

Inferring the Initial Condition for the BK Equation

based on:

Phys.Rev.D 109 (2024) 054018 arXiv:2311.10491 with M.Karhunen and H.Mäntysaari; and,
(work in progress) with H.Hänninen and H.Mäntysaari

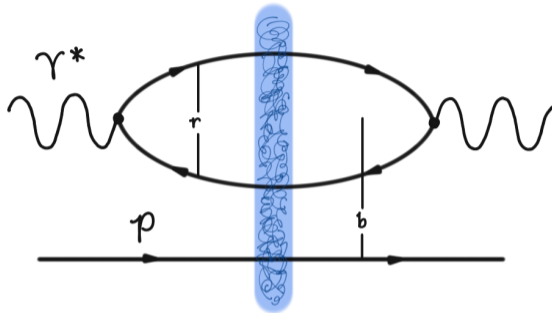
Speaker: Carlisle Casuga

Center of Excellence in Quark Matter, University of Jyväskylä

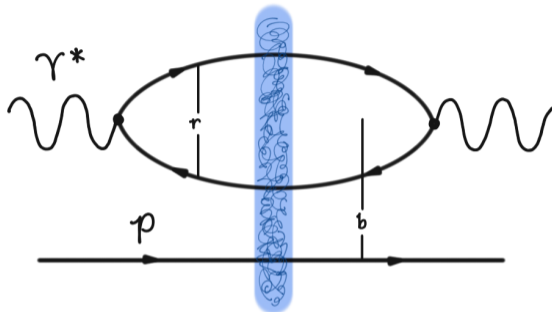
June 10 - 14, ECT* Trento



Deep Inelastic Scattering in the Dipole Picture

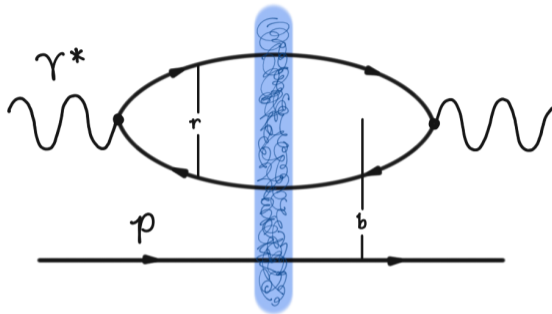


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$$\sigma_{T,L}^{\gamma^*p}(x, Q^2) \sim \frac{\sigma_0}{2} \otimes \mathcal{N}(r, x) \otimes \{\text{LCWF}\}$$

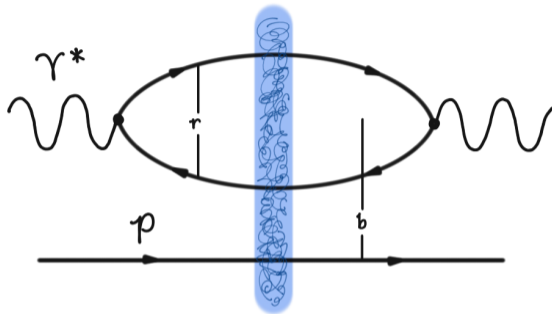
Deep Inelastic Scattering in the Dipole Picture



proton transverse area

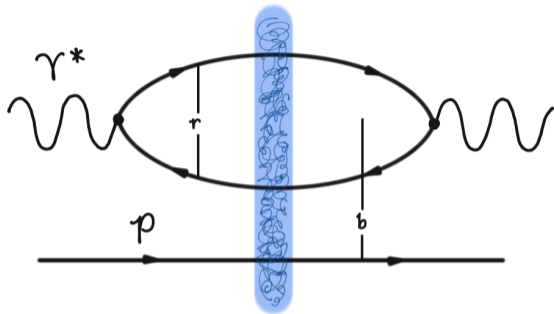
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$$\sigma_{T,L}^{\gamma^*p}(x, Q^2) \sim \frac{\sigma_0}{2} \otimes \mathcal{N}(r, x) \otimes \{\text{LCWF}\}$$

$$\text{rcBK: } \mathcal{N}(r, x = x_0; Q_{s0}^2, \gamma, e_c) \xrightarrow{C^2} \mathcal{N}(r, x)$$

Objectives

- Constrain model parameters, $[Q_{s0}^2, \gamma, e_c, C^2, \sigma_0/2]$ against combined HERA reduced cross section data ...

Some previous fits to HERA data:

- ✓ H.Mäntysaari, T. Lappi (2013): 1309.6963
- ✓ AAMQS Collaboration (2010) arXiv:1012.4408
- ✓ H.Hänninen et al. (2020) arXiv:2007.01645

... this work: provides uncertainty for the BK initial condition!

- Tool: Bayesian inference to extract posterior distribution.
- Account for correlated experimental uncertainties in HERA data.
- This talk: 2311.10491 Leading order acc. + light quarks
work in progress NLO + charm contribution

