Quantization of the sphere - KMS uniqueness results in the thermodynamic limit

joint work with Nicolò Drago and C.J.F. Van de Ven

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A Modern Odyssey: Quantum Gravity meets Quantum Collapse at Atomic and Nuclear physics energy scales in the Cosmic Silence Trento-ECT

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Introduction

- Modelling infinite volume systems $\to C^*-$approach: $\mathfrak{A}_{\mathbb{Z}^d} = \overline{\cup_{\Lambda \in \mathbb{Z}^d} \mathfrak{A}_\Lambda}$
- Thermal equilibrium $\to$ Kubo-Martin-Schwinger (KMS)

<u>Quantum</u>: $\omega(A\tau_{i\beta}(B)) = \omega(BA)$, <u>Classical</u>: $\omega(\{a,b\}) = \beta\omega(b\delta(a))$

Ref. -G.Gallavotti, E. Verboven (1967)
-C.J.F. Van de Ven -arXiv.2211.01755 (2023)

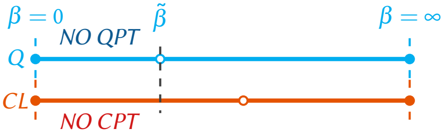
Q : Relating $\omega_{\text{Q-KMS}} \longleftrightarrow \omega_{\text{CL-KMS}}$,

A$_1$: $Q_j^{\mathbb{Z}^d} : \mathfrak{A}^{CL}_{\mathbb{Z}^d} \to \mathfrak{A}^{Q}_{\mathbb{Z}^d}$,

Ref. - F. A. Berezin, CMP (1975)
-S. Murro, C.J.F. Van de Ven, MPAG(2022)

A$_2$: $\omega_{Q-KMS} \circ Q_j^\Gamma \xrightarrow{*, j\to\infty} \omega_{CL-KMS}$,

A$_3$: $\beta \in [0, \widetilde{\beta})$ $\exists!$ $\omega_{\text{CL-KMS}} \Rightarrow \exists! \omega_{\text{Q-KMS}}$



Outline of the talk

- Berezin quantization: scope and properties
- Quantum and classical KMS-condition
- Absence of CPT implies absence of QPT
- Conclusions

SDQ setting in a nutshell

Berezin quantization: scope and properties

Def: $(\Gamma := \mathbb{Z}^d,\ \Lambda \Subset \Gamma,\ x \in \Gamma) \to \mathfrak{A}_x \ni I_x\,,\ \mathfrak{A}_\Lambda := \overline{\bigotimes_{x\in\Lambda} \mathfrak{A}_x}$

$$\iota^\Lambda : \mathfrak{A}_\Lambda \ni \mathfrak{a}_\Lambda \ \to\ \mathfrak{a}_\Lambda \otimes_{x\in\Lambda^c} I_x \in \mathfrak{A}_\Gamma$$

Def$_{CL}$: $-\mathfrak{A}_x = B_\infty := C(\mathbb{S}^2) \to B_\infty^\Lambda \simeq C((\mathbb{S}^2)^{\otimes\Lambda}) \to B_\infty^\Gamma \simeq C((\mathbb{S}^2)^{\otimes\Gamma})$,

$-\dot{B}_\infty^\Lambda := C^\infty((\mathbb{S}^2)^{\otimes\Lambda}) \ \to\ \dot{B}_\infty^\Gamma := \cup_{\Lambda\Subset\Gamma}\iota^\Lambda \dot{B}_\infty^\Lambda$,

$-\{\ ,\ \}_\Lambda : \dot{B}_\infty^\Lambda \times \dot{B}_\infty^\Lambda \to \dot{B}_\infty^\Lambda$

Def$_Q$: $-\mathfrak{A}_x = B_j := M_{2j+1}(\mathbb{C})\ (j \in \mathbb{Z}_+/2) \to B_j^\Gamma := \overline{\cup_{\Lambda\Subset\Gamma}\iota^\Lambda B_j^\Lambda}$,

