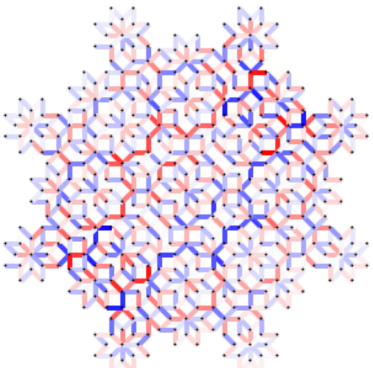
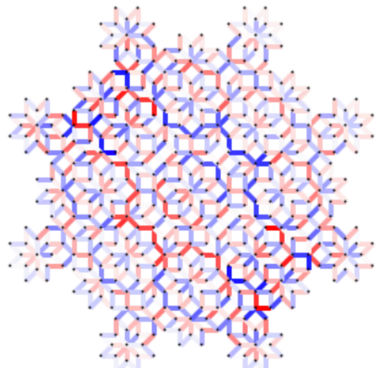
Operator content of quantum and classical lattice models from lossy compression theory

Maciej Koch-Janusz





University of Zurich UZH

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Zohar Ringel | Sebastian Huber | Doruk Efe Gokmen | Matthew Schmitt | Michel Fruchart | Vincenzo Vitelli






האוניברסיטה העברית בירושלים
THE HEBREW UNIVERSITY OF JERUSALEM

ETH zürich

THE UNIVERSITY OF CHICAGO

- **MKJ** and Zohar Ringel, *Nature Physics* ***14****, 578-582 (2018)*
- P. Lenngenhager, D.E. Gokmen, Z. Ringel, S.D. Huber and **MKJ**, *Phys. Rev. X* ***10****, 011037 (2020)*
- A. Gordon, A. Banerjee, **MKJ** and Z. Ringel, *Phys. Rev. Lett.* ***126****, 240601 (2021)*
- D.E. Gokmen, Z. Ringel, S.D. Huber and **MKJ**, *Phys. Rev. Lett.* ***127****, 240603 (2021)*
- D.E. Gokmen, Z. Ringel, S.D. Huber and **MKJ**, *Phys. Rev. E* ***104****, 064106 (2021)*

- D.E. Gokmen, S. Biswas, S.D. Huber, Z. Ringel, F. Flicker and **MKJ**, *arXiv:2301:11934 (2023)* *“ML for physical sciences” @NeurIPS 2023*
- L. Oppenheim, **MKJ**, S. Gazit, Z. Ringel *arXiv:2311.17994, "ML for physical sciences" @NeurIPS 2023*
- M. Schmitt, **MKJ**, M. Fruchart, D. Seara, V. Vitelli *arXiv:2312.06608, “ML for physical sciences” @NeurIPS 2023*

https://github.com/RSMI-NE/RSMI-NE

Outline

- Intro and motivation: RG, compression, and ML

- Information Bottleneck lossy compression
- Equivalence of “relevance” in IB and in RG

- The RSMI coarse-graining
- Example 1: operators and symmetries with RSMI
- Example 2: RG for quasi-periodic tilings
- Example 3: Dynamical systems

Effective descriptions

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- We seek effective descriptions of **complex** systems in terms of **few variables**

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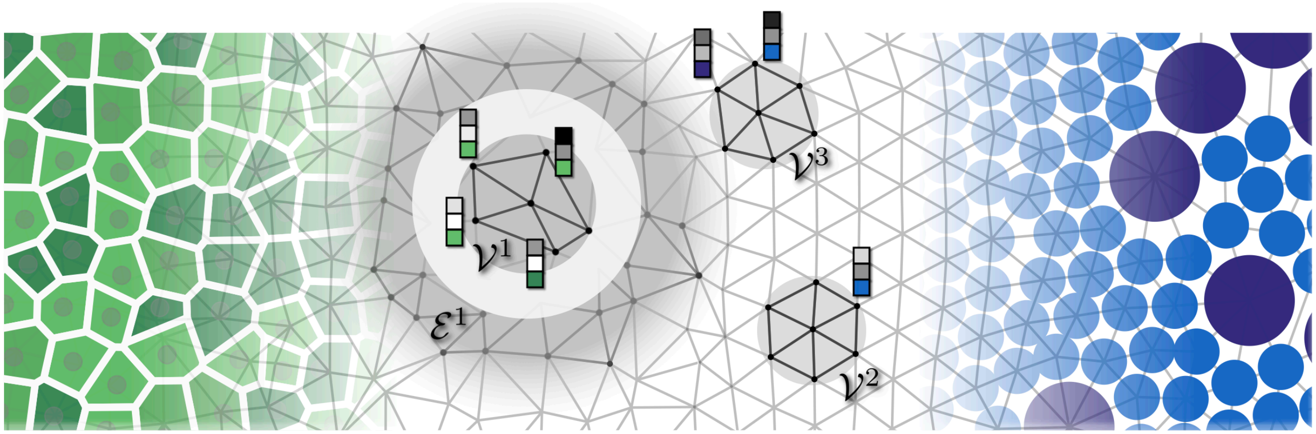

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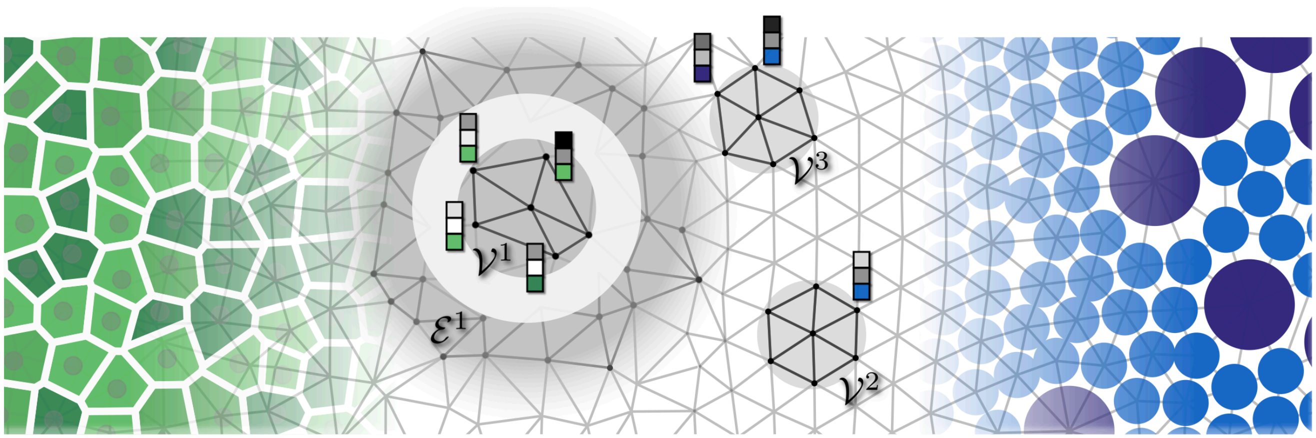
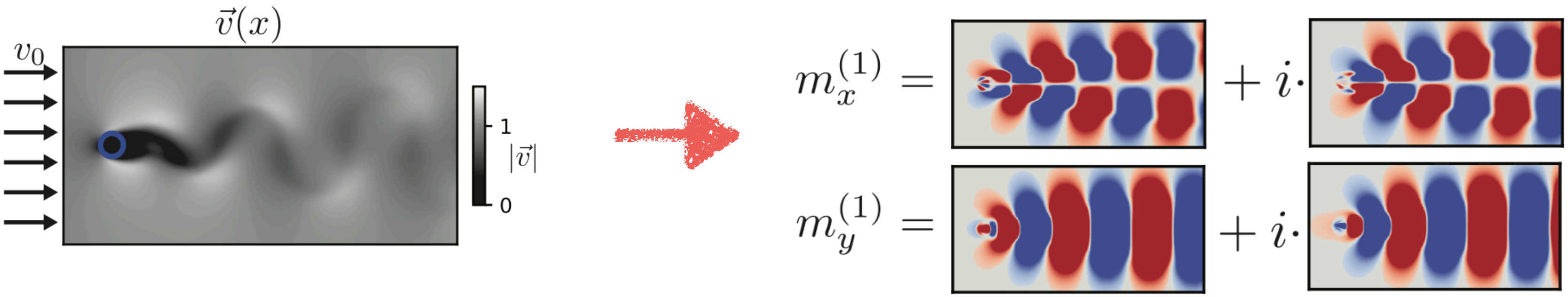


- Dynamical systems' model reductions



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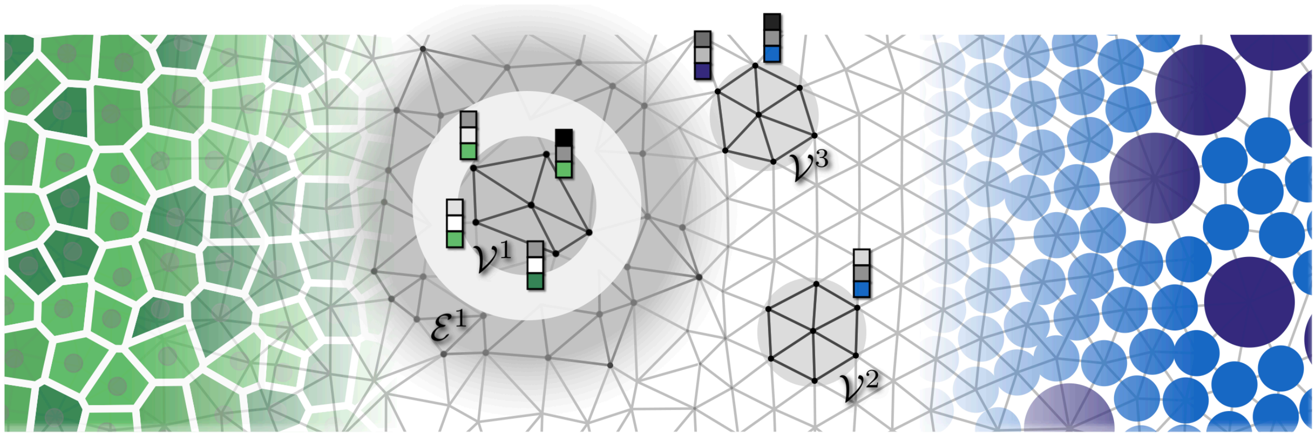

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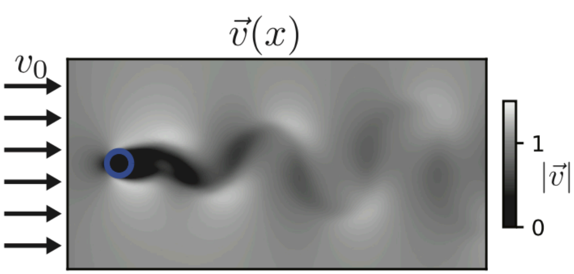
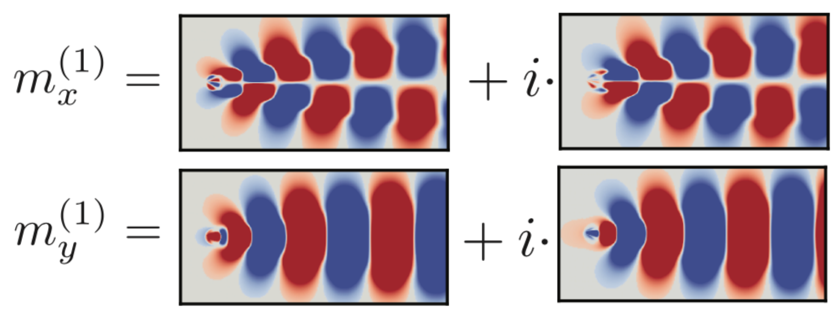
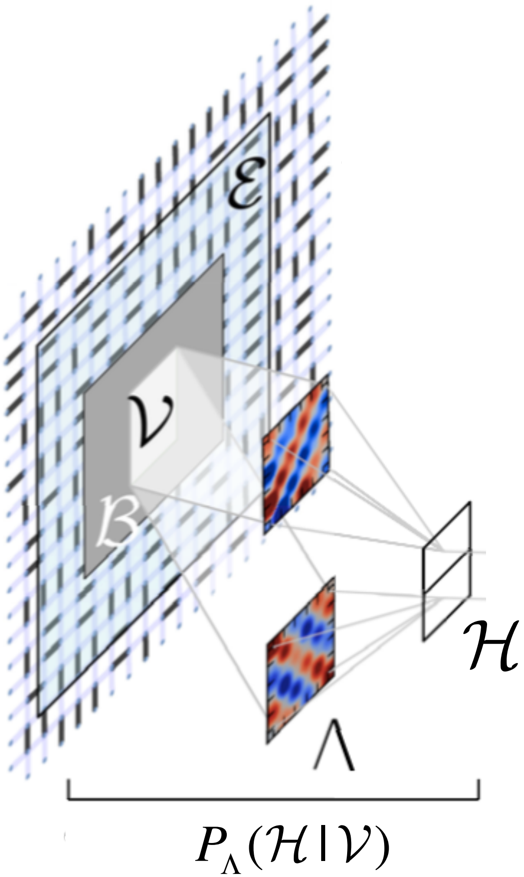




- Can one **systematically** derive an effective theory?

Motivating example: real-space RG



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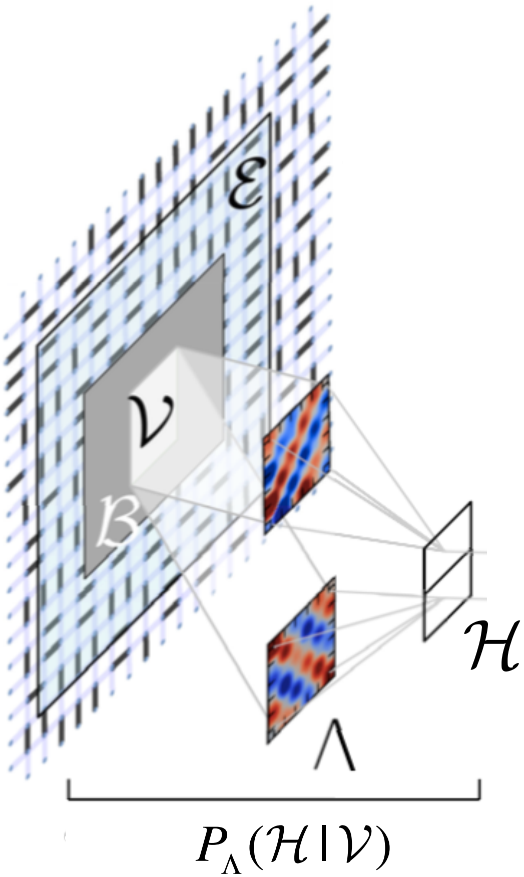

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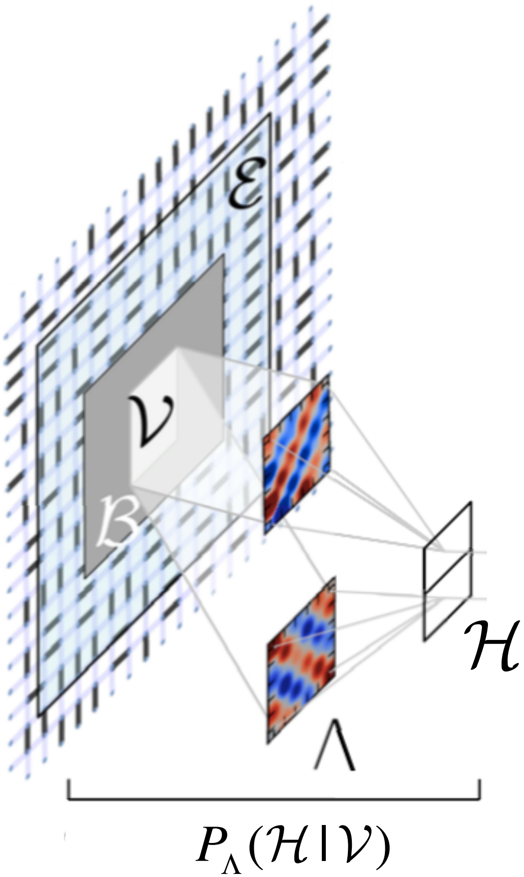

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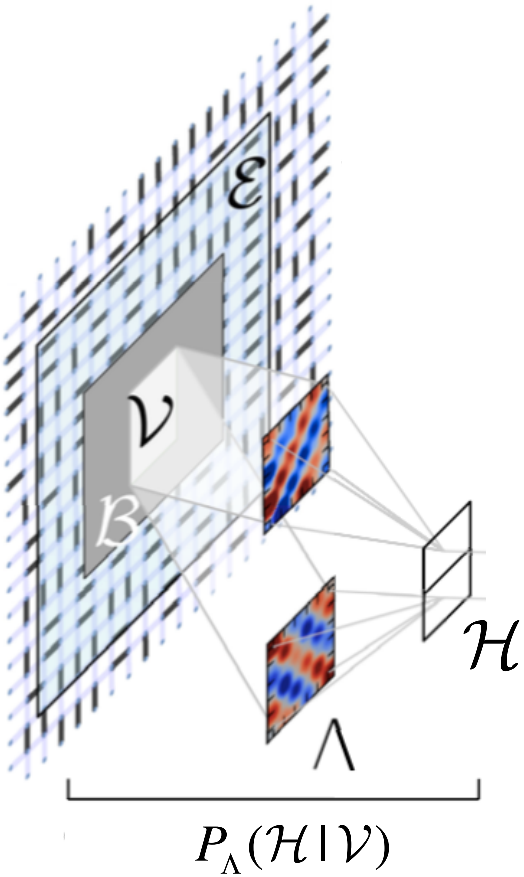

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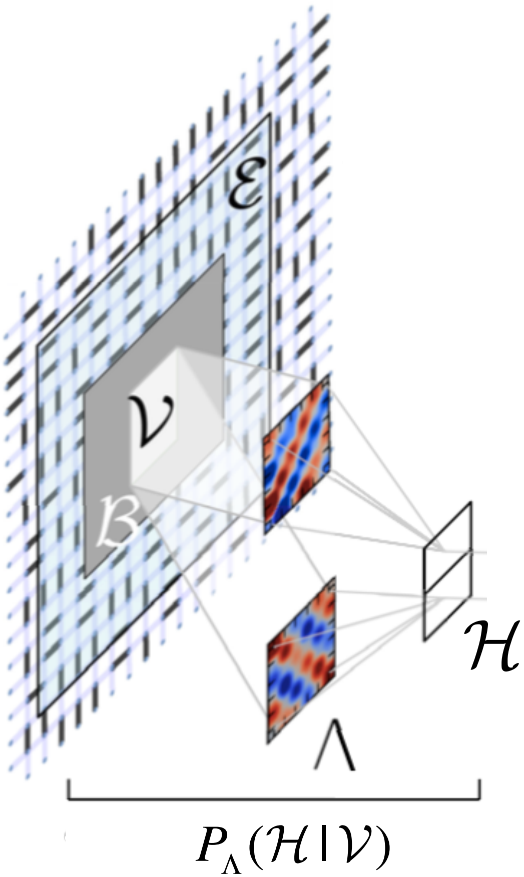

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The Real-Space Mutual Information (RSMI)

Nature Physics ***14****, 578-582 (2018)*

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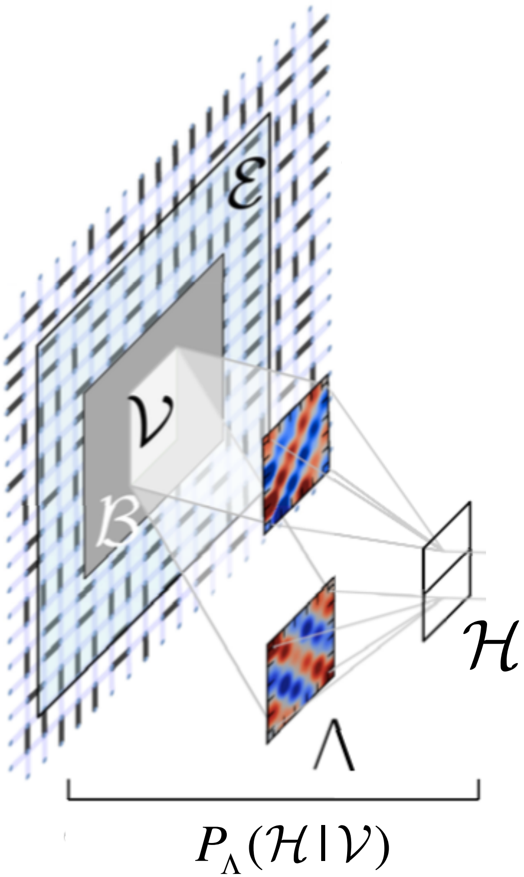