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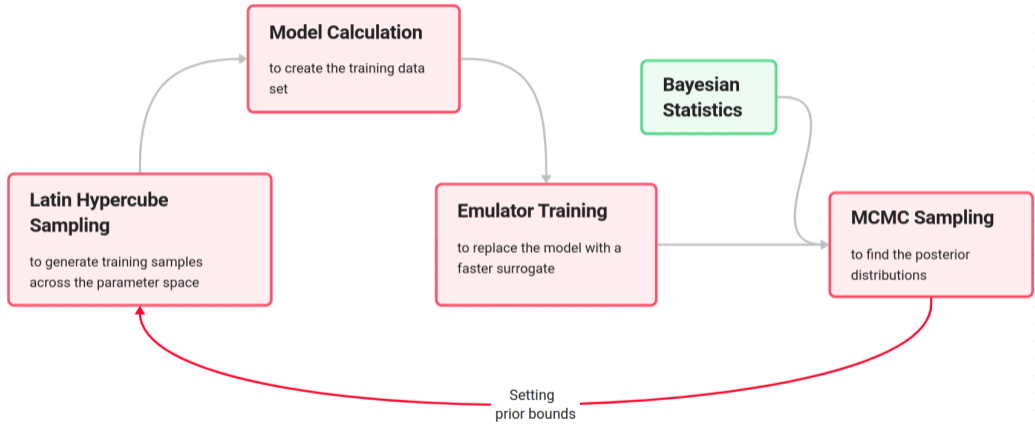
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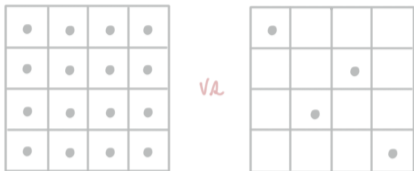
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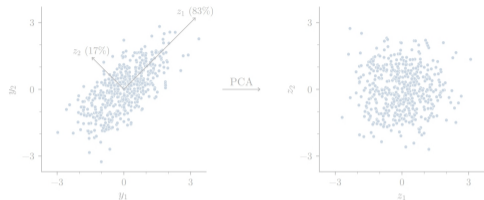
Typical Bayesian Workflow



Latin Hypercube Sampling



Principal Component Analysis



*Image: J.Bernhard PhD Thesis (2018)

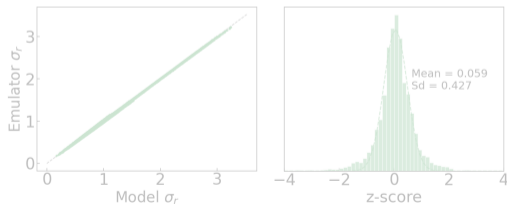
Gaussian Process Emulator

GPs learn the parameter dependence of the model!

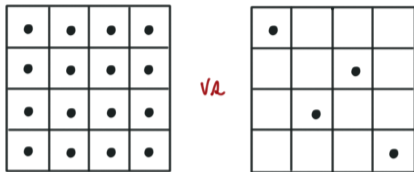


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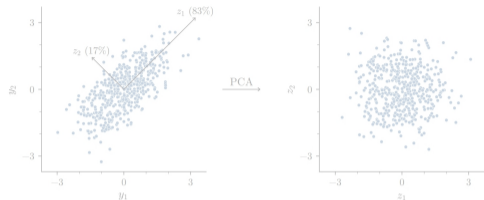
Validation



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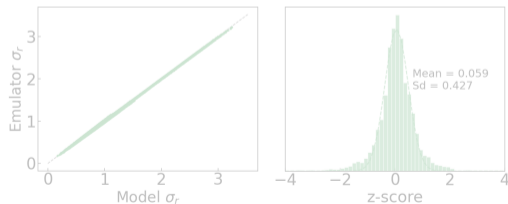


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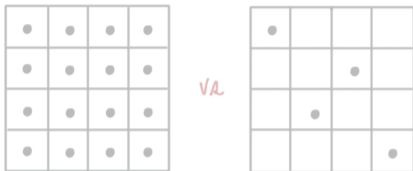
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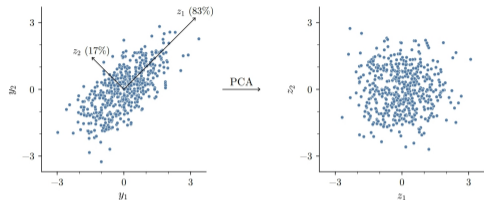
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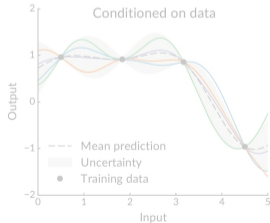
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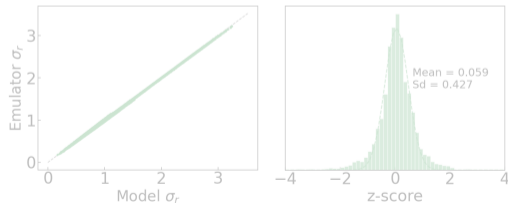
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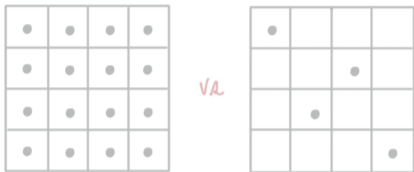


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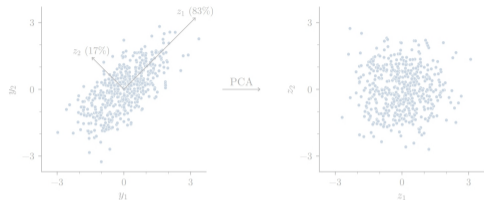
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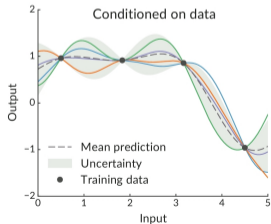


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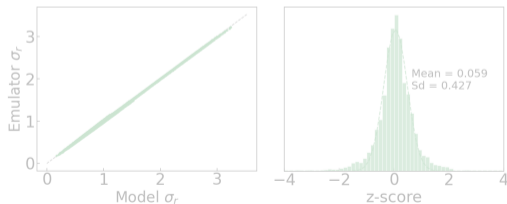


