma^\lambda g F_{-\sigma}(z_3) \psi(z_2) \right| 0 \right\rangle_{z_{1,2,3}^+ = 0}$$

Light-cone collinear factorization

- Most general two-body amplitude

$$\mathcal{A}_2 = \int \frac{d^4 k}{(2\pi)^4} \int d^4 z e^{-ik \cdot z} \langle M(p_M) | \bar{\psi}_\alpha^i(z) \psi_\beta^j(0) | 0 \rangle H_{2,\alpha\beta}^{ij}$$



- The separation is chosen as $z^\mu = \lambda n_2^\mu \implies$ no Wilson line in the n_2 **light-cone gauge**
- Sudakov decomposition: $k = k^+ n_1 + q = xp_M + q$ and **small q expansion**
- Two-body amplitude after Fierz decomposition

$$\mathcal{A}_2 = \frac{1}{4N_c} p_M^+ \int \frac{dx}{2\pi} \int \frac{dq^-}{2\pi} \int \frac{d^d \mathbf{q}}{(2\pi)^d} \int d^D z e^{-ixp_M^+ z^- - iq^- z^+ + i(\mathbf{q} \cdot \mathbf{z})} \\ \times \langle M(p_M) | \bar{\psi}(z) \Gamma_\lambda \psi(0) | 0 \rangle \text{tr} [H_2(xp_M + q) \Gamma^\lambda]$$

Light-cone collinear factorization

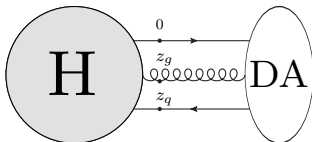
- **Taylor expansion** of the hard part

$$H_2(xp_M + q) = H_2(xp_M) + q_{\perp\mu} \left[\frac{\partial}{\partial q_{\perp\mu}} H_2(xp_M + q) \right]_{k=xp_M} + \text{h.t.}$$

- Two-body amplitude factorized form up to the **twist-3**

$$\mathcal{A}_2 = \frac{1}{4N_c} \int dx p_M^+ \int \frac{dz^-}{2\pi} e^{-ixp_M^+ z^-} \left\{ \langle M(p) | \bar{\psi}(z^-) \Gamma_\lambda \psi(0) | 0 \rangle \text{tr} [H_2(xp_M) \Gamma^\lambda] \right. \\ \left. + i \langle M(p_M) | \bar{\psi}(z^-) \overleftrightarrow{\partial}_{\perp\mu} \Gamma_\lambda \psi(0) | 0 \rangle \text{tr} [\partial_{\perp}^\mu H_2(xp_M) \Gamma^\lambda] \right\}$$

- *Gauge invariance* is broken \implies need to include a **3-body contribution**



- Three-body contribution factorized

$$\mathcal{A}_3 = \frac{1}{2(N_c^2 - 1)} \int dx_q dx_g (p_M^+)^2 \int \frac{dz_q^-}{2\pi} \frac{dz_g^-}{2\pi} e^{-ix_q p_M^+ z_q^- - ix_g p_M^+ z_g^-} \\ \times \langle M(p) | \bar{\psi}(z_q^-) \Gamma_\lambda g A_\mu(z_g^-) \psi(0) | 0 \rangle \text{tr} [t^b H_3^{\mu,b}(x_q p_M, x_g p_M) \Gamma^\lambda]$$

Light-cone collinear factorization

- Light-cone collinear factorization

[Ellis, Furmanwski, Petronzio (1982)] [Anikin, Teryaev (2002)]

- Factorization around the dominant **light-cone direction** is naturally implemented in **momentum space**
- **Overcomplete set of distributions** must be reduced exploiting QCD equations of motion

$$\langle i(\hat{D}(0)\psi(0))_\alpha \bar{\psi}_\beta(z) \rangle = 0 \quad \langle i\psi_\alpha(0)(\bar{\psi}(z)\overleftarrow{D}(z))_\beta \rangle = 0$$

- Invariance of the amplitude under **rotation on the light-cone**
[Anikin, Ivanov, Pire, Szymanowski, Wallon (2009)]

- i. Independence of the amplitude from the choice of n
- ii. Given a “natural” choice n_0 , we can define

$$n^\mu = \alpha p^\mu + \beta n_0^\mu + n_\perp^\mu$$

- iii. Imposing $p \cdot n = 1$ and $n^2 = 0 \rightarrow \beta = 1, \alpha = -n_\perp^2/2$
- iiii. The freedom is parametrized in terms of the transverse component

$$\frac{\partial \mathcal{A}}{\partial n_\perp^\mu} = 0$$

Covariant collinear factorization

- Covariant collinear factorization

[Braun, Filyanov (1990)]

[Ball, Braun, Koike, Tanaka (1998)]