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Insight: the optimal $P_\Lambda(\mathcal{H}|\mathcal{V})$ give access to the RG-relevant operators

Phys.Rev.Lett.126, 240601 (2021)

The three ingredients

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- The physical principle:
  lossy compression
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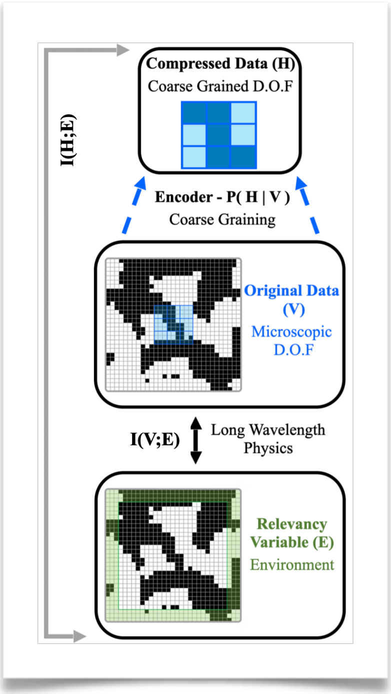

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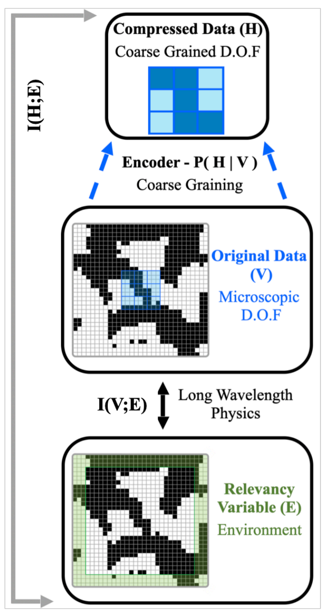
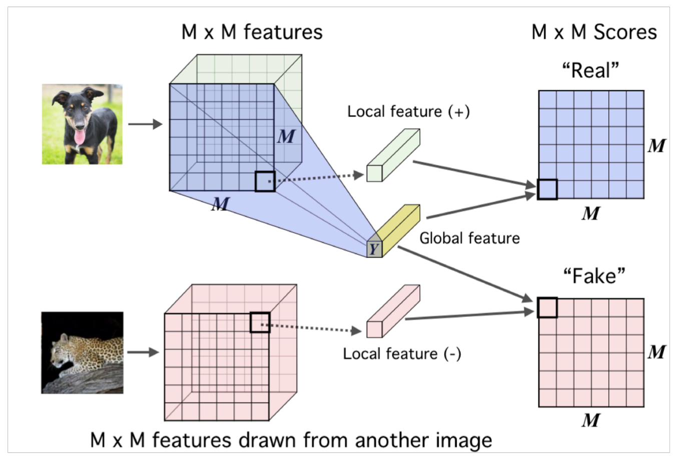


- The estimator of mutual information



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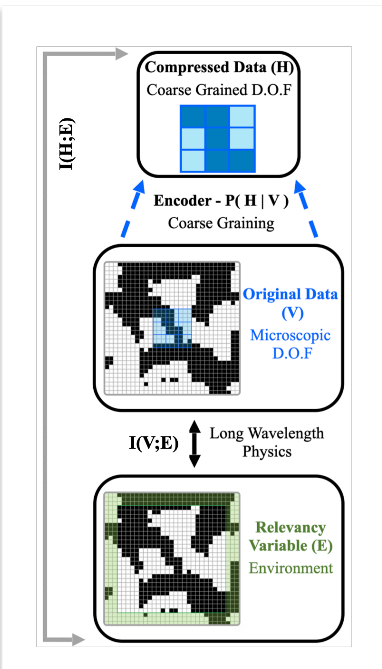
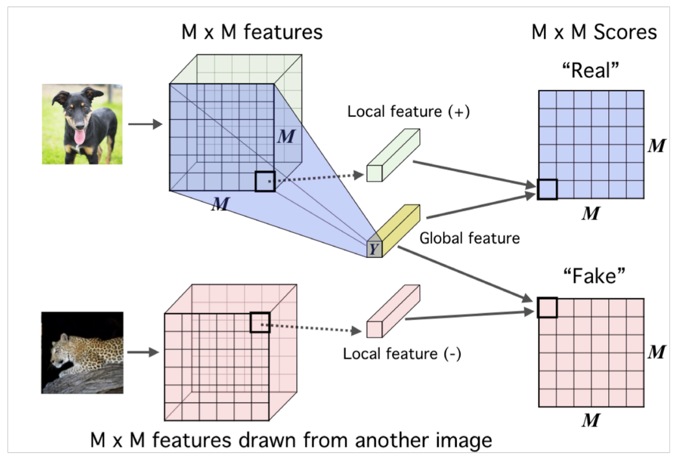
The three ingredients

- The physical principle: **lossy compression** maximising **I(H:E)**
- The estimator of mutual information
- The coarse-graining ansatz family **P(H I V)**





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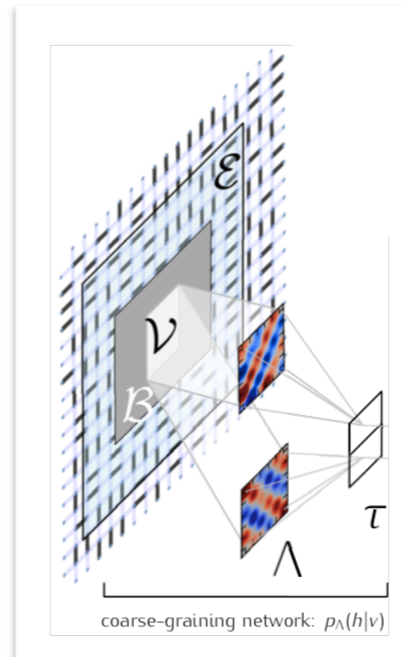

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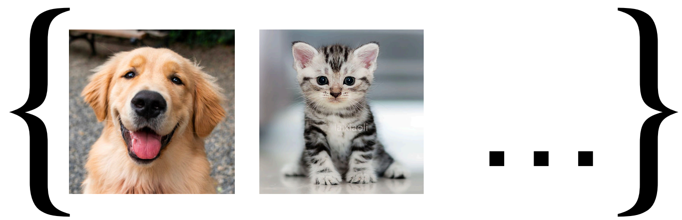

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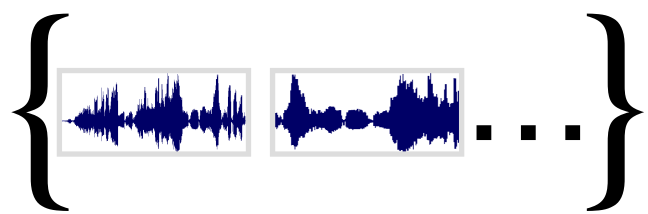
dim $d = 4 \ll N_{px}$

So that ***H*** retains **relevant** information for the down-stream task, **implicitly defined** by correlations with ***Y***



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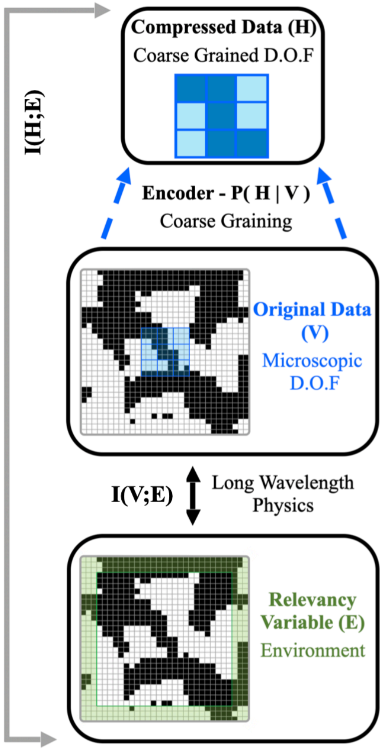

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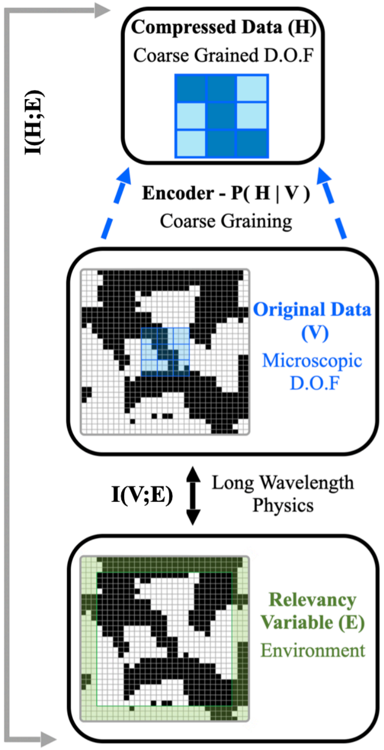
$$\min_{P(H|V)} \mathcal{L}_{IB}[P(H|V)] \equiv \min_{P(H|V)} I(V;H) - \beta I(H;E)$$



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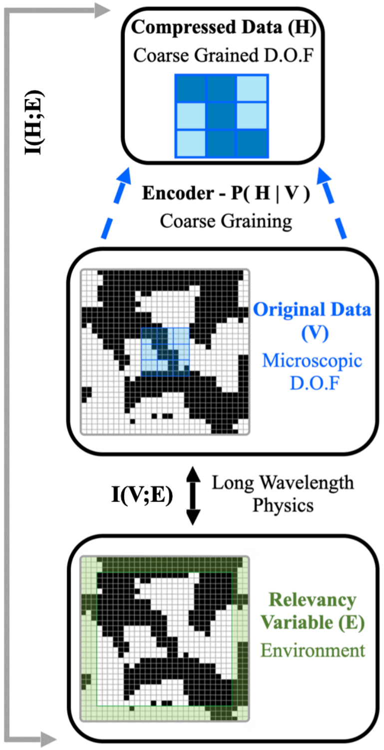

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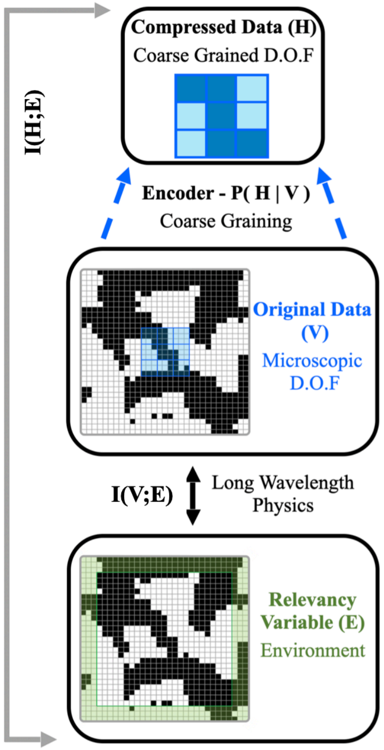

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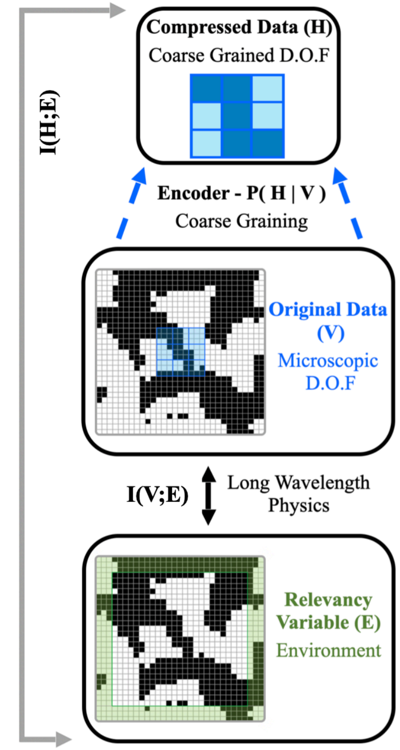

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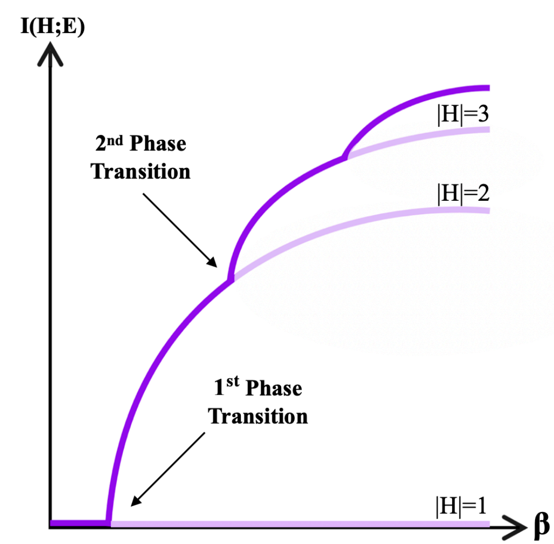

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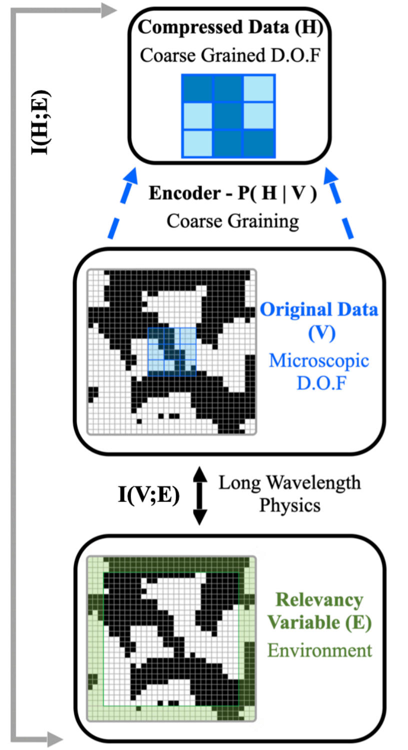

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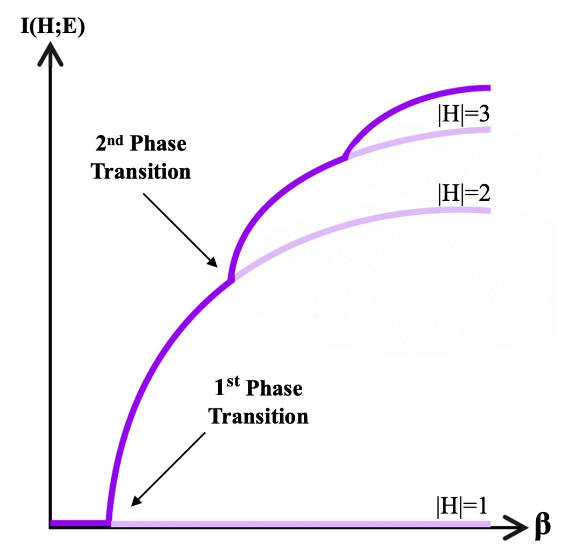
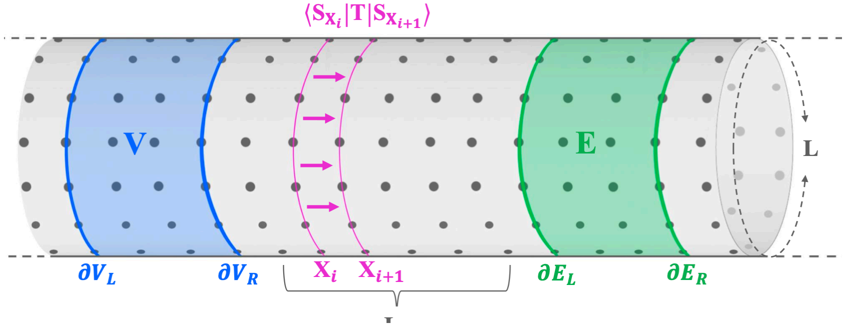
- RSMI arises in the infinite β limit, and finite alphabet



Equivalence of the IB and RG relevance



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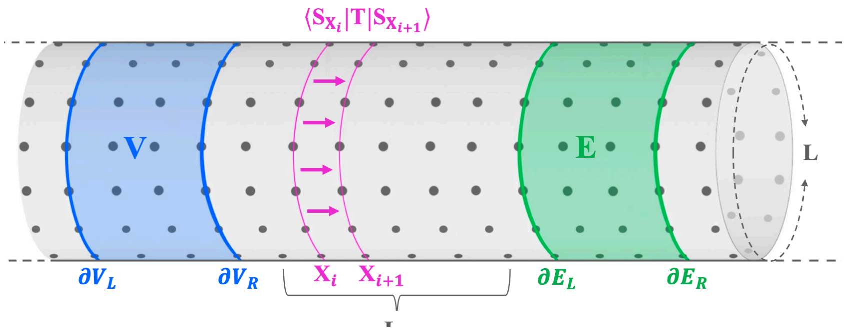

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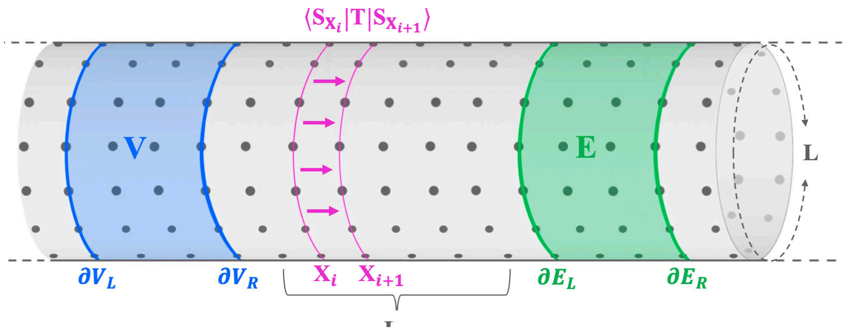

- Distributions in IB equations can be written using **transfer matrices** $Z = \mathrm{Tr}\left[\mathbb{T}^{L_\infty}\right]$
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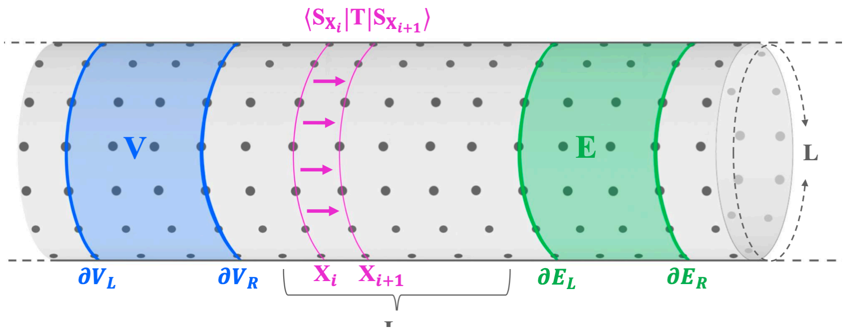

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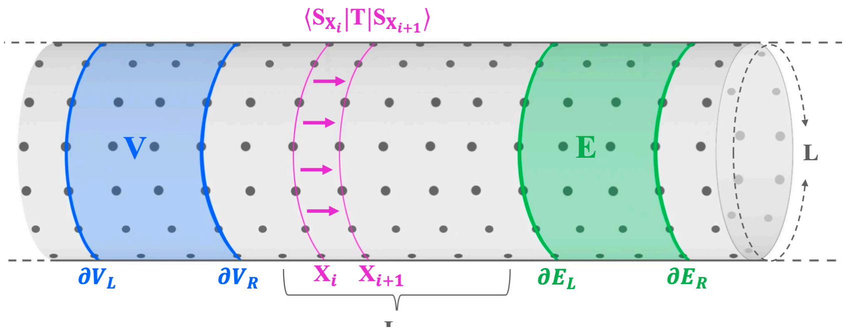