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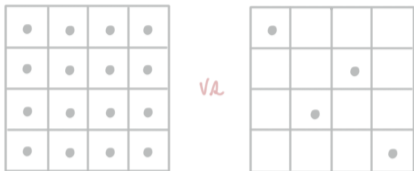
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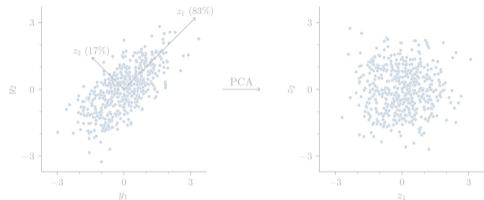
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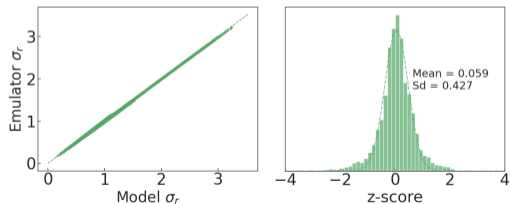
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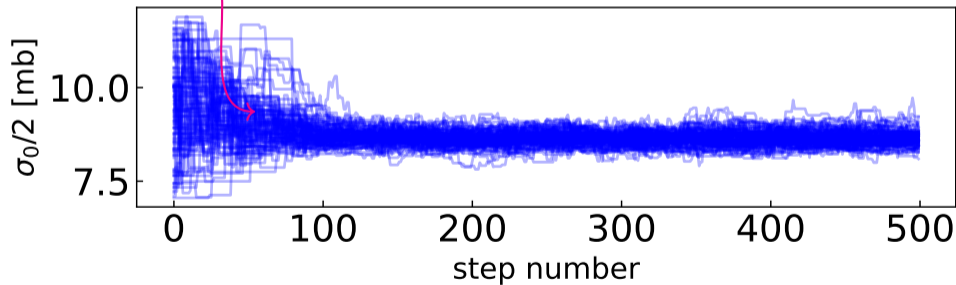
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Acceptance probability:

$$\alpha = \frac{P(\theta_{X+1})}{P(\theta_X)}$$



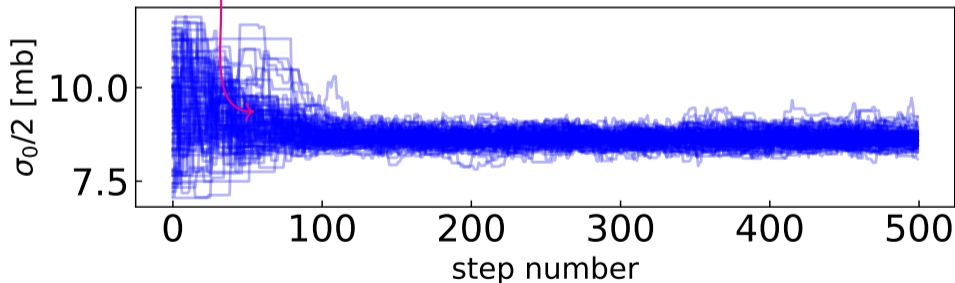
Bayesian Statistics

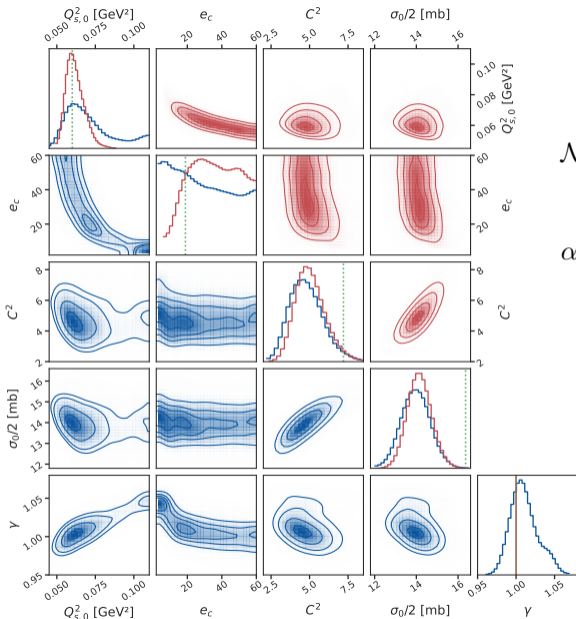
$$P(\theta) = \text{posterior} = \text{likelihood} \times \text{prior}$$

- Likelihood: how well data matches the model at θ
- Prior: bounds of the parameter space

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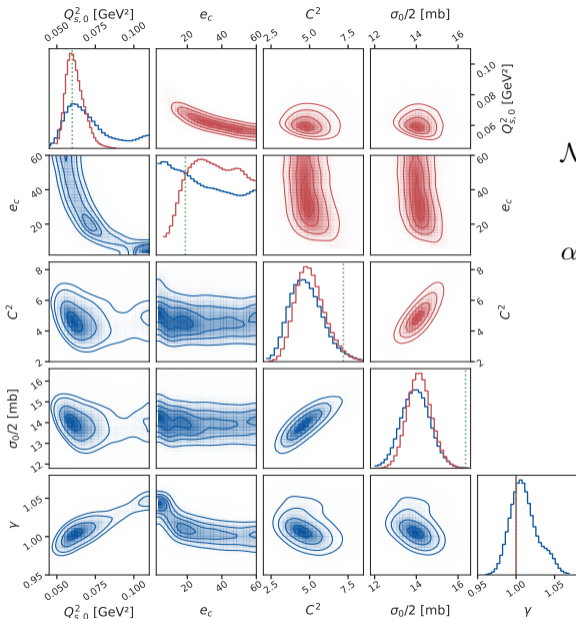
Posterior Distributions

$$\mathcal{N}(\mathbf{r}, \mathbf{x}_0) = 1 - \exp \left[-\frac{(\mathbf{r}^2 Q_{s,0}^2)^\gamma}{4} \ln \left(\frac{1}{|\mathbf{r}| \Lambda_{\text{QCD}}} + e_c \cdot e \right) \right]$$

$$\alpha_s(\mathbf{r}) = \frac{12\pi}{(33 - N_f) \log \left(\frac{4C^2}{\mathbf{r}^2 \Lambda_{\text{QCD}}^2} \right)}$$