- i.* Minimal basis of *independent distributions*
 - ii.* Minimal numbers of parameters
 - iii.* Easy to perform the calculation directly into coordinate space
- Idea: two- and three-body operators in gauge invariant form

$$\begin{aligned} & \langle M(p_M) | \bar{\psi}(z) \Gamma_\lambda [z, 0] \psi(0) | 0 \rangle \\ & \langle M(p_M) | \bar{\psi}(z) \gamma_\lambda [z, tz] g F^{\mu\nu}(tz) [tz, 0] \psi(0) | 0 \rangle \\ & \langle M(p_M) | \bar{\psi}(z) \gamma_\lambda [z, tz] g \tilde{F}^{\mu\nu}(tz) [tz, 0] \psi(0) | 0 \rangle \end{aligned}$$

where

$$[z, 0] = \mathcal{P}_{\text{exp}} \left[ig \int_0^1 dt A^\mu(tz) z_\mu \right]$$

- The expansion for $z^2 \rightarrow 0$ should be taken
- Expansion of **string operators** in powers of deviation from the light-cone

[Balitsky, Braun (1989)]

Covariant collinear factorization

- Matrix elements without gauge links need to be related to the fully gauge invariant one within **twist-3** accuracy
- Three-body matrix element

$$\begin{aligned} & \left\langle M(p) \left| \bar{\psi}(z_q) \gamma^\lambda g F^{\mu\nu}(z_g) \psi(z_{\bar{q}}) \right| 0 \right\rangle \\ & \simeq \left\langle M(p) \left| \bar{\psi}(z_q) [z_q, z_g] \gamma^\lambda g F^{\mu\nu}(z_g) [z_g, z_{\bar{q}}] \psi(z_{\bar{q}}) \right| 0 \right\rangle_{z_i^+ = 0} \end{aligned}$$

- Two-body matrix element

$$\left\langle M(p_M) \left| \bar{\psi}(r) [r, 0] \Gamma^\lambda \psi(0) \right| 0 \right\rangle_{r^+ = 0} \longleftrightarrow \left\langle M(p_M) \left| \bar{\psi}(r) \Gamma^\lambda \psi(0) \right| 0 \right\rangle_{r^+ = 0}$$

- Expansion of the gauge link

$$[z, 0] = \mathcal{P} \exp \left[ig \int_0^1 dt A^\mu(tz) z_\mu \right] \simeq 1 + ig \int_0^1 dt A^\mu(tz) z_\mu + \text{h.t.}$$

- In a given n light-cone gauge

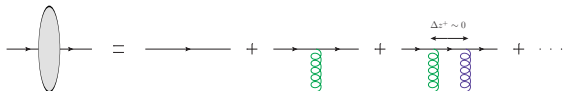
$$A^\mu(z) = \int_0^\infty d\sigma e^{-\epsilon\sigma} n_\nu F^{\mu\nu}(z + \sigma n)$$

- The difference between the two matrix elements is parameterized in terms of a three-body contribution

Dilute regime: two-body contribution

- **Reggeon** definition [**Caron-Huot (2013)**] $R^a(z) \equiv \frac{f^{abc}}{gC_A} \ln(U_z^{bc})$
- Expansion of the *Wilson line* in Reggeons

$$V_{z_1} = 1 + igt^a R^a(z_1) - \frac{1}{2} g^2 t^a t^b R^a(z_1) R^b(z_1) + O(g^3)$$



- **BFKL k_T -factorization**

$$\mathcal{A}_2^{\text{dilute}} = \frac{g^2}{4N_c} (2\pi)^d \delta^d(\mathbf{q} - \mathbf{p}_M - \mathbf{\Delta}) \int \frac{d^d \ell}{(2\pi)^d} \mathcal{U}(\ell) \int_0^1 dx$$

$$\times \underbrace{\left[\Phi_2 \left(x, \ell - \frac{x - \bar{x}}{2} \mathbf{\Delta} \right) + \Phi_2 \left(x, -\ell - \frac{x - \bar{x}}{2} \mathbf{\Delta} \right) - \Phi_2(x, \bar{x} \mathbf{\Delta}) - \Phi_2(x, -x \mathbf{\Delta}) \right]}_{\Phi_{2,\text{BFKL}}(x, \mathbf{l}, \mathbf{\Delta})}$$

- $\mathcal{U}(\ell) \rightarrow k_T$ -**unintegrated gluon density** (UGD) in the BFKL sense

$$\mathcal{U}(\ell) \equiv \int d^d \mathbf{v} e^{-i(\ell \cdot \mathbf{v})} \left\langle P(p') \left| R^a \left(\frac{\mathbf{v}}{2} \right) R^a \left(-\frac{\mathbf{v}}{2} \right) \right| P(p) \right\rangle ,$$