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$$r_v = \frac{\langle 1|\partial V_R\rangle}{\langle 0|\partial V_R\rangle} = \frac{\langle 0|\phi_{\Delta_1}|\partial V_R\rangle}{\langle 0|\partial V_R\rangle}$$
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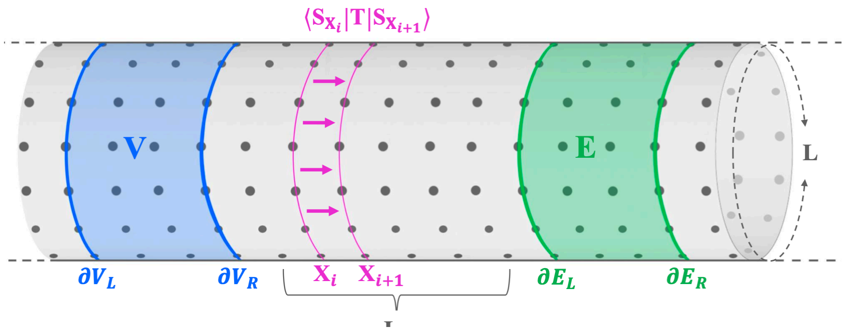

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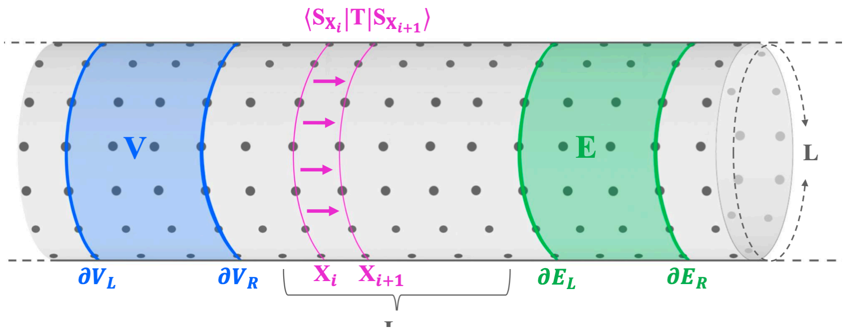
Gordon, Banerjee, **MKJ,** Ringel, *PRL **126**, 240601 (2021)*

Equivalence of the IB and RG relevance



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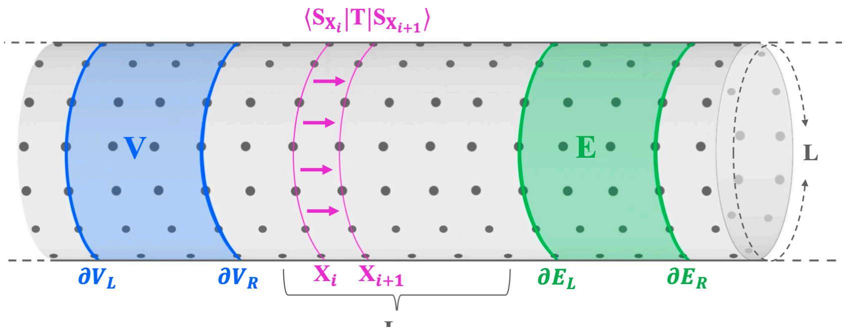

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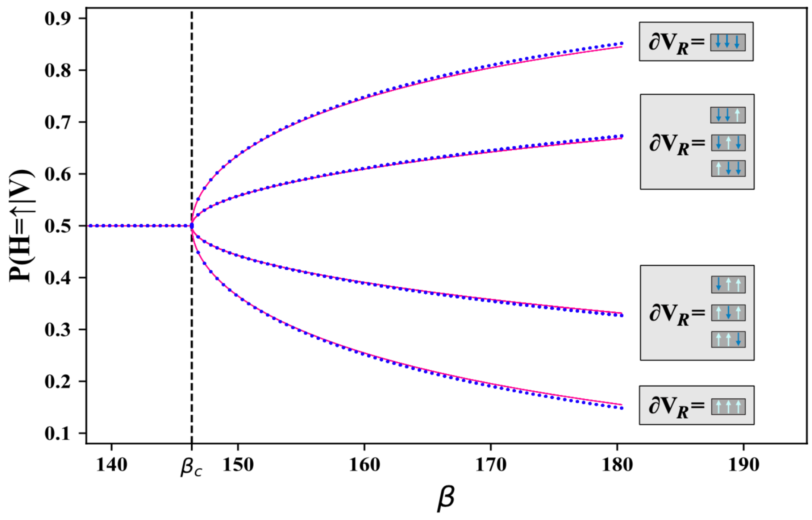

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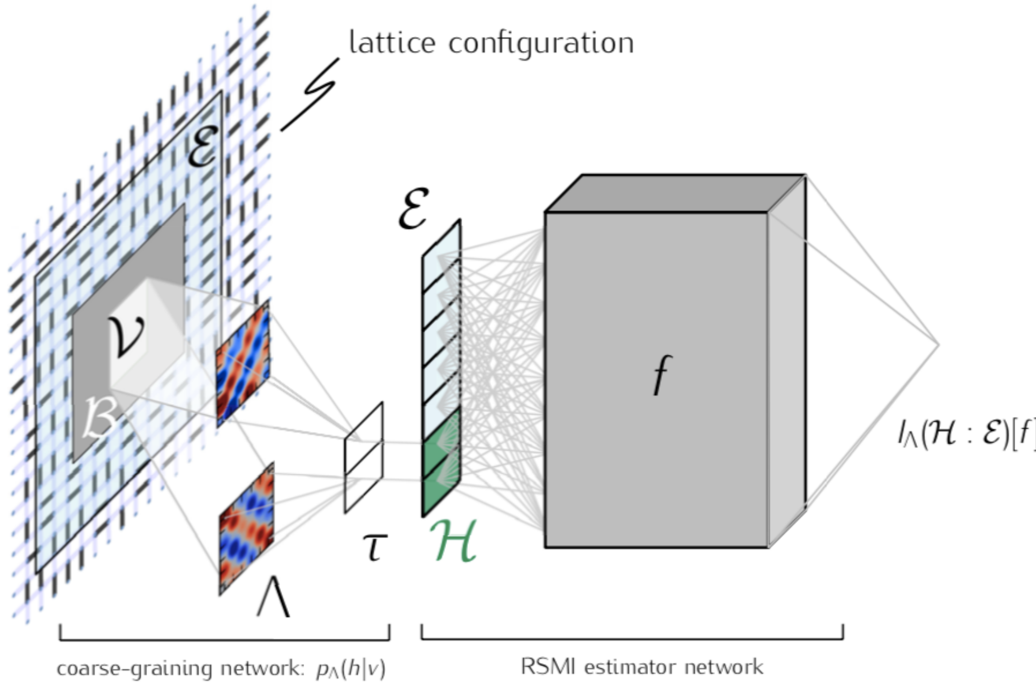
- We get a **parametric, differentiable**, and **tight** lower bound on MI

The RSMI-NE network



https://github.com/RSMI-NE/RSMI-NE

Phys.Rev. E, 104, 064106 (2021)
Phys.Rev.Lett 127, 240603 (2021)

The RSMI-NE network

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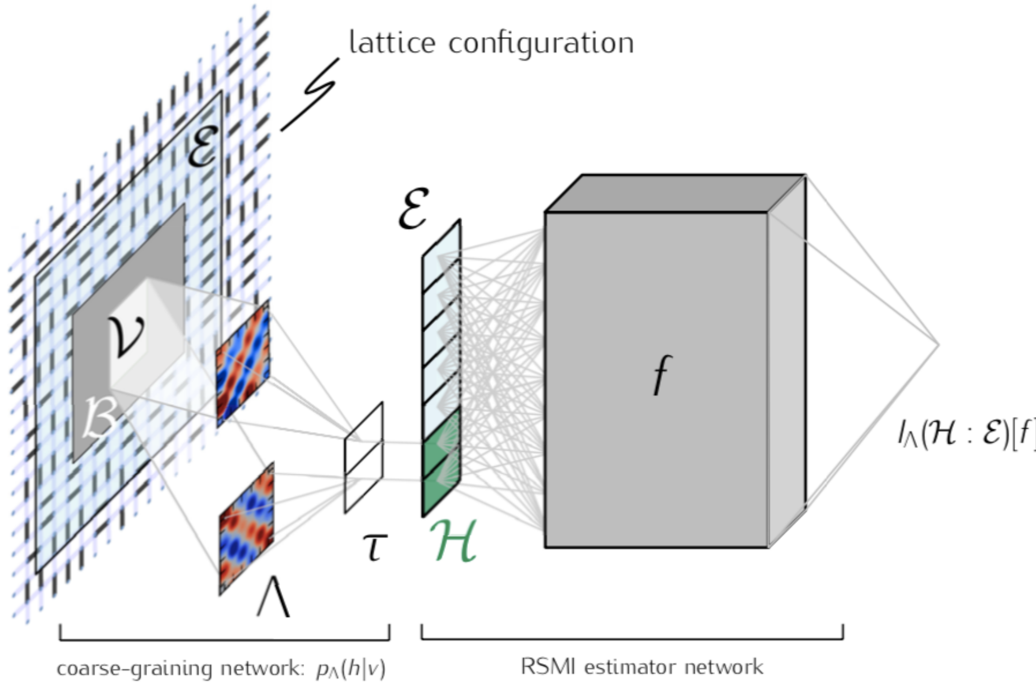

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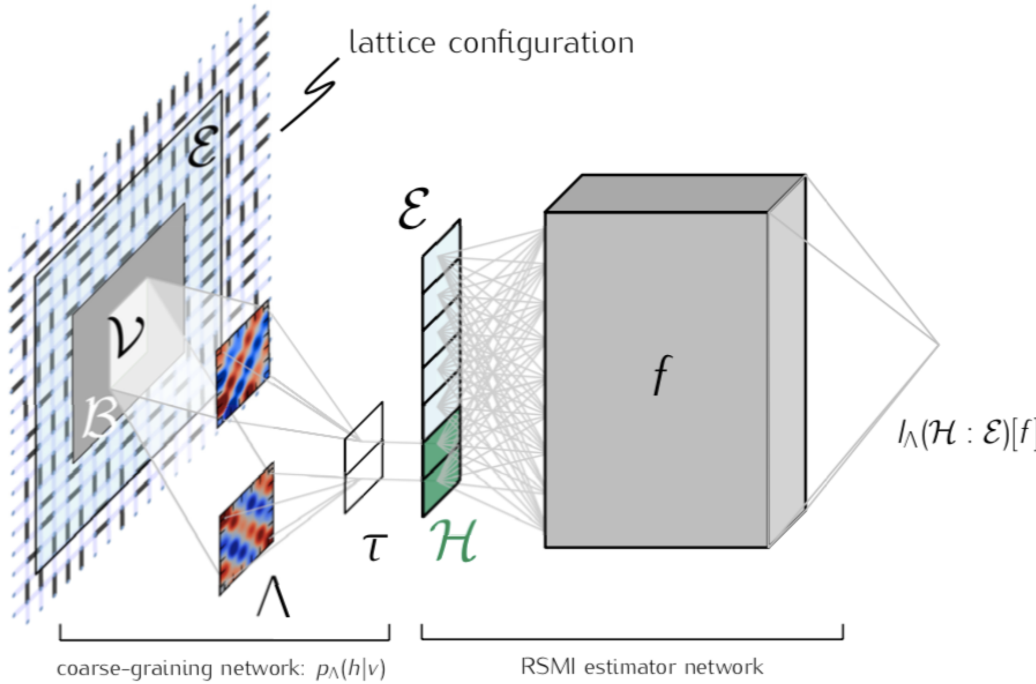

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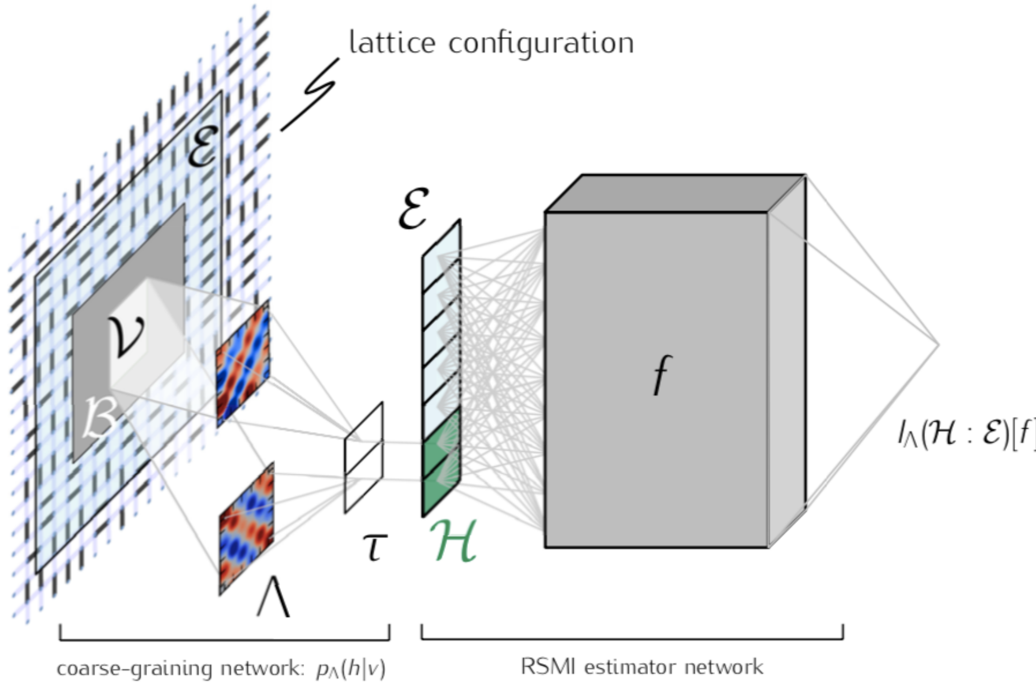

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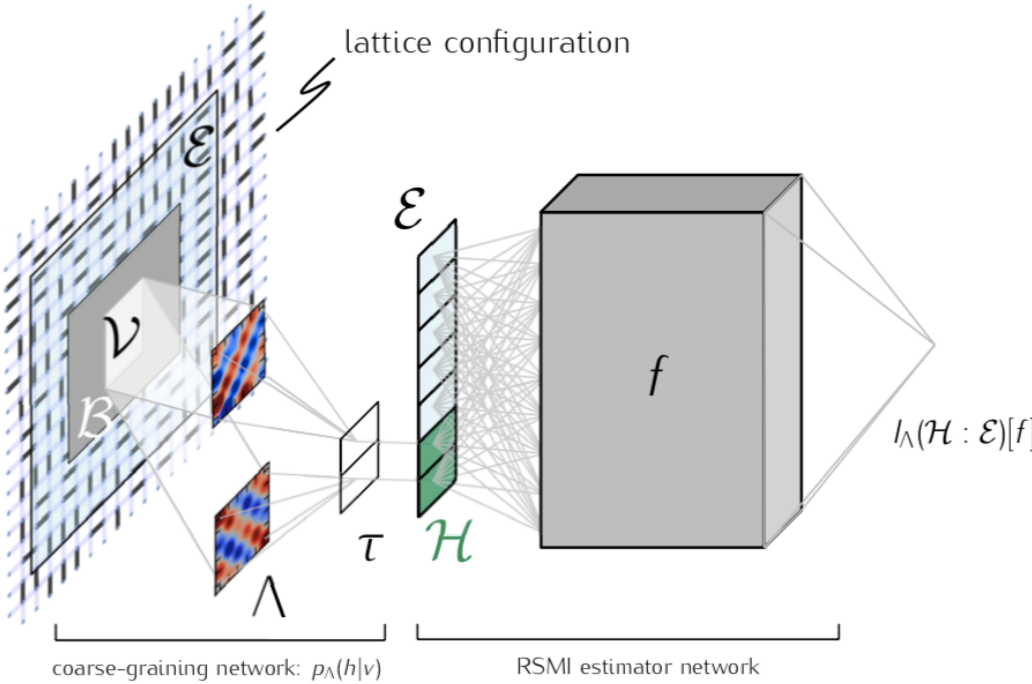

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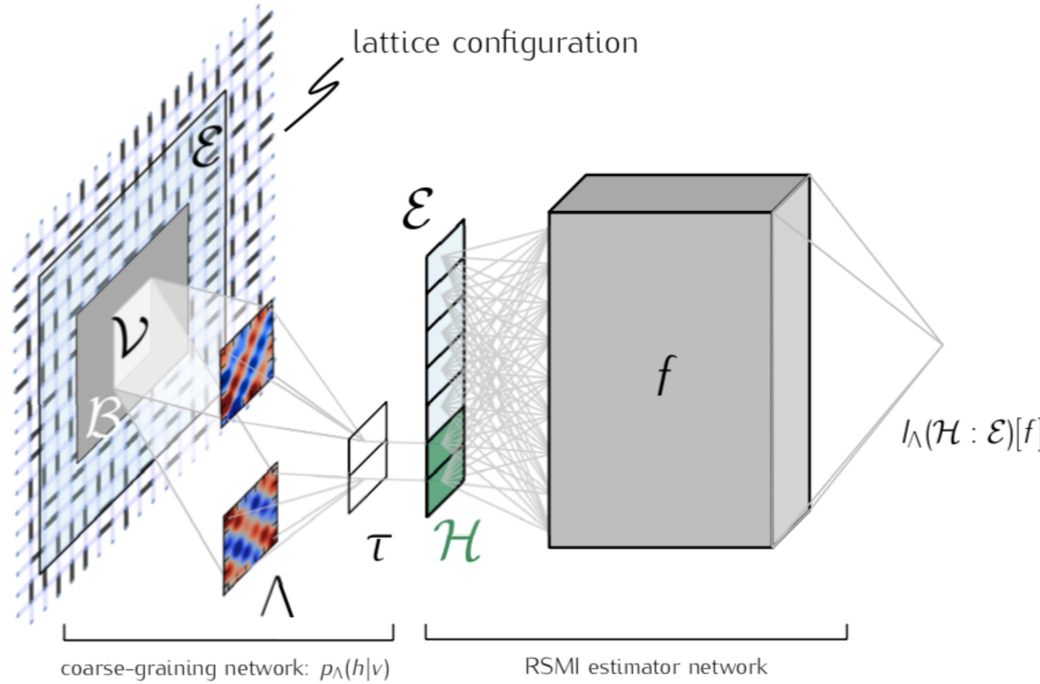

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- Co-trained with SGD as a single network **(differentiable, upper bounded!)**

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Example: the interacting dimer model

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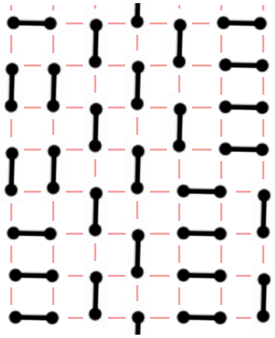

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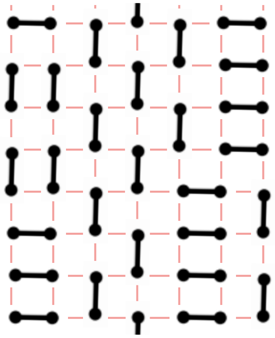

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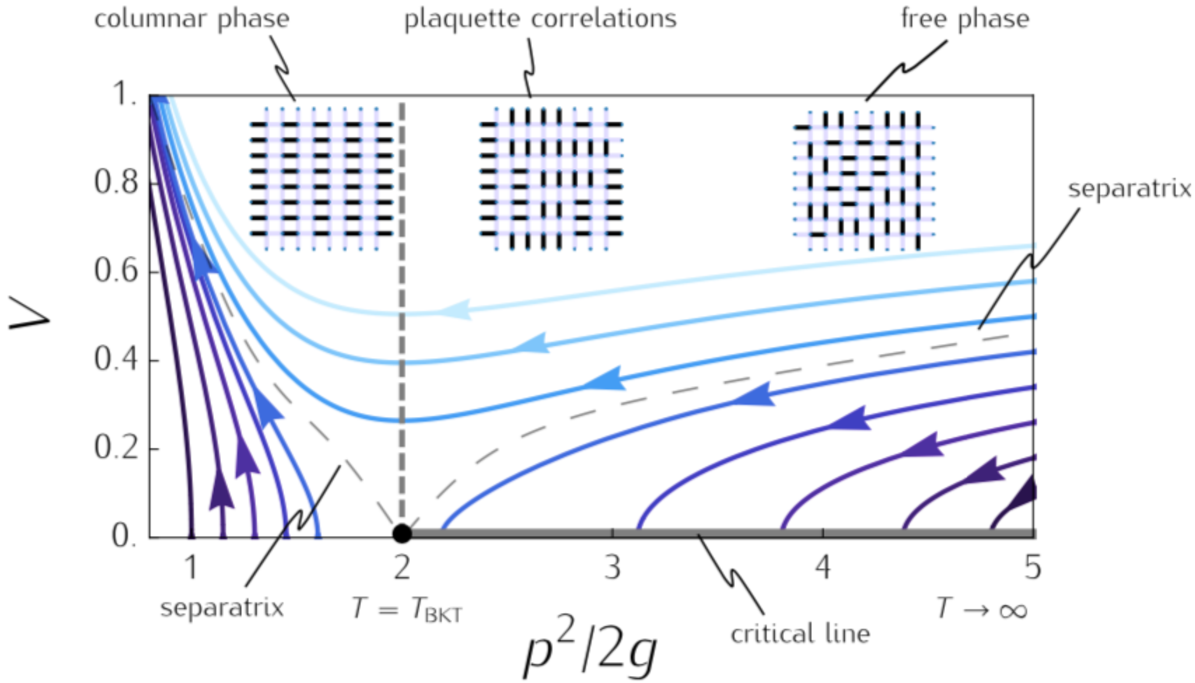