- 1 HERA data prefers a $\gamma \approx 1$
- 2 With γ as a free parameter, we obtain a wider posterior distribution
- 3 Off-diagonal plots = correlations

— $[Q_{s,0}^2, \gamma, e_c, C^2, \sigma_0/2]$ — $\gamma = 1$ — MV^e [H.M. & T.L. (2013)]



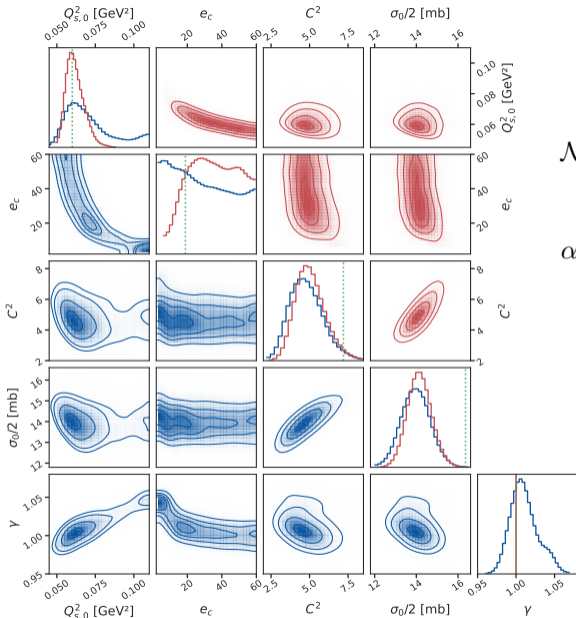
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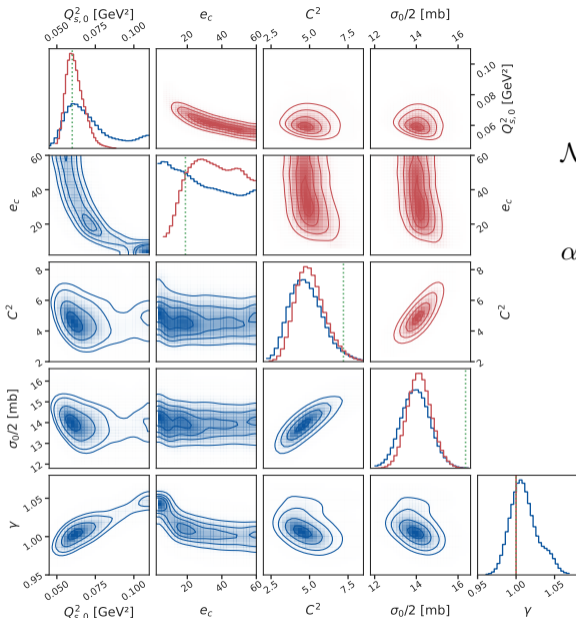
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C.C, M. Karhunen, H. Mäntysaari, 2311.10491

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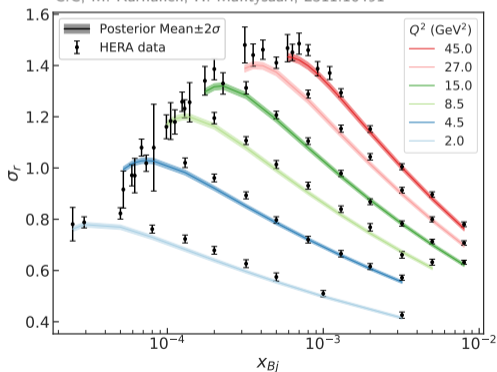
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Posterior Samples, Median and MAP

5 - parameter	$Q_{s0}^2 [GeV^2]$	γ	e_c	C^2	$\sigma_0/2[mb]$	χ^2/dof	Q_s^2
median	0.067	1.01	27.5	4.72	14.0	1.016	0.288
MAP	0.077	1.01	15.6	4.47	13.9	1.012	0.289

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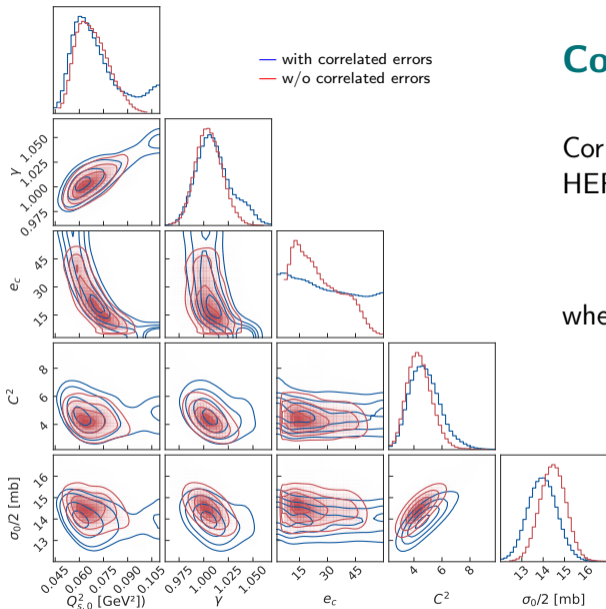
! Good agreement with HERA data

Correlated systematic errors

Correlated systematic uncertainties from the HERA data are input to:

$$P(\theta) \sim -\Delta\mathbf{y}(\theta)^T \Sigma^{-1}(\theta) \Delta\mathbf{y}(\theta)$$

where $\Sigma = \Sigma_{\text{emu}} + \Sigma_{\text{exp}}$.