$-\dot{B}_j^\Lambda := B_j^\Lambda \ \to\ \dot{B}_j^\Gamma := \cup_{\Lambda\Subset\Gamma}\iota^\Lambda B_j^\Lambda$

! Find $Q_j^\Gamma : \dot{B}_\infty^\Gamma \to \dot{B}_j^\Gamma$ **positive**, **surjective**, with **continuity properties** e.g.:

$$\left\| \tfrac{2j+1}{i}\left[Q_j^\Gamma(a_\Gamma), Q_j^\Gamma(b_\Gamma)\right] - Q_j^\Gamma(\{a_\Gamma, b_\Gamma\}) \right\|_{B^\Gamma_{2j+1}} \xrightarrow{j\to\infty} 0$$

Explicit quantization I

Berezin quantization: scope and properties

→ Focus on single site:

Def$_1$: $D^{(j)}: SU(2) \to M_{2j+1}(\mathbb{C})$, $|j,\sigma\rangle := D^{(j)}(e^{-i\phi(\sigma)J_z}e^{-i\theta(\sigma)J_y})|j,j\rangle \leftarrow$ Coherent state

-A.M. Perelomov, CMP (1972)

$$\boxed{1 = \int_{\mathbb{S}^2} P_{j,\sigma}\mathrm{d}\mu_j(\sigma)}, \quad P_{j,\sigma} := |j,\sigma\rangle\langle j,\sigma|, \quad \mathrm{d}\mu_j(\sigma) := \frac{2j+1}{4\pi}\mathrm{d}\Omega$$

Def$_2$: $Q_j : B_\infty \to B_j$, $Q_j(a) := \int_{\mathbb{S}^2} a(\sigma)P_{j,\sigma}\mathrm{d}\mu_j(\sigma)$,

→ $\Lambda \Subset \Gamma$-S.Murro, C.J.F. Van de Ven MPAG(2022) : $\boxed{Q_j^\Lambda =: \bigotimes_{x\in\Lambda} Q_j^{\{x\}} : B_\infty^\Lambda \to B_j^\Lambda}$

→ Theorem (Drago, Van de Ven, P.): $Q_j^\Gamma : \dot{B}_\infty^\Gamma \to \dot{B}_j^\Gamma$, $Q_j^\Gamma(a_\Lambda) := \begin{cases} Q_j^\Lambda(a_\Lambda), & j \in \mathbb{Z}_+/2 \\ a_\Lambda, & j = \infty \end{cases}$

with $\{B_j^\Gamma\}_{j\in\overline{\mathbb{Z}_+/2}}$, $\{Q_j^\Gamma\}_{j\in\overline{\mathbb{Z}_+/2}}$ define a **SDQ**.

Explicit quantization II

Berezin quantization: scope and properties

Def$_3$: $\check{a}_j(\sigma) := \langle j,\sigma|Q_j(a)|j,\sigma\rangle \to \frac{1}{2J+1}\,\mathsf{Tr}\big[Q_j(a)Q_j(b)\big] = \int_{\mathbb{S}^2} a(\sigma)\check{b}_j(\sigma)\mathrm{d}\mu_0(\sigma)$

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Def$_4$: $\tilde{D}^j : SU(2) \ni R \to \tilde{D}^j(R) \in \mathscr{B}(M_{2j+1}(\mathbb{C}))$, $\tilde{D}^j(R)A := D^j(R)AD^j(R)^*$ NOT Irr.

Def$_5$: $\hat{R} : SU(2) \ni R \to \hat{R} \in \mathscr{B}(\mathbb{L}^2(\mathbb{S}^2))$, $\hat{R}a := a \circ R^{-1}$ NOT Irr.