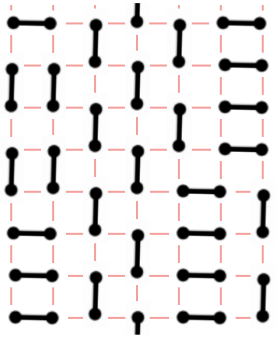

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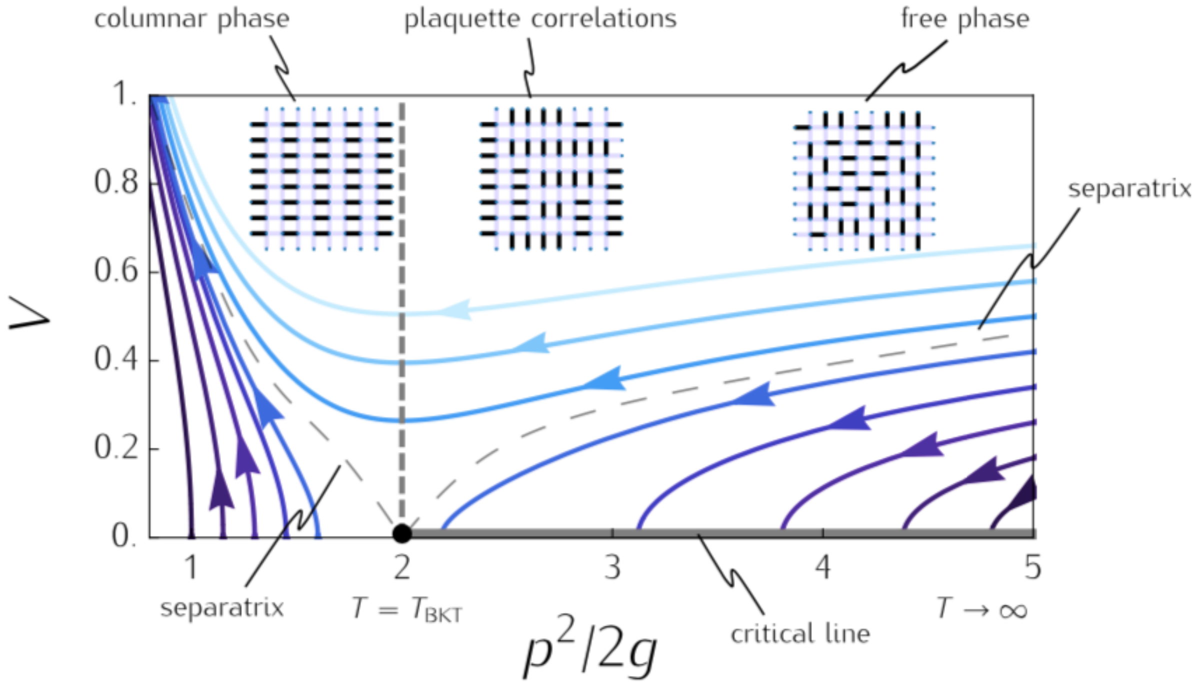

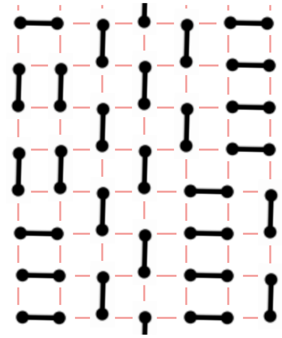

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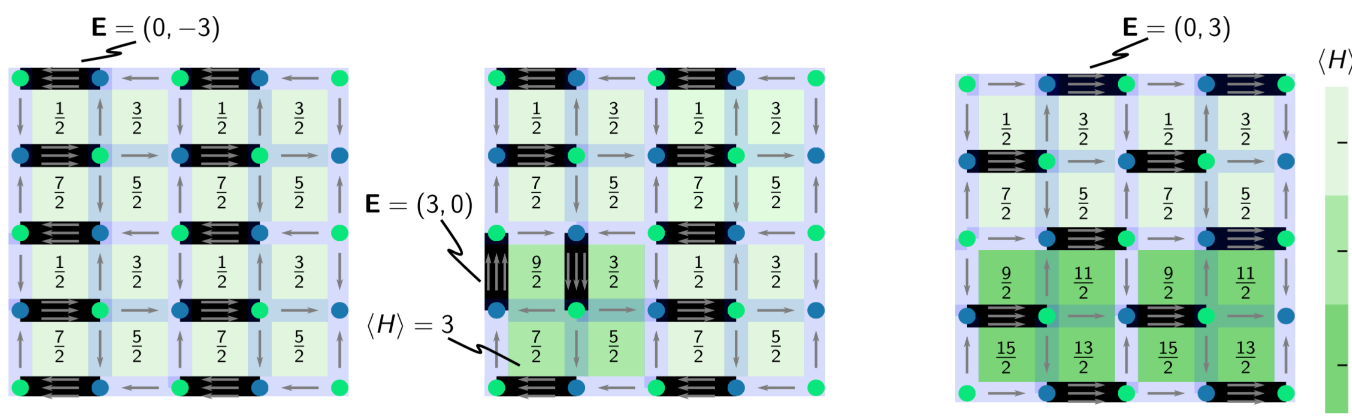
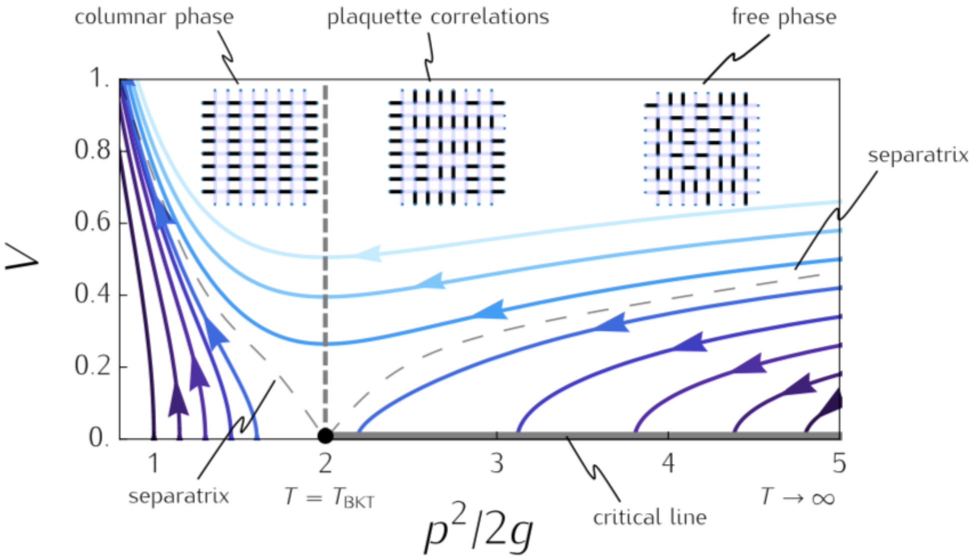
RG of dimer model:
mapping to height field





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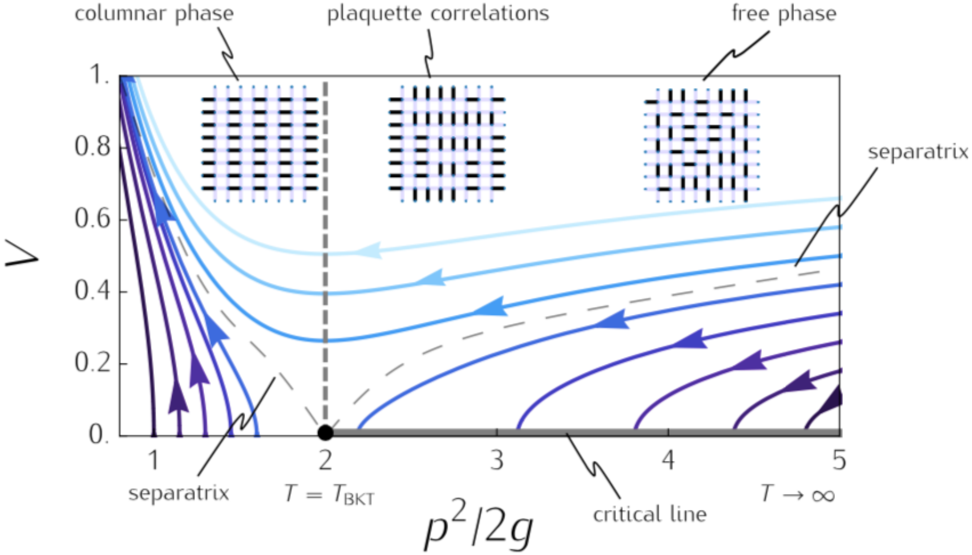
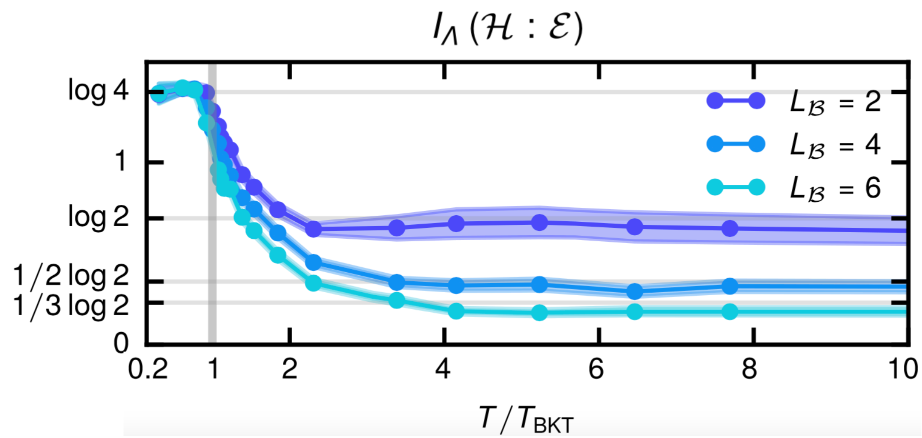


- Total RSMI with the *optimal* filter



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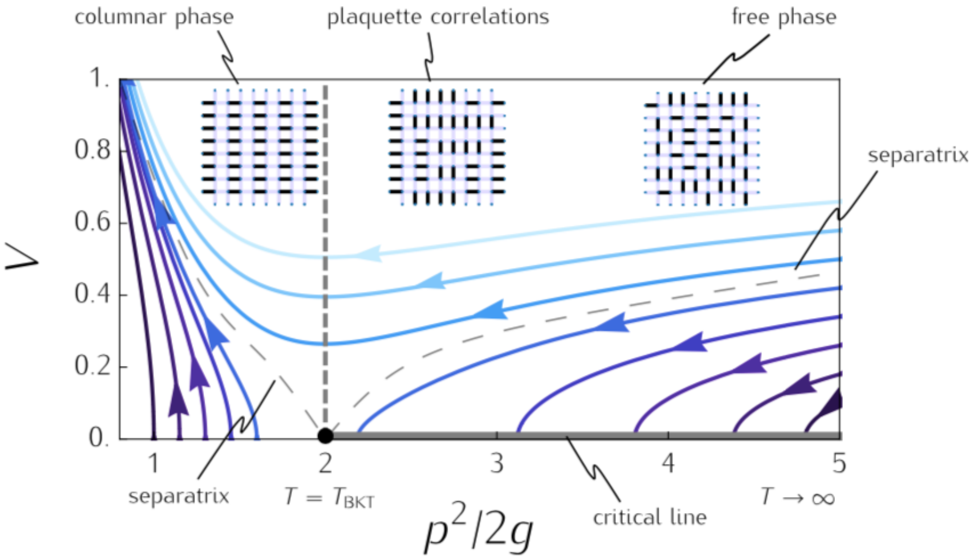

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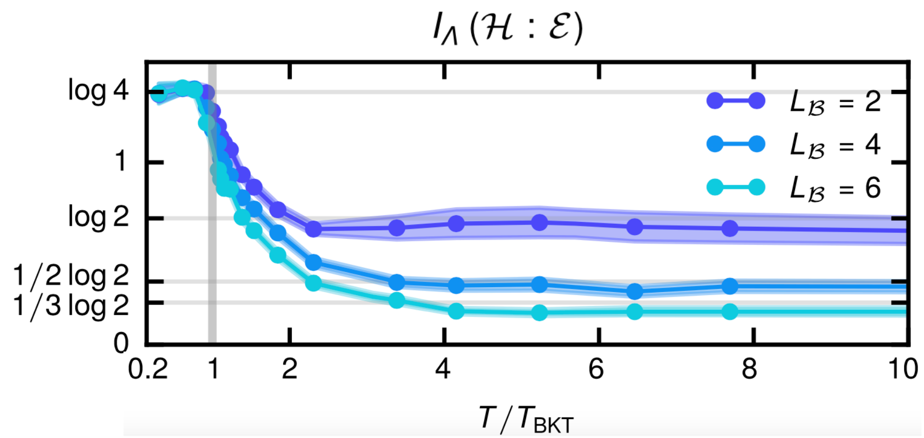
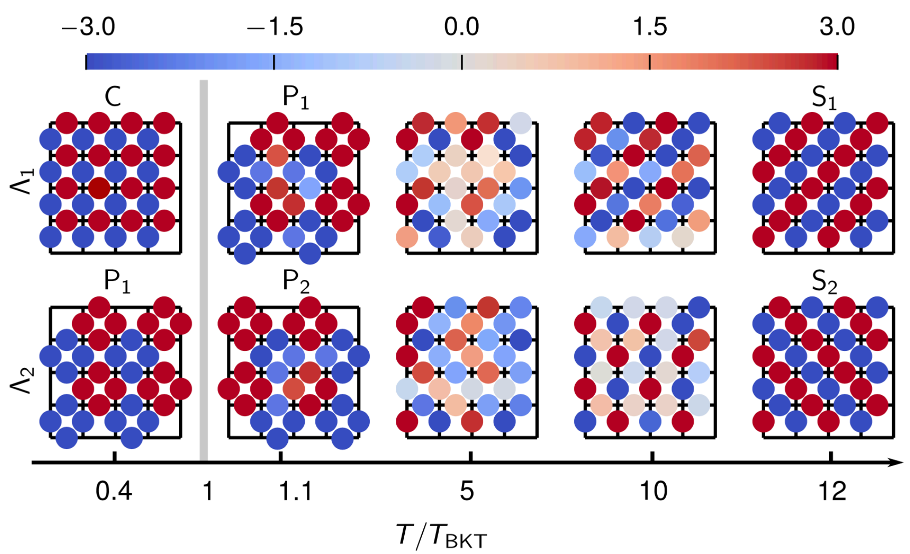


- The optimal filters ***depend*** on T



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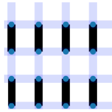
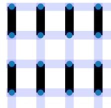
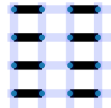
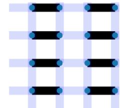
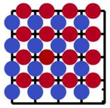
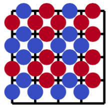
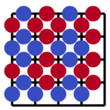
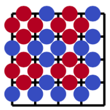
Phys.Rev.Lett 127, 240603 (2021)

- Pairs of C/P filters **label broken symmetry** states

$C(\mathbf{r})$				
Λ_{C}  Λ_{P1} 	$(-1,-1)$	$(-1,+1)$	$(+1,-1)$	$(+1,+1)$
Λ_{P1}  Λ_{P2} 	$(-1,-1)$	$(+1,+1)$	$(-1,+1)$	$(+1,-1)$

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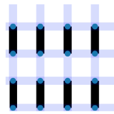
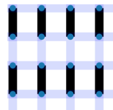
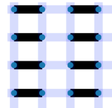
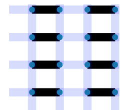
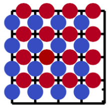
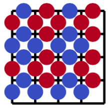
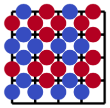
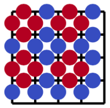
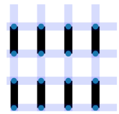
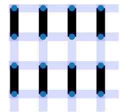
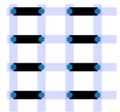
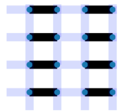
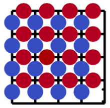
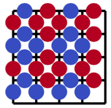
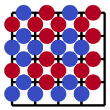
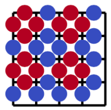
$C(\mathbf{r})$				
Λ_{C}  Λ_{P1} 	$(-1,-1)$	$(-1,+1)$	$(+1,-1)$	$(+1,+1)$
Λ_{P1}  Λ_{P2} 	$(-1,-1)$	$(+1,+1)$	$(-1,+1)$	$(+1,-1)$

- Filters **define** order parameters:

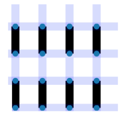
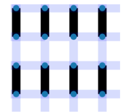
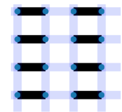
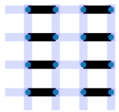
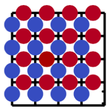
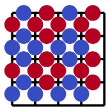
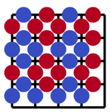
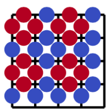
- Pairs of C/P filters **label broken symmetry states**

$C(\mathbf{r})$				
Λ_{C}  Λ_{P1} 	$(-1,-1)$	$(-1,+1)$	$(+1,-1)$	$(+1,+1)$
Λ_{P1}  Λ_{P2} 	$(-1,-1)$	$(+1,+1)$	$(-1,+1)$	$(+1,-1)$

- Filters **define** order parameters:

$$D_i := \mathbb{E}\left[\frac{1}{N_{\mathcal{V}}}\sum_k \tau \circ (\Lambda_i \cdot \mathcal{V}_k)\right]$$

- Pairs of C/P filters **label broken symmetry** states

$C(\mathbf{r})$				
Λ_C  Λ_{P1} 	$(-1,-1)$	$(-1,+1)$	$(+1,-1)$	$(+1,+1)$
Λ_{P1}  Λ_{P2} 	$(-1,-1)$	$(+1,+1)$	$(-1,+1)$	$(+1,-1)$

- Filters **define** order parameters:

$$D_i := \mathbb{E}\left[\frac{1}{N_{\mathcal{V}}}\sum_k \tau \circ (\Lambda_i \cdot \mathcal{V}_k)\right]$$

$$\mathrm{DSB} := \mathbb{E}\left[\sum_k \tau \circ (\Lambda_{\mathrm{C}} \cdot \mathcal{V}_k)\right]$$

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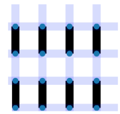
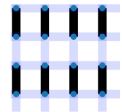
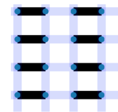
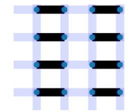
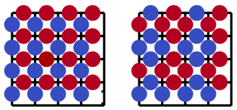
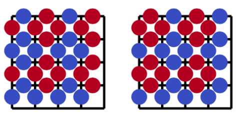
- Pairs of C/P filters **label broken symmetry states**

$C(\mathbf{r})$				
Λ_C Λ_{P1} 	$(-1,-1)$	$(-1,+1)$	$(+1,-1)$	$(+1,+1)$
Λ_{P1} Λ_{P2} 	$(-1,-1)$	$(+1,+1)$	$(-1,+1)$	$(+1,-1)$