Explicit two-body term in the dilute and $\Delta = 0$ limit

- **BK impact factor***

$$\Phi_{2,\Delta=0}(x, \mathbf{l}) = 2\pi m_M f_M e_q \delta \left(1 - \frac{p_M^+}{q^+} \right) \\ \times \left[\frac{2\mathbf{l}^2}{[\mathbf{l}^2 + x\bar{x}Q^2]^2} T_{f.} \phi_{2,f.}(x) - \frac{x\bar{x}Q^2}{[\mathbf{l}^2 + x\bar{x}Q^2]^2} T_{n.f.} \phi_{2,n.f.}(x) \right]$$

- Helicity structures

$$T_{f.} = \frac{(\boldsymbol{\varepsilon}_q \cdot \mathbf{l})(\boldsymbol{\varepsilon}_M^* \cdot \mathbf{l})}{\mathbf{l}^2} - \frac{\boldsymbol{\varepsilon}_q \cdot \boldsymbol{\varepsilon}_M^*}{2} \quad T_{n.f.} = \boldsymbol{\varepsilon}_q \cdot \boldsymbol{\varepsilon}_M^*$$

- **Forward limit matching**

$$\Phi_{2,\Delta=0}^{\text{BFKL}}(x, \mathbf{l}) = 2 \left(\Phi_{2,\Delta=0}(x, \mathbf{l}) - \Phi_{2,\Delta=0}(x, \mathbf{0}) \right)$$

- **BFKL impact factor**

$$\Phi_{2,\Delta=0}^{\text{BFKL}}(x, \mathbf{l}) = 4\pi m_M f_M e_q \delta \left(1 - \frac{p_M^+}{q^+} \right) \\ \left[\frac{2\mathbf{l}^2}{[\mathbf{l}^2 + x\bar{x}Q^2]^2} T_{f.} \phi_f(x) + \frac{\mathbf{l}^2(\mathbf{l}^2 + 2x\bar{x}Q^2)}{x\bar{x}Q^2 [\mathbf{l}^2 + x\bar{x}Q^2]^2} T_{n.f.} \phi_{n.f.}(x) \right]$$

Dilute Regime: three-body contribution

- Linearization in the three-body case \rightarrow combination of more impact factors

$$\begin{aligned}
 A_3^{\text{dilute}} &= \left(\prod_{i=1}^3 \int dx_i \theta(x_i) \right) \delta(1 - x_1 - x_2 - x_3) \frac{-g^2}{4N_c} \int \frac{d^d \mathbf{l}}{(2\pi)^d} \mathcal{U}(\mathbf{l}) \\
 &\left\{ \Phi'_3(\{x\}, \bar{x}_1 \Delta, -x_2 \Delta, -x_3 \Delta) - \Phi'_3\left(\{x\}, \left(\frac{1-2x_1}{2}\right) \Delta - \mathbf{l}, -x_2 \Delta, \left(\frac{1-2x_3}{2}\right) \Delta + \mathbf{l}\right) \right. \\
 &- \Phi'_3\left(\{x\}, \left(\frac{1-2x_1}{2}\right) \Delta + \mathbf{l}, -x_2 \Delta, \left(\frac{1-2x_3}{2}\right) \Delta - \mathbf{l}\right) + \Phi'_3(\{x\}, -x_1 \Delta, -x_2 \Delta, \bar{x}_3 \Delta) \\
 &+ \Phi'_3(\{x\}, -x_1 \Delta, -x_2 \Delta, \bar{x}_3 \Delta) - \Phi'_3\left(\{x\}, -x_1 \Delta, \left(\frac{1-2x_2}{2}\right) \Delta + \mathbf{l}, \left(\frac{1-2x_3}{2}\right) \Delta - \mathbf{l}\right) \\
 &- \Phi'_3\left(\{x\}, -x_1 \Delta, \left(\frac{1-2x_2}{2}\right) \Delta - \mathbf{l}, \left(\frac{1-2x_3}{2}\right) \Delta + \mathbf{l}\right) + \Phi'_3(\{x\}, -x_1 \Delta, \bar{x}_2 \Delta, -x_3 \Delta) \\
 &- \frac{1}{N_c^2} \left[\Phi'_3(\{x\}, \bar{x}_1 \Delta, -x_2 \Delta, -x_3 \Delta) - \Phi'_3\left(\{x\}, \left(\frac{1-2x_1}{2}\right) \Delta - \mathbf{l}, \left(\frac{1-2x_2}{2}\right) \Delta + \mathbf{l}, -x_3 \Delta\right) \right. \\
 &\left. - \Phi'_3\left(\{x\}, \left(\frac{1-2x_1}{2}\right) \Delta + \mathbf{l}, \left(\frac{1-2x_2}{2}\right) \Delta - \mathbf{l}, -x_3 \Delta\right) + \Phi'_3(\{x\}, -x_1 \Delta, \bar{x}_2 \Delta, -x_3 \Delta) \right] \left. \right\}
 \end{aligned}$$

where

$$\Phi'_3(\{x\}, \{\mathbf{p} + x\mathbf{p}_M\}) \equiv \Phi_3(\{x\}, \{\mathbf{p}\})$$