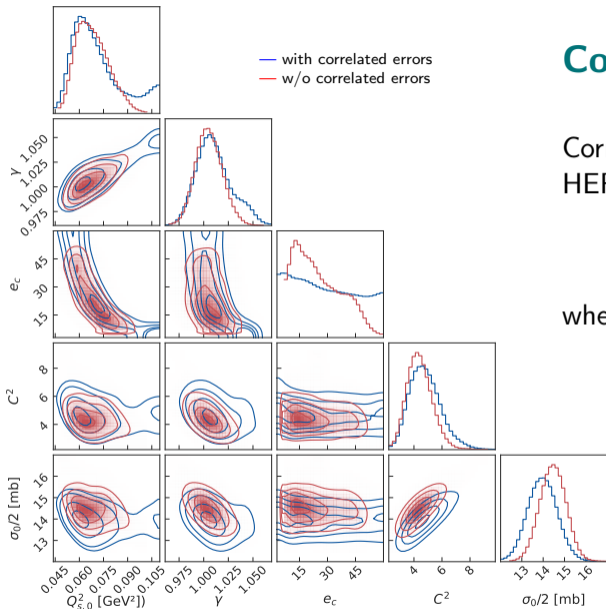
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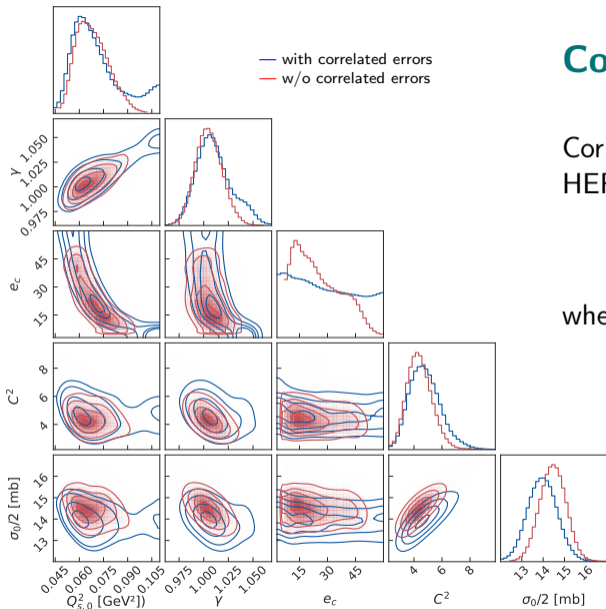
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Inclusive quark production

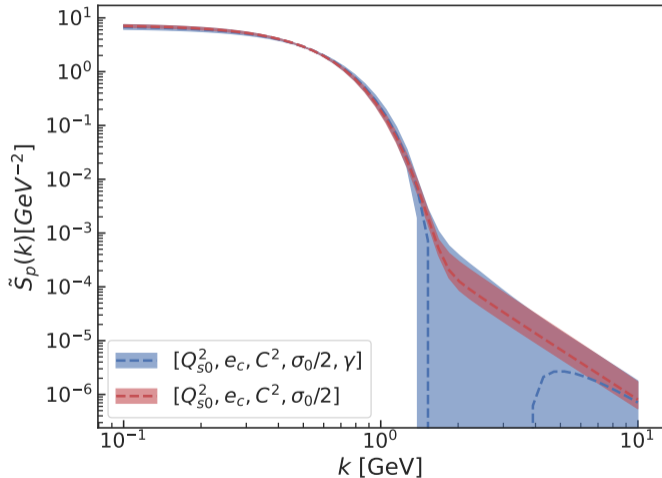
$$d\sigma^{q+A \rightarrow q+X} = xq(x, \mathbf{k}^2) \tilde{S}_p(\mathbf{k})$$

$\tilde{S}_p(\mathbf{k}) \rightarrow$ 2DFT of Dipole amplitude

$$= \int d^2\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} \\ \times [1 - \mathcal{N}(\mathbf{r}, x = x_0)]$$

- $\gamma > 1$ result to negative 2DFT values [B. Giraud et. al. (2016)]

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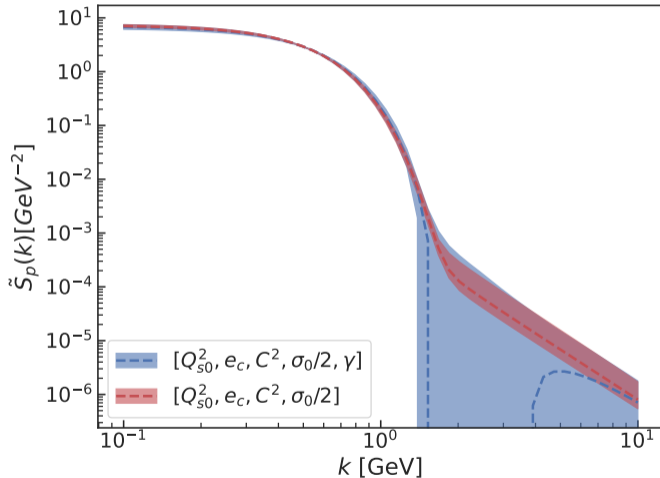
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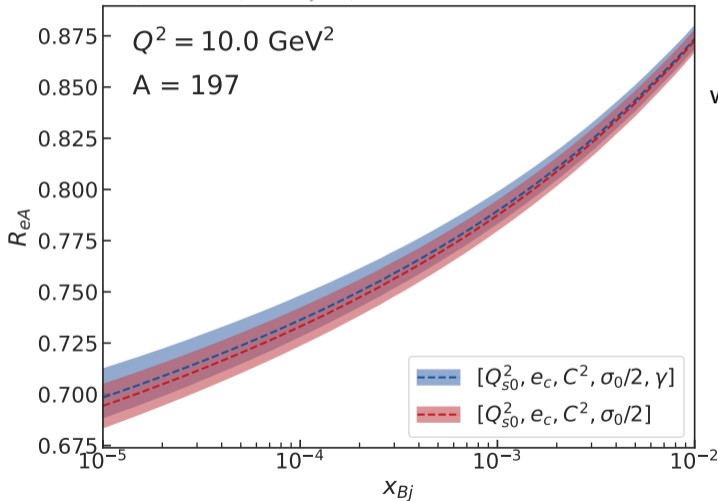
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Nuclear Modification factor at the EIC and beyond

