Stochastic Ricci flow: emerging complexity in quantum and analog gravity

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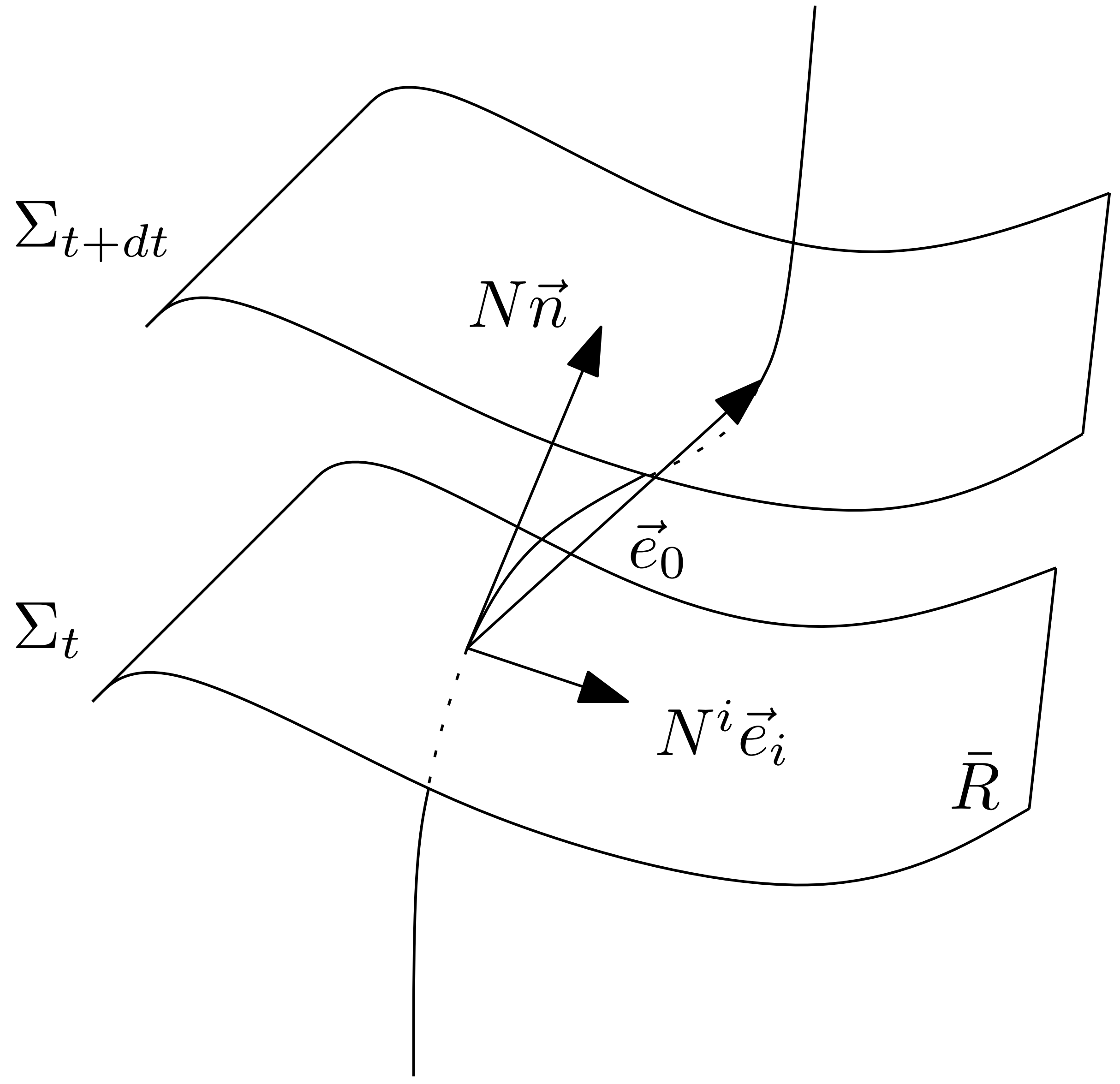
Stochastic quantization, symmetry breaking and gravity



$$\frac{\partial}{\partial s}\phi_A(x^\mu, s) = -\frac{\delta S[\phi]}{\delta \phi_A} + \eta_A(x^\mu, s)$$

Additive (white) noise associated

$$\langle \eta_A(x,s)\rangle = 0$$

$$\langle \eta_A(x,s)\,\eta_B(x',s)\rangle = \alpha_\eta \delta_{AB}\delta(x-x')\delta(s-s')$$

Symmetries breakdown out-of-equilibrium, effective RG-noise and relaxation toward equilibrium

$$R^T_{\mu\nu} = \frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]$$

$$\frac{\partial}{\partial s}g_{\mu\nu} = -2\left[R_{\mu\nu} - R^T_{\mu\nu}\right] + \eta_{\mu\nu}$$

$$= -2\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{8\pi G}{c^4}T_{\mu\nu}\right] - g_{\mu\nu}\left(R - T\right) + \eta_{\mu\nu}$$

Quantization Methods

All methods start from a classical description

Canonical Quantization - Dynamic Perspective
Considering a mechanical system…

$$L = L(q_i, \dot{q}_i)$$

$$p^i(\dot{q}_k) = \frac{\partial L}{\partial \dot{q}_i}$$

invert conjugate momenta

$$H = p^k \dot{q}_k - L$$

$$\delta S = \int \mathrm{d}t\, \delta\left(p^k \dot{q}_k - H\right) = 0$$

minimize the action

$$\dot{q}_i = [q, H]_{\mathrm{PB}} = \frac{\partial H}{\partial p}$$

$$\dot{p}_i = [p, H]_{\mathrm{PB}} = -\frac{\partial H}{\partial q}$$

$$[A, B]_{\mathrm{PB}} = \frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p^k} - \frac{\partial B}{\partial q_k}\frac{\partial A}{\partial p^k}$$

Poisson Brackets provide a "metric" (symplectic) structure in phase space

Replace PB with commutators

$$[A, B]_{\mathrm{PB}} \rightarrow -\frac{\imath}{\hbar}[\hat{A}, \hat{B}]$$

"*Any dynamical theory must first be put in the Hamiltonian form before one can quantize it.*"

P.A.M. Dirac, The theory of gravitation in Hamiltonian form, Proc. Roy. Soc. A, **246**, 1246 (1958)

Quantization Methods

All methods start from a classical description

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**
   Considering fields in four dimensions - Connection with Statistical Field Theory

Generating Functional, or Partition Function (Wick rotation)

$$Z\left[J^{\mathrm{A}}\right]=\int[D\phi]\exp\frac{\imath}{\hbar}\int \mathrm{d}^4x\left[\mathcal{L}(\phi_{\mathrm{A}},\partial_\mu\phi)+J^{\mathrm{A}}\phi_{\mathrm{A}}\right]$$

$$\langle\phi_{\mathrm{A}_1}(x_1)\cdots\phi_{\mathrm{A}_N}(x_N)\rangle=\frac{(-\imath)^N}{Z\left[J^{\mathrm{A}}\right]}\frac{\delta^N Z\left[J^{\mathrm{A}}\right]}{\delta J^{\mathrm{A}_1}(x_1)\cdots\delta J^{\mathrm{A}_N}(x_N)}\bigg|_{J=0}$$

GF of Connected Feynman diagrams, Free Energy (WR)

$$W\left[J^{\mathrm{A}}\right]=\log Z\left[J^{\mathrm{A}}\right]$$

$$\langle\phi_{\mathrm{A}_1}(x_1)\cdots\phi_{\mathrm{A}_N}(x_N)\rangle_c=(-\imath)^N\frac{\delta^N W\left[J^{\mathrm{A}}\right]}{\delta J^{\mathrm{A}_1}(x_1)\cdots\delta J^{\mathrm{A}_N}(x_N)}\bigg|_{J=0}$$

Quantization Methods

All methods start from a classical description

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**

Difficulties in handling Symmetries

Dynamics limited to a hypersurface in phase-space — Constraints Hypersurface $\chi_i = 0$

- **1. Dirac Prescription**
  define *physical states* as
  $$\chi_i|\psi\rangle = 0$$

- **2. Gribov copies - Faddeev Popov determinant**

Path integral

$$= \int [Dx^\mu] \prod_{t,\alpha} \delta(\chi_\alpha) \prod_t (\mathrm{sdet}\,[\chi_\alpha, \chi_\beta])^{1/2} \exp iS[x^\mu(t)]. \qquad (16.5)$$

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

Symmetries in GR

- **Generators of gauge symmetries** found from the **Hamiltonian** [1]

$$\mathcal{L}_{\rm ADM} = N\sqrt{h}\left[K^2 - K^{ij}K_{ij} - \bar{R}\right] \qquad \mathcal{H}_{\rm ADM} = \Pi\dot{N} + \Pi_i\dot{N}^i + \Pi^{ij}\dot{h}_{ij} - \mathcal{L}_{\rm ADM},$$

$$\Pi = \frac{\partial \mathcal{L}_{\rm ADM}}{\partial \dot{N}} = 0, \qquad \Pi_i = \frac{\partial \mathcal{L}_{\rm ADM}}{\partial \dot{N}^i} = 0,$$

$$[\Pi, \mathcal{H}_{\rm ADM}]_{\rm PB} = 0, \qquad [\Pi_i, \mathcal{H}_{\rm ADM}]_{\rm PB} = 0,$$

$$\mathcal{H} = G_{ijkl}\Pi^{ij}\Pi^{kl} - \sqrt{h}\bar{R} = 0, \qquad \mathcal{H}_i = 2h_{ij}\nabla^{(3)}_k\Pi^{kj} = 0,$$

Super-Hamiltonian **Super-Momentum**

Secondary constraints: eq. of motion

$$R^{0i} - \frac{1}{2}g^{0i}R = \frac{N\mathcal{H}^i + N^i\mathcal{H}}{2N^2\sqrt{h}} = 0,$$

$$R^{00} - \frac{1}{2}g^{00}R = -\frac{\mathcal{H}}{2N^2\sqrt{h}} = 0,$$

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

Symmetries in GR

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$$\mathcal{L}_{\rm ADM} = N\sqrt{h}\left[K^2 - K^{ij}K_{ij} - \bar{R}\right] \qquad \mathcal{H}_{\rm ADM} = \Pi\dot{N} + \Pi_i\dot{N}^i + \Pi^{ij}\dot{h}_{ij} - \mathcal{L}_{\rm ADM},$$

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$$\mathcal{H} = G_{ijkl}\Pi^{ij}\Pi^{kl} - \sqrt{h}\bar{R} = 0, \qquad \mathcal{H}_i = 2h_{ij}\nabla_k^{(3)}\Pi^{kj} = 0,$$

Super-Hamiltonian **Super-Momentum**

Generators of gauge transformation: time, space diffeomorphisms

$$\xi\,[\mathcal{H}, h_{ij}]_{\rm PB} = \delta_t h_{ij} = \mathcal{L}_{\xi\mathbf{n}} h_{ij}$$

$$\xi^k\,[\mathcal{H}_k, h_{ij}]_{\rm PB} = \delta_{\boldsymbol{\xi}} h_{ij} = \mathcal{L}_{\boldsymbol{\xi}} h_{ij}$$

$$\mathcal{H}_{\rm ADM} = \Pi\dot{N} + \Pi_i\dot{N}^i + N\mathcal{H} + N^i\mathcal{H}_i = 0,$$

The Hamiltonian vanishes

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

Symmetries in GR

Path-integral Quantization [1]:

Path integral

$$= \int [Dx^\mu] \prod_{t,\alpha} \delta(\chi_\alpha) \prod_t (\text{sdet}\,[\chi_\alpha, \chi_\beta])^{1/2} \exp iS[x^\mu(t)]. \quad (16.5)$$

Delta on the constraints

We are asking the path integral not to fluctuate around the constraints
In GR constraints generate **external** symmetries

Symmetries are imposed in the quantization of GR rather than being **allowed to emerge**

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

Quantization Methods - Continued

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**
3. **Stochastic Quantization**

Stochastic Dynamic Relaxation of the Ensemble Average

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PERTURBATION THEORY WITHOUT GAUGE FIXING

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Abstract

We propose to formulate the perturbative expansion for field theory starting from the Langevin equation which describes the approach to equilibrium. We show that this formulation can be applied to gauge theories to compute gauge invariant quantities without fixing the gauge. A very simple example is worked out in detail. We also discuss the speed of approaching to equilibrium of the solution of the Langevin equation in the framework of perturbation theory.

Quantization Methods - Continued

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**
3. **Stochastic Quantization**
   A. Introduce a "stochastic time" variable s and noise - Langevin dynamics

$$\frac{\partial}{\partial s}\phi_A\left(x^\mu, s\right) = -\frac{\delta S\left[\phi\right]}{\delta \phi_A} + \eta_A\left(x^\mu, s\right),$$

$$\left\langle \eta_A\left(x, s\right)\eta_B\left(x', s'\right)\right\rangle = 2\alpha\delta_{AB}\delta\left(x - x'\right)\delta\left(s - s'\right) \qquad \langle \eta_{\mathrm{A}}\rangle = 0$$

$$p\left(\eta\right) \sim \left[D\eta\right]\exp -\frac{1}{4\alpha}\int \mathrm{d}^4x\,\mathrm{d}s\,\left[\eta\left(x, s\right)\right]^2$$

The noise is **additive** and **Gaussian**: higher-order (even) correlations are functions of $\langle \eta_{\mathrm{A}}\eta_{\mathrm{B}}\rangle$

Quantization Methods - Continued

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**
3. **Stochastic Quantization**
   A. Introduce a "stochastic time" variable s and noise - Langevin dynamics
   B. Expectation values: (i) with respect to the noise, (ii) with respect to $P(\phi, s)$

Average over histories

$$\langle \phi_{\mathrm{A}_1,\eta}(x_1,s)\cdots\phi_{\mathrm{A}_N,\eta}(x_N,s)\rangle = \frac{\int [D\eta]\exp\left[-\frac{1}{4}\int \mathrm{d}^4x\,\mathrm{d}\sigma\,\eta^2(x,\sigma)\right]\phi_{\mathrm{A}_1,\eta}(x_1,s)\cdots\phi_{\mathrm{A}_N,\eta}(x_N,s)}{\int [D\eta]\exp\left[-\frac{1}{4}\int \mathrm{d}^4x\,\mathrm{d}\sigma\,\eta^2(x,\sigma)\right]}$$

Average over dynamic measure

$$\langle \phi_{\mathrm{A}_1,\eta}(x_1,s)\cdots\phi_{\mathrm{A}_N,\eta}(x_N,s)\rangle = \int [D\phi_{\mathrm{B}}]\; P(\phi,s)\;\phi_{\mathrm{A}_1}(x_1)\cdots\phi_{\mathrm{A}_N}(x_N)$$

The Fokker-Planck Eq., associated to the Langevin, dictates the dynamics of $P(\phi, s)$

Quantization Methods - Continued

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**
3. **Stochastic Quantization**
   A. Introduce a “stochastic time” variable s and noise - Langevin dynamics
   B. Expectation values: (i) with respect to the noise, (ii) with respect to $P(\phi, s)$
   C. The associated Fokker-Planck ensures the correct “equilibrium” limit of $P(\phi, s)$

$$\frac{\partial F}{\partial s} = -\frac{\partial F}{\partial \phi_{\rm A}} \frac{\delta S[\phi]}{\delta \phi_{\rm A}} + \frac{1}{2} \frac{\partial^2 F}{\partial \phi_{\rm A}^2} + \frac{\partial F}{\partial \phi_{\rm A}} \eta_{\rm A}$$

Stochastic Calculus

$$\frac{\partial P(\phi, s)}{\partial s} = \frac{\partial}{\partial \phi_{\rm A}} \left[P(\phi, s) \frac{\delta S[\phi]}{\delta \phi_{\rm A}} \right] + \frac{1}{2} \frac{\partial^2 P(\phi, s)}{\partial \phi_{\rm A}^2}$$