Q_j intertwines : $Q_j(\hat{R}a) = \tilde{D}^j(R)Q_j(a) \to \boxed{\check{Y}_{\ell m} = a_{j|\ell} Y_{\ell m}}$

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Def$_6$: $Y_{\ell m} \to \mathscr{Y}_{j|\ell m} := Q_j(Y_{\ell m}/\sqrt{a_{j|\ell}}),\ \langle A,B\rangle_{HS} := \mathrm{Tr}[A^*B]/(2j+1)$

1. $\|Y_{\ell m}\|_\infty \leq 1,\ \langle Y_{\ell' m'}, Y_{\ell m}\rangle_{\mathbb{L}^2(\mathbb{S}^2)} = \delta_{\ell\ell'}\delta_{mm'}4\pi/(2\ell+1)$

1' $\left\|\mathscr{Y}_{j|\ell m}\right\|_{B_j} \leq 1,\ \langle \mathscr{Y}_{j|\ell' m'}, \mathscr{Y}_{\ell m}\rangle_{HS} = \delta_{\ell\ell'}\delta_{mm'}/(2\ell+1)$

Generalized notions of equilibrium

Quantum and classical KMS-condition

Def$_Q$: $\mathfrak{A}$ non-commutative C^*, τ strongly cont, δ infinitesimal generator
$\omega \in S(\mathfrak{A})$ $(\beta, \delta)-$ **KMS quantum state** if

$$\omega(A\tau_{i\beta}(B)) = \omega(BA), \quad A, B\ \tau-\text{analytic}$$

Def$_{CL}$ $(\mathfrak{A}, \{\ ,\ \})$ commutative C^*, $\delta : D(\delta) \to \mathfrak{A}$, $*-$derivation. $\omega \in S(\mathfrak{A})$ $(\beta, \delta)-$**KMS classical state** if -G. Gallavotti, E. Verboven (1967)

$$\omega(\{a, b\}) = \beta\omega(b\delta(a)), \quad a, b \in D(\delta).$$

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Hyp: $\varphi := (\varphi_X)_{X\in\Gamma} \subset \dot{B}^\Gamma_\infty \quad \& \quad \sum_{m\geq 0} e^{\lambda m} \sup_{x\in\Gamma} \sum_{\substack{X\ni x \\ |X|=m+1}} \|\varphi_X\|_{C^1(\mathbb{S}^2)} < \infty$

δ^Γ_∞: $\dot{B}^\Gamma_\infty \to B^\Gamma_\infty$, $\delta^\Gamma_\infty(a_\Lambda) := \sum_{X\in\Gamma}\{a_\Lambda, \varphi_X\}$,

δ^Γ_j: $\dot{B}^\Gamma_j \to B^\Gamma_j$, $\delta^\Gamma_j(A_\Lambda) := i\sum_{X\in\Gamma}\left[Q^X_j(\varphi_X), A_\Lambda\right]$

Semi-classical limit

Quantum and classical KMS-condition

Q: given ω_j^Γ −KMS, $\omega_j^\Gamma \circ Q_j^\Gamma$ classical state $\to \lim_{j\to\infty} \omega_j^\Gamma \circ Q_j^\Gamma$ −Classical KMS?

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A: Theorem (Drago, Van de Ven, P.) : $\omega_j^\Gamma \in S(B_j^\Gamma)$ (β, δ_j^Γ)−KMS quantum state
any weak* limit point of $(\omega_j^\Gamma \circ Q_j^\Gamma)_{j\in\mathbb{Z}_+/2}$ is a $(\beta, \delta_\infty^\Gamma)$−KMS classical state

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! $\omega \in S(\mathfrak{A})$, $(\beta,\delta)$$-$KMS quantum state iff
-G. Roepstorff, H. Araki, G. Sewell (1976-1977)

$$-i\beta(A^*\delta(A)) \geq \omega(A^*A)\log\left(\frac{\omega(AA^*)}{\omega(A^*A)}\right), \quad A \in D(\delta)$$

! Lemma (Drago, Van de Ven, P.) : $\omega \in S(\mathfrak{A})$, $(\beta,\delta)$$-$KMS classical state iff

$$-i\beta\,\omega(a^*\delta(a)) \geq i\omega(\{a,a^*\}), \quad a \in D(\delta)$$

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Condition for quantum uniqueness