C.C. M. Karhunen, H. Mäntysaari, 2311.10491



$$R_{eA} = \frac{F_{2,A}}{AF_{2,p}}$$

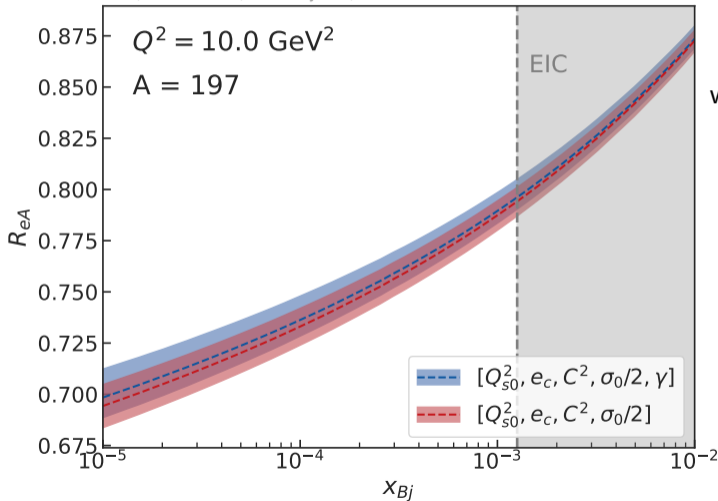
where $Q_{s,A}^2 = Q_{s,p}^2 \cdot \sigma_0/2 \cdot AT_A(b)$.

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Next-to-Leading Order Analysis

■ NLO Cross section:

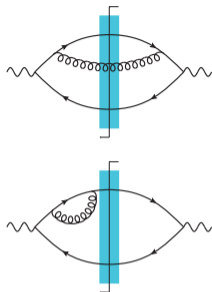
$$\sigma_{L,T}^{\text{NLO}} = \sigma_{L,T}^{\text{IC}} + \sigma_{L,T}^{\text{qg}} + \sigma_{L,T}^{\text{dip}}$$

- 1 Leading order term, amplitude evaluated at IC
- 2 Gluon term, a gluon is emitted into the final state
- 3 Dipole term, NLO $q\bar{q}$ contribution + virtual corrections

■ NLO BK:

- Evolution equation: ResumBK [E. Iancu et al., 1502.05642]
- Running coupling prescription: Balitsky + smallest dipole
- Model parameters: $IC \rightarrow [Q_{s0}^2, \gamma, C^2, \sigma_0/2]$ and charm mass $\rightarrow m_c$

- **First analysis with simultaneous constraint** against HERA total reduced cross section + charm quark production



* Image from H. Hänninen et al. 2211.03504

* set-up and formulations based mostly on [G. Beuf, H. Hänninen, T. Lappi, H. Mäntysaari \(2020\) 2007.01645](#)

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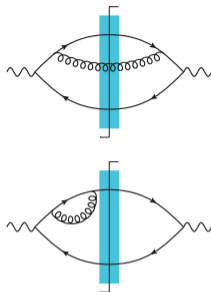
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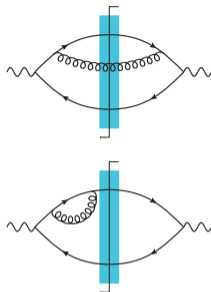
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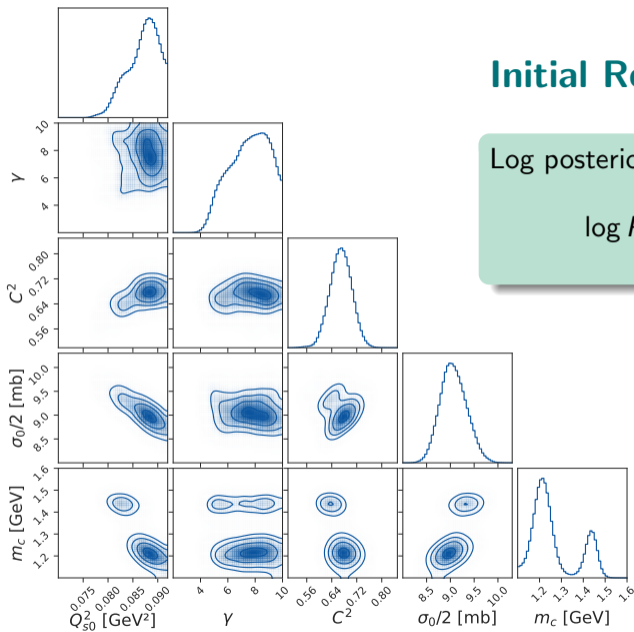
- Evolution equation: ResumBK [E. Iancu et. al., 1502.05642] * Image from H. Hänninen et al. 2211.03504
- Running coupling prescription: Balitsky + smallest dipole
- Model parameters: $IC \rightarrow [Q_{s0}^2, \gamma, C^2, \sigma_0/2]$ and charm mass $\rightarrow m_c$

- **First analysis with simultaneous constraint** against HERA total reduced cross section + charm quark production



* set-up and formulations based mostly on [G. Beuf, H. Hänninen, T. Lappi, H. Mäntysaari \(2020\) 2007.01645](#)