Fokker-Planck

$$\frac{\partial P(\phi, s)}{\partial s} = 0 \longrightarrow \lim_{s \to \infty} P(\phi, s) = P(\phi) \sim \exp[-S]$$

The Euclidean Path-Integral measure is recovered at equilibrium

$$\lim_{s \to +\infty} \langle \phi_{\rm A}(x, s)\, \phi_{\rm B}(x', s) \rangle_{p(s)} = \langle \phi_{\rm A}(x)\, \phi_{\rm B}(x') \rangle_{\rm E}$$

Quantization Methods - Continued

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**
3. **Stochastic Quantization**
   A. Introduce a "stochastic time" variable s and noise - Langevin dynamics
   B. Expectation values: (i) with respect to the noise, (ii) with respect to $P(\phi, s)$
   C. The associated Fokker-Planck ensures the correct "equilibrium" limit of $P(\phi, s)$
   D. Gauge theories: the **propagator splits** in a **gauge-invariant** part (**finite**) and a **gauge-dependent** part (**divergent)**

$$\langle\phi_{\mathrm{A}}(x,s)\,\phi_{\mathrm{B}}(x',s)\rangle_{p(s)} = \underbrace{\langle\phi_{\mathrm{A}}(x,s)\,\phi_{\mathrm{B}}(x',s)\rangle}_{\text{Convergent to gauge-invariant propagator}}{}_{\text{gauge-inv}} + \underbrace{\langle\phi_{\mathrm{A}}(x,s)\,\phi_{\mathrm{B}}(x',s)\rangle}_{\text{Divergent in eq. limit and gauge-dependent}}{}_{\text{not-gauge-inv}},$$

No need to use Faddeev-Popov approach when applied to gauge theories

Stochastic Quantization of General Relativity à la Ricci Flow

- Let's consider the presence of matter $S = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}R + \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_{\mathrm{M}},$

We propose the Langevin equation [1] with scalar multiplicative noise

$$\frac{\partial}{\partial s} g_{\mu\nu}(x,s) = \imath\mathcal{G}_{\alpha\beta\mu\nu}(\lambda=-1)\frac{\delta S}{\delta g_{\alpha\beta}} + g_{\mu\nu}(x,s)\, e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\,\tilde{\eta}(x,s)$$

$$\langle \tilde{\eta}(x,s)\,\tilde{\eta}(x',s')\rangle = \delta(s-s')\,\delta^{(4)}(x-x')$$

- Starting point: seminal Rumpf work [2] with additive tensorial noise

$$\frac{\partial}{\partial s} g_{\mu\nu}(x,s) = \imath\mathcal{G}_{\alpha\beta\mu\nu}(\lambda)\frac{\delta S}{\delta g_{\alpha\beta}} + \eta_{\mu\nu}(x,s)$$

$$\langle \eta_{\alpha\beta}(x,s)\,\eta_{\mu\nu}(x',s')\rangle = \mathcal{G}_{\alpha\beta\mu\nu}(\lambda)\,\delta(s-s')\,\delta^{(4)}(x-x')$$

- DeWitt Supermetric: special choice [2] $\lambda = -1$
- Strong link to Horava-Lifshitz gravity

$$\mathcal{G}^{\alpha\beta\mu\nu}(\lambda) = \frac{\sqrt{-g}}{2\kappa}\left[g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - \lambda g^{\alpha\beta}g^{\mu\nu}\right]$$

$$\mathcal{G}_{\alpha\beta\mu\nu}(\lambda) = \frac{2\kappa}{\sqrt{-g}}\left[g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu} - \frac{\lambda}{2\lambda+1}g_{\alpha\beta}g_{\mu\nu}\right]$$

[1] A. M. Lulli, A. Marciano, X. Shan, arXiv:2112.01490 (2021)

[2] H. Rumpf, Phys. Rev. D, 33, 4, 1986

Stochastic Quantization of General Relativity à la Ricci Flow

- Let's consider the presence of matter $S=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}R+\int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_{\mathrm{M}},$

We propose the Langevin equation [1] with scalar multiplicative noise

$$\frac{\partial g_{\mu\nu}}{\partial s}=-2\imath\left[R_{\mu\nu}-\kappa\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)\right]+g_{\mu\nu}e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\,\tilde{\eta}$$

$$\langle\tilde{\eta}\left(x,s\right)\tilde{\eta}\left(x',s'\right)\rangle=\delta\left(s-s'\right)\delta^{(4)}\left(x-x'\right)$$

- Zero-noise limit: looks like a Ricci-flow with a target fixed point
- Noise amplitude to be determined at the saddle-point of the equilibrium

[1] A. M. Lulli, A. Marciano, X. Shan, arXiv:2112.01490 (2021)

[2] H. Rumpf, Phys. Rev. D, 33, 4, 1986

Emergent Cosmological Constant

$$\frac{\partial g_{\mu\nu}}{\partial s} = -2\imath \left[R_{\mu\nu} - \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \right] + g_{\mu\nu} e^{\imath \frac{\gamma}{2}} \sqrt{2\alpha}\, \tilde{\eta},$$

- We need to "interpret" the Langevin equation: Itô calculus [1]
- Different variables coupled to the same noise
- Need to compute Itô rules for this case: for a generic F

$$\frac{\partial F}{\partial s} = -2\imath \frac{\partial F}{\partial g_{\mu\nu}} \left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu} \right] + \alpha e^{\imath\gamma} g_{\mu\nu} g_{\alpha\beta} \frac{\partial^2 F}{\partial g_{\mu\nu} \partial g_{\alpha\beta}} + e^{\imath \frac{\gamma}{2}} \sqrt{2\alpha} \frac{\partial F}{\partial g_{\mu\nu}} g_{\mu\nu} \tilde{\eta}$$

- The associated Fokker-Planck reads

$$\partial_s p = -\frac{\partial}{\partial g_{\mu\nu}} \left(-2\imath \left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu} \right] p \right) + \alpha e^{\imath\gamma} \frac{\partial^2}{\partial g_{\mu\nu} \partial g_{\alpha\beta}} \left(g_{\mu\nu} g_{\alpha\beta}\, p \right)$$

[1] H. Rumpf, Phys. Rev. D, 33, 4, 1986

Emergent Cosmological Constant

- We can look for the steady state solution of the Fokker-Planck

$$\partial_s p = -\frac{\partial}{\partial g_{\mu\nu}}\left(-2\imath\left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right] p\right) + \alpha e^{\imath\gamma}\frac{\partial^2}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\left(g_{\mu\nu}g_{\alpha\beta}\, p\right)$$

- After setting $\partial_s p = 0$ one obtains

Number of independent metric components

$$p\left(g_{\mu\nu}\right) \simeq \mathcal{C}\exp\left\{\frac{1}{D\alpha e^{i\gamma}}\int^{g^{\mu\nu}}\mathcal{D}\bar{g}^{\mu\nu}\left[-2\imath\left(R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right) - \alpha e^{\imath\gamma}\left(1+n_g\right)\bar{g}_{\mu\nu}\right]\right\}$$

- The saddle-point $\frac{\delta p}{\delta g^{\mu\nu}} = 0$ condition reads $R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu} - \frac{1}{2}\alpha\left(1+n_g\right)e^{\imath\left(\gamma+\frac{\pi}{2}\right)}g_{\mu\nu} = 0$

- After setting $\frac{1}{2}\alpha\left(1+n_g\right) = \Lambda, \qquad \alpha = \frac{2\Lambda}{1+n_g},$

The cosmological constant
appears as a consequence of the noise at the saddle point at equilibrium (!)

$$\boxed{R_{\mu\nu} - \Lambda e^{\imath\left(\gamma-\frac{\pi}{2}\right)}g_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)}$$

Stochastic Quantization à la Ricci-Flow in ADM

Let us go from the metric tensor to the ADM variables

We need to follow the Itô transformation rule

$$\frac{\partial F}{\partial s} = -2\imath \frac{\partial F}{\partial g_{\mu\nu}} \left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu} \right] + \alpha e^{\imath\gamma} g_{\mu\nu} g_{\alpha\beta} \frac{\partial^2 F}{\partial g_{\mu\nu} \partial g_{\alpha\beta}} + e^{\imath\frac{\gamma}{2}} \sqrt{2\alpha} \frac{\partial F}{\partial g_{\mu\nu}} g_{\mu\nu} \tilde{\eta}$$

The final set of equations reads

$$\frac{\partial N}{\partial s} = -\frac{N}{2} \left[\imath \left(\frac{\mathcal{H}}{\sqrt{h}} + R \right) + \frac{\Lambda e^{-\imath\gamma}}{1+n_g} + e^{\imath\frac{\gamma}{2}} \sqrt{\frac{4\Lambda}{1+n_g}} \tilde{\eta} \right]$$

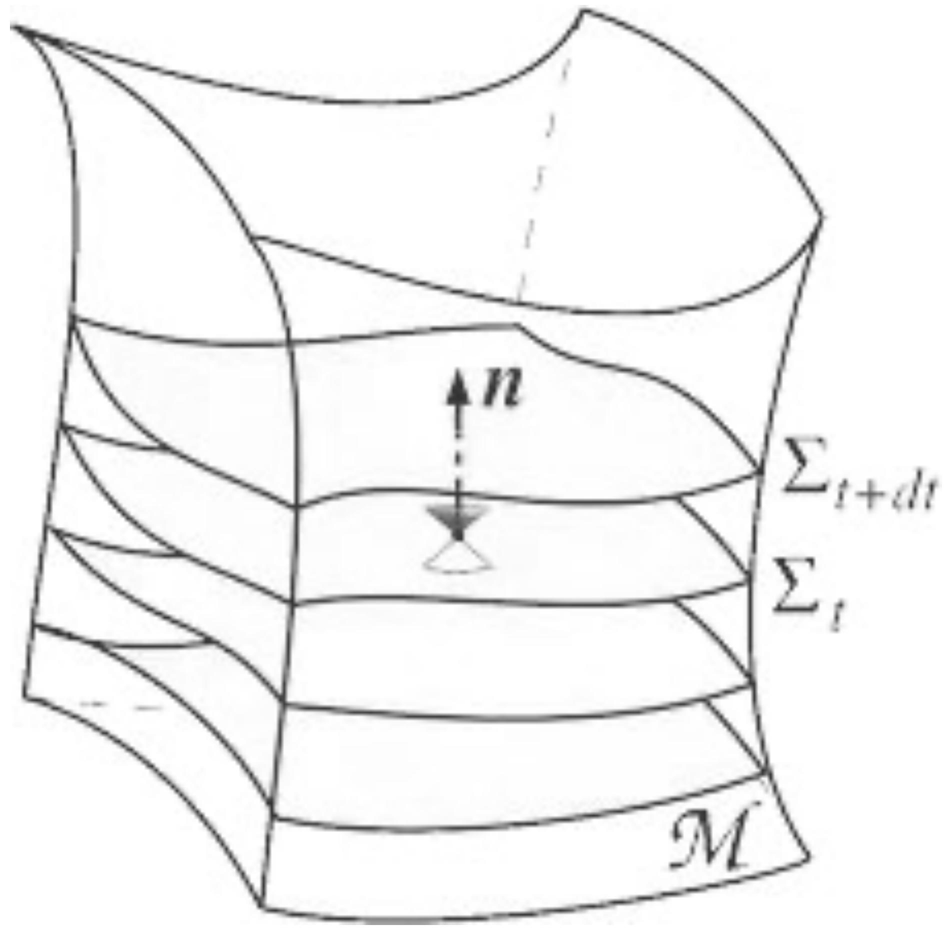
$$\boxed{\frac{\partial N^k}{\partial s} = \frac{\imath N \mathcal{H}^k}{\sqrt{h}}}$$

$$\frac{\partial h_{ij}}{\partial s} = \frac{1}{N} \mathcal{L}_{Nn} \left[\mathcal{H}, h_{ij} \right] + \left[\mathcal{H}, \left[\mathcal{H}, h_{ij} \right] \right] + \frac{h_{ij}\mathcal{H}}{2\sqrt{h}} - h_{ij} e^{\imath\frac{\gamma}{2}} \sqrt{\frac{4\Lambda}{1+n_g}} \tilde{\eta}$$

Thermal time and conformal transformation

$$\varepsilon^2(s) = \exp\left[2\int_{\tau_0}^{\tau} d\bar{\tau}\varphi(\bar{\tau}, s)\right]$$

The norm of the normal is not constant



$$\delta s = \frac{\mathrm{d}s}{\mathrm{d}\tau}\delta\tau = \ell(s)\sqrt{\frac{1}{c^2\varepsilon^2(s)}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}$$

$$s \to \tau$$

From having introduced a projective term in the connection

$$\frac{dg_{\mu\nu}(\tau, s)}{d\tau} = -2\varphi(\tau, s)g_{\mu\nu}(\tau, s)$$

Projective connection

$$\bar{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + \mathcal{C}^{\gamma}_{\alpha\beta}$$

$$\mathcal{C}^{\gamma}_{\alpha\beta} = \lambda_1 \delta^{\gamma}_{\alpha} u_{\beta} + \lambda_2 u_{\alpha} \delta^{\gamma}_{\beta} + \lambda_3 w_{\alpha\beta} u^{\gamma} + \lambda_4 u_{\alpha} u_{\beta} u^{\gamma}$$

$$\sqrt{-g}\bar{R} = \sqrt{-g}R + \sqrt{-g}g^{\beta\delta}\left(\mathcal{C}^{\mu}_{\beta\delta}\mathcal{C}^{\alpha}_{\mu\alpha} - \mathcal{C}^{\mu}_{\beta\alpha}\mathcal{C}^{\alpha}_{\mu\delta}\right)$$

Cosmological term induced in the action

$$\sqrt{-g}g^{\beta\delta}\left(\mathcal{C}^{\mu}_{\beta\delta}\mathcal{C}^{\alpha}_{\mu\alpha} - \mathcal{C}^{\mu}_{\beta\alpha}\mathcal{C}^{\alpha}_{\mu\delta}\right)$$
$$= \sqrt{-g}\left[\left(\lambda_2^2 + \lambda_3^2\right)(D-1)\, u_{\mu}u^{\mu}\right]$$

Breakdown of the conformal symmetry

$$h_{\mu\nu} = h^{\perp}_{\mu\nu} + \partial_\mu a^{\perp}_\nu + \partial_\nu a^{\perp}_\mu + \left(\partial_\mu\partial_\nu - \frac{1}{4}\eta_{\mu\nu}\Box\right) a + \frac{1}{4}\eta_{\mu\nu}\varphi$$

$$\Phi = \varphi - \bar{\Box} a$$

Einstein-Hilbert expanded on dS or AdS backgrounds

$$\mathcal{S}^{(2)}_{\mathrm{EH}} = \frac{c^3}{16\pi G}\int d^4x\sqrt{-\bar{g}}\left[\frac{1}{4}h^{\mu\nu}_{\perp}\left(\bar{\Box} - \frac{\bar{R}}{6}\right)h^{\perp}_{\mu\nu} - \frac{3}{32}\Phi\left(\bar{\Box} + \frac{\bar{R}}{3}\right)\Phi\right]$$

Residual gauge transformation: conformal Killing vector and disappearance of the ghost

$$h^{\perp}_{\mu\nu} \to h^{\perp}_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu\,, \qquad \Phi \to \Phi + 2\nabla^\mu k_\mu$$