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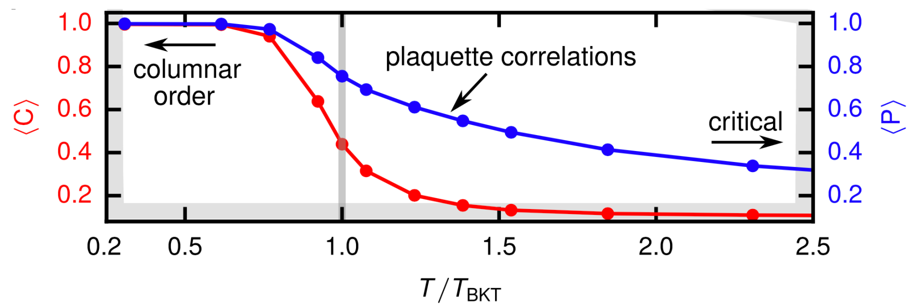
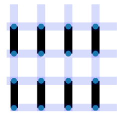
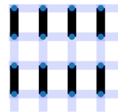
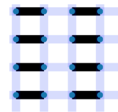
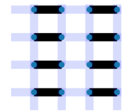
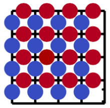
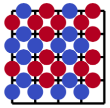
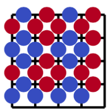
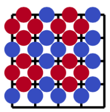


- Pairs of C/P filters **label broken symmetry states**

$C(\mathbf{r})$				
Λ_C  Λ_{P1} 	$(-1,-1)$	$(-1,+1)$	$(+1,-1)$	$(+1,+1)$
Λ_{P1}  Λ_{P2} 	$(-1,-1)$	$(+1,+1)$	$(-1,+1)$	$(+1,-1)$

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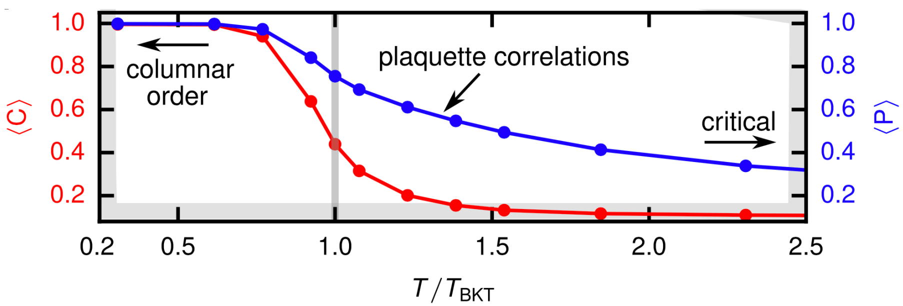
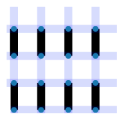
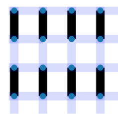
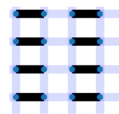
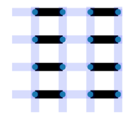
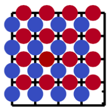
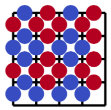
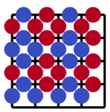
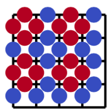


- Filters **are** relevant operators:

- Pairs of C/P filters **label broken symmetry states**

$C(\mathbf{r})$				
Λ_C  Λ_{P1} 	$(-1,-1)$	$(-1,+1)$	$(+1,-1)$	$(+1,+1)$
Λ_{P1}  Λ_{P2} 	$(-1,-1)$	$(+1,+1)$	$(-1,+1)$	$(+1,-1)$

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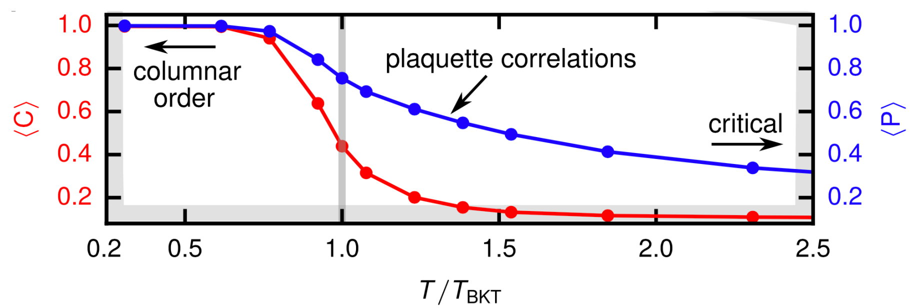

- Filters **are** relevant operators:

$$\mathcal{O}_n(\varphi) = (\cos(n\varphi), \sin(n\varphi))$$

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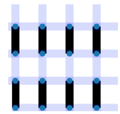
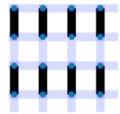
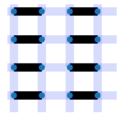
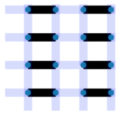
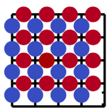
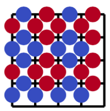
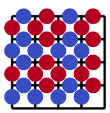
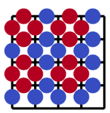
- Pairs of C/P filters **label broken symmetry** states

$C(\mathbf{r})$				
Λ_C  Λ_{P1} 	$(-1,-1)$	$(-1,+1)$	$(+1,-1)$	$(+1,+1)$
Λ_{P1}  Λ_{P2} 	$(-1,-1)$	$(+1,+1)$	$(-1,+1)$	$(+1,-1)$
$\varphi(\mathbf{r})$	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	π	0
$\mathcal{O}_1(\varphi) = (\cos\varphi, \sin\varphi)$	$(0,1)$	$(0,-1)$	$(-1,0)$	$(+1,0)$
$\mathcal{O}_2(\varphi) = \cos(2\varphi)$	-1	-1	$+1$	$+1$

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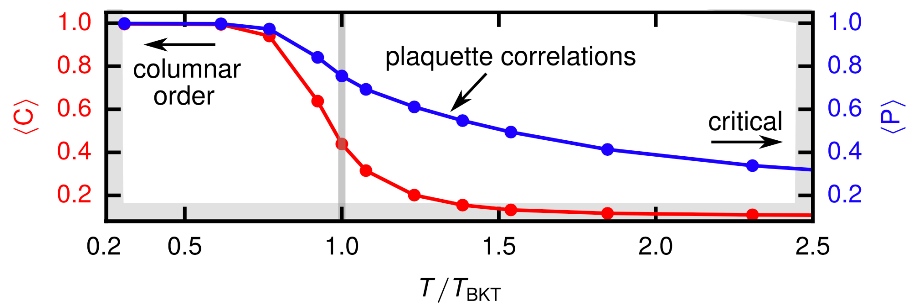

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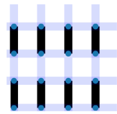
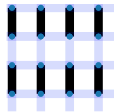
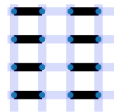
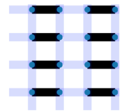
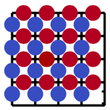
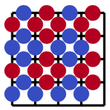
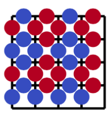
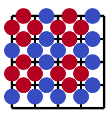
Papanikolaou et al. **PRB 76**, 134514 (2007)

- Pairs of C/P filters **label broken symmetry states**

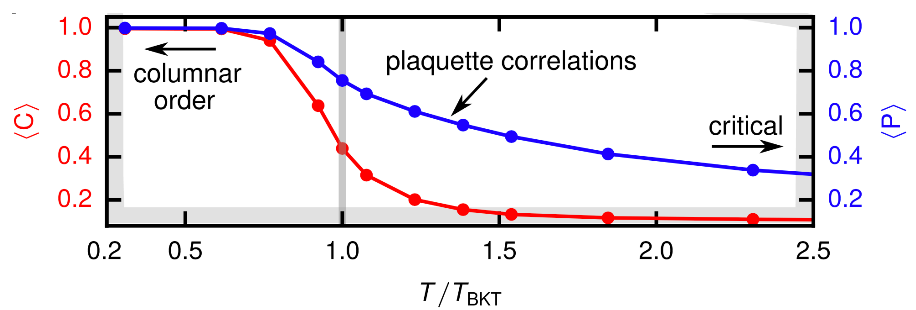
$C(\mathbf{r})$				
Λ_C  Λ_{P1} 	$(-1,-1)$	$(-1,+1)$	$(+1,-1)$	$(+1,+1)$
Λ_{P1}  Λ_{P2} 	$(-1,-1)$	$(+1,+1)$	$(-1,+1)$	$(+1,-1)$
$\varphi(\mathbf{r})$	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	π	0
$\mathcal{O}_1(\varphi) = (\cos\varphi, \sin\varphi)$	$(0,1)$	$(0,-1)$	$(-1,0)$	$(+1,0)$
$\mathcal{O}_2(\varphi) = \cos(2\varphi)$	-1	-1	$+1$	$+1$

- Filters **define** order parameters:

$$D_i := \mathbb{E}\left[\frac{1}{N_\mathcal{V}}\sum_k \tau \circ (\Lambda_i \cdot \mathcal{V}_k)\right]$$

$$\mathrm{DSB} := \mathbb{E}\left[\sum_k \tau \circ (\Lambda_\mathrm{C} \cdot \mathcal{V}_k)\right]$$

Alet et al. **PRE 74**, 041124 (2006)



- Filters **are** relevant operators:

$$\mathcal{O}_n(\varphi) = (\cos(n\varphi), \sin(n\varphi))$$

Papanikolaou et al. **PRB 76**, 134514 (2007)

$$(\Lambda_\mathrm{P1}, \Lambda_\mathrm{P1}) \circ \varphi = (\cos(\varphi + \pi/4), \sin(\varphi + \pi/4))$$

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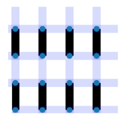
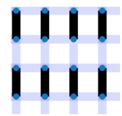
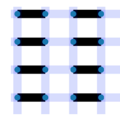
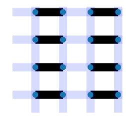
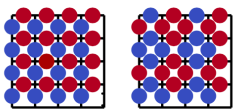
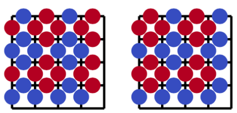
$C(\mathbf{r})$				
Λ_{C} Λ_{P1} 	$(-1,-1)$	$(-1,+1)$	$(+1,-1)$	$(+1,+1)$
Λ_{P1} Λ_{P2} 	$(-1,-1)$	$(+1,+1)$	$(-1,+1)$	$(+1,-1)$
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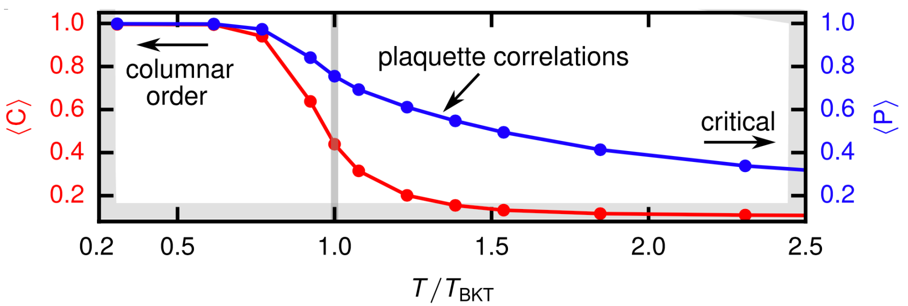

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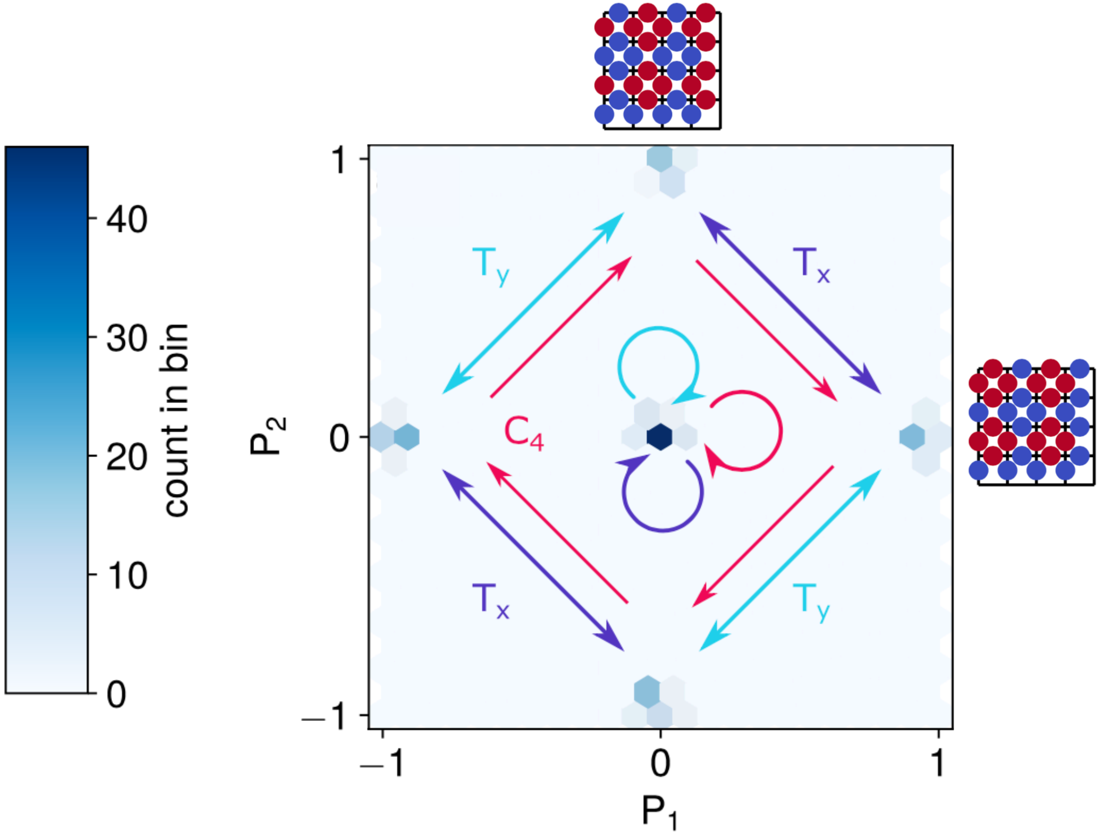
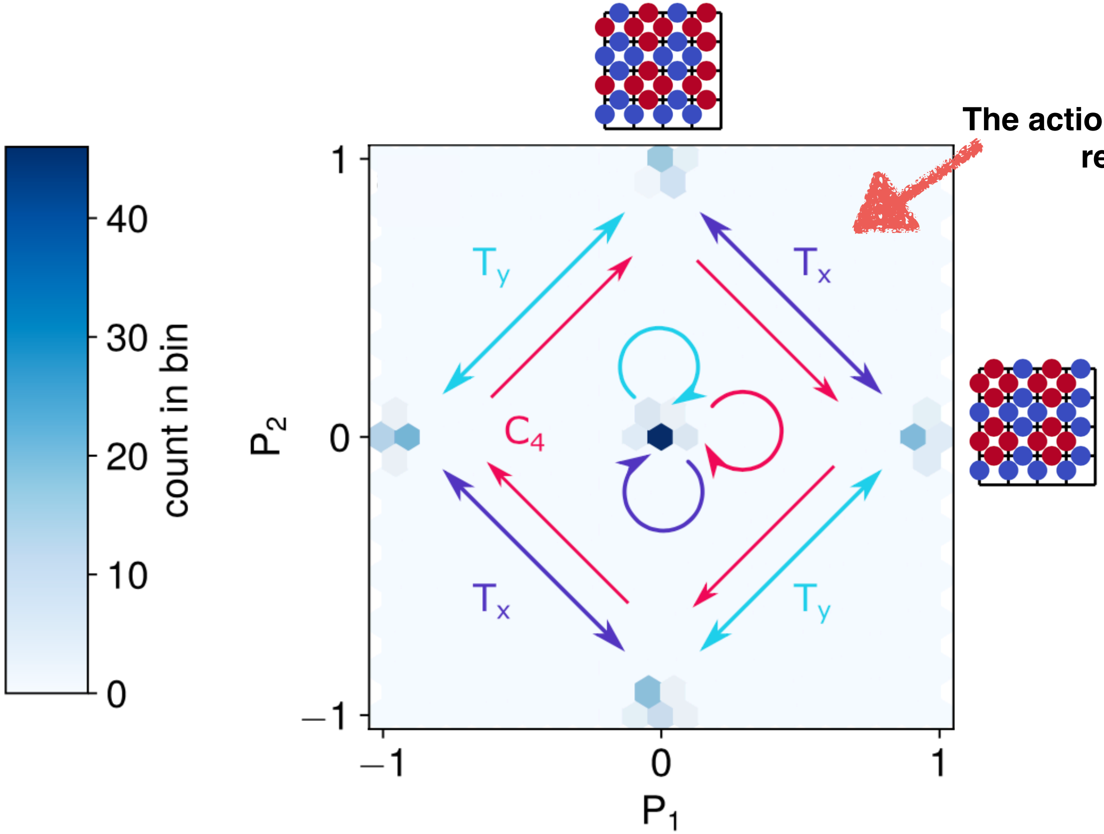
(Also: staggered filters are gradients of the height field)

Symmetries in the RSMI ensemble



PRE, 104, 064106 (2021)
PRL 127, 240603 (2021)

Symmetries in the RSMI ensemble



The action of symmetries of the physical state are represented on the space of filters

PRE, 104, 064106 (2021)
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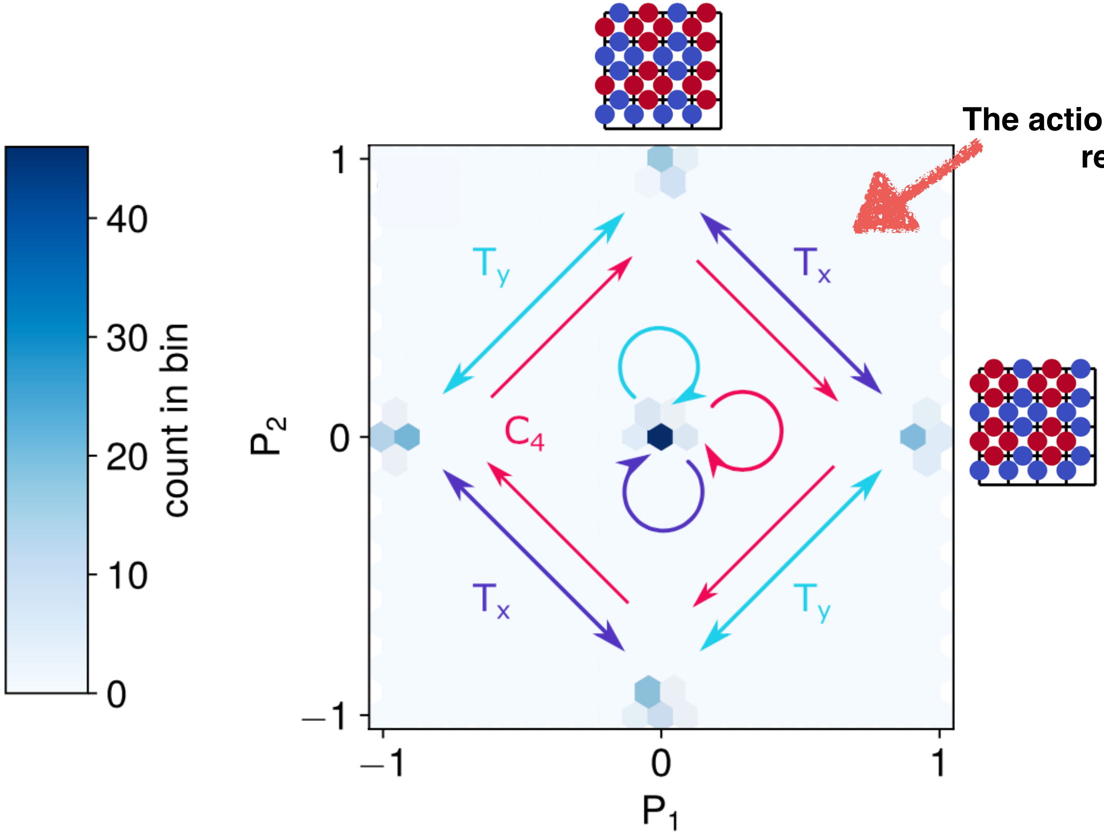