Initial Results of NLO analysis

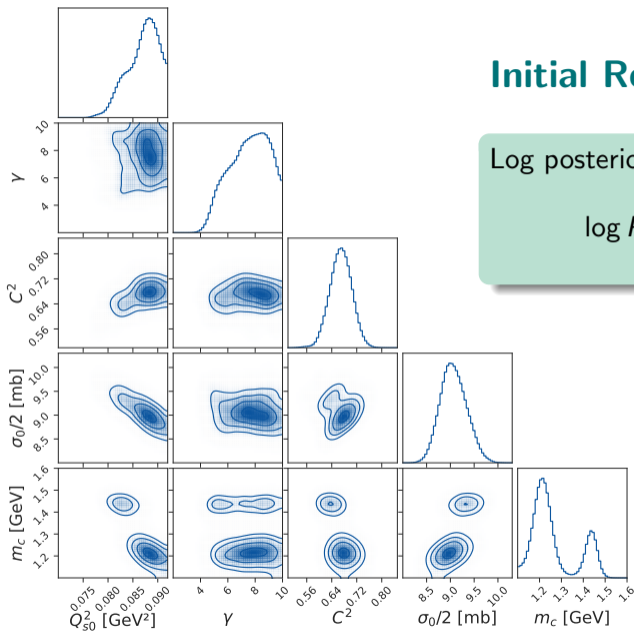


Log posterior for multiple constraints:

$$\log P(\theta|X) = \sum_{i=\text{datasets}} \log P(\theta|X_i)$$

- Close to $C^2 = e^{-2\gamma E} \approx 0.315$ as derived in Y. Kovchegov, H. Weigert (2007) [hep-ph/0609090](https://arxiv.org/abs/hep-ph/0609090)
- High values of γ
- Effectively wider posterior... tension between datasets?

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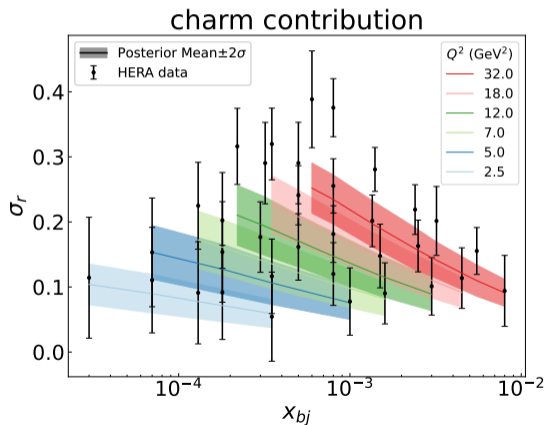
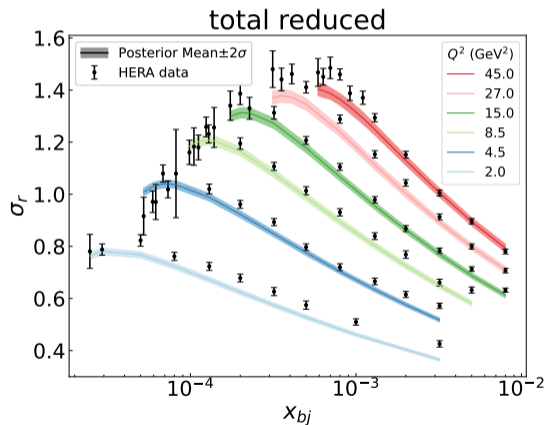
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Description of charm data

- $\chi^2/\text{d.o.f.}$ over 100 samples: 1.70 (total reduced) and 1.39 (charm)



Summary & Outlook

- Posterior distribution for model parameters of BK IC, a necessary input to LO and NLO CGC calculations

! First time:

- Method for propagating the uncertainties BK IC
- Accounted for correlated errors in HERA data
- Global fit including heavy quark cross section data

Further work

- Other IC parametrizations and NLO BK (approximate) prescriptions
- Constrain against beauty production data

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Back up Slides

Parameters

- $\sigma_0/2$, half of normalization to the cross section, proton transverse area, $2 \int d^2b \rightarrow \sigma_0$
- $Q_{s,0}^2$, **related** to the saturation scale at initial x . Previous fits used GBW parametrization:

$$\mathcal{N}(\mathbf{r}, x_0) = 1 - \exp \left[-\frac{(\mathbf{r}^2 Q_{s,0}^2)^\gamma}{4} \right]$$

- C^2 , connects the running coupling in \mathbf{r} to its Fourier transform
- γ , anomalous dimension, controlling the steepness of the cross section related to its fall-off for small dipoles
- e_c , infrared cut-off in the MV model

F_2 Structure Function for Nucleus

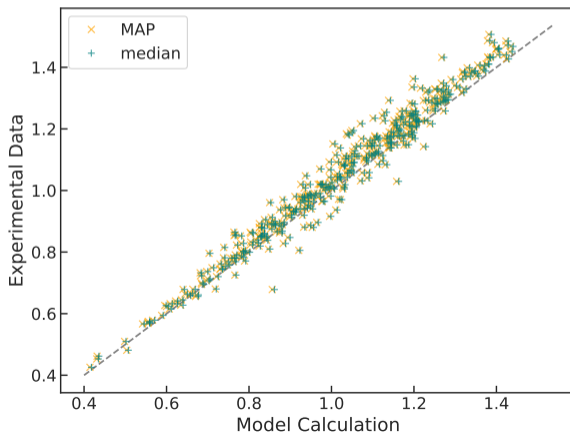
$$N_A(\mathbf{r}, \mathbf{b}, x = x_0) = 1 - \exp \left[-AT_A(\mathbf{b}) \frac{\sigma_0}{2} \frac{(\mathbf{r}^2 Q_{s,0}^2)^\gamma}{4} \ln \left(\frac{1}{|\mathbf{r}| \Lambda_{\text{QCD}}} + e_c \cdot e \right) \right]$$

where $T_A(\mathbf{b})$ is the transverse thickness function of the nucleus of mass number A , obtained through integrating the Woods-Saxon distribution

$$\rho_A(\mathbf{b}, z) = \frac{n}{1 + \exp \left[\frac{\sqrt{\mathbf{b}^2 + z^2} - R_A}{d} \right]}$$