Hamiltonian analysis of the Ricci flow

$$\eta_{\mu\nu} = \eta \, g_{\mu\nu}$$

Multiplicative choice of the noise source entails additivity in the Hamiltonian ADM picture

$$\frac{\partial N}{\partial s} = -\frac{N}{2}\left[\imath\left(\frac{\mathcal{H}}{\sqrt{h}} + R\right) + \frac{\Lambda e^{-\imath\gamma}}{1+n_g} + e^{\imath\frac{\gamma}{2}}\sqrt{\frac{4\Lambda}{1+n_g}}\tilde{\eta}\right]$$

$$\frac{\partial N^k}{\partial s} = \frac{\imath N \mathcal{H}^k}{\sqrt{h}}$$

$$\frac{\partial h_{ij}}{\partial s} = \frac{1}{N}\mathcal{L}_{Nn}\left[\mathcal{H}, h_{ij}\right] + \left[\mathcal{H}, \left[\mathcal{H}, h_{ij}\right]\right] + \frac{h_{ij}\mathcal{H}}{2\sqrt{h}} - h_{ij}e^{\imath\frac{\gamma}{2}}\sqrt{\frac{4\Lambda}{1+n_g}}\tilde{\eta}$$

Shift-vector and Navier-Stokes

- Let's consider the shift-vector equation

$$\frac{\partial N^k}{\partial s} = \frac{\imath N \mathcal{H}^k}{\sqrt{h}}$$

- **The noise term cancels out**
- The fixed point solution automatically implements super-Momentum constraint

$$\mathcal{H}_i = 2h_{ij}\nabla_k^{(3)}\Pi^{kj} = 0,$$

- In the non-relativistic limit the super-momentum reduces to Navier-Stokes [1]

$$\Pi^{ij} = \sqrt{h}(Kh^{ij} - K^{ij}) = \frac{1}{2}\sqrt{h}T^{ij}$$ ⟶ Brown-York stress tensor

Its divergence yields Navier-Stokes $r_c^{3/2}\partial^k T_{ki} = \partial_t v_i - \zeta\partial^2 v_i + \partial_i P + v^k\partial_k v_i = 0$

Incompressibility $r_c^{3/2}\partial_k T^{kt} = \partial_k v^k = 0$

[1] I. Bredberg, C. Keeler, V. Lysov, A. Strominger, JHEP, 2012, 146 (2012)

Shift-vector and Navier-Stokes

- It reduces to a forced incompressible Navier-Stokes equation (for Euclidean signature)

$$\frac{\partial N^k}{\partial s} = \frac{\imath N \mathcal{H}^k}{\sqrt{h}}$$

$$\boxed{\partial_t v^i - \zeta \partial^2 v^i + \partial^i P + v^k \partial_k v^i = -\frac{\imath}{N}\frac{\partial N^i}{\partial s}}$$

$$r_c^{3/2} \partial_k T^{kt} = \partial_k v^k = 0$$

- **The noise term cancels out**
- The fields can enter a turbulent regime
- Turbulent fluctuations are **intermittent**
- Possibility to investigate the multi-fractal hypothesis [1] in GR

The only equation that loses noise responds with intermittency!

Possible role of turbulence in cosmological first-order phase transitions (!)

[1] U. Frisch, G. Parisi; R Benzi, G Paladin, G Parisi, A Vulpiani Journal of Physics A: Mathematical and General 17 (18), 3521

Black Holes and KPZ

- Let us consider a "static" spherical symmetric metric $\mathrm{d}s^2 = -e^{\nu(r)}\mathrm{d}t^2 + e^{\mu(r)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$
- After performing Itô transformation and taking the **quasi-equilibrium limit** $\mu = -\nu$

$$\frac{\partial\nu}{\partial s} = \frac{\mathrm{d}\nu}{\mathrm{d}g_{00}}\frac{\partial g_{00}}{\partial s} + e^{\imath\gamma}\alpha\, g_{00}^2\frac{\mathrm{d}^2\nu}{\mathrm{d}g_{00}^2}\,,$$

$$\frac{\partial\nu}{\partial s} = -\,\imath\left[\frac{\mathrm{d}^2\nu}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\nu}{\mathrm{d}r} + \left(\frac{\mathrm{d}\nu}{\mathrm{d}r}\right)^2\right]e^{\nu} - \frac{1}{2}e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\,\left(\tilde{\eta} + e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\right)$$

- Let's normalise the equation to 2α

$$\nu = \nu_0(r) + \bar{\nu}(r,s) \qquad \frac{\bar{\nu}}{\sqrt{2\alpha}} = \bar{h} \qquad s\sqrt{2\alpha} = t \qquad \bar{\eta} = -\frac{1}{2}\frac{e^{\imath\frac{\gamma}{2}}}{\sqrt{2\alpha}}\tilde{\eta} \qquad h = \bar{h} + \frac{1}{2}e^{\imath\frac{\gamma}{2}}t$$

$$\frac{1}{2\alpha}\frac{\partial\bar{\nu}}{\partial s} = \frac{\partial\bar{h}}{\partial t} = \imath\left[\frac{e^{\nu_0}}{\sqrt{2\alpha}}\left(\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\bar{h}}{\mathrm{d}r}\right) + e^{\nu_0}\left(\frac{\mathrm{d}\bar{h}}{\mathrm{d}r}\right)^2\right] + \bar{\eta} - \frac{1}{2}e^{\imath\frac{\gamma}{2}}$$

$$\frac{\partial h}{\partial t} = \imath\left[\nu_{\mathrm{KPZ}}(r,\alpha)\,\nabla^2 h(r,s) + \lambda_{\mathrm{KPZ}}(r)\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2\right] + \bar{\eta}$$

Kardar-Parisi-Zhang Eq. [1]
for a spherical surface

$$\nu_{\mathrm{KPZ}}(r,\alpha) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \qquad \lambda_{\mathrm{KPZ}}(r) = e^{\nu_0(r)}$$

[1] M. Kardar, G. Parisi, Y.C. Zhang, PRL 56, 9, 889–892 (1986)

Black Holes and KPZ

$$\frac{\partial h}{\partial t} = \imath \left[\nu_{\mathrm{KPZ}}(r,\alpha) \nabla^2 h(r,s) + \lambda_{\mathrm{KPZ}}(r) \left(\frac{\mathrm{d}h}{\mathrm{d}r} \right)^2 \right] + \bar{\eta}$$

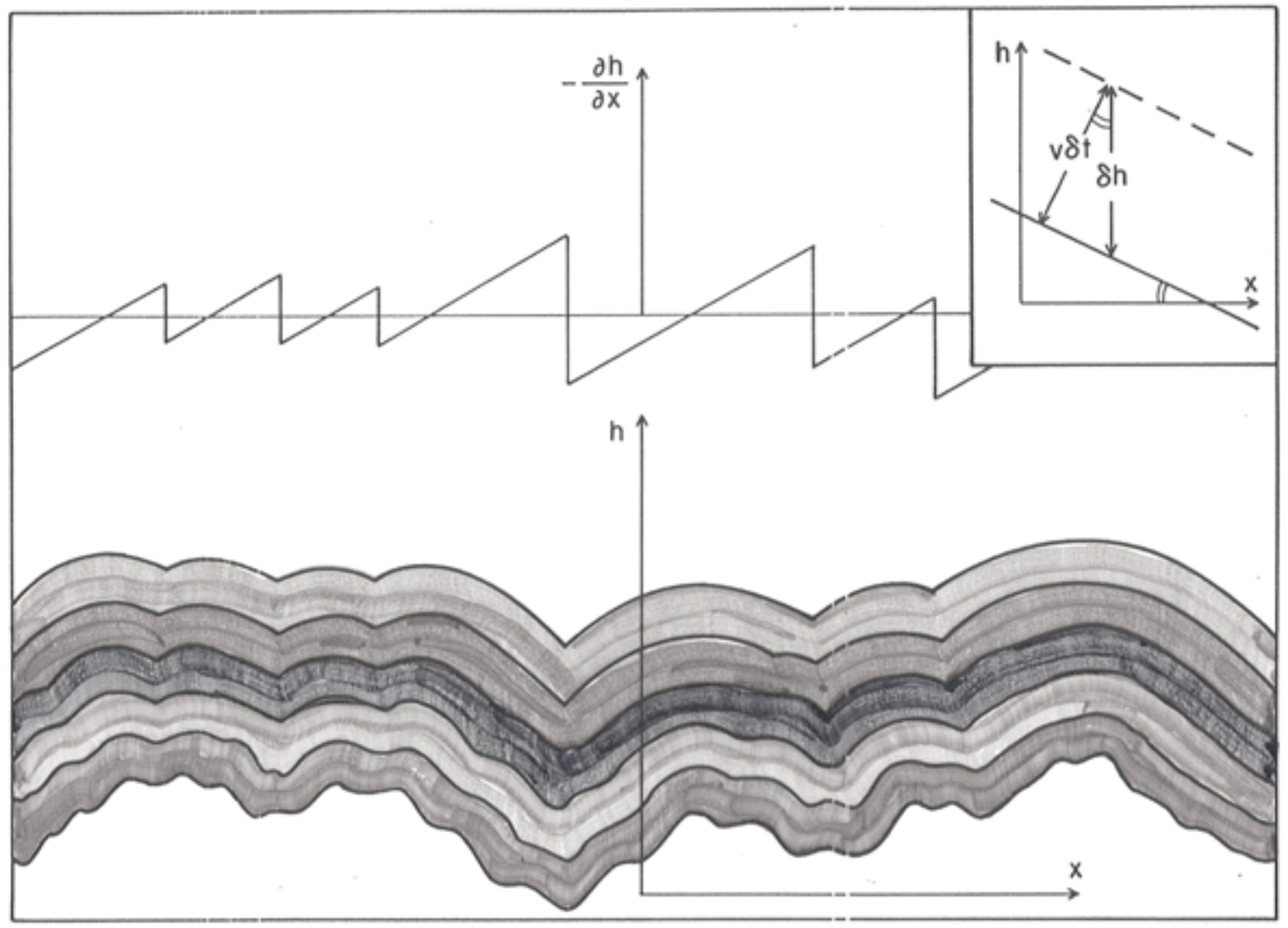
Kardar-Parisi-Zhang Eq. [1]
for a spherical surface

$$\nu_{\mathrm{KPZ}}(r,\alpha) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \qquad \lambda_{\mathrm{KPZ}}(r) = e^{\nu_0(r)}$$

- KPZ defines its own **dynamic universality class** $z = 3/2$
- It models **surface growth** by deposition: **surface height** h
- It is related to Burgers' equation via a change of variable

$$u = -\lambda \frac{\partial h}{\partial x} \longrightarrow \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

- Both models exhibit **intermittency** for small dissipation
- The small dissipation limit is obtained near the horizon



$$\lim_{r\to r_s} \nu_{\mathrm{KPZ}}(r,\alpha) = \lim_{r\to r_s} \lambda_{\mathrm{KPZ}}(r) \propto \lim_{r\to r_s} g_{00} = \lim_{r\to r_s} \left(1 - \frac{r_s}{r}\right) = 0$$

[1] M. Kardar, G. Parisi, Y.C. Zhang, PRL 56, 9, 889–892 (1986)

Black Holes and KPZ

$$\frac{\partial h}{\partial t} = \imath \left[\nu_{\mathrm{KPZ}}(r,\alpha) \nabla^2 h(r,s) + \lambda_{\mathrm{KPZ}}(r) \left(\frac{\mathrm{d}h}{\mathrm{d}r} \right)^2 \right] + \bar{\eta}$$

Kardar-Parisi-Zhang Eq. [1]
for a spherical surface

$$\nu_{\mathrm{KPZ}}(r,\alpha) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \qquad \lambda_{\mathrm{KPZ}}(r) = e^{\nu_0(r)}$$

- Gravitational waves good probe for near-horizon physics: **no screening**
- Black holes mergers: possibly use GW right before merger and final part of "ringdown" [2]
- Proper-time intermittency: include **BH spin**

$$\mathrm{d}s^2 \simeq -e^{\nu_0(r)} \left(1 + \sqrt{2\alpha} h\right) \mathrm{d}t^2 + e^{\mu_0(r)} \left(1 - \sqrt{2\alpha} h\right) \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2$$

- **Low dissipation**: far-from-horizon (linear) but need no large $\sqrt{2\alpha}$
- Possible accumulated effects during gravitational waves propagations [3]

[1] M. Kardar, G. Parisi, Y.C. Zhang, PRL 56, 9, 889–892 (1986)

[2] V. Cardoso, E. Franzin, P. Pani, PRL, 116, 171101 (2016)

[3] M. Lulli, A. Marcianò and N. Yunes, work in progress

Stochastic Ricci RG flow of Λ

$$ds^2 = -N^2 dt^2 + a^2(t)\left[\frac{(dr)^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

FLRW background

$$S = 6\int d^4x N a^3 R + \int d^4x N a^3 (D-1)\lambda_2^2 \epsilon(\lambda) \qquad R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}^2}{a^2}\right) + \frac{k}{a^2}\right)$$

$$\frac{\partial a}{\partial s} = -\frac{2\imath}{N^2}\left(a\dot{H} + 3aH^2 + \varepsilon N^2 \lambda_2^2\right) + a\eta\,,$$

$$\frac{\partial N}{\partial s} = -2\imath\left(\frac{3}{2N}(\dot{H} + H^2) + \frac{1}{16}N\left(\Lambda_0 + 8\lambda_2^2\right)\right) - N\eta\,,$$

$$\frac{\partial \lambda_2}{\partial s} = \imath\left(-2\varepsilon - \imath\eta\right)\lambda_2\,,$$

Ricci flow equations

Hubble tension: a macroscopic QG effect?

$$\langle \lambda_2^k(s) \rangle = \exp\left[\left(i(-2\varepsilon) + \frac{\Lambda_0}{2}\right) s\right] \langle \lambda_2^k(0) \rangle$$

Thermal time oriented as the proper time implies mild increase of Λ

[1] M. Lulli, A. Marciano, X. Shan, arXiv:2112.01490 (2021)

[2] M. Lulli, A. Marciano, L. Visinelli, in preparation

Cosmological measurements $67.4 \pm 1.4 \quad (\mathrm{km/s})/\mathrm{Mpc}$

Astronomical measurements $74.03 \pm 1.42 \quad (\mathrm{km/s})/\mathrm{Mpc}$

DP model from SRF: main message

DP quantum collapse master equation as a
the non-relativistic limit of a QG model based on
stochastic quantisation and the Ricci flow

Running of the lapse function in a stochastic thermal time

Strategy: describe a discrete system of masses undergoing
gravitational interaction

First principle discussion and RG flow induced by a stochastic gradient Ricci flow

M. Lulli, A. Marciano & X. Shan, arXiv:2112.01490v2

Derivation of the DP quantum collapse master equation from the stochastic Ricci
flow and discussion of the parameter space

M. Lulli, A. Marciano & K. Piscicchia, arXiv:2307.10136

Stochastic Ricci flow & quantum collapse I

Consider a discrete system of gravitationally interacting bodies

Ideal gas approximation for the energy-momentum tensor

Use the non-relativistic semi-classical limit of

$$\frac{\partial N}{\partial s} = -\frac{N}{2}\left[\imath\left(\frac{\mathcal{H}}{\sqrt{h}} + R\right) + \frac{\Lambda e^{-\imath\gamma}}{1+n_g} + e^{\imath\frac{\gamma}{2}}\sqrt{\frac{4\Lambda}{1+n_g}}\tilde{\eta}\right]$$

$$d|\psi_t\rangle = \left[-\frac{i}{\hbar}H\,dt + \sqrt{\frac{G}{\hbar}}\int d\mathbf{x}\,\left(\mathcal{M}(\mathbf{x}) - \langle\mathcal{M}(\mathbf{x})\rangle_t\right)dW(\mathbf{x},t) - \frac{G}{2\hbar}\int d\mathbf{x}\int d\mathbf{y}\,\frac{\left(\mathcal{M}(\mathbf{x}) - \langle\mathcal{M}(\mathbf{x})\rangle_t\right)\left(\mathcal{M}(\mathbf{y}) - \langle\mathcal{M}(\mathbf{y})\rangle_t\right)}{|\mathbf{x}-\mathbf{y}|}\,dt\right]|\psi_t\rangle,$$