Dilute regime: three-body contribution

- Fourier transform

$$\Phi_3(\{x\}, \{\mathbf{p}\}) = \left(\prod_{j=1}^3 \int d^2 \mathbf{z}_j e^{-i \mathbf{z}_j \mathbf{p}_j} \right) \Psi_3(\{x\}, \{\mathbf{z}\})$$

- **Momentum space impact factor** (after twist-expansion)

$$\begin{aligned} \Phi_3(\{x\}, \{\mathbf{p}\}) &= \frac{e_q m_M}{4} c_f \left(\varepsilon_{q\rho} - \frac{\varepsilon_q^+}{q^+} q_\rho \right) \left(\varepsilon_M^{*\beta} - \frac{p_M^\beta}{p_M^+} \varepsilon_M^{*+} \right) \delta \left(1 - \frac{p_M^+}{q^+} \right) \\ &\times \left(\prod_{j=1}^3 \frac{\theta(1-x_j)\theta(x_j)}{x_j} \right) \frac{(2\pi)^3 \delta^{(2)} \left(\sum_{i=1}^3 \mathbf{p}_i + x_i \mathbf{p}_M \right)}{\left[Q^2 + \sum_{i=1}^3 (\mathbf{p}_i + x_i \mathbf{p}_M)^2 / x_i \right]} \left\{ g_{\beta\sigma} f_{3M}^V V(x_1, x_2) \left(4g_{\perp\perp}^{\rho\sigma} \frac{x_1 x_2}{1-x_2} \right. \right. \\ &+ \tilde{T}_1^{\sigma\rho\nu}(\{x\}) \Big|_{\mathbf{k}_i = -x_i \mathbf{p}_M} \frac{x_1 x_2 (p_3 + x_3 p_M)_{\perp\nu} - x_1 x_3 (p_2 + x_2 p_M)_{\perp\nu}}{(\mathbf{p}_1 + x_1 \mathbf{p}_M)^2 + x_1 (1-x_1) Q^2} \Big) - \epsilon_{-\sigma\beta} f_{3M}^A A(x_1, x_2) \\ &\times \left(4 \frac{x_1 x_2}{1-x_2} \epsilon^{\sigma\rho\alpha\gamma} + i \tilde{T}_2^{\sigma\rho\nu}(\{x\}) \Big|_{\mathbf{k}_i = -x_i \mathbf{p}_M} \frac{x_1 x_2 (p_3 + x_3 p_M)_{\perp\nu} - x_1 x_3 (p_2 + x_2 p_M)_{\perp\nu}}{(\mathbf{p}_1 + x_1 \mathbf{p}_M)^2 + x_1 (1-x_1) Q^2} \right) \Big\} \\ &+ (1 \leftrightarrow 2) \end{aligned}$$

Explicit three-body term in the dilute and $\Delta = 0$ limit

- Transverse to transverse transition in the **forward** and **dilute** limit

$$\begin{aligned}
 \mathcal{A}_{3T, \Delta=0}^{\text{dilute}} &= e_q m_M \frac{g^2}{N_c} (2\pi) \delta \left(1 - \frac{p_M^+}{q^+} \right) (2\pi)^2 \delta^2(\mathbf{q} - \mathbf{p}_M) \int \frac{d^d \ell}{(2\pi)^d} \mathcal{U}(\ell) \\
 &\times \left(\prod_{i=1}^3 \int_0^1 \frac{dx_i}{x_i} \right) \frac{\delta(1 - x_1 - x_2 - x_3)}{x_3} \frac{\ell^2}{Q^2} \left\{ T_{\text{f.}} \left[f_{3M}^V V(x_1, x_2) - f_{3M}^A A(x_1, x_2) \right] \right. \\
 &\times 2x_1 \left(\frac{x_3 c_f}{\ell^2 + \frac{x_2 x_3}{x_2 + x_3} Q^2} + \frac{x_3 c_f}{\ell^2 + \frac{x_1 x_3}{x_1 + x_3} Q^2} - \frac{\bar{x}_3 (1 - c_f)}{\ell^2 + \frac{x_1 x_2}{x_1 + x_2} Q^2} + \frac{x_2 - \bar{x}_1 c_f}{\ell^2 + x_1 \bar{x}_1 Q^2} + \frac{x_1 - \bar{x}_2 c_f}{\ell^2 + x_2 \bar{x}_2 Q^2} \right) \\
 &\quad \left. - T_{\text{n.f.}} \left[f_{3M}^V V(x_1, x_2) + f_{3M}^A A(x_1, x_2) \right] \right. \\
 &\times \left. \left(\frac{(1 - c_f) x_1 \bar{x}_3}{\bar{x}_3 \ell^2 + x_1 x_2 Q^2} - \frac{c_f x_3^2}{\bar{x}_1 \ell^2 + x_2 x_3 Q^2} - \frac{(x_2 - \bar{x}_1 c_f) x_1 x_2}{\bar{x}_1 (\ell^2 + x_1 \bar{x}_1 Q^2)} - \frac{(x_1 - \bar{x}_2 c_f) \bar{x}_2}{(\ell^2 + x_2 \bar{x}_2 Q^2)} \right) \right\}
 \end{aligned}$$