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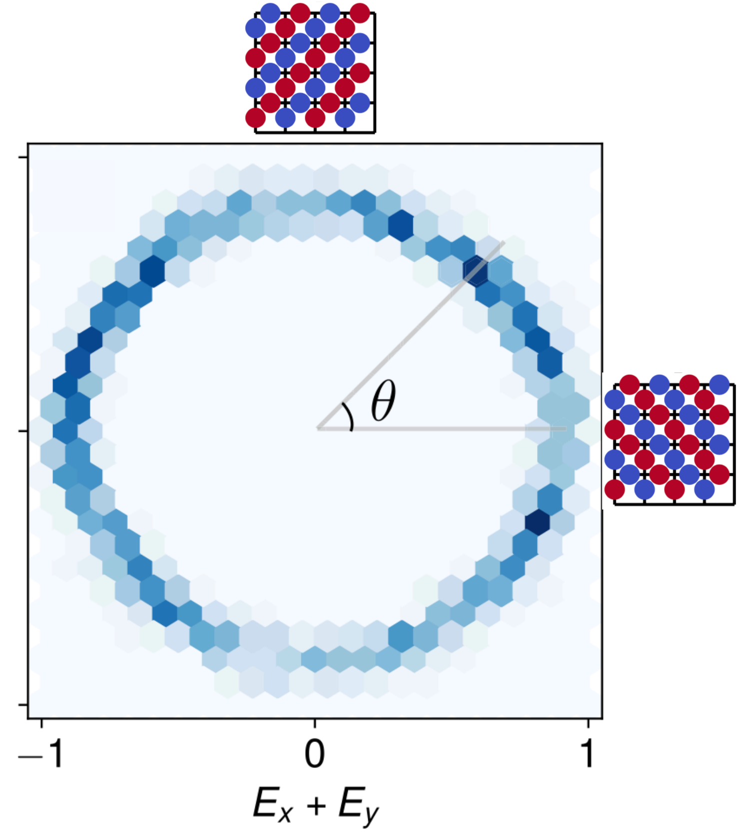

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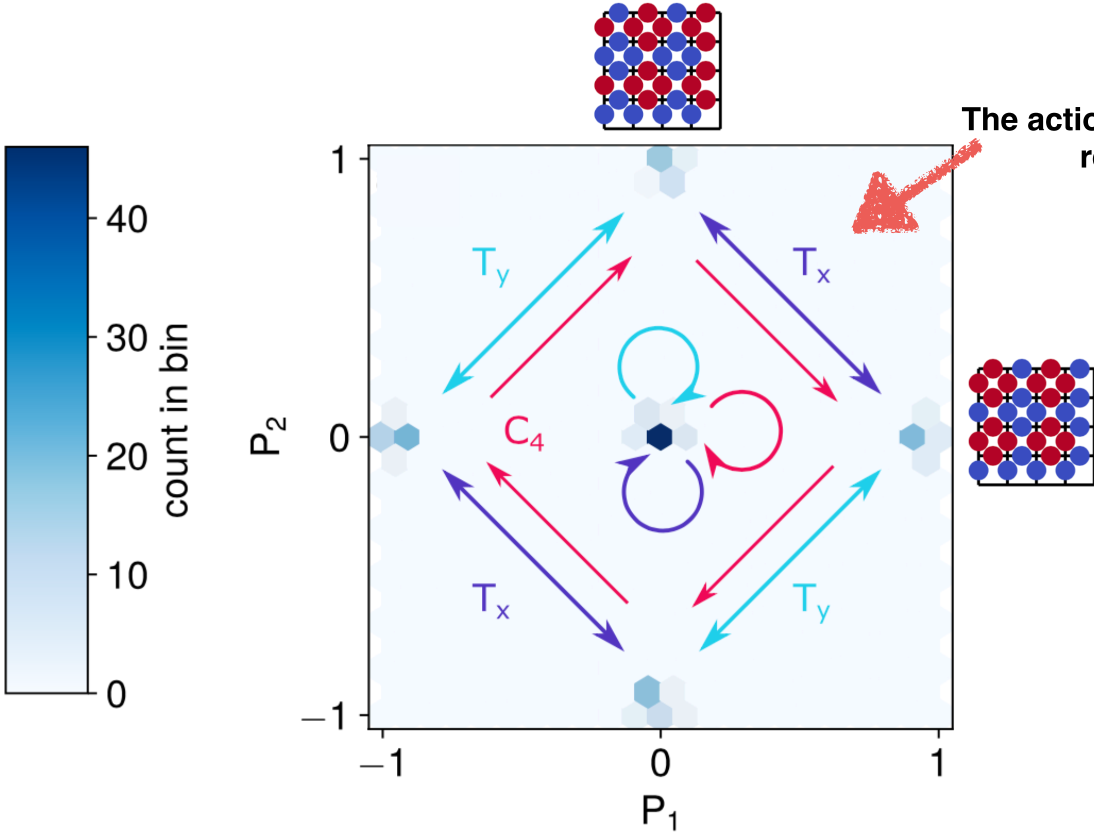
Symmetries in the RSMI ensemble



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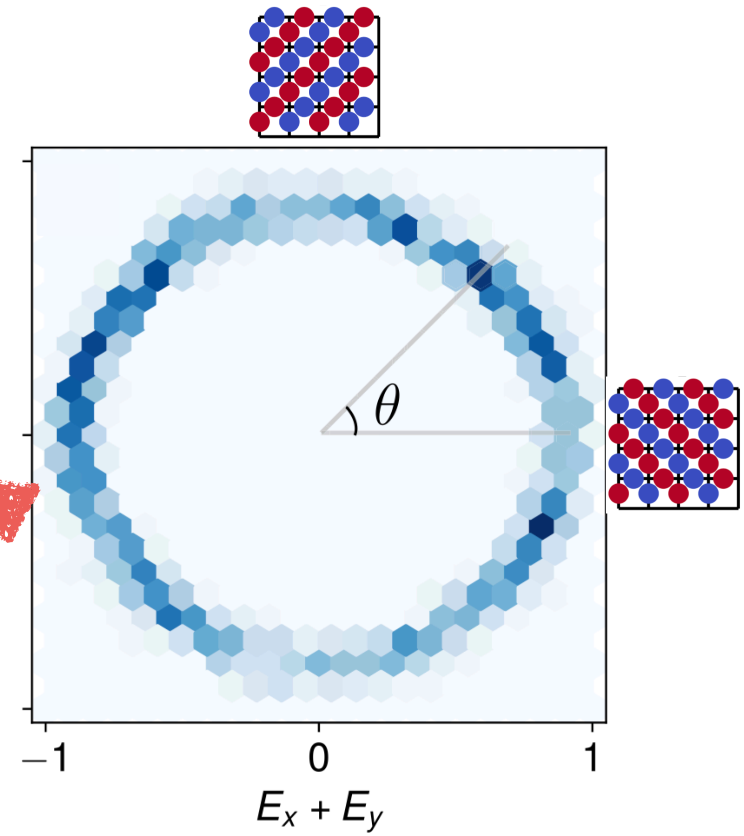

Emergent U(1):

$$R(\vartheta)(\Lambda_{S_1}, \Lambda_{S_2}) \sim [\cos(\vartheta)E_x - \sin(\vartheta)E_y, \sin(\vartheta)E_x + \cos(\vartheta)E_y]$$

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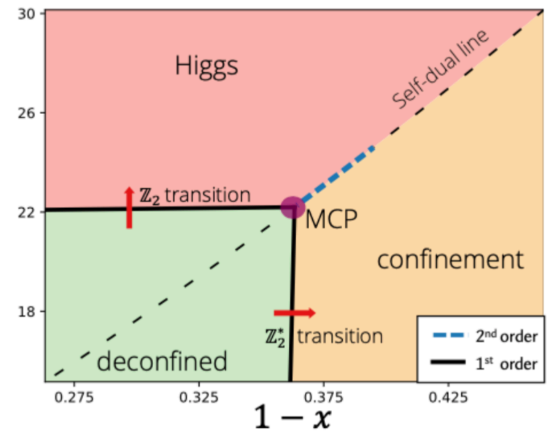
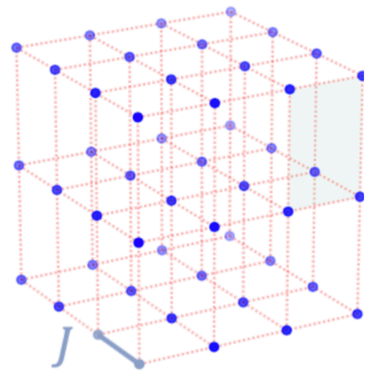
Lattice gauge theories

L. Oppenheim, **MKJ**, S. Gazit, Z. Ringel *arXiv:2311.17994*

Lattice gauge theories

$$\mathcal{S}_{\text{SD-IHG}} = K \sum_{\square} \prod_{\langle i,j \rangle \in \square} \sigma_{ij} + J \sum_{\langle i,j \rangle} \tau_i \sigma_{ij} \tau_j$$





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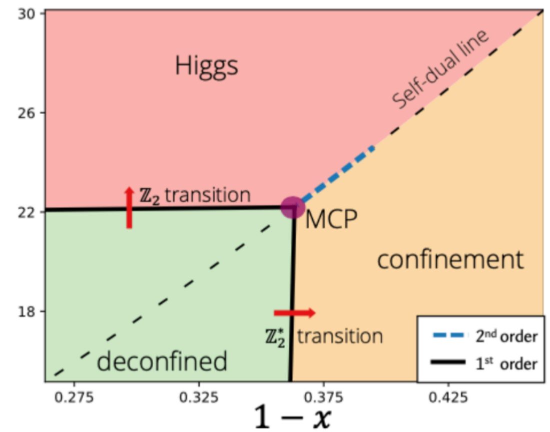
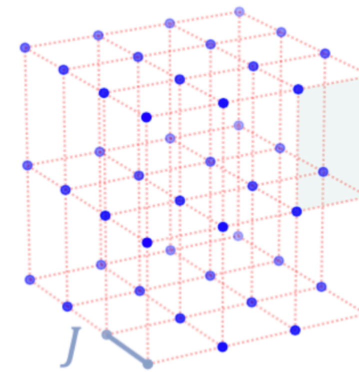
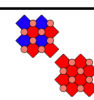
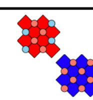
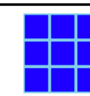
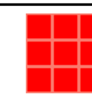
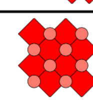
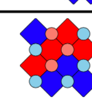
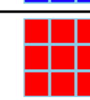
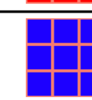
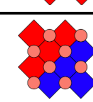
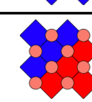
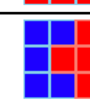
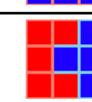




- We can identify subleading operators (or the absence of the expected ones)

AT-TFI				SD-IHG			
RSMI-NE Scaling Dimension {Expected[46]}	Analytic Operator {Deg.}	Neural Operator Projection: Maximum	Neural Operator Projection: Minimum	RSMI-NE Scaling Dimension {Expected[17]}	Analytic Operator {Deg.}	Neural Operator Projection: Maximum	Neural Operator Projection: Minimum
1.24(1) 1.22(1) {1.23629}	$\langle\sigma\rangle^2 - \langle\tau\rangle^2$ $\langle\sigma\rangle\langle\tau\rangle$ {2}			1.24(1) {1.222}	$\langle A\rangle$ {1}		
1.49(2) {1.51136}	$\langle\sigma\rangle^2 + \langle\tau\rangle^2$ {1}			1.54(2) {1.502}	$\langle S\rangle$ {1}		
2.02(3) {2.0}	$\langle\sigma\rangle\langle\partial\tau\rangle - \langle\tau\rangle\langle\partial\sigma\rangle$ {3}			2.20(6) {2.222}	$\langle\partial A\rangle$ {3}		

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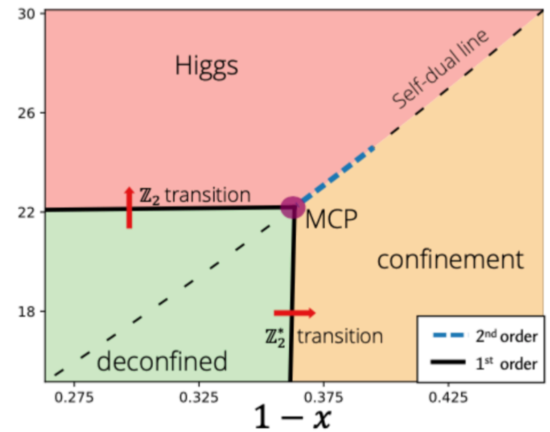

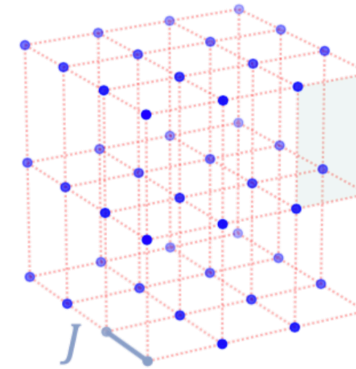

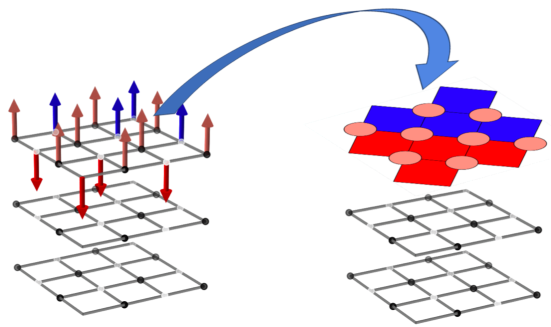

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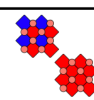
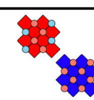
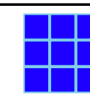
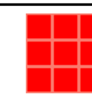
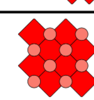
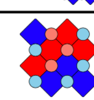
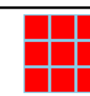
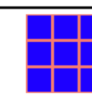
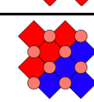
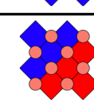
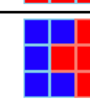
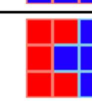
1.24(1) 1.22(1) {1.23629}	$\langle\sigma\rangle^2 - \langle\tau\rangle^2$ $\langle\sigma\rangle\langle\tau\rangle$ {2}			1.24(1) {1.222}	$\langle A \rangle$ {1}		
1.49(2) {1.51136}	$\langle\sigma\rangle^2 + \langle\tau\rangle^2$ {1}			1.54(2) {1.502}	$\langle S \rangle$ {1}		
2.02(3) {2.0}	$\langle\sigma\rangle\langle\partial\tau\rangle - \langle\tau\rangle\langle\partial\sigma\rangle$ {3}			2.20(6) {2.222}	$\langle \partial A \rangle$ {3}		

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RG for quasi-periodic systems

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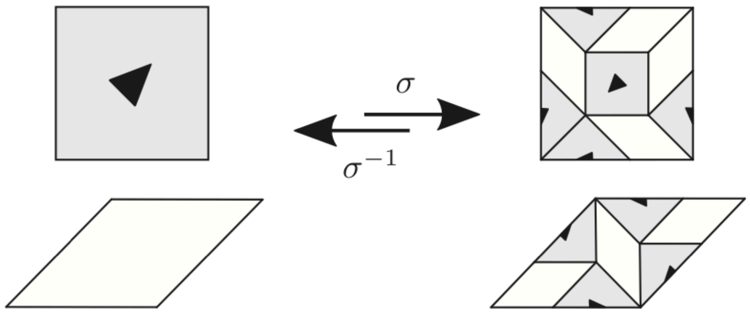

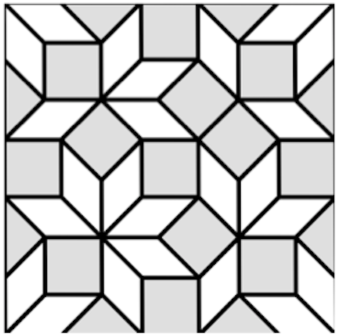

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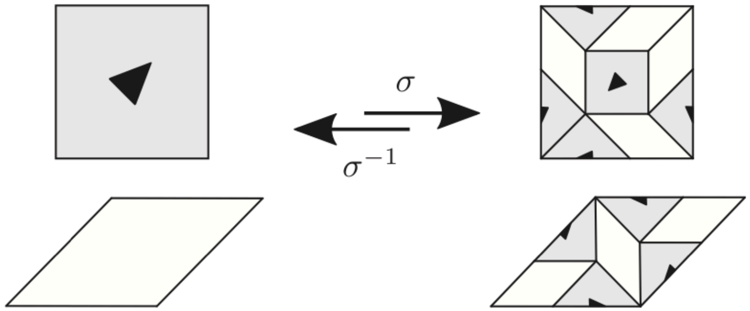
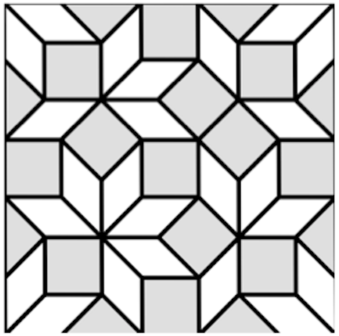
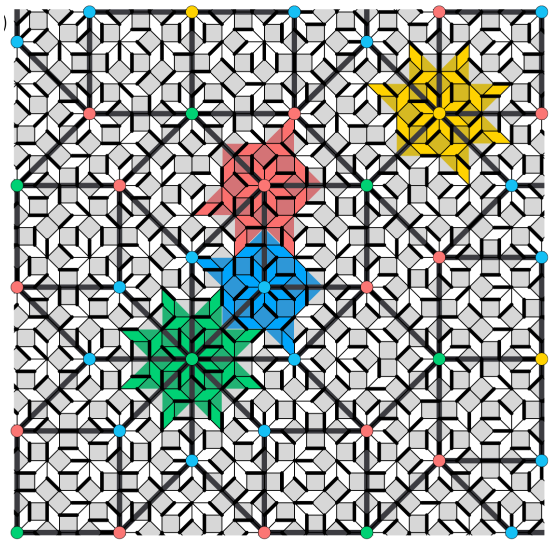
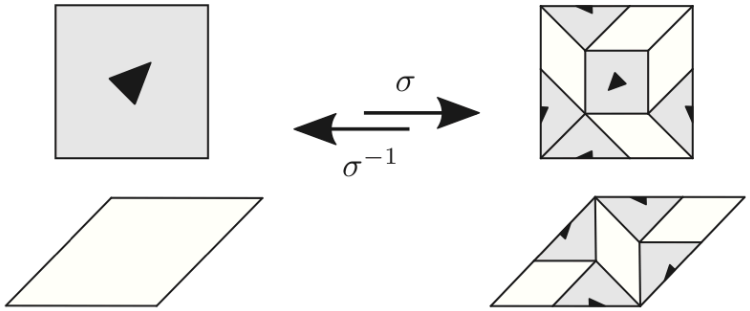


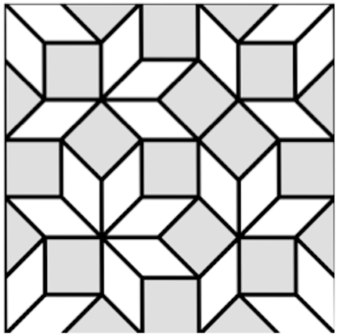




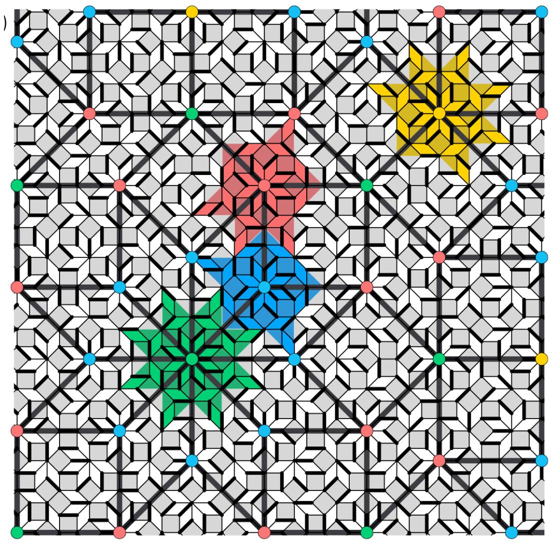
RG for quasi-periodic systems

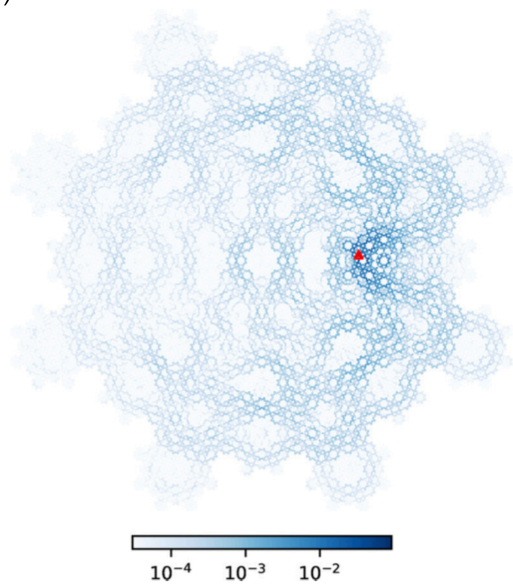
- The quasiperiodic Amman Beenker (AB) tiling is generated hierarchically:





- AB tiling admits perfect dimer covering, and shows evidence of power-law dimer correlations: **Phys. Rev. B** 106, 094202 (2020)





RG for quasi-periodic systems

D.E. Gokmen, S. Biswas, S.D. Huber, Z. Ringel, F. Flicker and **MKJ**, *arXiv:2301:11934 (2023)*

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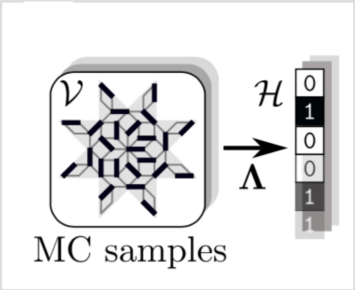

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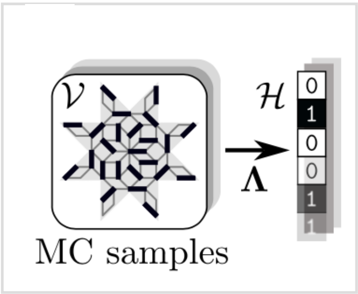

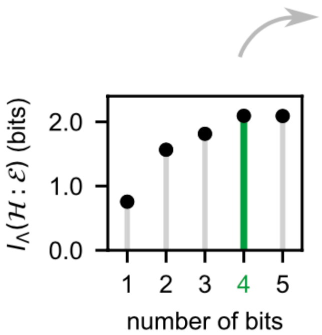

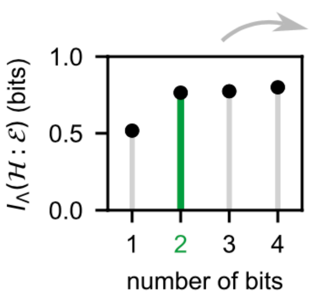

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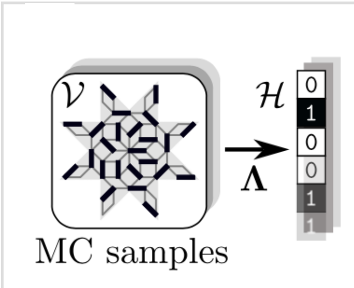

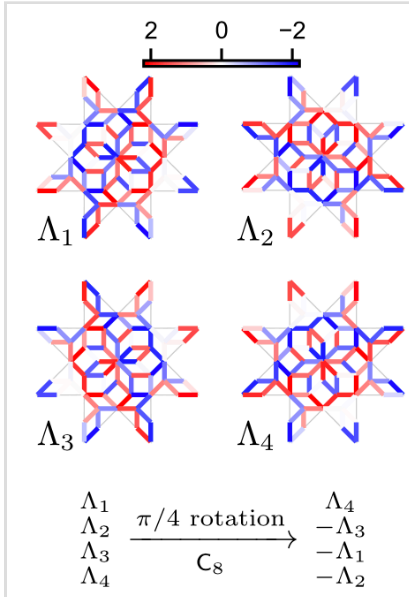

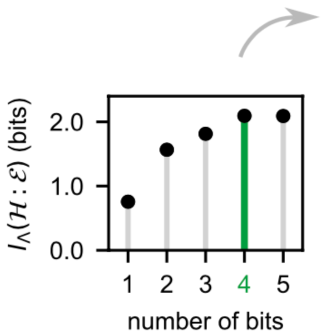

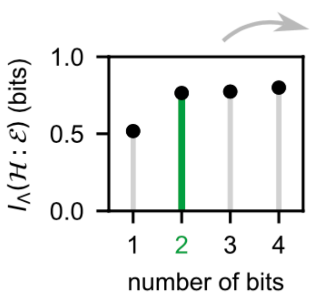

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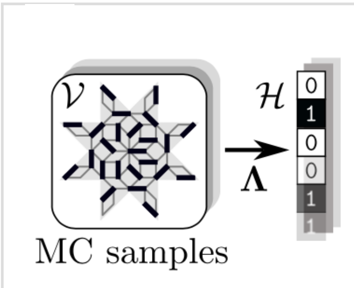

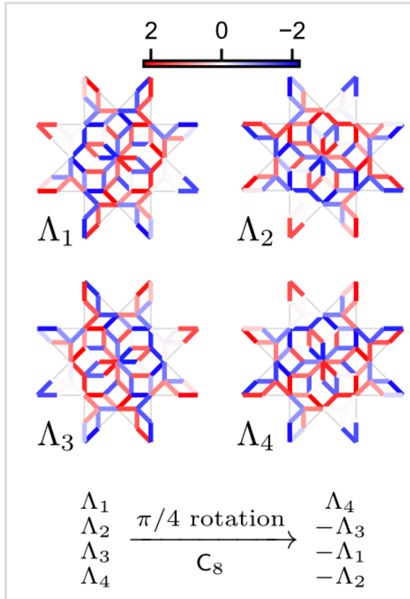

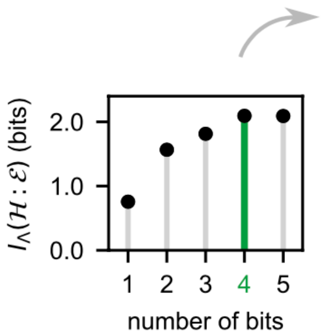

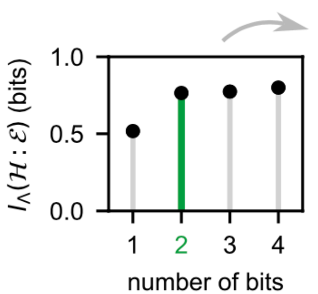

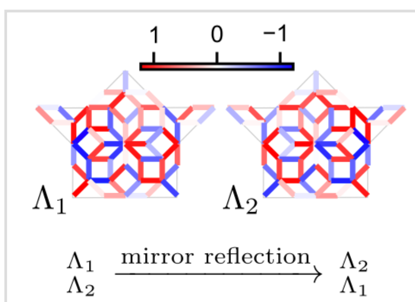

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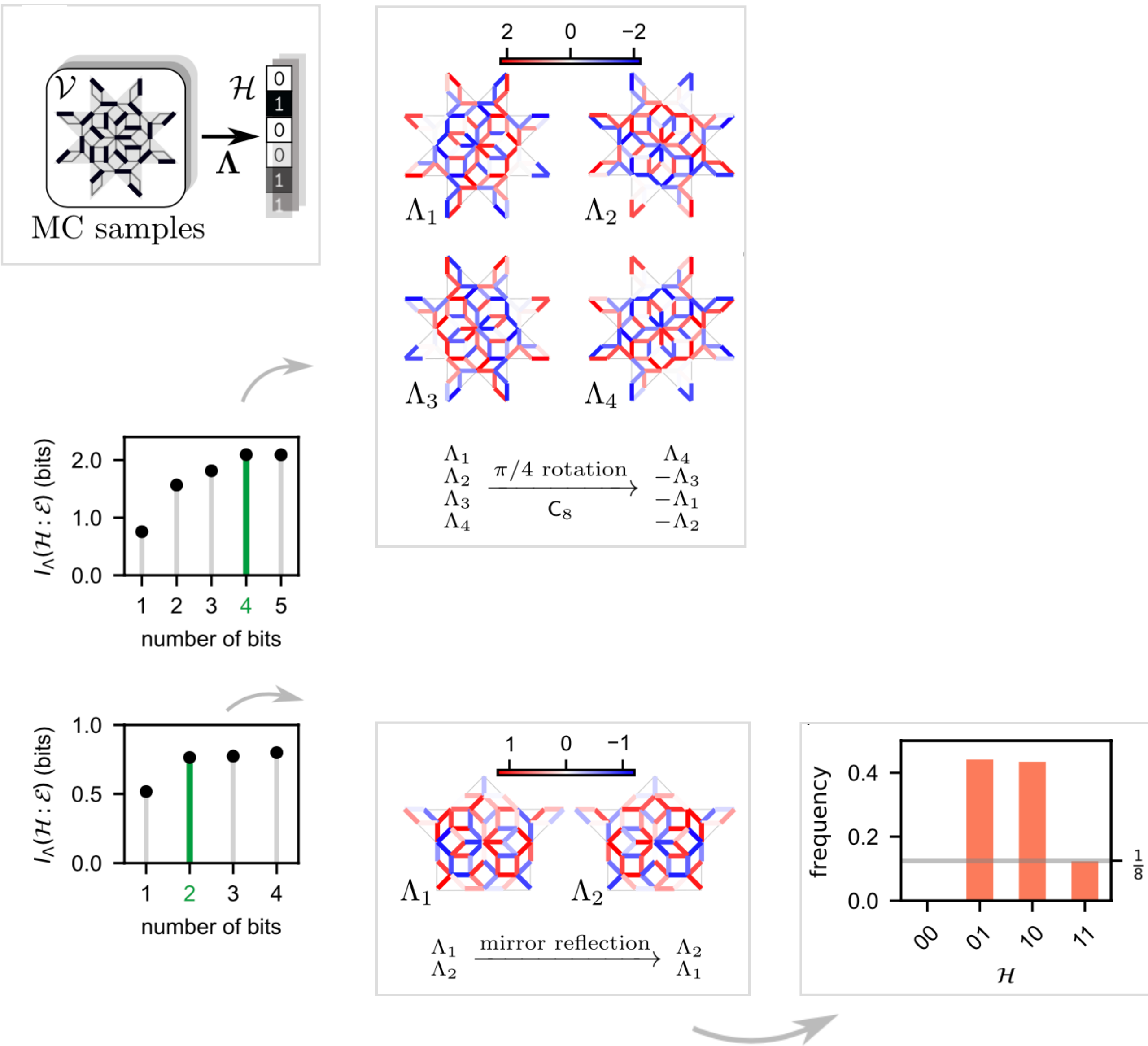

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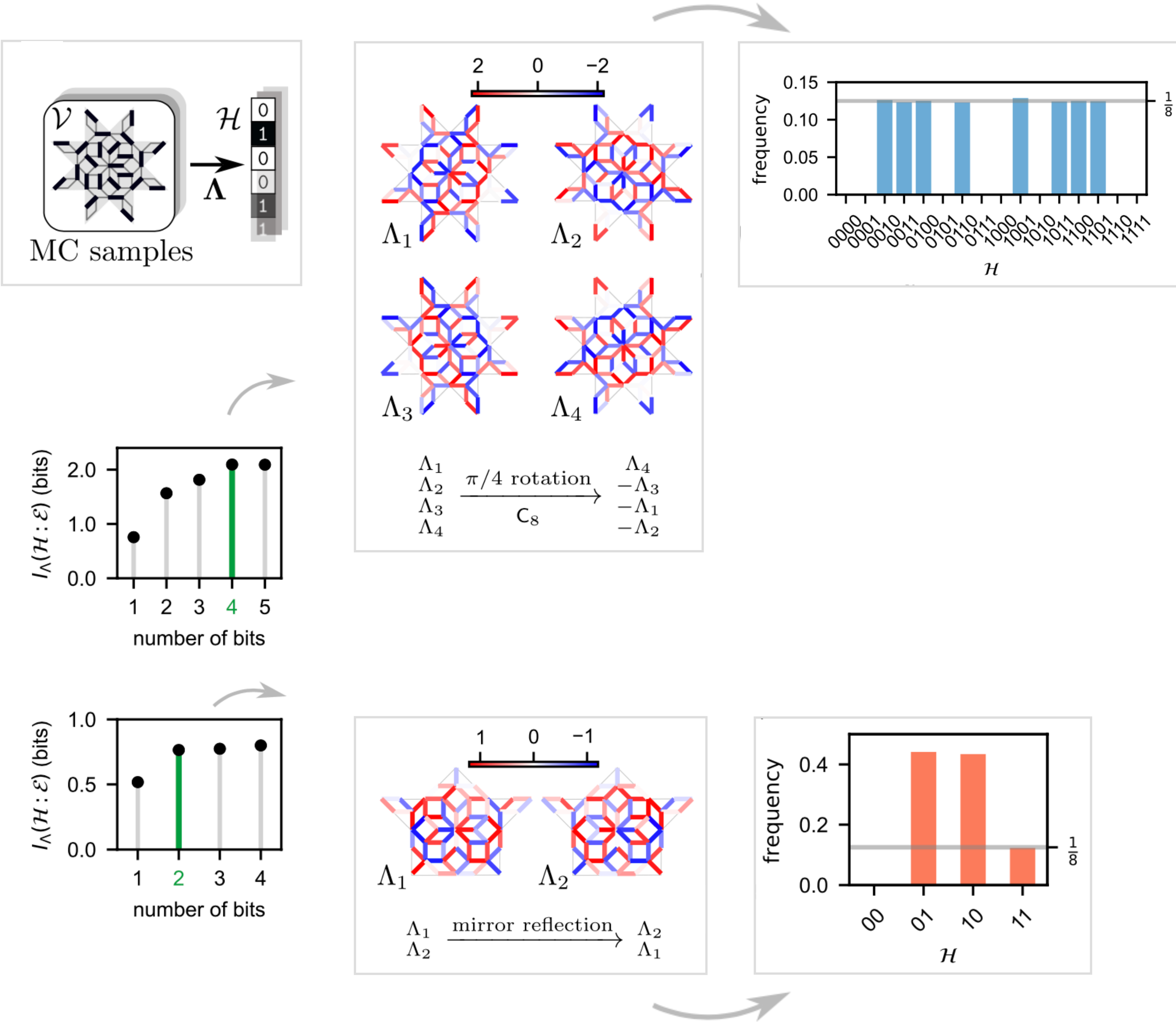

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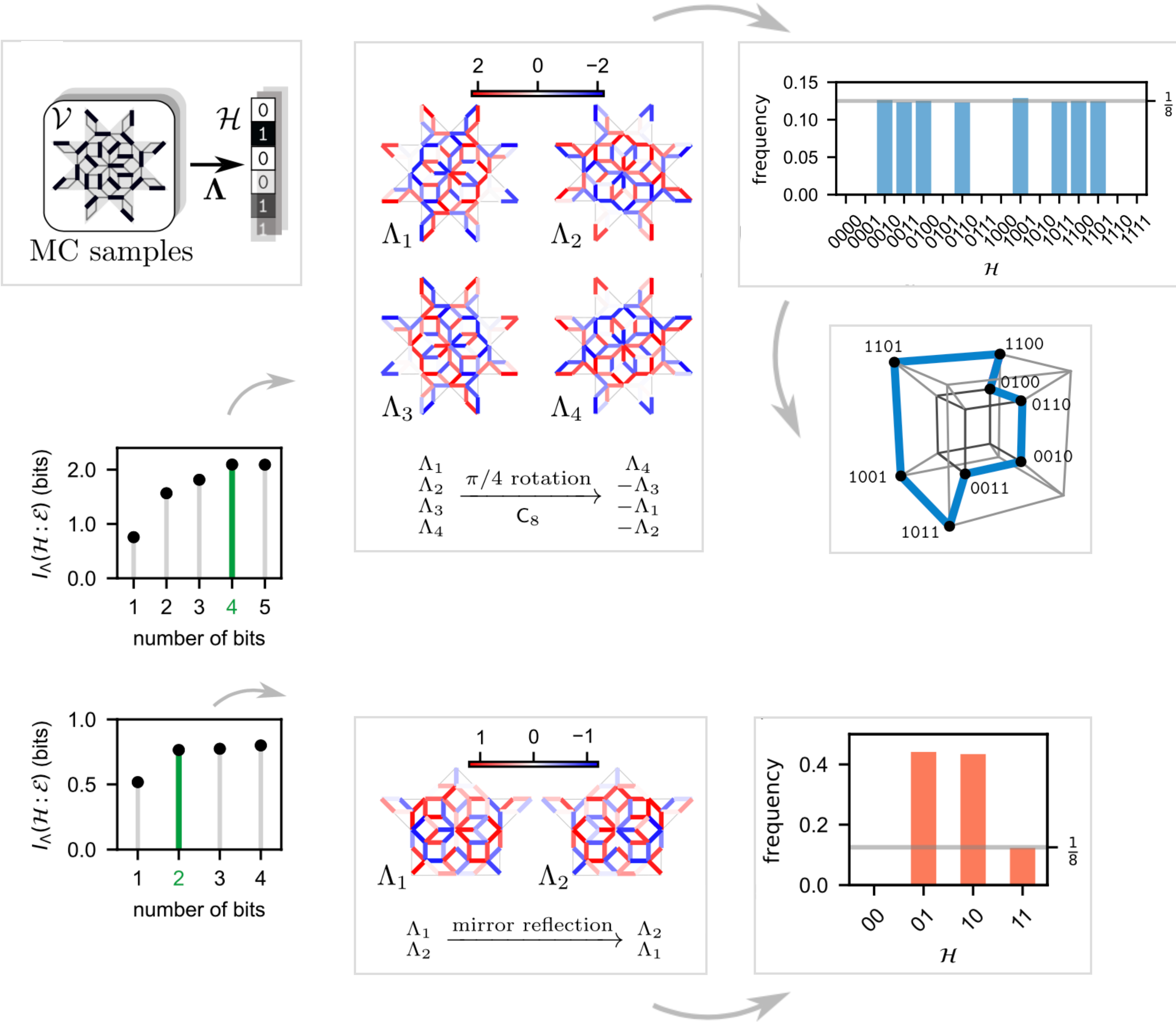

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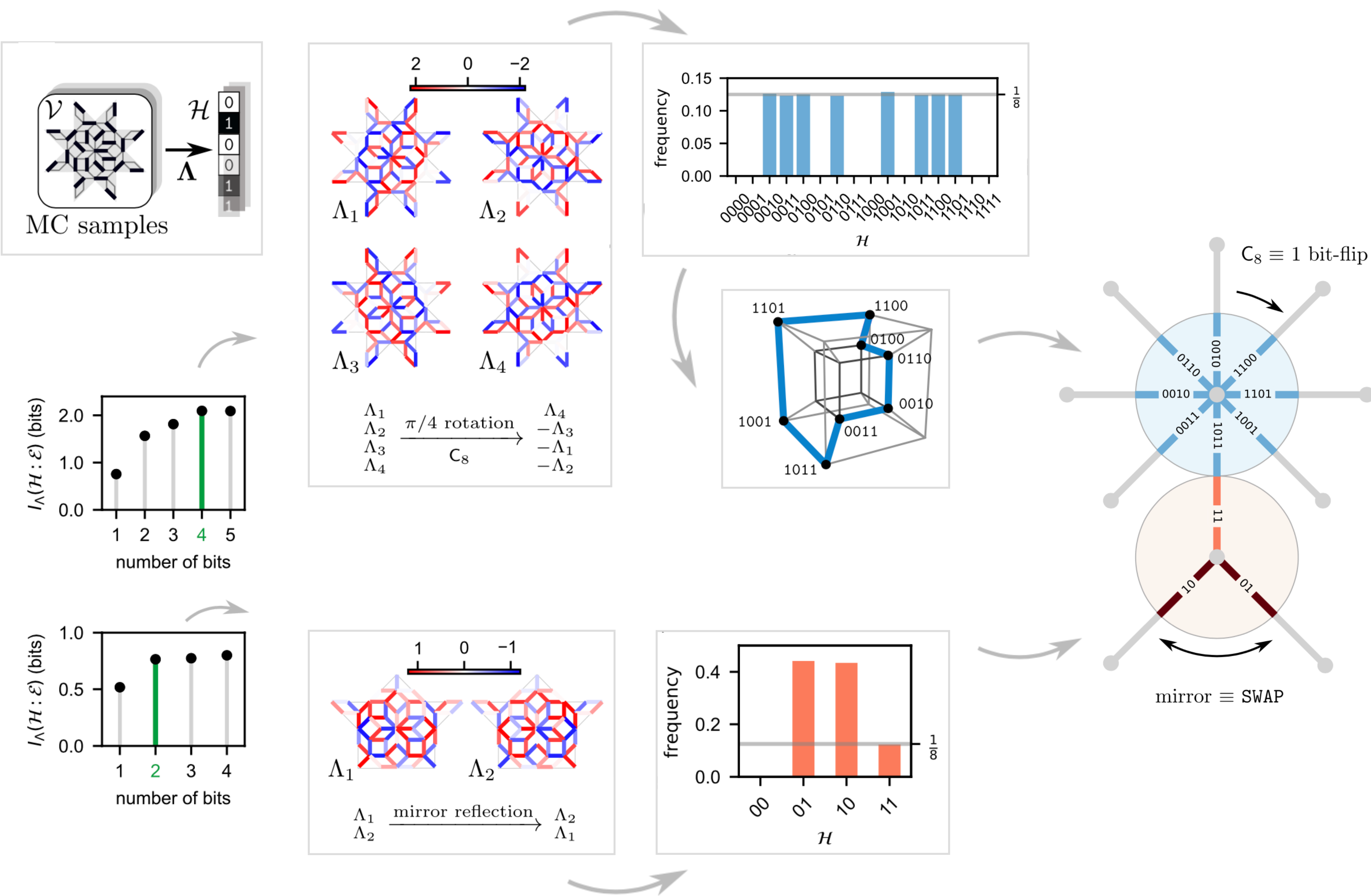