M. Lulli, A. Marciano & K. Piscicchia, arXiv:2307.10136

Stochastic Ricci flow & quantum collapse II

DP quantum collapse master equation as out-of-equilibrium relaxation described through the stochastic Ricci flow

Similarity: both realizations have multiplicative noise

New task: explain the role of the lapse, recalling that N=1+V+…

Tools: use the Ito calculus to account for the variation of the lapse

M. Lulli, A. Marciano & K. Piscicchia, arXiv:2307.10136

Stochastic Ricci flow & quantum collapse III

Langevin equation for the shift

$$\frac{\partial N}{\partial s} = -2\imath \frac{\partial N}{\partial g_{\mu\nu}} \left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu} \right] + \alpha e^{\imath\gamma} g_{\mu\nu} g_{\alpha\beta} \frac{\partial^2 N}{\partial g_{\mu\nu} \partial g_{\alpha\beta}} + e^{\imath\gamma/2} \sqrt{2\alpha} \frac{\partial N}{\partial g_{\mu\nu}} g_{\mu\nu} \tilde{\eta}$$

$$\frac{1}{2}\alpha \left(1 + n_g\right) = \Lambda, \qquad \alpha = \frac{2\Lambda}{1 + n_g},$$

with matter Ricci target

$$R^{\mathrm{T}}_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

$$T^{\mu}_{\nu} = \mathrm{diag}\left\{ -\rho, \rho\sigma T, \rho\sigma T, \rho\sigma T \right\}$$

M. Lulli, A. Marciano & K. Piscicchia, arXiv:2307.10136

Stochastic Ricci flow & quantum collapse IV

Using the Ito calculus

$$\mathrm{d}N = \left\{ -\frac{\imath}{2N}\left[\nabla^2 N^2 + \kappa N^2 \rho\left(3\sigma T + 1\right)\right] - \frac{\alpha}{4} e^{\imath\gamma} N \right\} \mathrm{d}s + e^{\imath\gamma/2}\sqrt{\frac{\alpha}{2}} N \mathrm{d}W$$

$$\mathrm{d}N = a\mathrm{d}s + b\mathrm{d}W$$

$$df = \int \mathrm{d}^4x\, m_x \left[\frac{\delta f}{\delta N_x} a_x + \frac{1}{2}\int \mathrm{d}^4y\, n_y\, b_x b_y\, k_\varepsilon\left(x,y\right) \frac{\delta}{\delta N_x}\frac{\delta}{\delta N_y} f\right] \mathrm{d}s + \int \mathrm{d}^4x\, m_x \frac{\delta f}{\delta N_x} b_x \mathrm{d}W_x$$

for a functional f=f[N]

$$\langle \frac{\mathrm{d}f}{\mathrm{d}s} \rangle = -\int \mathrm{d}^4x\, \mathrm{d}N_x\, m_x\, f \frac{\delta}{\delta N_x}\left[\left(a_x p\left[N_x\right]\right) - \frac{1}{2}\int \mathrm{d}^4y \mathrm{d}^4z\, n_y \frac{\delta}{\delta N_y}\left(b_x b_z p\left[N_x\right]\right) k_\varepsilon\left(x,y\right) k_\varepsilon\left(y,z\right)\right]$$

M. Lulli, A. Marciano & K. Piscicchia, arXiv:2307.10136

Stochastic Ricci flow & quantum collapse V

Stationary Fokker-Planck

$$a_x p\left[N_x\right] - \frac{1}{2}\int \mathrm{d}^4 y \mathrm{d}^4 z\, n_y \,\frac{\delta}{\delta N_y}\left(b_x b_z p\left[N_x\right]\right)\, k_\varepsilon\left(x,y\right) k_{\varepsilon'}\left(y,z\right) = 0$$

$$p\left[N_x\right] = B\exp\left\{\int^{N_x}\prod_y \mathrm{d}N_y \frac{4}{e^{\imath\gamma}\alpha N_y^2}\left[a_y - \frac{e^{\imath\gamma}\alpha}{2}N_y\right]\right\} = B\exp\left[\frac{\imath}{\hbar}S_0\right]$$

M. Lulli, A. Marciano & K. Piscicchia, arXiv:2307.10136

Stochastic Ricci flow & quantum collapse VI

Stochastic Ricci-flow equation for a wave functional depending on N

$$\mathrm{d}\Psi = \left[\int \mathrm{d}^4x\, m_x\, \frac{\imath}{\hbar}\frac{\delta S_0}{\delta N_x} a_x \mathrm{d}s + \int \mathrm{d}^4x\, m_x\, \frac{\imath}{\hbar}\frac{\delta S_0}{\delta N_x} b_x \mathrm{d}W_x\right]\Psi$$
$$+\frac{1}{2}\int \mathrm{d}^4x\, m_x \int \mathrm{d}^4y\, n_y\, b_x b_y\, k_\varepsilon\left(x,y\right)\frac{\delta}{\delta N_x}\left(\Psi\frac{\imath}{\hbar}\frac{\delta S_0}{\delta N_y}\right)\mathrm{d}s$$

WKB ansatz for the wave functional

$$\Psi = \exp\left[\frac{\imath}{\hbar}S_0\right] = \exp\left[\int^{N_x}\prod_y \mathrm{d}N_y \frac{4}{e^{\imath\gamma}\alpha N_y^2}\left[a_y - \frac{e^{\imath\gamma}\alpha}{2}N_y\right]\right], \qquad b_x = e^{\imath\gamma/2}\sqrt{\frac{\alpha}{2}}N_x$$

$$a_x = -\frac{\imath}{2}\left[2\left(N_x\left(\nabla\log N_x\right)^2 + \nabla^2 N_x\right) + \kappa N_x \rho_x\left(3\sigma T + 1\right)\right] - \frac{\alpha}{4}e^{\imath\gamma}N_x$$

M. Lulli, A. Marciano & K. Piscicchia, arXiv:2307.10136

Stochastic Ricci flow & quantum collapse VII

Non-relativistic weak-gravity limit

$$d|\psi_t\rangle = \Bigg[-\frac{i}{\hbar} H\, dt + \sqrt{\frac{G}{\hbar}} \int d\mathbf{x} \left(\mathcal{M}(\mathbf{x}) - \langle \mathcal{M}(\mathbf{x})\rangle_t\right) dW(\mathbf{x}, t) - \frac{G}{2\hbar} \int d\mathbf{x} \int d\mathbf{y} \frac{\left(\mathcal{M}(\mathbf{x}) - \langle \mathcal{M}(\mathbf{x})\rangle_t\right)\left(\mathcal{M}(\mathbf{y}) - \langle \mathcal{M}(\mathbf{y})\rangle_t\right)}{|\mathbf{x} - \mathbf{y}|} dt \Bigg] |\psi_t\rangle,$$

$$\Psi \int \mathrm{d}^4 x\, m_x \left[2\imath\kappa\rho_x \left(2 + \sigma T\right) + \frac{\alpha}{4} e^{\imath\gamma}\right] \mathrm{d}s$$

$$\Psi e^{-\imath\gamma/2} \sqrt{\kappa} \sqrt{\frac{\kappa}{2\alpha}} \int \mathrm{d}^4 x\, m_x \left[-2\imath\rho_x \left(3\sigma T + 2\right) - 3 e^{\imath\gamma} \frac{\alpha}{\kappa}\right] \mathrm{d}W_x$$

M. Lulli, A. Marciano & K. Piscicchia, arXiv:2307.10136

Stochastic Ricci flow & quantum collapse VIII

Non-relativistic weak-gravity limit

$$d|\psi_t\rangle = \Bigg[-\frac{i}{\hbar}H\,dt + \sqrt{\frac{G}{\hbar}}\int d\mathbf{x}\,\big(\mathcal{M}(\mathbf{x}) - \langle\mathcal{M}(\mathbf{x})\rangle_t\big)\,dW(\mathbf{x},t)$$
$$- \frac{G}{2\hbar}\int d\mathbf{x}\int d\mathbf{y}\,\frac{\big(\mathcal{M}(\mathbf{x}) - \langle\mathcal{M}(\mathbf{x})\rangle_t\big)\big(\mathcal{M}(\mathbf{y}) - \langle\mathcal{M}(\mathbf{y})\rangle_t\big)}{|\mathbf{x}-\mathbf{y}|}\,dt\Bigg]|\psi_t\rangle,$$

$$\frac{1}{2}\int \mathrm{d}^4x\, m_x \int \mathrm{d}^4y\, n_y\, \frac{1}{2}\, k_\varepsilon\,(x,y)\, \frac{16}{e^{\imath\gamma}\alpha}\mathrm{d}s\Psi$$
$$\times\left[-\frac{\imath}{2}\left[\kappa\rho_y + \kappa\rho_y\,(3\sigma T + 1)\right] - \frac{3}{4}\alpha e^{\imath\gamma}\right]$$
$$\times\left[-\frac{\imath}{2}\left[\kappa\rho_x + \kappa\rho_x\,(3\sigma T + 1)\right] - \frac{3}{4}\alpha e^{\imath\gamma}\right]$$

M. Lulli, A. Marciano & K. Piscicchia, arXiv:2307.10136

Stochastic Ricci flow & quantum collapse IX

Non-unitary but trace-preserving and positive

$$\begin{aligned}
\mathrm{d}\Psi = & \left\{\int \mathrm{d}^4x\, m_x \left[2\imath\kappa\rho_x\,(2+3\sigma T) + \frac{\alpha}{4}e^{\imath\gamma}\right]\mathrm{d}s\right\}\Psi \\
& - \left\{\frac{e^{-\imath\frac{\gamma}{2}}}{\sqrt{2\alpha}}\int \mathrm{d}^4x\, m_x[2\imath\kappa\rho_x\,(3\sigma T+2)+3\alpha e^{\imath\gamma}]\mathrm{d}W_x\right\}\Psi \\
& + \left\{\frac{1}{2}\int \mathrm{d}^4x\, m_x \int \mathrm{d}^4y\, n_y\, \frac{1}{2}\, k_\varepsilon\,(x,y)\, \frac{1}{e^{\imath\gamma}\alpha}\mathrm{d}s\, \times \right. \\
& \left. [2\imath\kappa\rho_x\,(2+3\sigma T)+3\alpha e^{\imath\gamma}]\,[2\imath\kappa\rho_y\,(2+3\sigma T)+3\alpha e^{\imath\gamma}]\right\}\Psi
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\rho_\Psi \equiv & \left\langle \frac{\mathrm{d}}{\mathrm{d}s}\rho_\Psi\right\rangle_{\tilde{\eta}} = \imath\int d^4x\, m_x\,[\mathcal{H}_x,\rho_\Psi] \\
& + \int d^4x\, m_x \int d^4y\, n_y\, \widetilde{k}_\varepsilon\,(x,y)\;\mathcal{L}_x\rho_\Psi\mathcal{L}_y^\dagger \\
& - \frac{1}{2}\int d^4x\, m_x \int d^4y\, n_y\, \widetilde{k}_\varepsilon\,(x,y)\;\{\mathcal{L}_x^\dagger\mathcal{L}_y,\rho_\Psi\}
\end{aligned}$$

Generalization of GKLS Equation

M. Lulli, A. Marciano & K. Piscicchia, arXiv:2307.10136

Stochastic Ricci Flow & quantum collapse X

The DP master equation is recovered, modulo a judicious choice of the regulator functions/kernels

Same terms of the DP master equation recovered, but propagation of the gravitationally interacting bodies in the stochastic quantum gravitational foam now induces an extra cosmological energy density term

Address phenomenological relevance of the new parameters, the temperature of the system and the cosmological constant, related to the amplitude of the stochastic noise in quantum gravity

Developing a geometric intuition on the RG flow

$$S = \alpha' \int_{\mathcal{M}} d^2\sigma \sqrt{h} h^{ab}(\sigma) g_{ij}(X) \frac{\partial X^i}{\partial \sigma^a} \frac{\partial X^j}{\partial \sigma^b}$$

The case of the non-linear sigma models

$$\frac{\partial g_{ij}}{\partial \lambda} = -\alpha' R_{ij} - \frac{\alpha'^2}{2} R_{iklm} R_j{}^{klm} + \cdots$$

$$\frac{\partial}{\partial \lambda} g_{\mu\nu} = -2R_{\mu\nu}$$

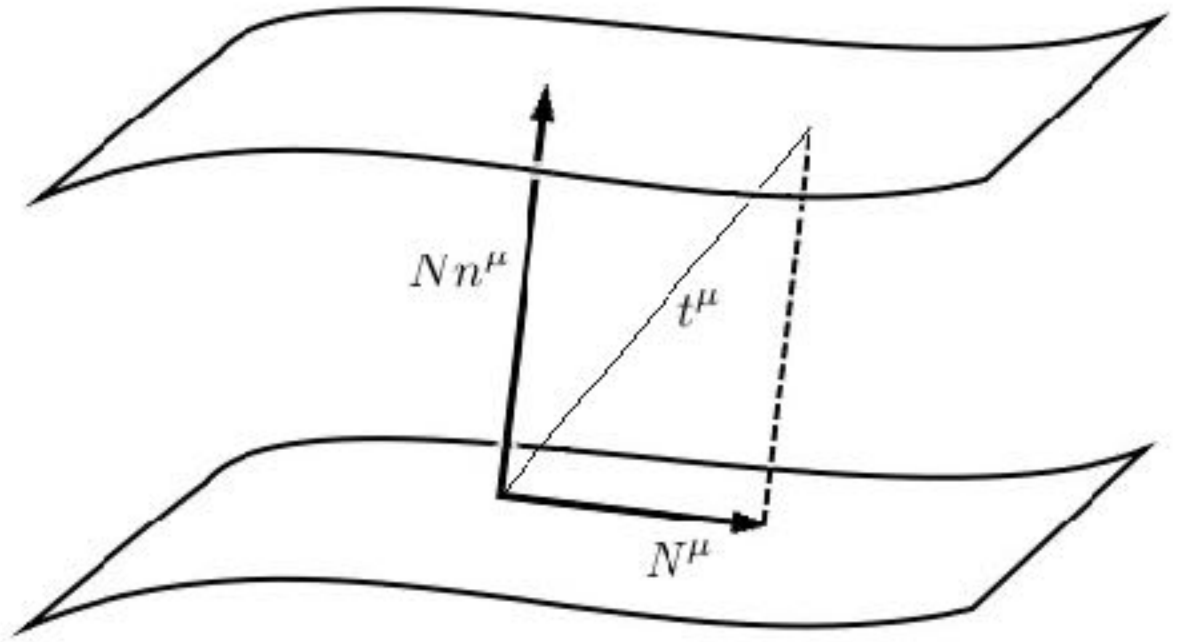
Hamilton

Stochastic Ricci Flow and geometric phase

$$R^T_{\mu\nu} = \frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right]$$