- Helicity structures

$$T_{\text{f.}} = \frac{(\boldsymbol{\varepsilon}_q \cdot \mathbf{l})(\boldsymbol{\varepsilon}_M^* \cdot \mathbf{l})}{l^2} - \frac{\boldsymbol{\varepsilon}_q \cdot \boldsymbol{\varepsilon}_M^*}{2} \quad T_{\text{n.f.}} = \boldsymbol{\varepsilon}_q \cdot \boldsymbol{\varepsilon}_M^*$$

- **The forward and dilute limit matches the previous result**

[Anikin, Ivanov, Pire, Szymanowski, Wallon (2009)]

- i. BFKL language
- ii. Light-cone collinear factorization

Summary and outlook

Summary

- **Transversally polarized light vector meson production**
[Boussarie, M.F., Szymanowski, Wallon (to appear)]
- DVMP in the **non-linear** regime in the transversely polarized case
- Both **forward** and **non-forward** results
- **Beyond SCHC** amplitudes
- **Coordinate** and **momentum space** representations
- Linearization [Caron-Huot (2013)] \implies **BFKL results**

Outlook

- Phenomenological studies
- Full NLO calculation
- Extension to different exclusive processes

Thanks for your attention

Backup

Balitsky-JIMWLK evolution equations

- **Balitsky-JIMWLK evolution equations** for the dipole
 [Balitsky — Jalilian-Marian, Iancu, McLerran, Weigert, Kovner, Leonidov]

$$\frac{\partial \mathcal{U}_{12}^\eta}{\partial \eta} = \frac{\alpha_s N_c}{2\pi^2} \int d^2 \vec{z}_3 \left(\frac{\vec{z}_{12}^2}{\vec{z}_{23}^2 \vec{z}_{31}^2} \right) \left[\underbrace{\mathcal{U}_{13}^\eta + \mathcal{U}_{32}^\eta - \mathcal{U}_{12}^\eta}_{\text{BFKL}} - \mathcal{U}_{13}^\eta \mathcal{U}_{32}^\eta \right]$$

$$\frac{\partial \mathcal{U}_{13}^\eta \mathcal{U}_{32}^\eta}{\partial \eta} = \dots$$

← Balitsky hierarchy

⋮

- **Double dipole contribution** and **Dipole contribution**



- **Dipole contribution**



Balitsky-Kovchegov evolution equation

- Large- N_c limit

[G. 't Hooft (1974)]

$$= \frac{1}{2} \begin{array}{c} j \longrightarrow k \\ i \longleftarrow l \end{array} - \frac{1}{2N_c} \begin{array}{c} j \\ i \end{array} \begin{array}{c} k \\ l \end{array}$$

$$t_{ij}^a t_{kl}^a = \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right)$$

- Double dipole \rightarrow Dipole \times dipole

$$\langle \mathcal{U}_{13}^\eta \mathcal{U}_{32}^\eta \rangle \rightarrow \langle \mathcal{U}_{13}^\eta \rangle \langle \mathcal{U}_{32}^\eta \rangle$$

- Hierarchy of equations broken \rightarrow closed non-linear **BK-equation**

[I. I. Balitsky (1995)] [Y. V. Kovchegov (1999)]

$$\frac{\partial \langle \mathcal{U}_{12}^\eta \rangle}{\partial \eta} = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z_3 \left(\frac{z_{12}^2}{z_{23}^2 z_{31}^2} \right) [\langle \mathcal{U}_{13}^\eta \rangle + \langle \mathcal{U}_{32}^\eta \rangle - \langle \mathcal{U}_{12}^\eta \rangle - \langle \mathcal{U}_{13}^\eta \rangle \langle \mathcal{U}_{32}^\eta \rangle]$$

with $\langle \mathcal{U}_{12}^\eta \rangle \equiv \langle P' | \mathcal{U}_{12}^\eta | P \rangle$

Diffractive dijet/hadron production

- Precise predictions to detect **saturation** effects at both the EIC or LHC

[Iancu, Mueller, Triantafyllopoulos (2022)]

- Possibility of studying multi-dimensional **gluon tomography**

[Feng's talk] [Hatta, Xiao, Yuan (2022)]

[Hauksson, Iancu, Mueller, Triantafyllopoulos (2024)]

- **Diffractive dijet/hadron(s) production at NLO**

$$\gamma^{(*)}(p_\gamma) + P(p_0) \rightarrow j_1(p_{h_1}) + j_2(p_{h_2}) + P(p'_0)$$