Absence of CPT implies absence of QPT

Def: Given $\mathfrak{A}, \tau, \beta \to$ if $\exists!\ \omega\ (\beta, \tau)$−KMS state: **No Phase Transition**

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Conditions for **absence of QPT**:

$$\|\Phi\|_\lambda := \sum_{m \geq 0} e^{\lambda m} (2j+1)^{2m} \left(\sup_{x \in \Gamma} \sum_{\substack{X \ni x \\ |x| = m+1}} \|\Phi_X\|_{B_j^X} \right) < \infty$$

$$\beta \|\Phi\|_\lambda < \frac{\lambda}{2} \left(1 + \frac{e^\lambda (2j+1)^3}{2j+1} \right)^{-1}$$

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✓ $\|\Phi_X\|_{B_j^X} = \left\|Q_j(\varphi_X)\right\|_{B_j^X} \leq \|\varphi_X\|_{B_\infty^X}$

× j dependence $\to$ ~~classical limit~~

Derivation of a classical condition

Absence of CPT implies absence of QPT

Theorem (Drago, Van de Ven, P.):

$\varphi := (\varphi_X)_{X \Subset \Gamma}, \ \varphi_X \in C^{2s}(\mathbb{S}^2_X), \ s > 7/4$

$$\|\varphi\|_s := \sum_{m \geq 0} (2K_s C^s_\Delta)^m \left(\sup_{x \in \Gamma} \sum_{\substack{|X|=m+1 \\ X \ni x}} \|\varphi_X\|_{C^{2s}(\mathbb{S}^2_X)} \right) < +\infty$$

$$\beta(s) := \frac{\log 2}{2K_s C^s_\Delta \|\varphi\|_s}$$

$\exists! \ \omega^\Gamma_\infty$ KMS if $\beta \in [0, \beta(s))$

Basic ideas

Absence of CPT implies absence of QPT

Note: $\ell_\Lambda \in \mathbb{N}_\Lambda,\ y \in \Lambda,\ m_y \in [-\ell_y, \ell_y]$; $\quad Y_{\ell_\Lambda, m_\Lambda} = \bigotimes_{y\in\Lambda} Y_{\ell_y m_y}$

! $a_\Lambda \in \dot{B}^\Lambda_\infty \to a_\Lambda = \sum_{\ell_\Lambda \in \mathbb{Z}^+_\Lambda} \sum_{m_\Lambda} \hat{a}(\ell_\Lambda, m_\Lambda) Y_{\ell_\Lambda m_\Lambda} \to \omega^\Gamma_\infty$ determined by $\omega^\Gamma_\infty(Y_{\ell_\Lambda m_\Lambda})$

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! $\underline{\omega}^\Gamma_\infty(\ell_\Lambda, m_\Lambda) := \underline{\omega}^\Gamma_\infty(Y_{\ell_\Lambda m_\Lambda}) \to (\underline{1} - \underline{L}_\beta)\underline{\omega}^\Gamma_\infty = \underline{\delta},$

$$\left\|\underline{L}_\beta\right\|_{\mathscr{B}(\underline{X})} < 1 \quad\Longrightarrow\quad \exists!\ \omega^\Gamma_\infty = \sum_{n\geq 0} \underline{L}^n_\beta \underline{\delta}, \quad \left\|\underline{f}\right\|_{\underline{X}} := \sup_{\Lambda\Subset\Gamma} \sup_{\ell_\Lambda, m_\Lambda} |f_\Lambda(\ell_\Lambda, m_\Lambda)|$$

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- Lemma (Drago, Van de Ven, P.):