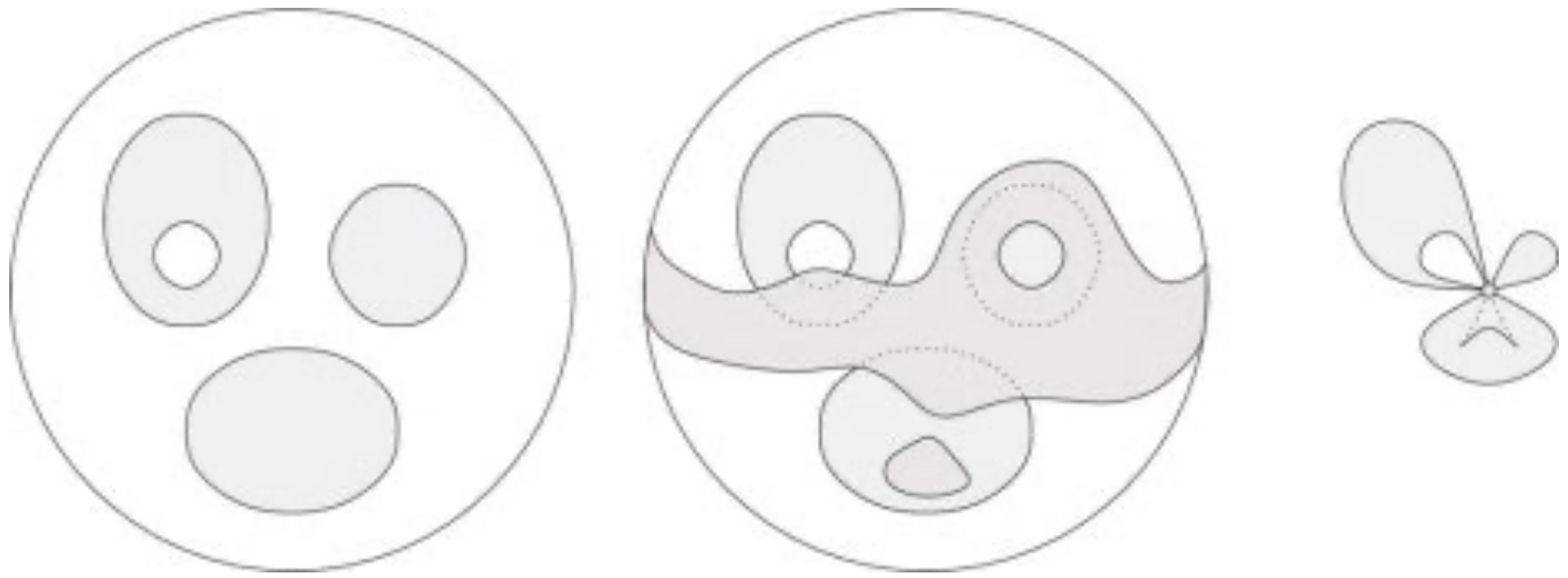
$$\begin{aligned}\frac{\partial}{\partial s} g_{\mu\nu} &= -\,2\left[R_{\mu\nu} - R^T_{\mu\nu}\right] \\ &= -\,2\left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{8\pi G}{c^4} T_{\mu\nu}\right] - g_{\mu\nu}\left(R - T\right)\end{aligned}$$

Changes of topologies through defects are induced by singularities in the Ricci flow



Topological features of vacua

The Ricci flow allows for topology changes from equilibrium



Geometrical interpretation of ground-states

Topological charges label ground-states structures and are related to the characterization of the matter content —e.g. Atiyah-Singer Index theorem

Stochastic dynamics and the Ricci RG flow

Holography in 4+1D and dynamics in the stochastic time parameter

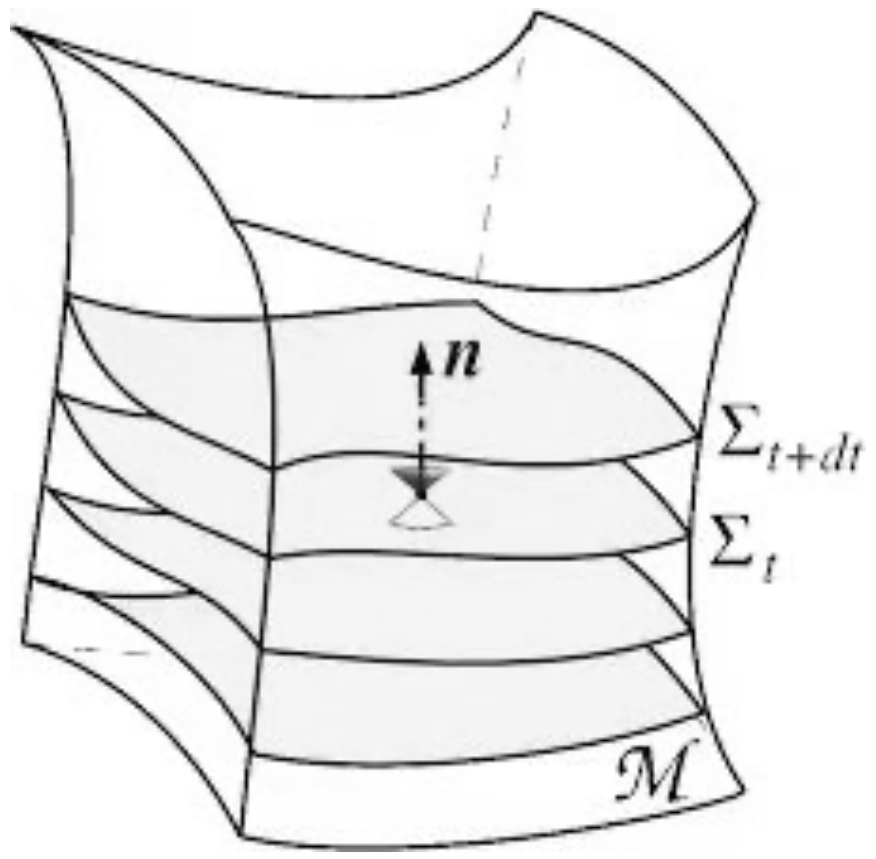
The Ricci flow amounts to a conformal transformation of the 3D-hypersurfaces

The Langevin equation and the probability distributions for manifolds with Lorentzian signature and complex structure

Manifolds with Lorentzian signature enable to fully take into account dynamics of out-of-equilibrium systems and relaxations features

Vortices and turbulences (both for fluids and gauge fields) can be addressed as a by-product of the Ricci flow driven relaxation processes

Out-of-equilibrium TQNNs



$$P = \text{“}\prod_{x\in\ \Sigma} \delta(\hat{F}(A))\text{”} = \int D[N]\ \exp(i\int_{\Sigma} \mathrm{Tr}[N\hat{F}(A)])$$

↓

$$P = \text{“}\prod_{x\in\Sigma} \delta(\hat{F}(\omega) + \Lambda\hat{e}\wedge\hat{e})\text{”}$$
$$= \int \mathcal{D}[N] \exp i \int_{\Sigma} \mathrm{Tr}[N(F(\omega) + \Lambda\hat{e}\wedge\hat{e})]$$

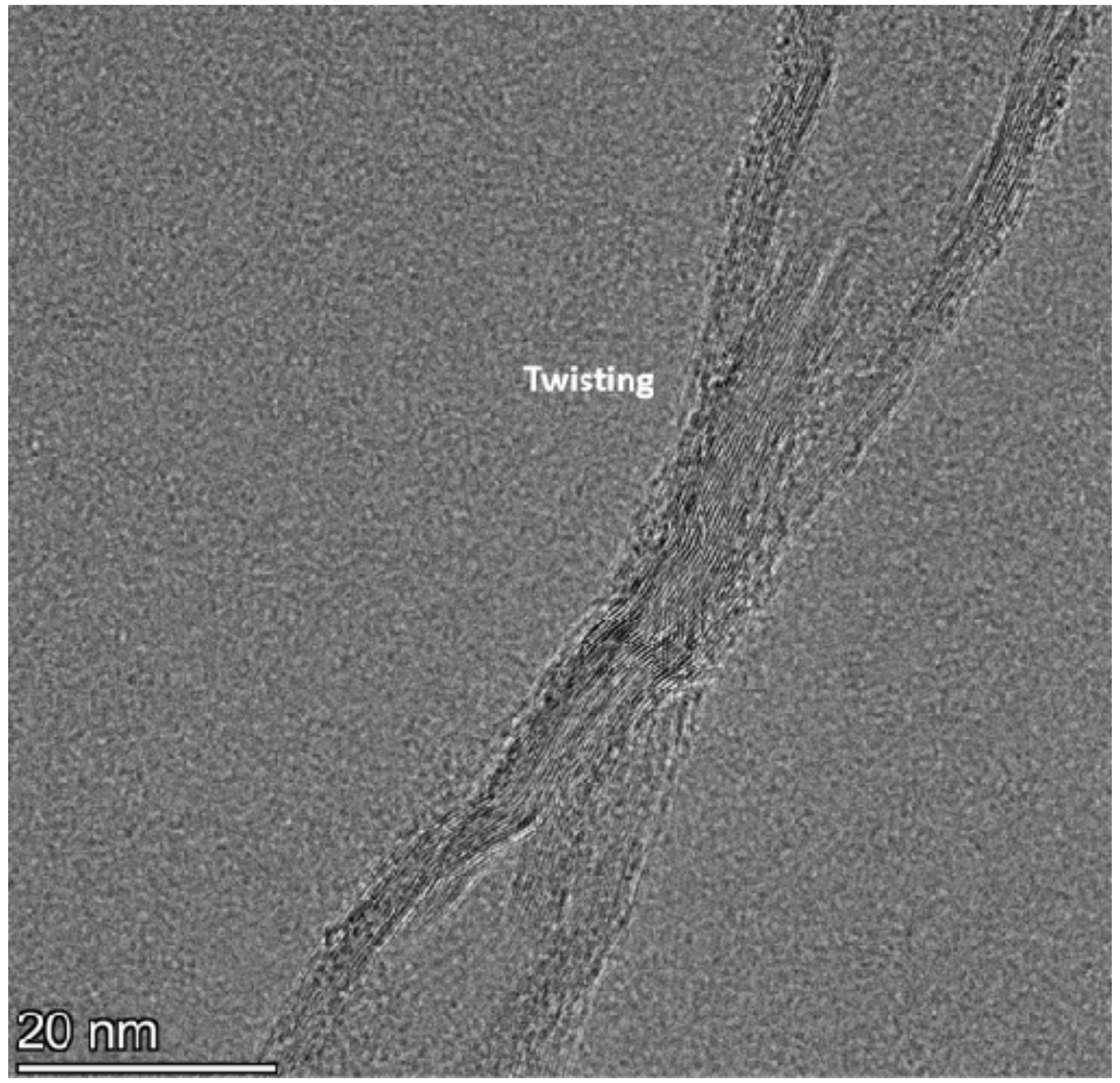
[1] M. Lulli, A. Marciano and E. Zappala, in preparation

Ricci solitons and topology changes in nanotubes

Twisting and quantum groups in TQFT

$$\begin{aligned}\mathcal{S}_{\mathrm{AdS}} &= \int \langle B \wedge F(A) - \ell^{-2} B \wedge B \wedge B \rangle \\ &= \mathcal{S}_{\mathrm{CS}}(A^{+}) - \mathcal{S}_{\mathrm{CS}}(A^{-}) , \\ \mathcal{S}_{\mathrm{CS}}(A) &= \int A \wedge dA + \frac{2}{3} A \wedge A \wedge A ,\end{aligned}$$

Topology changes and Ricci solitons as dS/ adS phases



Diffusion Fick's equation for dopant distribution

$$\frac{dc}{dt} = D\Delta c$$

[1] A. Addazi, F. Boi and A. Marciano, in preparation

Outlook and conclusions

Stochastic quantisation and out-of-equilibrium breakdown of symmetries

Geometric RG flow complemented with stochastic (multiplicative) noise

Consequences of the SRF: astrophysics, cosmology, particle physics and foundational aspects

DP model from the Stochastic Ricci flow in non-relativistic limit and a following WKB approximation

Analog gravity applications to out-of-equilibrium systems: symmetry braking & topology changes

谢谢

Thank you!

Grazie!

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Quantization Methods

All methods start from a classical description

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**

Difficulties in handling Symmetries

"Gauge" symmetries always pertain the equations of motion [1]

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0, \qquad \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j$$

$$C^{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial p^j}{\partial \dot{q}_i}$$

C^{ij} **not invertible** → **"gauge" symmetries**

Hamiltonian perspective: Constraints (EOM) generate symmetries (first-class)

In **EM** one has the **Gauss constraint** $F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu, \qquad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$

$$\chi = \partial_k \Pi^k = \partial_k E^k = 0, \qquad \delta A^i = \varepsilon\left[\partial_k \Pi^k, A^i\right] = \partial^i \varepsilon$$

[1]: Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

Quantization Methods - Continued

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**
3. **Stochastic Quantization**
   A. Introduce a "stochastic time" variable s and noise - Langevin dynamics
   B. Expectation values: (i) with respect to the noise, (ii) with respect to $P(\phi, s)$
   C. The associated Fokker-Planck ensures the correct "equilibrium" limit of $P(\phi, s)$

$$\mathrm{d}\phi_{\mathrm{A}}(x,s) = -\frac{\delta S[\phi]}{\delta \phi_{\mathrm{A}}}\mathrm{d}s + \eta_{\mathrm{A}}(x,s)\,\mathrm{d}s = -\frac{\delta S[\phi]}{\delta \phi_{\mathrm{A}}}\mathrm{d}s + \mathrm{d}W_{\mathrm{A}}(x,s)$$

$$\eta_{\mathrm{A}}(x,s)\,\mathrm{d}s = \mathrm{d}W_{\mathrm{A}}(x,s)\,,$$ **Wiener process (RW)**

$$\mathrm{d}W_{\mathrm{A}}(x,s)\,\mathrm{d}W_{\mathrm{B}}(x,s) = \delta_{\mathrm{AB}}\left[\mathrm{d}W_{\mathrm{A}}(x,s)\right]^2$$ **(independence)**

Variable transformation: first order in $\mathrm{d}s$

$$\mathrm{d}F = F(\phi_{\mathrm{A}} + \mathrm{d}\phi_{\mathrm{A}}) - F(\phi_{\mathrm{A}}) \simeq \frac{\partial F}{\partial \phi_{\mathrm{A}}}\mathrm{d}\phi_{\mathrm{A}} + \frac{1}{2}\frac{\partial^2 F}{\partial \phi_{\mathrm{A}}\partial \phi_{\mathrm{B}}}\mathrm{d}\phi_{\mathrm{A}}\mathrm{d}\phi_{\mathrm{B}}$$

$$\simeq \frac{\partial F}{\partial \phi_{\mathrm{A}}}\mathrm{d}\phi_{\mathrm{A}} + \frac{1}{2}\frac{\partial^2 F}{\partial \phi_{\mathrm{A}}\partial \phi_{\mathrm{B}}}\delta_{\mathrm{AB}}\left[\mathrm{d}W_{\mathrm{A}}(x,s)\right]^2$$

$$\mathrm{d}W_{\mathrm{A}}(x,s)\,\mathrm{d}s \to 0,$$ **higher-order**

$$\left[\mathrm{d}W_{\mathrm{A}}(x,s)\right]^2 = \mathrm{d}s$$ **variance**

Quantization Methods - Continued

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**
3. **Stochastic Quantization**
   A. Introduce a “stochastic time” variable s and noise - Langevin dynamics
   B. Expectation values: (i) with respect to the noise, (ii) with respect to $P(\phi, s)$
   C. The associated Fokker-Planck ensures the correct “equilibrium” limit of $P(\phi, s)$

$$\frac{\partial F}{\partial s} = -\frac{\partial F}{\partial \phi_A}\frac{\delta S[\phi]}{\delta \phi_A} + \frac{1}{2}\frac{\partial^2 F}{\partial \phi_A^2} + \frac{\partial F}{\partial \phi_A}\eta_A$$

Stochastic Calculus

$$\begin{aligned}
\langle\frac{\partial F}{\partial s}\rangle &= \langle -\frac{\partial F}{\partial \phi_A}\frac{\delta S[\phi]}{\delta \phi_A} + \frac{1}{2}\frac{\partial^2 F}{\partial \phi_A^2} + \frac{\partial F}{\partial \phi_A}\eta_A\rangle \\
\langle\frac{\partial F}{\partial s}\rangle &= \int [D\phi_A] F(\phi)\frac{\partial P(\phi, s)}{\partial s} = \langle -\frac{\partial F}{\partial \phi_A}\frac{\delta S[\phi]}{\delta \phi_A} + \frac{1}{2}\frac{\partial^2 F}{\partial \phi_A^2}\rangle \\
&= \int [D\phi_A]\left[-\frac{\partial F}{\partial \phi_A}\frac{\delta S[\phi]}{\delta \phi_A} + \frac{1}{2}\frac{\partial^2 F}{\partial \phi_A^2}\right] P(\phi, s) \\
&= \int [D\phi_A] F(\phi)\left\{\frac{\partial}{\partial \phi_A}\left[P(\phi, s)\frac{\delta S[\phi]}{\delta \phi_A}\right] + \frac{1}{2}\frac{\partial^2 P(\phi, s)}{\partial \phi_A^2}\right\}
\end{aligned}$$

$$\langle\frac{\partial F}{\partial \phi_A}\eta_A\rangle = \langle\frac{\partial F}{\partial \phi_A}\rangle\langle\eta_A\rangle = 0$$