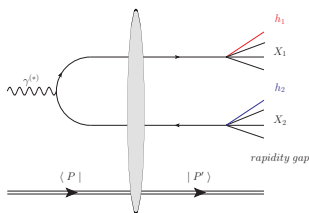
[Boussarie, Grabovsky, Szymanowski, Wallon (2016)]

$$\gamma^{(*)}(p_\gamma) + P(p_0) \rightarrow h_1(p_{h_1}) + h_2(p_{h_2}) + X + P(p'_0) \quad (X = X_1 + X_2)$$

[M. F., Grabovsky, Li, Szymanowski, Wallon (2023)]

$$\gamma^{(*)}(p_\gamma) + P(p_0) \rightarrow h_1(p_{h_1}) + X + P(p'_0)$$

[M. F., Grabovsky, Li, Szymanowski, Wallon (2024)]



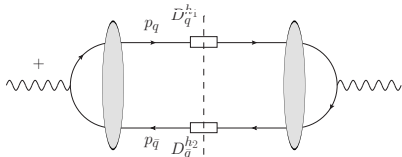
i. General kinematics (t, Q^2) and photon polarization

ii. Rapidity gap between $(h_1 h_2 X)$ and P'

iii. $\vec{p}_{h_1}^2, \vec{p}_{h_2}^2 \gg \Lambda_{\text{QCD}}^2$

- Sudakov decomposition for the momenta: $p_i^\mu = x_i p_\gamma^+ n_1^\mu + \frac{\vec{p}_\perp^2}{2x_i p_\gamma^+} n_2^\mu + p_\perp^\mu$

$$p_\gamma = \begin{pmatrix} + & - & \perp \\ p_\gamma^+ & -\frac{Q^2}{2p_\gamma^+} & \vec{0} \end{pmatrix}$$



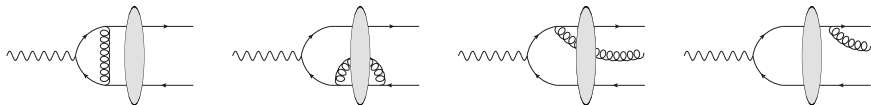
- Collinearity $(p_q^+, \vec{p}_q) = (x_q/x_{h_1})(p_{h_1}^+, \vec{p}_{h_1})$ and $(p_{\bar{q}}^+, \vec{p}_{\bar{q}}) = (x_{\bar{q}}/x_{h_2})(p_{h_2}^+, \vec{p}_{h_2})$

$$\frac{d\sigma_{0JI}^{h_1 h_2}}{dx_{h_1} dx_{h_2} d^d \vec{p}_{h_1} d^d \vec{p}_{h_2}} = \sum_q \int_{x_{h_1}}^1 \frac{dx_q}{x_q} \int_{x_{h_2}}^1 \frac{dx_{\bar{q}}}{x_{\bar{q}}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d$$

$$D_q^{h_1} \left(\frac{x_{h_1}}{x_q}\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}\right) \frac{d\hat{\sigma}_{JI}}{dx_q dx_{\bar{q}} d^d \vec{p}_q d^d \vec{p}_{\bar{q}}} + (h_1 \leftrightarrow h_2)$$

$J, I \rightarrow$ photon polarization for respectively the complex conjugated amplitude and the amplitude.

Treatment of UV and rapidity divergences



Rapidity divergences ($x_g \rightarrow 0$)

- Coming from Φ_{V_2} (double dipole part of the virtual contribution)
- Regularized by **longitudinal cut-off**: $|p_g^+| = |x_g|p_\gamma^+ > \alpha p_\gamma^+ \implies \ln \alpha$ term
- B-JIMWLK evolution from the non-physical cutoff α to the rapidity e^η

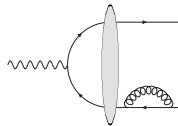
$$U_{\vec{x}}^\alpha = U_{\vec{x}}^{e^\eta} + \int_{e^\eta}^\alpha d\rho \left(\frac{\partial U_{\vec{x}}^\rho}{\partial \rho} \right) \implies \Phi_{V_2} \longrightarrow \tilde{\Phi}_{V_2} = \Phi_{V_2} + 2 \ln \left(\frac{e^\eta}{\alpha} \right) \mathcal{K}_{\text{BK}} \otimes \Phi_0 \tilde{\mathcal{W}}_{123}$$

UV-divergences ($\vec{p}_g^2 \rightarrow \infty$)

- Just dressing of the external quark lines

$$\Phi_{\text{dress}} \propto \left(\frac{1}{2\epsilon_{IR}} - \frac{1}{2\epsilon_{UV}} \right)$$

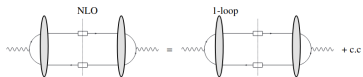
- $\epsilon_{IR} = \epsilon_{UV}$ turns **UV** into **IR** divergences