  $\omega^\Gamma_\infty \in S(B^\Gamma_\infty)$, $(\beta, \delta^\Gamma_\infty)$-KMS $\to\ \omega^\Gamma_\infty(a_\Lambda) = \omega^\Gamma_\infty\left(e^{\beta \sum_{X\ni x}(1-\hat{R}_x)\varphi_X} \hat{R}_x a_\Lambda\right)$
- Schur: $\int_{SU(2)} \hat{R}_x Y_{\ell_x m_x} \mathrm{d}R_x = 0$

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- $\underline{\omega}^\Gamma_\infty(\ell_\Lambda, m_\Lambda) := \underline{\omega}^\Gamma_\infty(Y_{\ell_\Lambda m_\Lambda}) \to (\underline{1} - \underline{L}_\beta)\underline{\omega}^\Gamma_\infty = \underline{\delta}$,

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$\to\ \underline{\omega}^\Gamma_\infty(\ell_\Lambda, m_\Lambda) = \omega^\Gamma_\infty\left(\int_{SU(2)} \left(1 - e^{\beta \sum_{X\ni x}(1-\hat{R}_x)\varphi_X}\right) Y_{\ell_\Lambda m_\Lambda} \mathrm{d}R_x \right), \quad \underline{\omega}^\Gamma_\infty(\ell_\varnothing, m_\varnothing) := 1$

Expansion of the equation

Absence of CPT implies absence of QPT

! Three main steps:

- $\left(1 - e^{\beta \sum_{X \ni x}(1-\hat{R}_x)\varphi_X}\right) Y_{\ell_\Lambda m_\Lambda} = -\sum_{n \geq 0} \frac{\beta^n}{n!} \left[\prod_{i=1}^n \sum_{X_i \ni x} (1 - \hat{R}_x)\varphi_{X_i}\right] Y_{\ell_\Lambda, m_\Lambda}$
- $(1 - \hat{R}_x)\varphi_{X_i} = \sum_{\ell_{X_i}, m_{X_i}} C_{X_i, R_x}(\ell_{X_i}, m_{X_i}) Y_{\ell_{X_i} m_{X_i}}$
- Using the product expansion of Y's $(S_n := X_1 \cup \cdots \cup X_n,)$

$$Y_{\ell_{X_1}, m_{X_1}} \cdots Y_{\ell_{X_n}, m_{X_n}} Y_{\ell_\Lambda, m_\Lambda} = \left(\prod_{y \in \Lambda \cap S_n^c} Y_{\ell_{\Lambda,y}, m_{\Lambda,y}} \right) \sum_{h=1}^{N(\ell_{X_1}, \ldots, \ell_{X_n})} C'(\ell_{S_n}^h, m_{S_n}^h) Y_{\ell_{S_n}^h, m_{S_n}^h} ,$$

✓ Uniformity in Λ

✓ $|C'(\ell_{S_n}^h, m_{S_n}^h)| \leq 1$

One big ri-expansion

Absence of CPT implies absence of QPT

$$! \ \underline{\omega}^{\Gamma}_{\infty}(\ell_\Lambda, m_\Lambda) = -\sum_{n\geq 1}\frac{\beta^n}{n!}\sum_{\substack{X_1,\dots,X_n\\ x\in X_1\cap\ldots\cap X_n}}\sum_{\substack{\ell_{X_1},\dots,\ell_{X_n}\\ m_{X_1},\dots,m_{X_n}}}\int_{SU(2)}\prod_{i=1}^{n} C_{X_i,R_x}(\ell_{X_i}, m_{X_i})\mathrm{d}R_x$$

$$\sum_{h=1}^{N(\ell_{X_1},\dots,\ell_{X_n})} C'(\ell^h_{S_n}, m^h_{S_n})\omega^{\beta,\Gamma}_{\infty}\left(Y_{\ell^h_{S_n},m^h_{S_n}}\prod_{y\in\Lambda\cap S^c_n} Y_{\ell_{\Lambda,y},m_{\Lambda,y}}\right) =: (\underline{L}_\beta\underline{\omega}^{\Gamma}_{\infty})(\ell_\Lambda, m_\Lambda),$$