Non-anticipating Function

Stochastic Quantization - Examples

Abelian gauge field $$\frac{\partial}{\partial s} A_\mu = \partial^2 A_\mu - \partial_\mu \partial_\nu A_\nu + \eta_\mu$$

Gauge symmetry broken by boundary condition (restored at equilibrium) $$A_\nu(x,s)|_{s=0} = 0$$

Solution again a Gaussian random variable

$$A_\mu(k,s) = \int_0^s ds' G_{\mu\nu}(k, s-s')\eta_\nu(k,s')$$

Retarded Green function

$$G_{\mu\nu}(k, s-s') = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) e^{-k^2(s-s')} + \frac{k_\mu k_\nu}{k^2}$$

Correlation function

$$D_{\mu\nu}(k,s,s') = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \frac{e^{-k^2|s-s'|} - e^{-k^2|s+s'|}}{k^2} + \frac{k_\mu k_\nu}{k^2} 2\min(s,s')$$

$$D_{\mu\nu}(k,s,s) = \underbrace{(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \frac{1}{k^2}}_{\text{Feynman propagator Landau gauge}} + \underbrace{2s \frac{k_\mu k_\nu}{k^2}}_{\text{longitudinal term secular divergence}}$$

Decompose in transverse A_μ^{T} and longitudinal part

$$A_\mu(x,s) = A_\mu^{\mathrm{T}}(x,s) + \partial_\mu \alpha(x,s) \qquad \partial_\mu A_\mu^{\mathrm{T}}(x,s) = 0$$

Equilibrium Limit $$\langle A_\mu^{\mathrm{T}}(k,s) A_\mu^{\mathrm{T}}(-k,s)\rangle = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}$$

Slower evolution: random walk $$\langle \alpha(k,s)\alpha(-k,s)\rangle = \frac{2}{k^2} s$$

Stochastic Quantization - Examples

Massive scalar field with potential

$$S[\phi] = \int \mathrm{d}^4 x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{3}g\phi^3\right]$$

The stochastic equations

$$\frac{\partial \phi}{\partial s} = \partial^2 \phi - m^2 \phi + g\phi^2 + \eta$$

$$\langle \eta(x,s)\eta(x',s')\rangle = 2\delta(s-s')\,\delta(x-x')$$

Non-interacting case $g = 0$

$$\frac{\partial}{\partial s} G(x,s) - (\partial^2 - m^2)\, G(x,s) = \delta(x)\,\delta(s)$$

$$G(x,s) = 0, \qquad (\text{for } s < 0)$$

$$G(x,s) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \exp\left[-s\left(k^2+m^2\right) + \imath \mathbf{k}\cdot\mathbf{x}\right] \theta(s)$$

Associated Green function - **No FT on stoch. time**

$$\phi(x,s) = \int_0^s \mathrm{d}\tau \int \mathrm{d}^4 y\, G(x-y, s-\tau)\, \eta(y,\tau)$$

The field is a Gaussian variable

correlation function

$$\langle \phi(x,s)\,\phi(x',s')\rangle \equiv D(x-x';t,t')$$

$$= \langle \int_0^\infty \mathrm{d}\tau\, \mathrm{d}\tau' \int \mathrm{d}^4 y\, \mathrm{d}^4 y'\, G(x-y, s-\tau)\, G(x'-y', s'-\tau')\, \eta(y,\tau)\, \eta(y',\tau')\rangle$$

$$= 2\int_0^\infty \mathrm{d}\tau \int \mathrm{d}^4 y\, G(x-y, s-\tau)\, G(x'-y, s'-\tau)$$

Non-Eq. FT $D(k;t,t') = \dfrac{\exp-\left(k^2+m^2\right)(t-t')}{k^2+m^2}\left[1 - \exp -2\left(k^2+m^2\right)t'\right]$

Equilibrium limit $\lim_{t\to\infty} D(k;t,t) = \dfrac{1}{k^2+m^2}$

Stochastic Quantization - Examples

Massive scalar field with potential

$$S[\phi] = \int \mathrm{d}^4 x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} g \phi^3\right]$$

The stochastic equations

$$\frac{\partial \phi}{\partial s} = \partial^2 \phi - m^2 \phi + g \phi^2 + \eta$$

$$\langle \eta(x,s) \eta(x',s') \rangle = 2\delta(s-s')\,\delta(x-x')$$

Non-interacting case $g = 0$

$$\frac{\partial}{\partial s} G(x,s) - (\partial^2 - m^2)\, G(x,s) = \delta(x)\,\delta(s)$$

$$G(x,s) = 0, \qquad (\text{for } s < 0)$$

$$G(x,s) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \exp\left[-s\left(k^2 + m^2\right) + \imath \mathbf{k} \cdot \mathbf{x}\right] \theta(s)$$

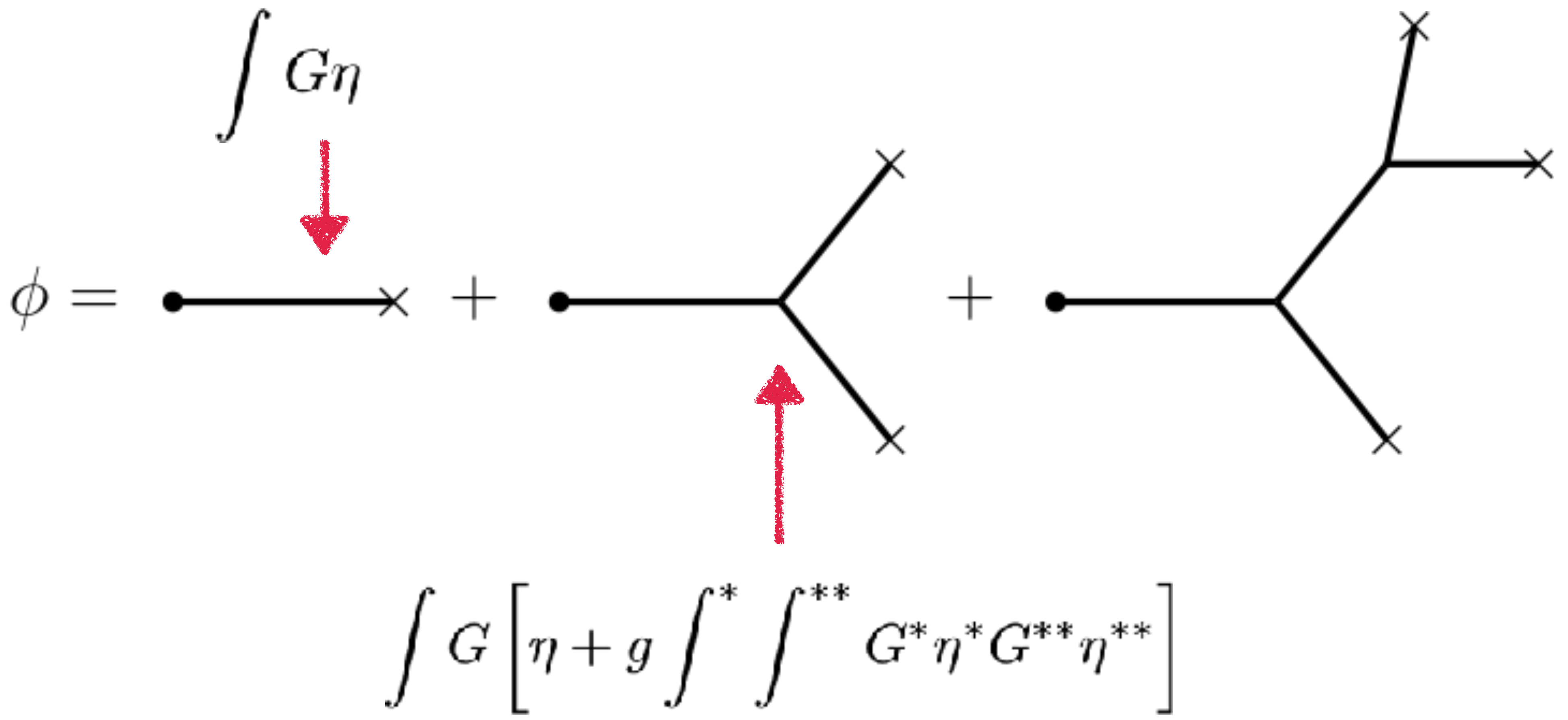
Associated Green function - **No FT on stoch. time**

$$\phi(x,s) = \int_0^s \mathrm{d}\tau \int \mathrm{d}^4 y\, G(x-y, s-\tau)\; \eta(y,\tau)$$

The field is a Gaussian variable

Interacting case $g \neq 0$

$$\phi(x,s) = \int_0^s \mathrm{d}\tau \int \mathrm{d}^4 y\, G(x-y, s-\tau)\; \left[\eta(y,\tau) + g\phi^2(y,\tau)\right]$$



Stochastic time and scale transformation

- Let us look for a link between the proper time and the stochastic time
- Assume stochastic-time dependence in space-time coordinates

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\partial}{\partial s} + \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\nabla_\mu$$

- Assume that the total derivatives are proportional to the normal to the hyper surface

$$\ell(s)\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = n^\mu = -g^{\mu\alpha}N\partial_\alpha t$$

- The normalisation reads

$$g_{\mu\nu}n^\mu n^\nu = g_{\mu\nu}\ell^2(s)\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = -g_{\mu\nu}\ell(s)\frac{\mathrm{d}x^\nu}{\mathrm{d}s}g^{\mu\alpha}N\partial_\alpha t \quad \rightarrow \quad \varepsilon(s)\frac{\delta s}{\ell(s)} = -N\delta t = -\delta\tau$$

$$\boxed{\frac{\mathrm{d}\tau}{\mathrm{d}s} = -\frac{\varepsilon(s)}{\ell(s)}} \quad \longrightarrow \quad \boxed{\delta s = \frac{\mathrm{d}s}{\mathrm{d}\tau}\delta\tau = \ell(s)\sqrt{\frac{1}{c^2\varepsilon^2(s)}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}}$$

Stochastic time flow - Effective scale transformation
Stochastic dynamics related to RG group transformation