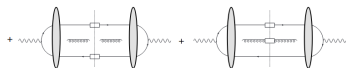
NLO cross-section in a nutshell

- Different fragmentation mechanisms

- i. Quark/anti-quark fragmentation
- ii. Quark/gluon fragmentation
- iii. Anti-quark/gluon fragmentation

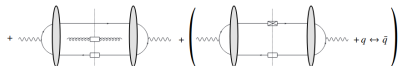


(a) : soft + collinear



(b) : soft + collinear

(c) : collinear



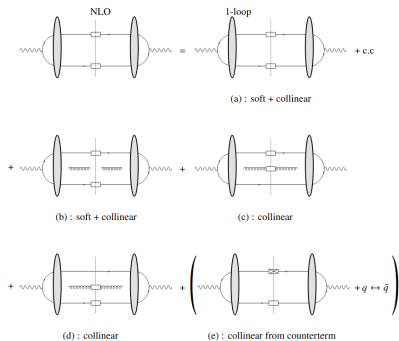
(d) : collinear

(e) : collinear from counterterm

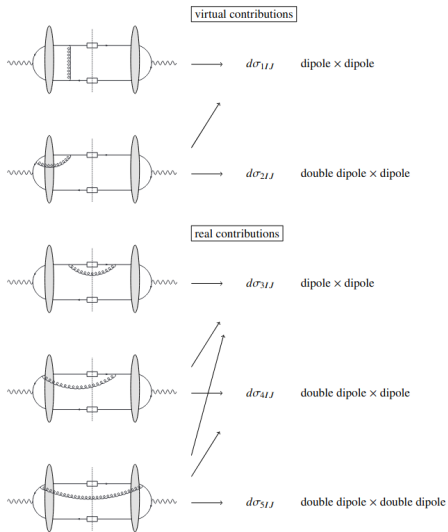
NLO cross-section in a nutshell

- Different fragmentation mechanisms

- i. Quark/anti-quark fragmentation
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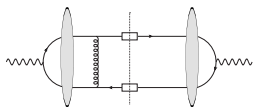


- Operator structure classification

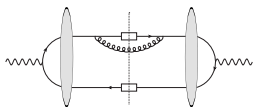


IR singularities: Quark/anti-quark fragmentation

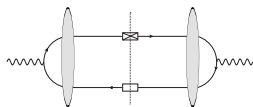
- Divergent contributions



$d\sigma_{1IJ}$

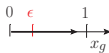
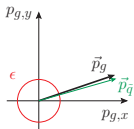


$d\sigma_{3IJ}$



$d\sigma_{\text{counter}}$

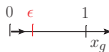
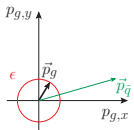
- Collinear divergence



i. $\vec{p}_g \rightarrow \vec{p}_{\bar{q}} = \frac{x_g}{x_q} \vec{p}_q$

ii. x_g generic

- Soft divergence



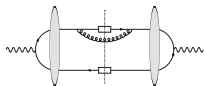
i. $\vec{p}_g \equiv x_g \vec{u}$

ii. $x_g \rightarrow 0$ and \vec{u} generic

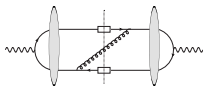
- Soft and collinear divergence ($x_g \rightarrow 0$ and $\vec{u} \rightarrow \frac{\vec{p}_q}{x_q}$)

IR singularities

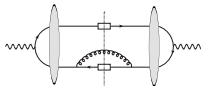
- Divergences: $q\bar{q}$ -fragmentation



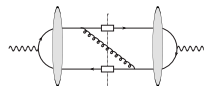
(1): soft + collinear (qq)



(2): soft



(3): soft + collinear (qg)

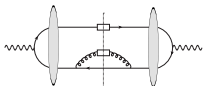


(4): soft

- Treatment of divergences in a nutshell

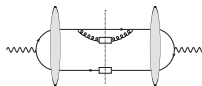
$$d\sigma_1 + \underbrace{d\sigma_{3,\text{soft}} + (d\sigma_3^{(1)} - d\sigma_{3,\text{soft}}^{(1)})}_{d\sigma_{3,\text{collinear}}^{(1)}} + (d\sigma_3^{(2)} - d\sigma_{3,\text{soft}}^{(2)}) + \underbrace{(d\sigma_3^{(3)} - d\sigma_{3,\text{soft}}^{(3)})}_{d\sigma_{3,\text{collinear}}^{(3)}} + ((d\sigma_3^{(4)} - d\sigma_{3,\text{soft}}^{(4)})) + d\sigma_{\text{counter}}$$