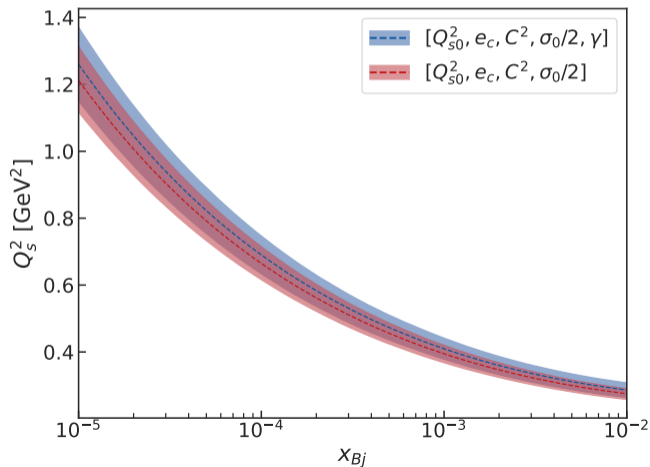
Emulator vs Model

$$\frac{\chi^2}{\text{d.o.f}} = \frac{1}{N - p} \Delta \mathbf{y}(\boldsymbol{\theta})^T \boldsymbol{\Sigma}_{\text{exp}}^{-1} \Delta \mathbf{y}(\boldsymbol{\theta}),$$

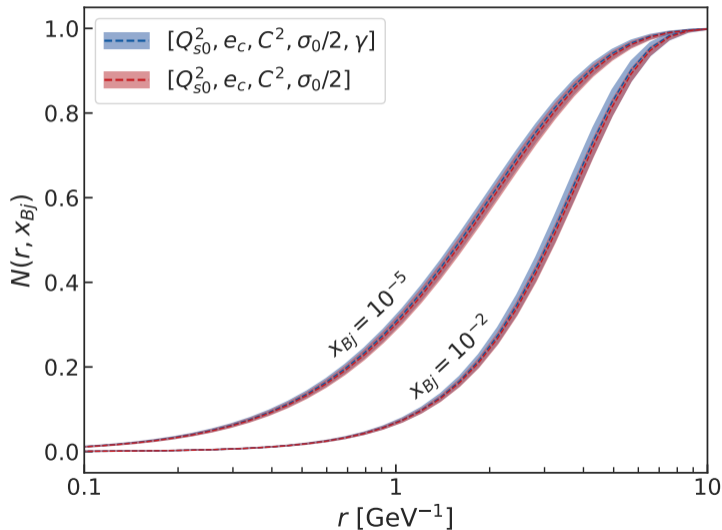


Initial Saturation Scale

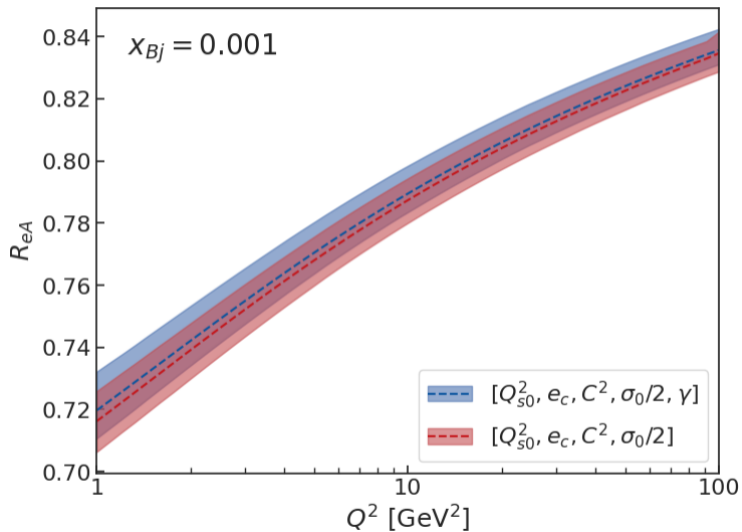
$$N(r^2 = 2/Q_s^2) = 1 - e^{-1/2}$$



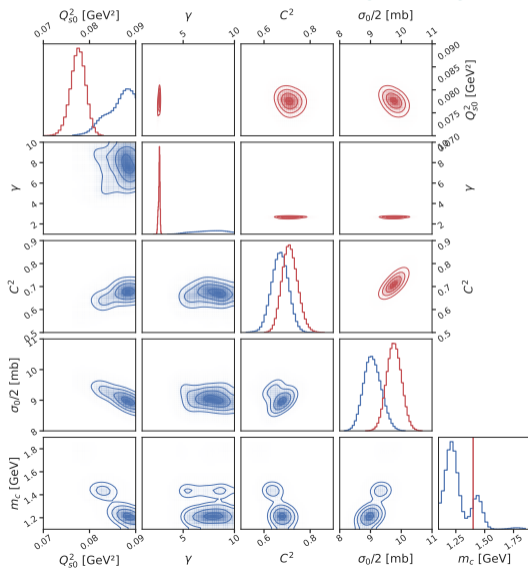
Initial and Evolved $\mathcal{N}(r, y)$



Nuclear Modification Ratio



Comparison to 4-parameter set up (NLO)



Total reduced cross section constraint only (NLO)