Def$_1$ $(\underline{L}_\beta \underline{f})(\ell_\varnothing, m_\varnothing) := 0$

Def$_2$ $\underline{\delta}(\ell_\varnothing, m_\varnothing) := 1$ $\underline{\delta}(\ell_\Lambda, m_\Lambda) := 0$ if $\Lambda \neq \varnothing$

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! Convergent series : $\langle f, Y_{lm}\rangle_{\mathbb{L}^2(\mathbb{S}^2)} = \frac{1}{[1+\ell(\ell+1)]^s}\langle(1-\Delta_{\mathbb{S}^2})^s f, Y_{lm}\rangle_{\mathbb{L}(\mathbb{S})^2}$,

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! $\underline{\omega}^{\Gamma}_{\infty}(\ell_\Lambda, m_\Lambda) = -\sum_{n\geq 1} \frac{\beta^n}{n!} \sum_{\substack{X_1,\ldots,X_n \\ x\in X_1\cap\ldots\cap X_n}} \sum_{\substack{\ell_{X_1},\ldots,\ell_{X_n} \\ m_{X_1},\ldots,m_{X_n}}} \int_{SU(2)} \prod_{i=1}^{n} C_{X_i,R_x}(\ell_{X_i}, m_{X_i}) \mathrm{d}R_x$

$$\sum_{h=1}^{N(\ell_{X_1},\ldots,\ell_{X_n})} C'(\ell^h_{S_n}, m^h_{S_n})\omega^{\beta,\Gamma}_{\infty}\left(Y_{\ell^h_{S_n},m^h_{S_n}} \prod_{y\in\Lambda\cap S^c_n} Y_{\ell_{\Lambda,y},m_{\Lambda,y}}\right) =: (\underline{L}_\beta\underline{\omega}^{\Gamma}_{\infty})(\ell_\Lambda, m_\Lambda),$$

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⋆ $|(\underline{L}_\beta \underline{f})(\ell_\Lambda, m_\Lambda)| \leq \left\|\underline{f}\right\|_{\underline{X}}(\exp\left[2C^s_\Delta K_s\beta\|\varphi\|_s\right] - 1)$

⋆⋆ $(\exp\left[2C^s_\Delta K_s\beta\|\varphi\|_s\right] - 1) < 1 \iff \beta < \beta(s) = \frac{\log 2}{2K_sC^s_\Delta\|\varphi\|_s} \implies \left\|\underline{L}_\beta\right\|_{\underline{X}} < 1$ □

The Quantum situation

Absence of CPT implies absence of QPT

(↺ Déjà vu) Theorem (Drago, Van de Ven, P.):

$\Phi := (Q_j^X(\varphi_X))_{X \in \Gamma}, \ \varphi_X \in C^{2s}(\mathbb{S}_X^2),$

$$\|\varphi\|_{s,\lambda} := \sum_{m \geq 0} (e^{\lambda} K_s C_{\Delta}^s)^m \sup_{y \in \Gamma} \sum_{\substack{|X| = m+1 \\ X \ni y}} \|\varphi_X\|_{C^{2s}(\mathbb{S}_X^2)} < +\infty,$$

$\exists! \ \omega_j^{\Gamma}$ KMS if $\beta \in [0, \beta(s, \lambda))$

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! $A_\Lambda \in B_j^{\Lambda} \to A_\Lambda = \sum_{\ell_\Lambda, m_\Lambda} \hat{A}(\ell_\Lambda, m_\Lambda) \mathscr{Y}_{j|\ell_\Lambda, m_\Lambda} \to \underline{\omega}_j^{\Gamma}(\ell_\Lambda, m_\Lambda) := \omega_j^{\Gamma}(\mathscr{Y}_{j|\ell_\Lambda m_\Lambda})$

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! (Schur + KMS) $\underline{\omega}_j^{\Gamma}(\ell_\Lambda, m_\Lambda) = \omega_j^{\Gamma}\left(\int_{SU(2)} \mathscr{Y}_{j|\ell_\Lambda m_\Lambda} D_x^j(R)^* (I - \tau^{\Gamma}_{i\beta}) D_x^j(R) \mathrm{d}R \right)$,