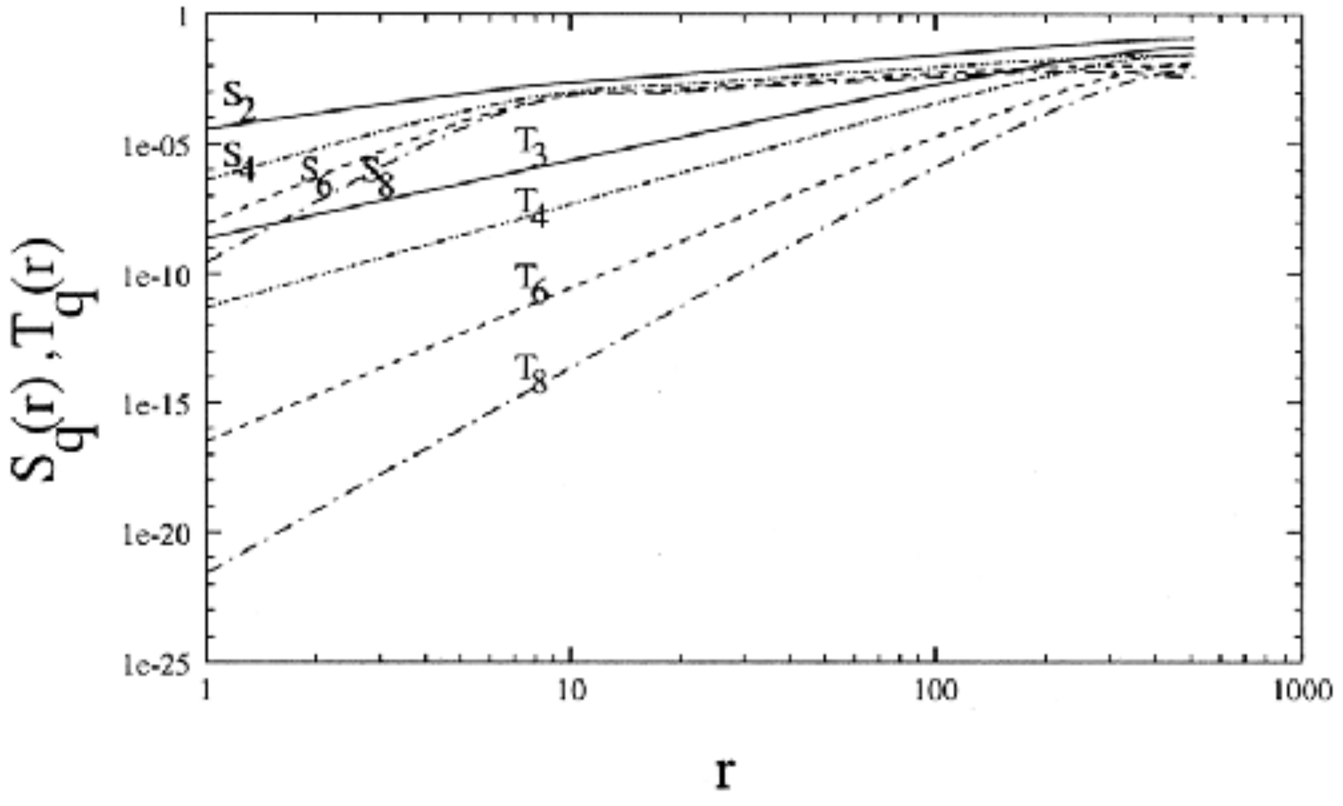
Black Holes and KPZ

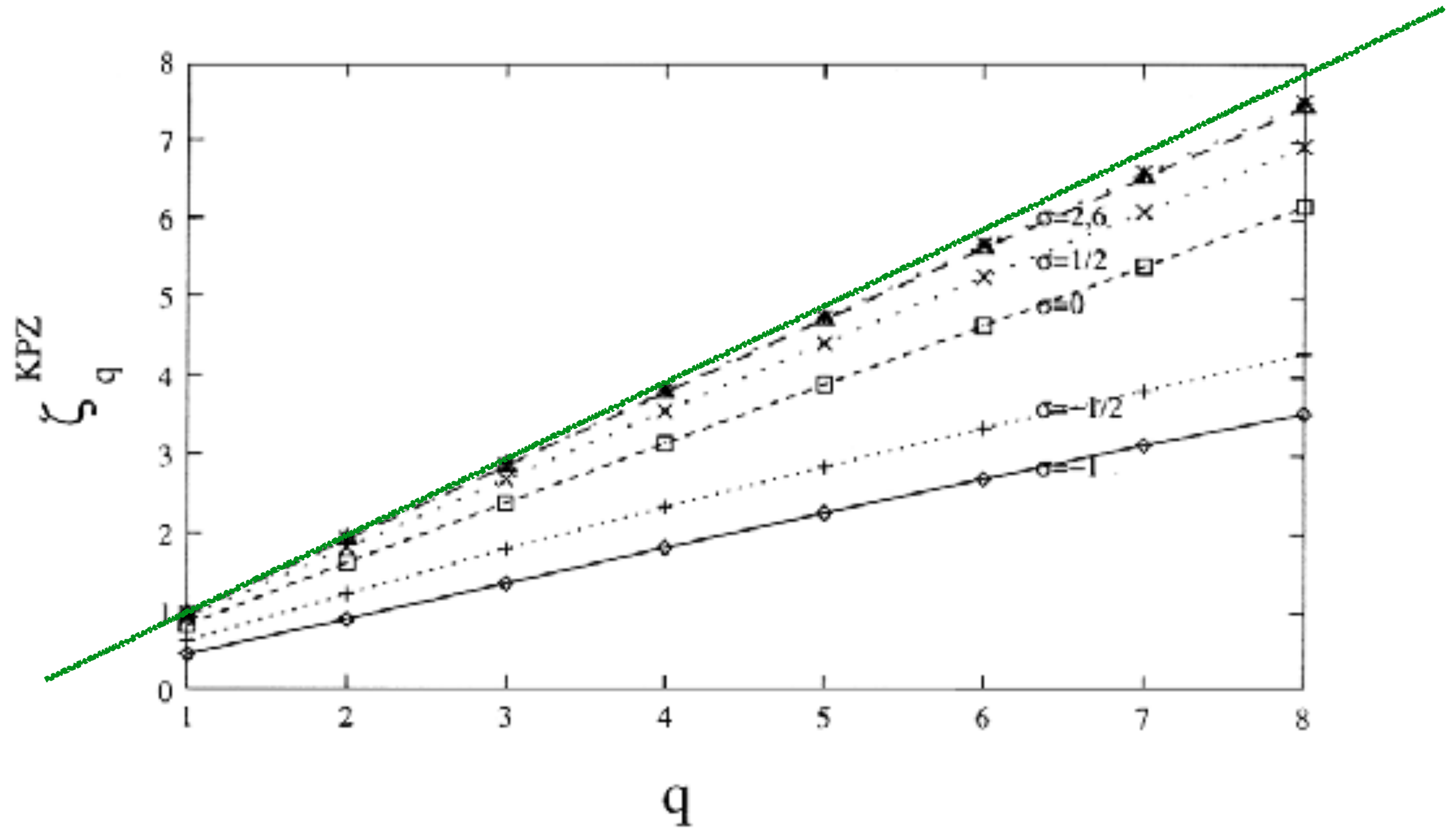
$$\frac{\partial h}{\partial t} = \imath \left[\nu_{\mathrm{KPZ}}\left(r,\alpha\right) \nabla^2 h\left(r,s\right) + \lambda_{\mathrm{KPZ}}\left(r\right) \left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2 \right] + \bar{\eta}$$

Kardar-Parisi-Zhang Eq.
[1] for a spherical surface

$$\nu_{\mathrm{KPZ}}\left(r,\alpha\right) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \qquad \lambda_{\mathrm{KPZ}}\left(r\right) = e^{\nu_0(r)}$$

- What is exactly **intermittency?**
- The KPZ structure function is defined as $T_q\left(r\right) = \langle |\Delta h\left(r\right)|^q \rangle \qquad \Delta h\left(r\right) = h\left(x+r\right) - h\left(x\right)$
- Display power-law scaling $T_q\left(r\right) = \langle |\Delta h\left(r\right)|^q \rangle \propto r^{\zeta_q}$
- For **Gaussian statistics** one would expect **linear scaling of the exponents** $\zeta_q = \chi q$
- **Different probability distribution** at every **scale**





[1: M. Kardar, G. Parisi, Y.C. Zhang, PRL 56, 9, 889–892 (1986)]
2: M. K. Verma, Phys. A 277 (2000) 359-388]

Stochastic dynamics and the Ricci RG flow

Holography in 4+1D and dynamics in the stochastic time parameter

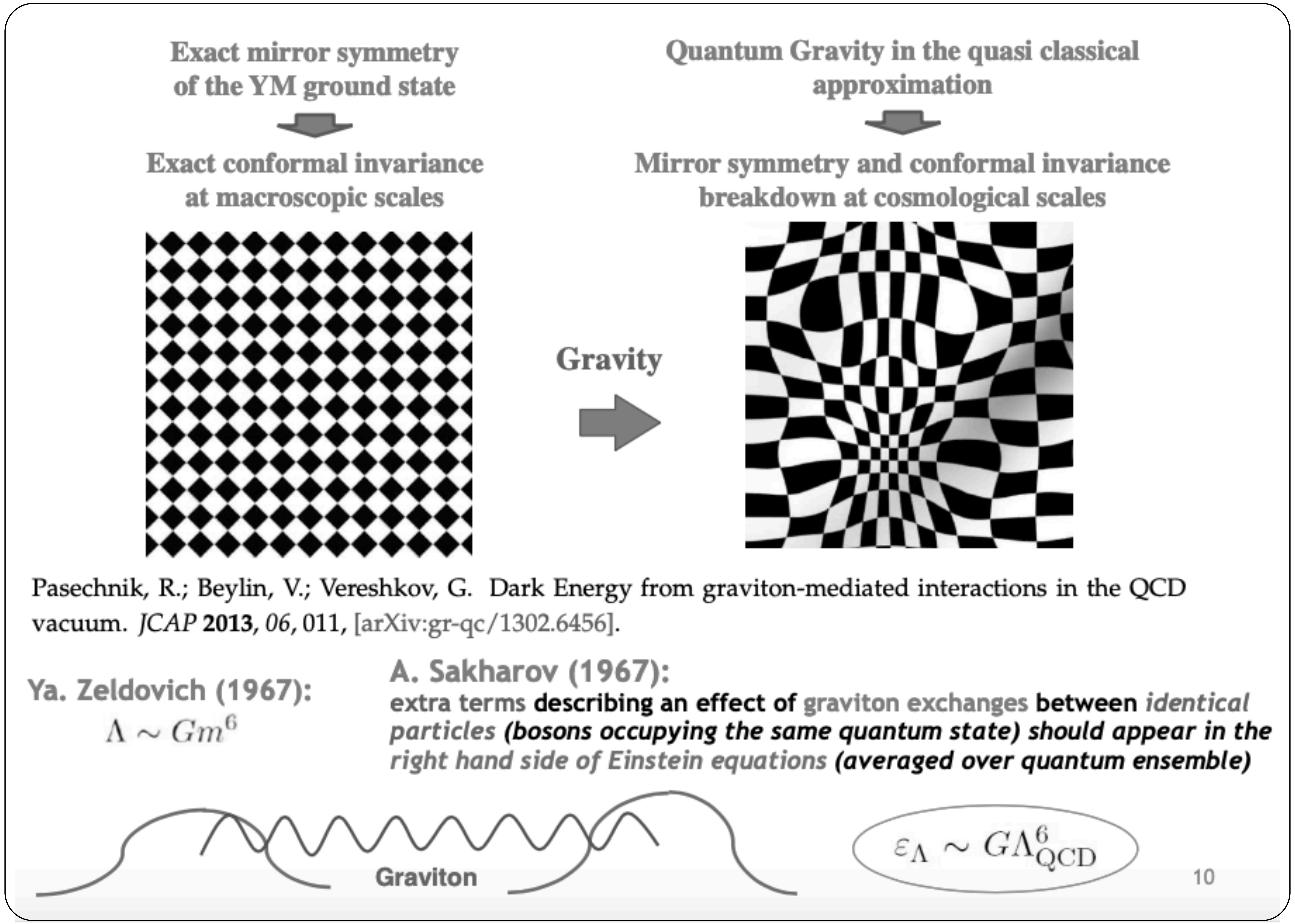
The Ricci flow amounts to a conformal transformation of the 3D-hypersurfaces

The Langevin equation and the probability distributions for manifolds with Lorentzian signature and complex structure

Manifolds with Lorentzian signature enable to fully take into account dynamics of out-of-equilibrium systems and relaxations features

Vortices and turbulences (both for fluids and gauge fields) can be addressed as a by-product of the Ricci flow driven relaxation processes

Gravitational back-reaction to YM



Credit to R. Pasechnik

Stochastic Quantization: trivial example

Field defined only in one point (reminiscent mini-superspace)

$$\hat{H} = \frac{1}{2}P^2 + U(Q)$$

The potential V(Q) increasing fast at infinity; H has a discrete spectrum

$$\hat{H}\psi_n(Q) = \lambda_n \psi_n(Q)\,, \qquad \lambda_{i+1} > \lambda_i$$

Correlation function at equal time

$$\langle Q(s)^k \rangle = \sum_{n=0}^{\infty} c_n e^{-2\lambda_n s} \int dQ\, Q^k\, \psi_n(Q)\psi_0(Q)$$

exp(-V/2) is an eigenvector of H, and its ground-state (function without zeros)

→ at large stochastic time $\langle Q(s)^k \rangle = \frac{\int dQ\, Q^k\, e^{-V(Q)}}{\int dQ\, e^{-V(Q)}} + \mathcal{O}[\exp(-2\lambda_1 s)]$

Stochastic Quantization: less trivial example

Field defined not restricted in one point

$$H = \int d^D x \left\{ \frac{1}{2}\pi(x)^2 + U(\phi(x)) \right\} , \qquad [\pi(x), \phi(y)] = -\imath\delta(x-y)$$

The solution of the Schroedinger functional with $\lambda_0 = 0$

$$\left\{ -\frac{1}{2}\frac{\delta^2}{\delta\phi(x)^2} + U(\phi(x)) \right\} \psi_0(\phi) = \lambda_0 \psi_0(\phi) \quad \longrightarrow \quad \psi_0(\phi) = e^{-\frac{1}{2}\int d^D x\, V(\phi(x))}$$

A more general Langevin equation can be considered

$$\frac{\partial}{\partial s}\phi(x,s) = -\int d^D y\, M(x,y)\, \frac{\delta V}{\delta\phi(y)} + \eta(x,s)$$

still equivalent to the previous one if the matrix M is positive and if

$$\langle \eta(x,s)\eta(y,s') \rangle = 2M(x,y)\delta(s-s')$$

Stochastic Quantization: diagrammatics I

For simplicity consider the case

$$V(\phi) = \int d^D x \left\{ \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} g \phi^3 \right\} , \qquad M(x,y) = \delta(x-y)$$

With associated Langevin equation and gaussian noise

$$\frac{\partial}{\partial s}\phi(x,s) = \partial^2 \phi - m^2 \phi + g\phi^2 + \eta(x,s) , \qquad \langle \eta(x,s)\eta(x',s')\rangle = 2\delta(x-x')\delta(s-s')$$

Solution for g=0 is gaussian stochastic

$$\phi(x,s) = \int_0^s d\sigma \int d^D y \, G(x-y, s-\sigma)\eta(y,\sigma)$$

G(x,s) retarded Green function

$$\frac{\partial}{\partial s} G(x,s) = (\partial^2 - m^2) G(x,s) + \delta(x)\delta(s)$$

$$G(x,s) = 0 \quad \text{for} \quad t < 0$$

$\longrightarrow$

$$G(x,s) = \int \frac{d^D k}{(2\pi)^D} \, e^{-s(k^2+m^2)+\imath k\cdot x} \, \theta(t)$$

Stochastic Quantization: diagrammatics II

Correlation function

$$\langle \phi(x,s)\phi(x',s')\rangle \equiv D(x-x';s,s') = 2\int_0^\infty d\sigma \int d^D y\, G(x-y,s-\sigma)G(x'-y,s'-\sigma)$$

In the momentum space, for s'<s $\quad D(x-x';s,s') = \frac{e^{-(k^2+m^2)(s-s')}}{k^2+m^2}\left(1-e^{-2(k^2+m^2)s'}\right)$

For s' that tends to infinity, second term negligible

At equal time we recover the propagator at equilibrium $\quad \frac{1}{k^2+m^2}$

Stochastic Quantization: diagrammatics III

$g \neq 0$ case

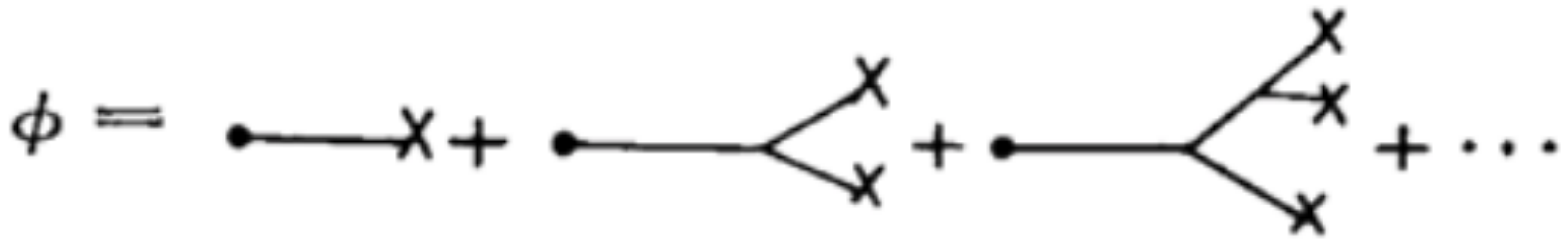
$$\phi(x,s) = \int_0^s d\sigma \int d^D y\, G(x-y, s-\sigma)\left[\eta(y,\sigma) + g\phi^2(y,\sigma)\right]$$

Denote Green function as a line, noise as a cross and the field as a point

Assign a factor g to each three-line vertex

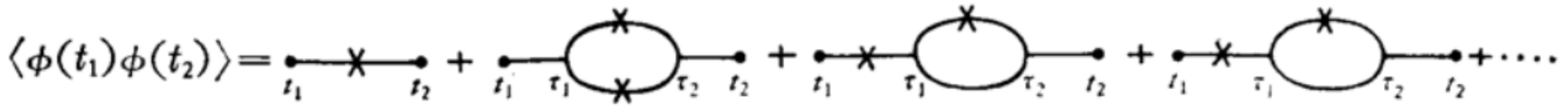
Integrate over stochastic times and coordinates of all crosses and vertices

Iterate expression for the stochastic self-interacting field



Crosses must coincide for the mean over the noise not to vanish

Stochastic Quantization: diagrammatics IV



(a) (b) (c) (d)

$$b = g^2 \int \frac{d^D k_1}{(2\pi)^D} \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 G(k;t_1-\tau_1)G(k;t_2-\tau_2) \times D(k_1;\tau_1,\tau_2)D(k-k_1;\tau_1,\tau_2),$$

$$c+d = g^2 \int \frac{d^D k_1}{(2\pi)^D} \int_0^{t_1} d\tau_1 \int_0^{t_1} d\tau_2 \{D(k-k_1;\tau_1,\tau_2) \times [D(k;t_1,\tau_1)G(k_1;\tau_2-\tau_1)G(k;t_2-\tau_2) + D(k;t_2,\tau_2)G(k_1;\tau_1-\tau_2)G(k;t_1-\tau_1)] + \text{terms obtained by } k_1 \rightleftarrows k-k_1\}.$$

Parisi & Wu

Correct equilibrium limit recovered for s=s' that tends to infinity

$$b = g^2 \frac{d^D k_1}{(2\pi)^D} \frac{1}{(k^2+m^2)(k_1^2+m^2)(k_2^2+m^2)(k^2+k_1^2+k_2^2+3m^2)},$$

$$c+d = g^2 \int \frac{d^D k_1}{(2\pi)^D} \left(\frac{1}{k_1^2+m^2} + \frac{1}{k_2^2+m^2}\right) \frac{1}{(k^2+m^2)^2(k^2+k_1^2+k_2^2+3m^2)};$$

$$b+c+d = g^2 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{(k_1^2+m^2)(k_2^2+m^2)(k^2+m^2)^2},$$

Stochastic Quantization: simple example I

Potential for n-dimensional vector $V(q) = -\frac{1}{2}\mu^2 q^2 + \frac{1}{4} g\,(q^2)^2\,, \qquad q^2 = \sum_{i=1}^{n} (q^i)^2$

Compute perturbatively in g, around the minimum $\langle q^2 \rangle \propto \int d[q]\, q^2\, e^{-V(q)}$