- Divergences: qg -fragmentation



(5): collinear $\rightarrow d\sigma_{3,\text{collinear}}^{\bar{q}g(5)}$

- Divergences: $\bar{q}g$ -fragmentation



(6): collinear $\rightarrow d\sigma_{3,\text{collinear}}^{\bar{q}g(6)}$

Renormalization of FFs and gluon fragmentation

- Renormalized quark FFs (similar for the anti-quark)

$$\tilde{D}_q^{h1} \left(\frac{x_{h1}}{x_q} \right) = D_q^{h1} \left(\frac{x_{h1}}{x_q}, \mu_F \right) - \frac{\alpha_s}{2\pi} \left(\frac{1}{\epsilon} + \ln \frac{\mu_F^2}{\mu^2} \right) \left[[P_{qq} \otimes D_q^{h1}] \left(\frac{x_{h1}}{x_q}, \mu_F \right) + [P_{gq} \otimes D_g^{h1}] \left(\frac{x_{h1}}{x_q}, \mu_F \right) \right]$$

Renormalization of FFs and gluon fragmentation

- Renormalized quark FFs (similar for the anti-quark)

$$\tilde{D}_q^{h_1} \left(\frac{x_{h_1}}{x_q} \right) = D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F \right) - \frac{\alpha_s}{2\pi} \left(\frac{1}{\hat{\epsilon}} + \ln \frac{\mu_F^2}{\mu^2} \right) \left[[P_{qq} \otimes D_q^{h_1}] \left(\frac{x_{h_1}}{x_q}, \mu_F \right) + [P_{gq} \otimes D_g^{h_1}] \left(\frac{x_{h_1}}{x_q}, \mu_F \right) \right]$$

↓

$$\begin{aligned} d\sigma_{LL}^{h_1 h_2} \Big|_{\text{ct}} &= \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^4 (d-1) N_c} \sum_q Q_q^2 \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}} x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}} \right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}} \right)^d \delta(1 - x_q - x_{\bar{q}}) \\ &\quad \times \mathcal{F}_{LL} \left(-\frac{\alpha_s}{2\pi} \right) \left(\frac{1}{\hat{\epsilon}} + \ln \frac{\mu_F^2}{\mu^2} \right) \left\{ \underbrace{[P_{qq} \otimes D_q^{h_1}] \left(\frac{x_{h_1}}{x_q}, \mu_F \right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F \right)}_{(1)} \right. \\ &\quad \left. + \underbrace{[P_{gq} \otimes D_g^{h_1}] \left(\frac{x_{h_1}}{x_q}, \mu_F \right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F \right)}_{(6)} + \left[(q, x_q, x_{h_1}) \leftrightarrow (\bar{q}, x_{\bar{q}}, x_{h_2}) \right] \right\} + (h_1 \leftrightarrow h_2) \end{aligned}$$

Renormalization of FFs and gluon fragmentation

- Renormalized quark FFs (similar for the anti-quark)

$$\bar{D}_q^{h_1} \left(\frac{x_{h_1}}{x_q} \right) = D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F \right) - \frac{\alpha_s}{2\pi} \left(\frac{1}{\hat{\epsilon}} + \ln \frac{\mu_F^2}{\mu^2} \right) \left[[P_{qq} \otimes D_q^{h_1}] \left(\frac{x_{h_1}}{x_q}, \mu_F \right) + [P_{gq} \otimes D_g^{h_1}] \left(\frac{x_{h_1}}{x_q}, \mu_F \right) \right]$$

↓

$$\begin{aligned} d\sigma_{LL}^{h_1 h_2} \Big|_{\text{ct}} &= \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^4 (d-1) N_c} \sum_q Q_q^2 \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}} x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}} \right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}} \right)^d \delta(1 - x_q - x_{\bar{q}}) \\ &\quad \times \mathcal{F}_{LL} \left(-\frac{\alpha_s}{2\pi} \right) \left(\frac{1}{\hat{\epsilon}} + \ln \frac{\mu_F^2}{\mu^2} \right) \left\{ \underbrace{[P_{qq} \otimes D_q^{h_1}] \left(\frac{x_{h_1}}{x_q}, \mu_F \right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F \right)}_{(1)} \right. \\ &\quad \left. + \underbrace{[P_{gq} \otimes D_g^{h_1}] \left(\frac{x_{h_1}}{x_q}, \mu_F \right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F \right)}_{(6)} + \left[(q, x_q, x_{h_1}) \leftrightarrow (\bar{q}, x_{\bar{q}}, x_{h_2}) \right] \right\} + (h_1 \leftrightarrow h_2) \end{aligned}$$

- Finite part of the cross sections

$$d\sigma_{h_1, h_2} = \sum_{(a,b)} D_a^{h_1} \otimes D_b^{h_2} \otimes d\hat{\sigma}_{ab} \quad (a, b) = \{(q, \bar{q}), (q, g), (g, \bar{q})\}$$

- Extension to the **semi-inclusive diffractive DIS (SIDDIS)** at the NLO
[M.F., Grabovsky, Li, Szymanowski, Wallon (2024)]