! $Q_j^X(\varphi_X) = \sum_{\ell_X, m_X} C_X(\ell_X, m_X) c_{j|\ell_X} \mathscr{Y}_{j|\ell_X m_X}$,

Quantum estimates

Absence of CPT implies absence of QPT

Def: more involved because of non-commutativity

$$(\underline{L}_j^{\beta}\underline{f})_{\Lambda}(\ell_{\Lambda}, m_{\Lambda})$$

$$= -\sum_{n\geq 1}\frac{(-\beta)^n}{n!}\sum_{\substack{X_1,\dots,X_n\\ X_q\cap S_{q-1}\neq\varnothing}}\sum_{p\in\{\pm 1\}^n}\sum_{\substack{\ell_{X_{1,p}},\dots,\ell_{X_{n,p}}\\ m_{X_{1,p}},\dots,m_{X_{n,p}}}}\int_{SU(2)}\prod_{i=1}^{n}C_{j|X_{k,p}}(\ell_{X_{k,p}}, m_{X_{k,p}})\mathrm{d}R$$

$$\sum_{h=1}^{N(\ell_{X_1},\dots,\ell_{X_n})} C'_{j,p}(\ell^h_{S_n}, m^h_{S_n})f_{X(n,\Lambda)}(\ell^h_{X(n,\Lambda),p}, m^h_{X(n,\Lambda),p})$$

⋆ $\left\|\underline{L}_j^{\beta}\right\|_{\underline{X}} \leq e^{\lambda}\sum_{n\geq 1}(2\lambda^{-1}\beta K_s C^s_{\Delta}\|\varphi\|_{\lambda,s})^n = \frac{2\lambda^{-1}\beta K_s C^s_{\Delta}\|\varphi\|_{\lambda,s}}{1-2\lambda^{-1}\beta K_s C^s_{\Delta}\|\varphi\|_{\lambda,s}}$,

⋆⋆ Again ($\lambda > 0$): $\beta < \beta(s,\lambda) := \frac{\lambda}{1+e^{\lambda}}\frac{1}{K_s C^s_{\Delta}\|\varphi\|_{\lambda,s}} \implies \left\|\underline{L}_j^{\beta}\right\|_{\underline{X}} < 1$ □

To conclude

Conclusions

Striking results

- ϟ Quantization on the whole lattice Γ
- ϟϟ if $\lambda = \log 2 \to \|\varphi\|_s < \infty \implies \|\varphi\|_{\log 2, s} < \infty \ldots$

  … Moreover

  $\beta \in [0, \beta(\log 2, s))$ *then* $\exists!$Classical KMS-state $\implies$ $\exists!$Quantum KMS-state
- ϟϟϟ ω_j^Γ – unique quantum KMS $\to$ $(\omega_j^\Gamma \circ Q_j^\Gamma) \xrightarrow{j\to\infty} \omega_\infty^\Gamma$ – unique classical KMS

Outlooks:

- Absence of quantum phase transitions $\implies$ Absence of Classical Phase transitions?
- Classical Phase Transitions $\Leftrightarrow$ Quantum Phase Transitions?

Thank you for your attention and enjoy the 🍝 !

Explicit quantization III

Conclusions

1. $\|Y_{\ell m}\|_\infty \leq 1$, $\langle Y_{\ell' m'}, Y_{\ell m}\rangle_{\mathbb{L}^2(\mathbb{S}^2)} = \delta_{\ell\ell'}\delta_{mm'}4\pi/(2\ell+1)$
2. $Y_{\ell m}Y_{\ell' m'} = \sum_{\bar\ell=|\ell-\ell'|}^{\ell+\ell'} \mathrm{CG}^{\bar\ell 0}_{\ell 0,\ell' 0}\mathrm{CG}^{\bar\ell m+m'}_{\ell m,\ell' m'} Y_{\bar\ell m+m'}$