O(n) symmetry → choose the minimum $q_0 \equiv \left(\sqrt{\frac{\mu^2}{g}}, 0, \ldots, 0 \right)$

Find the potential in the new variable, shifted around the vacuum $q = \left(\sqrt{\frac{\mu^2}{g}} + q_L, \mathbf{q}_T \right)$

$$V(q) = \mu^2 q_L^2 + \sqrt{\mu^2 g}\, q_L (q_L^2 + \mathbf{q}_T^2) + \frac{1}{4} g (q_L^2 + \mathbf{q}_T^2)^2 - \frac{1}{4g}\mu^4$$

Stochastic Quantization: simple example II

Usually, change to the radial variable $r = \sqrt{q^2}$ and the angular coordinate on S_{n-1}

$$\langle q^2 \rangle = \frac{\int dr\, r^{n-1}\, r^2\, e^{-(\frac{g}{4}r^4 - \frac{\mu^2}{2}r^2)}}{\int dr\, r^{n-1}\, e^{-(\frac{g}{4}r^4 - \frac{\mu^2}{2}r^2)}}$$

The Langevin equation can be used to by-pass non-linea transformation

We start from $\langle q^2 \rangle = \frac{\mu^2}{g} + A(n-1) + B$ and write the Langevin equation in power of g

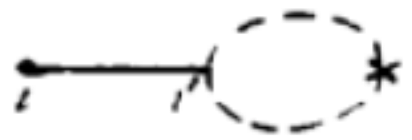
$$\dot{\mathbf{q}}_T = \eta_T + \mathcal{O}(g^{\frac{1}{2}})$$
$$q_L = -2\mu^2 q_L - \sqrt{\mu^2 g}\,(3q_L^2 + \mathbf{q}_T) + \mathcal{O}(g) + \eta_L$$

$\longrightarrow$ $\langle q_T^i(s) q_T^j(s') \rangle = 2\delta_{ij}\, \min(s, s')$

Use the out-equilibrium correlation function to compute q_L @ $\mathcal{O}(g^{\frac{1}{2}})$

Stochastic Quantization: simple example III

For the term proportional to (n-1) the only non-vanishing diagram is



$$q_L(s) = \mu g^{\frac{1}{2}} \int_0^s ds' \langle \mathbf{q}_T^2(s') \rangle e^{-2\mu^2(s-s')} = -(n-1)\sqrt{\frac{g}{\mu^2}}(s - \frac{1}{2\mu^2})$$

with dashed lines transverse propagator and neglecting vanishing terms for s infinity

Computing terms proportional to (n-1)

$$\langle q^2 \rangle = \frac{\mu^2}{g} + 2\sqrt{\frac{\mu^2}{g}}\langle q_L \rangle + \langle q_L^2 \rangle + \langle \mathbf{q}_T^2 \rangle$$
$$= -2(n-1)(s - \frac{1}{2\mu^2}) + 2(n-1)s = \frac{n-1}{\mu^2}$$

Terms proportional to s cancel each other:
finite contribution with correct result at the saddle-point

Stochastic Quantization: gauge theories I

Euclidean Hamiltonian in D-dimensions

$$V = \int d^D x \left\{ D_\mu \phi^\dagger D_\mu \phi + \frac{1}{2} \mathrm{Tr} F^2_{\mu\nu} \right\} , \qquad D_\mu \phi = (\partial_\mu - \imath e A_\mu)\phi \, , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \imath e [A_\mu, A_\nu] \, ,$$

$$A = A^a_\mu \tau_a \, , \qquad F_{\mu\nu} = F^a_{\mu\nu} \tau_a \, , \qquad \mathrm{Tr}(\tau_a \tau_b) = \frac{1}{2} \delta_{ab}$$

with associated Langevin equations

$$\frac{\partial}{\partial s} \phi = D^2 \phi + \eta_\phi \, , \qquad \frac{\partial}{\partial s} \phi^\dagger = D^2 \phi^\dagger + \eta^\dagger_\phi \, , \qquad \frac{\partial}{\partial s} A_\mu = D_\nu F_{\nu\mu} + J_\mu + \eta_\mu$$

complemented with interaction currents and gaussian noises

$$J_\mu = J^a_\mu \tau_a \, , \qquad J_\mu = J^a_\mu = \imath e \phi^\dagger \tau_a \partial_\mu \phi + e^2 \phi^\dagger \{\tau_a, A_\mu\} \phi \, ,$$

$$\langle \eta_\phi(x,s) \eta^\dagger_\phi(x',s') \rangle = 2\delta(x - x')\delta(s - s') \, ,$$

$$\langle \eta_\mu(x,s) \eta_\nu(x',s') \rangle = 2\delta_{\mu\nu}\delta(x - x')\delta(s - s')(\delta_{ab}\tau_a\tau_b)$$

Stochastic Quantization: gauge theories II

Abelian case

$$\frac{\partial}{\partial s} A_\mu = \partial^2 A_\mu - \partial_\mu \partial_\nu A_\nu + \eta_\mu$$

with boundary condition that break the gauge symmetry (restored at equilibrium)

$$A_\nu(x,s)|_{s=0} = 0$$

Solution again a gaussian random variable

$$A_\mu(k,s) = \int_0^s ds' G_{\mu\nu}(k, s-s')\eta_\nu(k,s')$$

with retarded Green function defined only for s>s'

$$G_{\mu\nu}(k, s-s') = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) e^{-k^2(s-s')} + \frac{k_\mu k_\nu}{k^2}$$

Correlation function

$$\rightarrow \quad D_{\mu\nu}(k,s,s') = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \frac{e^{-k^2|s-s'|} - e^{-k^2|s+s'|}}{k^2} + \frac{k_\mu k_\nu}{k^2} 2\mathrm{min}(s,s')$$

Stochastic Quantization: gauge theories III

At large equal stochastic times

$$D_{\mu\nu}(k,s,s) = \underbrace{(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\frac{1}{k^2}}_{\text{Feynman propagator in Landau gauge}} + \underbrace{2s\frac{k_\mu k_\nu}{k^2}}_{\text{Divergent in s longitudinal term}}$$

Decompose the connection in a gauge invariant transverse part and longitudinal one

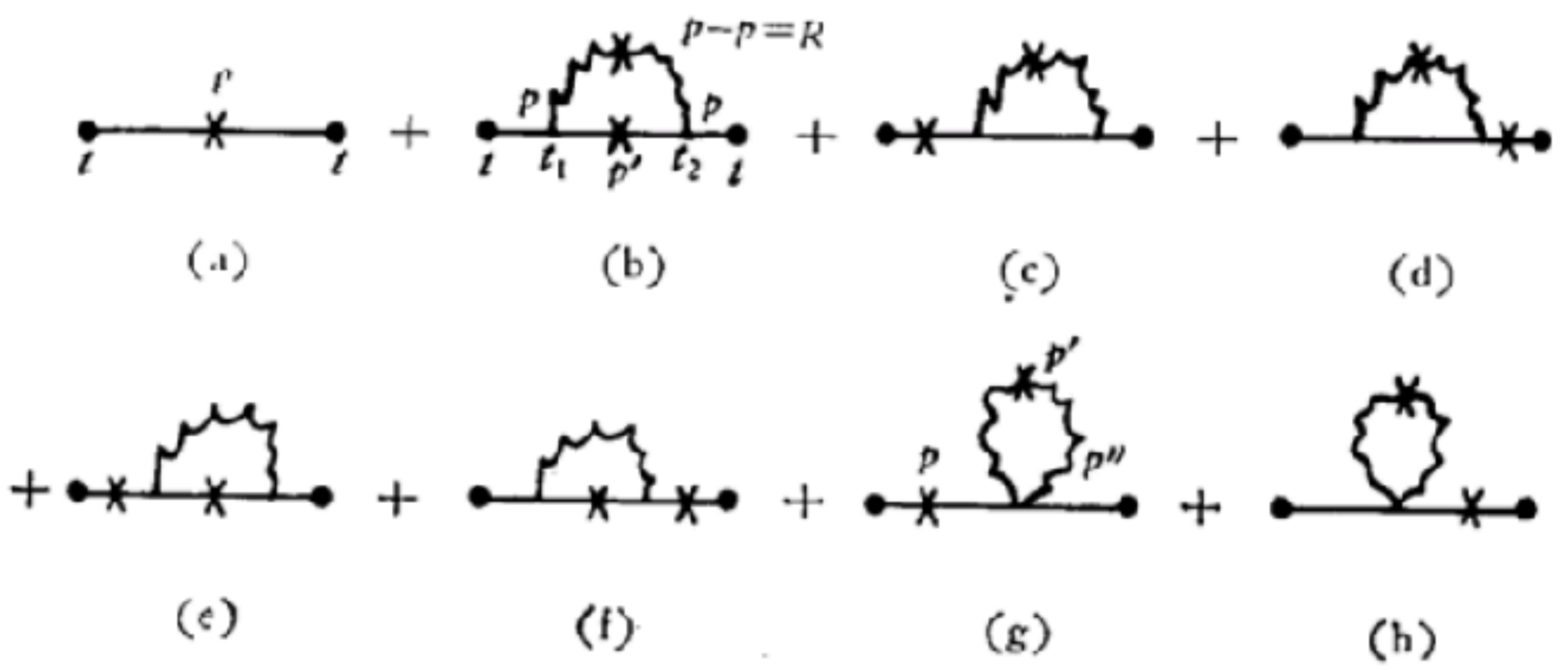
$$A_\mu(x,s) = A^T_\mu(x,s) + \partial_\mu \alpha(x,s) \qquad \text{with} \qquad \partial_\mu A^T_\mu(x,s) = 0$$

Faster evolution for gauge invariant quantities, random walk for not gauge-invariant ones

$$\langle A^T_\mu(k,s) A^T_\nu(-k,s)\rangle = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\frac{1}{k^2}\,, \qquad \langle \alpha(k,s)\alpha(-k,s)\rangle = \frac{2}{k^2}s$$

Stochastic Quantization: gauge theories IV

Scalar propagator at equal stochastic times



Same diagrammatic rules as before pause usual QED ones for $A_\mu \phi^\dagger \phi \, , A_\mu A_\nu \phi^\dagger \phi$

$$a = 2\int_0^t dt' \exp(-2p^2t') = \frac{1}{p^2},$$

contribution to transverse part

$$b = e^2 \int \frac{d^Dp'}{(2\pi)^D} \frac{1}{p^2p'^2k^2(p^2+p'^2+k^2)} \left[(p+p')^2 - \frac{(p^2-p'^2)^2}{k^2}\right]$$

$$+ e^2 \int \frac{d^Dp'}{(2\pi)^D} \frac{1}{p^2p'^2(p^2+p'^2)} \left[2t - \frac{2}{p^2+p'^2} - \frac{1}{p^2}\right] \frac{(p^2-p'^2)^2}{k^2},$$

contribution to longitudinal part

$$c + e = d + f = \frac{e^2}{2} \int \frac{d^Dp'}{(2\pi)^D} \frac{1}{p^2} \left(\frac{1}{p'^2} + \frac{1}{k^2}\right) \frac{1}{p^2+p'^2+k^2} \left[(p+p')^2 - \frac{(p^2-p'^2)^2}{k^2}\right]$$

$$+ \frac{e^2}{2} \int \frac{d^Dp'}{(2\pi)^D} \frac{1}{p^4(p^2+p'^2)} \left[\frac{1}{p'^2} + 2t - \frac{2}{p^2+p'^2} - \frac{1}{p^2}\right] \frac{(p^2-p'^2)^2}{k^2},$$

$$g + h = -3e^2 \int \frac{d^Dp'}{(2\pi)^D} \frac{1}{p^4p'^2} - e^2 \int \frac{d^Dp'}{(2\pi)^D} \frac{1}{p^4} \left[2t - \frac{1}{p^2}\right].$$

Parisi & Wu

Contributions at equal and large stochastic times clarified at k=p-p'

Stochastic Quantization: gauge theories V

Summing all the contribution of the transverse part, we recover result in Landau gauge

Contribution of the longitudinal part at large stochastic times equals variation of the equilibrium propagator induced by the adding to the connection propagator the gauge term $s\frac{k_\mu k_\nu}{k^2}$

At large stochastic time the charged field propagator tends to zero, for $s >> 1/e^2\omega(x-y)$

$$\langle \phi^\dagger(x,s)\phi(y,s)\rangle \sim_{s\to\infty} \langle \phi^\dagger(x)\phi(y)\rangle_{\rm free} e^{-e^2 s\,\omega(x-y)}\,, \qquad \omega(x) \simeq \frac{1}{|x|^{D-2}}$$

Using standard theorem for the variation of the of Green functions under gauge transformations

Consider now the gauge-invariant quantity $\phi^\dagger(x,s)\phi(x,s)$

Contribution proportional to s, corresponding to a gauge-transformation, is vanishing

Other non-vanishing contribution give the propagator in the Landau gauge

At the lowest order quantities are gauge-invariant at equilibrium, while not gauge-invariant quantitates converge dynamically (in the stochastic time) to zero

At the higher order, need for dimensional or lattice regularisation of diagrams and M

Stochastic Quantization: gauge theories VI

Non-abelian case

As in the abelian, we can regularize the matrix M and a) either send stochastic time to infinity before removing the (dimensional or lattice) regulator; b) viceversa

One can show that gauge-invariant quantities are approached uniformly in g and in s, so that the Taylor expansion in g and the limit for s to infinity can be exchanged

$$\partial_\mu A^T_\mu(x,s) = 0\,, \qquad \imath e A^T_\mu(x,s) = e^{-\imath e\alpha(x,s)}(\partial_\mu - \imath e A_\mu)e^{\imath e\alpha(x,s)}\,, \qquad \phi^T(x,s) = e^{-\imath e\alpha(x,s)}\phi(x,s)$$

Langevin equations for longitudinal component and the others decoupled

$$\frac{\partial}{\partial s}A^T_\mu(x,s) = F_1(A^T_\mu, \phi^T, \eta^T)\,, \qquad \frac{\partial}{\partial s}\phi^T(x,s) = F_2(A^T_\mu, \phi^T, \eta^T)$$

Approach toward equilibrium controlled by smallest positive eigenvalues of H

No term linear in s appears in the expectation value of gauge-invariant quantities

Correct results found, with contribution of Faddeev-Popov ghost

Fokker-Planck and cosmological constant

Langevin equation with complex multiplicative noise

$$\frac{\partial g_{\mu\nu}}{\partial s} = \imath \mathcal{G}_{\alpha\beta\mu\nu} \frac{\delta S}{\delta g_{\alpha\beta}} + g_{\mu\nu}\eta \qquad \eta = \sigma_{\tilde{\eta}}\tilde{\eta}$$

Related Fokker-Planck within the Ito differential calculus

$$\frac{\partial p}{\partial s} = -\frac{\delta}{\delta g_{\mu\nu}}\left[\mathcal{G}_{\alpha\beta\mu\nu}\frac{\delta S}{\delta g_{\alpha\beta}}\, p\right] + \frac{\delta^2}{\delta g_{\mu\nu}\delta g_{\rho\sigma}}\left[g^2_{\mu\nu}\, p\right]$$

$$p \simeq \frac{D}{g^2_{\mu\nu}} \exp\left[2\int^{g_{\mu\nu}} \mathcal{D}g_{\alpha\beta}\frac{\mathcal{G}_{\rho\sigma\alpha\beta}\frac{\delta S}{\delta g_{\rho\sigma}}}{\Lambda_0 g^2_{\alpha\beta}}\right] \longrightarrow \mathcal{G}_{\rho\sigma\mu\nu}\frac{\delta S}{\delta g_{\rho\sigma}} - \imath\Lambda_0 g_{\mu\nu} = 0$$

$$\sigma_{\tilde{\eta}} = \sqrt{\Lambda_0} \to e^{-\imath\frac{\pi}{4}}\sqrt{\Lambda_0}$$