Def$_6$: $Y_{\ell m} \to \mathscr{Y}_{j|\ell m} := Q_j(Y_{\ell m}/\sqrt{a_{j|\ell}})$, $\langle A, B\rangle_{HS} := \mathrm{Tr}[A^* B]/(2j+1)$

1' $\left\|\mathscr{Y}_{j|\ell m}\right\|_{B_j} \leq 1$, $\langle \mathscr{Y}_{j|\ell' m'}, \mathscr{Y}_{\ell m}\rangle_{HS} = \delta_{\ell\ell'}\delta_{mm'}/(2\ell+1)$

2' $\mathscr{Y}_{j|\ell m}\mathscr{Y}_{j|\ell' m'} = e^{i\phi}\sum_{\bar\ell=|\ell-\ell'|}^{\ell+\ell'} CG^{\bar\ell m+m'}_{\ell m,\ell' m'}\sqrt{2j+1}\sqrt{2\bar\ell+1}\begin{Bmatrix} j & j & \bar\ell \\ \ell & \ell' & j\end{Bmatrix} \leftarrow 6j-\text{symbol}$

!!! $\sum_{\ell_2} six_{\ell_1\ell_2}\mathrm{six}_{\ell_2\ell_3} = \delta_{\ell_1\ell_3}$, $\quad \mathrm{six}_{\ell_1\ell_2} := \sqrt{2\ell_1+1}\sqrt{2\ell_2+1}\begin{Bmatrix} j & j & \ell_2 \\ \ell & \ell' & \ell_1\end{Bmatrix}$

$\to$ $|\mathrm{six}_{\ell\ell'}| \leq 1$

Details on product formula

Conclusions

- Extension: $\ell_{X_i,y} \to \tilde{\ell}_{X_i,y} := \begin{cases} 0 \text{ if } y \in S_n/X_i \\ \ell_{X_i} \text{ if } y \in X_i \end{cases}$, $\dot{B}_\infty^{X_i} \ni Y_{\ell_{X_i},m_{X_i}} \hookrightarrow Y_{\tilde{\ell}_{X_i},\tilde{m}_{X_i}} \in \dot{B}_\infty^{S_n}$
- $\sum_{s_{y,1},\dots,s_{y,n}}$ truly means $\sum_{s_{y,1}=|\tilde{\ell}_{\Lambda,y}-\tilde{\ell}_{X_1,y}|}^{\tilde{\ell}_{\Lambda,y}+\tilde{\ell}_{X_1,y}} \sum \cdots \sum_{s_{y,n}=|\tilde{\ell}_{X_{n-1},y}-\tilde{\ell}_{X_n,y}|}^{\tilde{\ell}_{X_{n-1},y}+\tilde{\ell}_{X_n,y}}$
- From $\sum_{c=|a-b|}^{a+b} 1 \leq 2\min\{a,b\} + 1 - \dots$

  $\dots \to N(\ell_{X_1}, \dots \ell_{X_n}) := \prod_{y \in S_n} \sum_{s_{y,1}=|\tilde{\ell}_{\Lambda,y}-\tilde{\ell}_{X_1,y}|}^{\tilde{\ell}_{\Lambda,y}+\tilde{\ell}_{X_1,y}} \sum \cdots \sum_{s_{y,n}=|\tilde{\ell}_{X_{n-1},y}-\tilde{\ell}_{X_n,y}|}^{\tilde{\ell}_{X_{n-1},y}+\tilde{\ell}_{X_n,y}} 1$

  $\leq \prod_{y \in S_n} \prod_{i=1}^{n} (2\tilde{\ell}_{X_i,y} + 1) = \prod_{i=1}^{n} \prod_{y \in X_i} (2\ell_{X_i,y} + 1)$

- ✓ Uniformity in Λ
- ✓ $|C'(\ell_{S_n}^h, m_{S_n}^h)| \leq 1$