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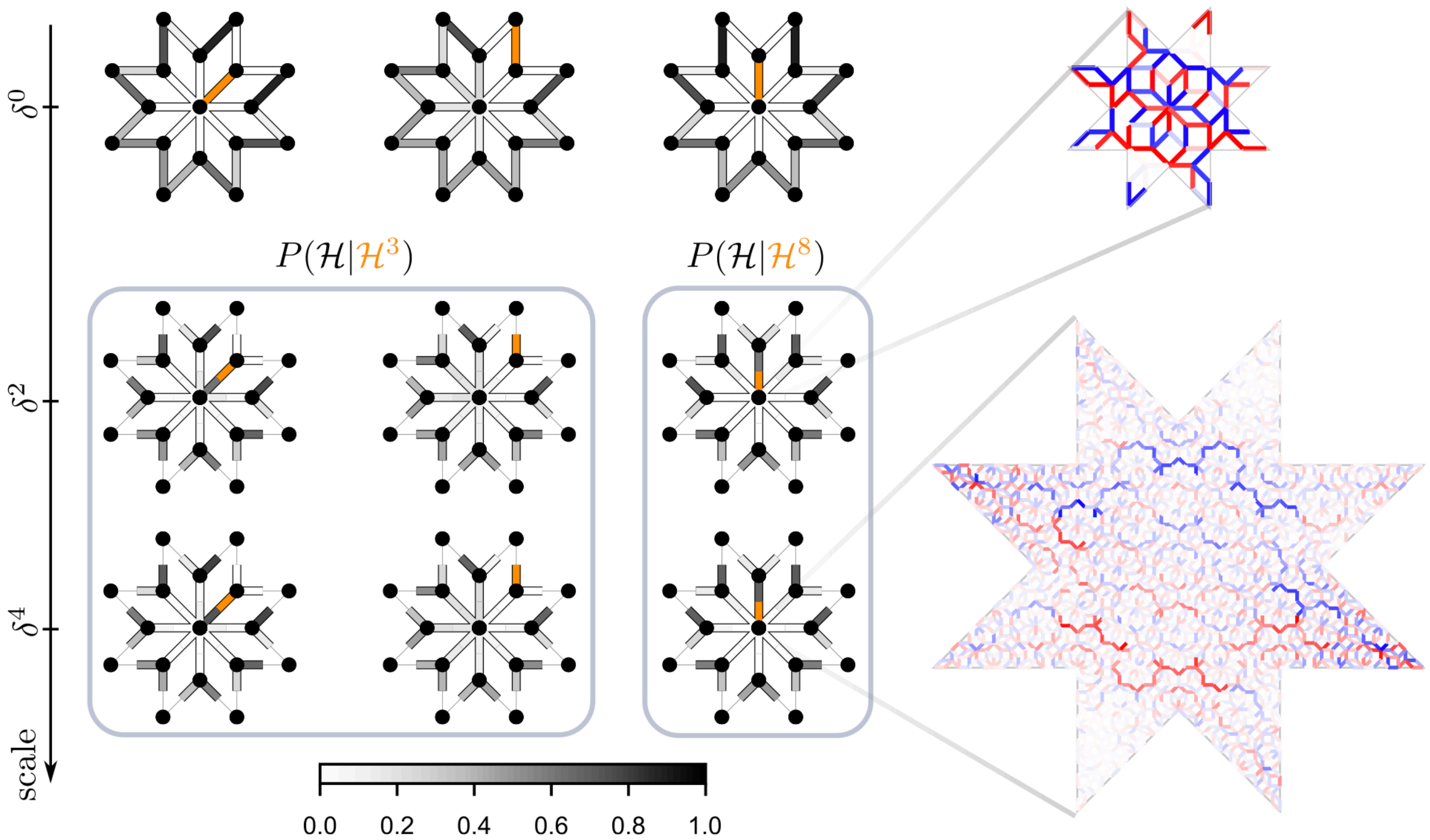
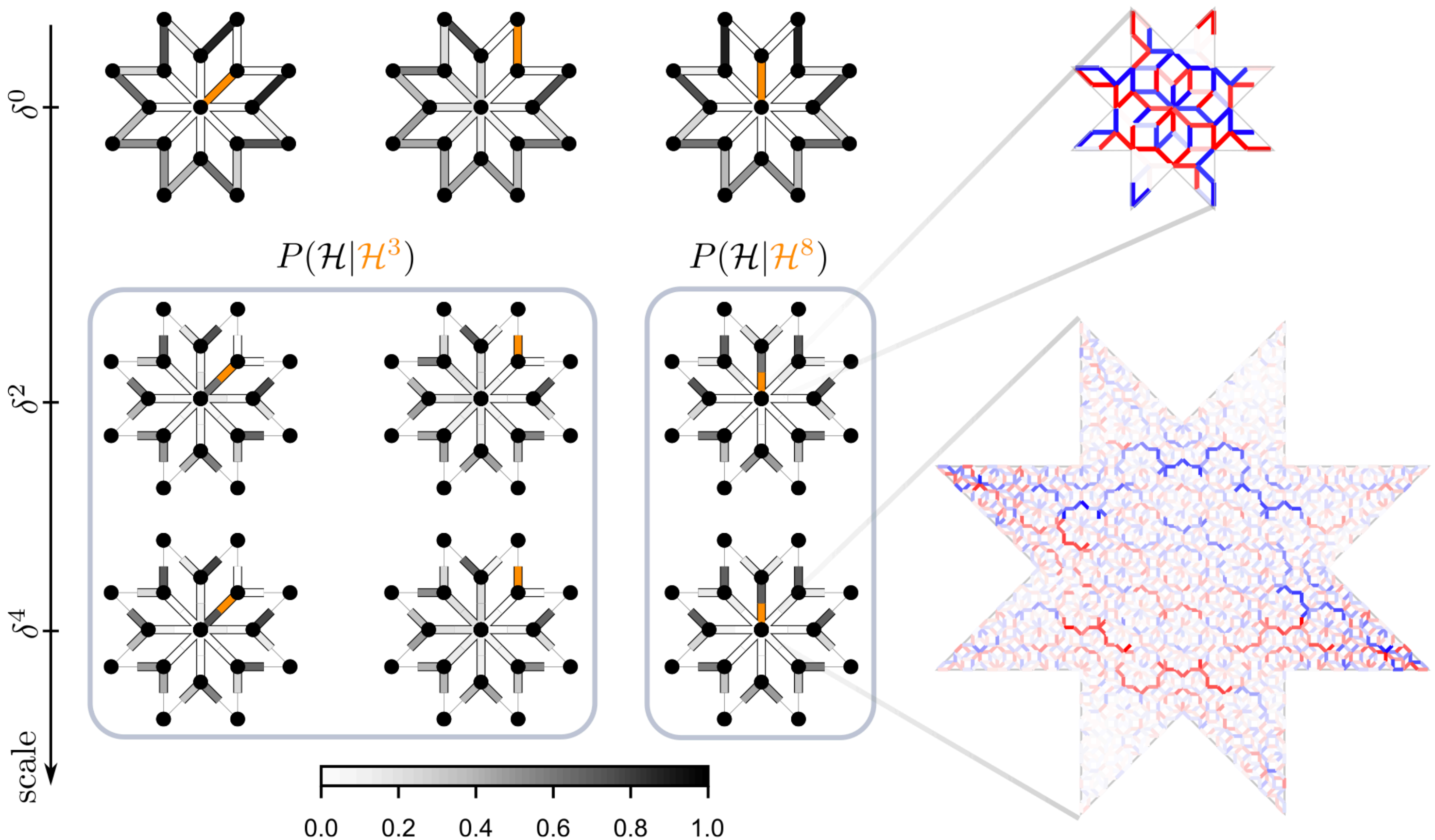


RG for quasi-periodic systems

- The compression map reveals effective **super-dimers** on a larger scale:



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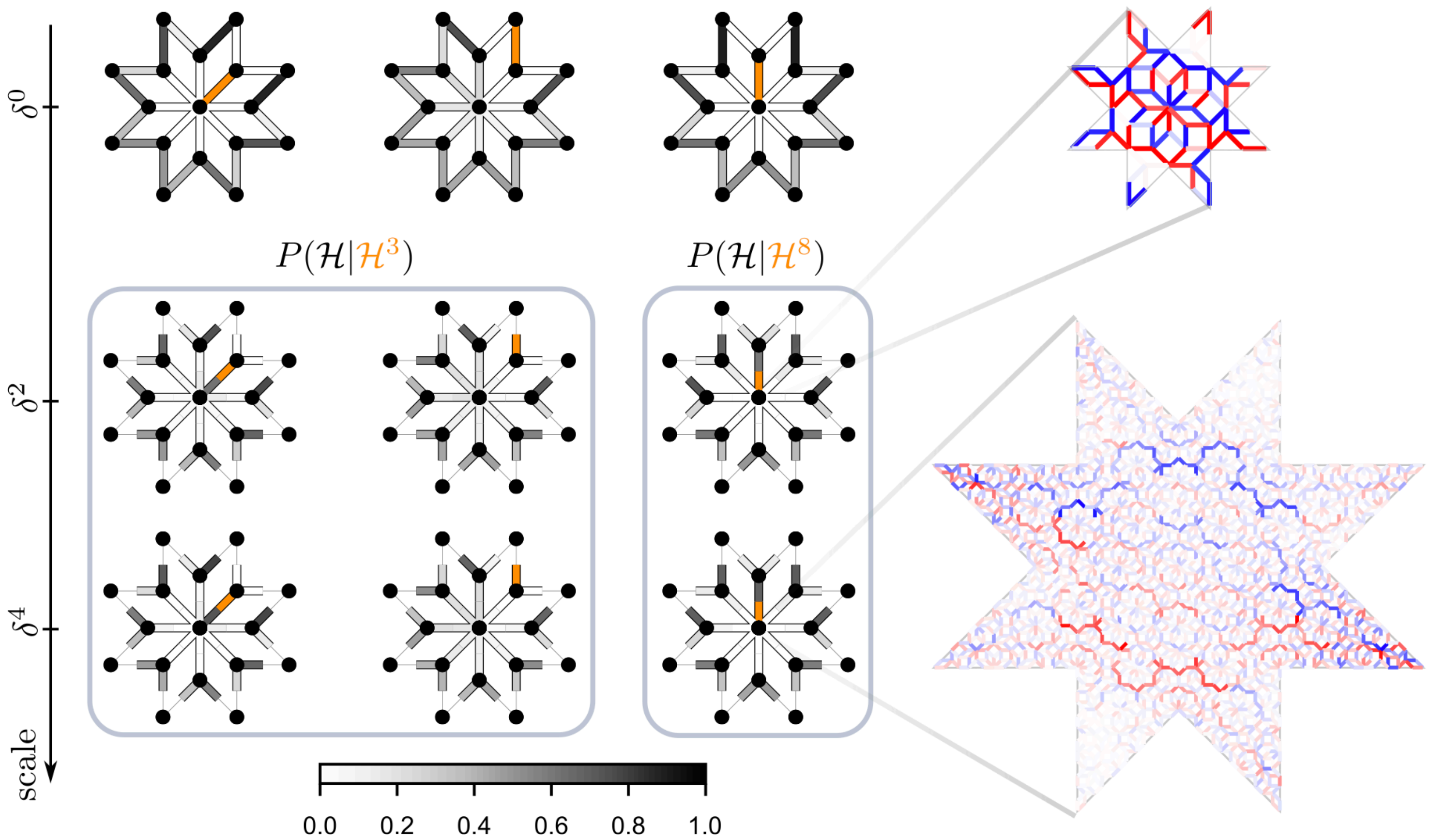
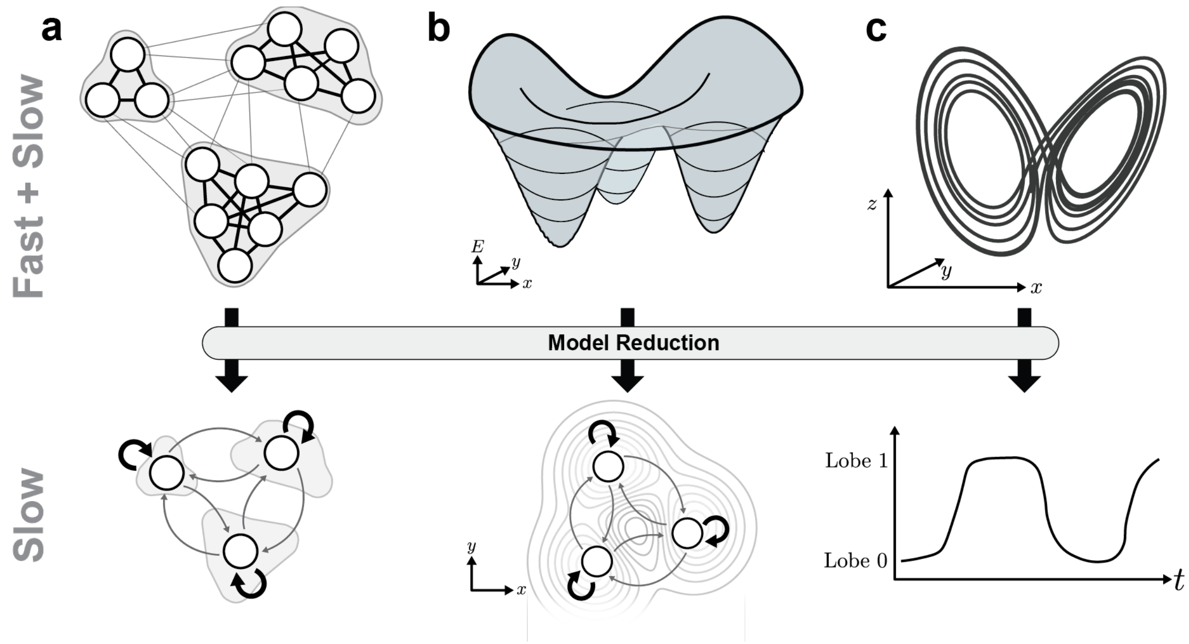


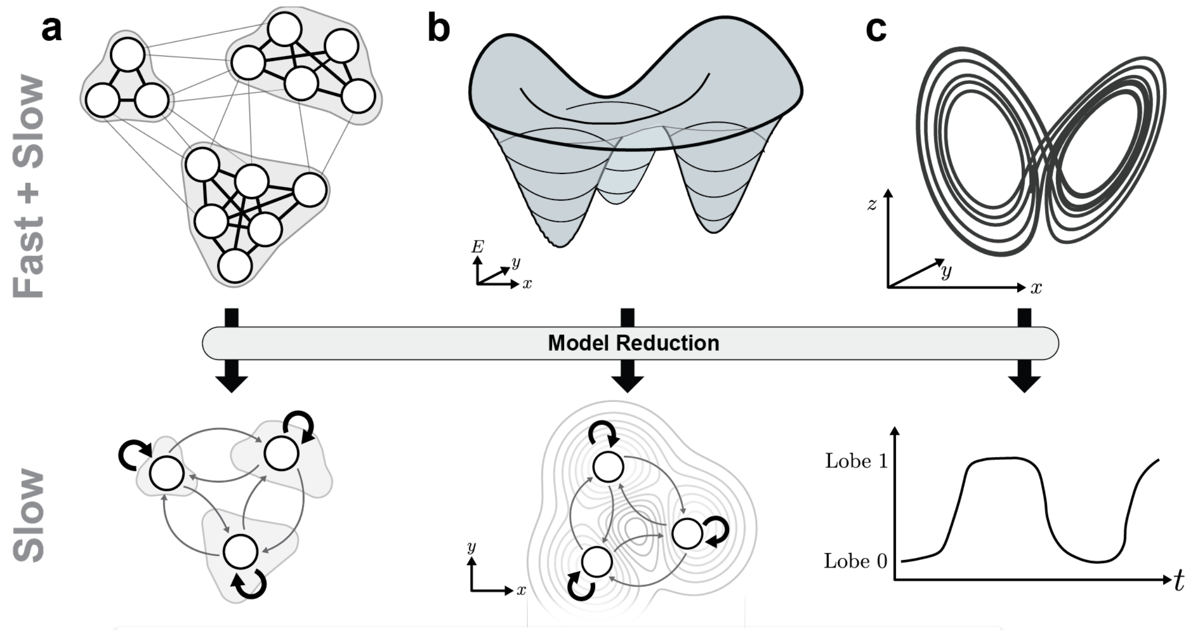
- The same compression maps persist across **multiple scales**

Dynamical systems



$$p^*_\beta(h|x) = \frac{1}{\mathcal{N}(x)}\, p^*_\beta(h)\, \exp\left[\beta \sum_n \mathrm{e}^{\lambda_n \Delta t} \phi_n(x) f_n(h)\right]$$

Dynamical systems



$$\mathcal{L}_{\mathrm{IB}}[p_{H_t|X_t}] = I(X_t, H_t) - \beta I(X_{t+\Delta t}, H_t)$$

- Compress to preserve information about the **future state** of the system.

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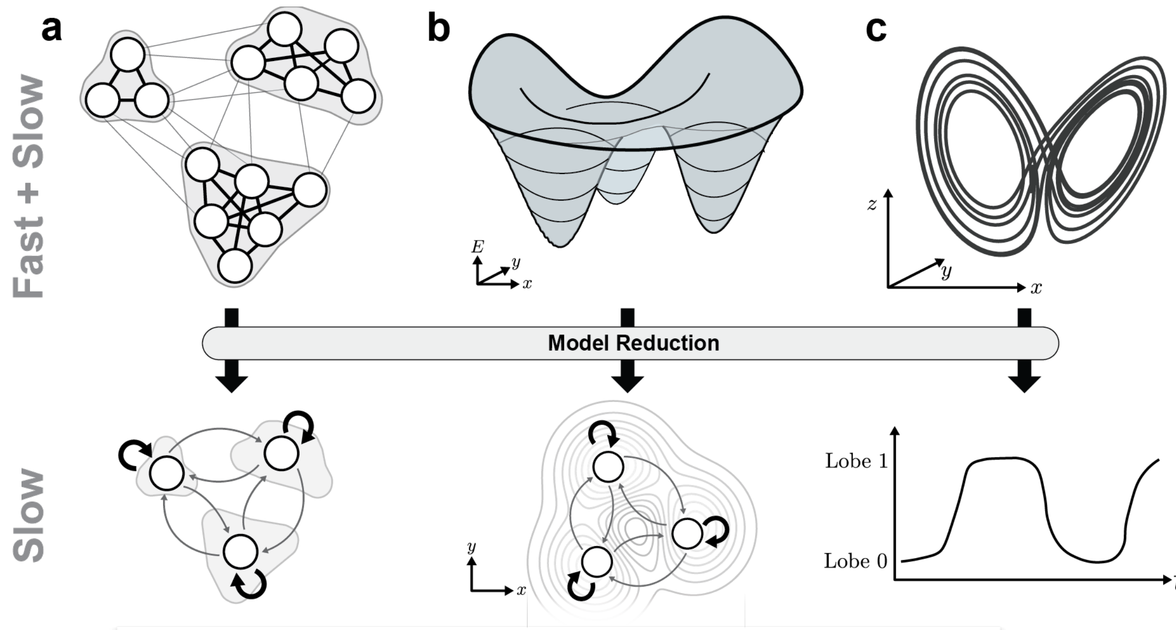

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- The **IB-optimal** encoder is determined by the eigenmodes of the transfer operator

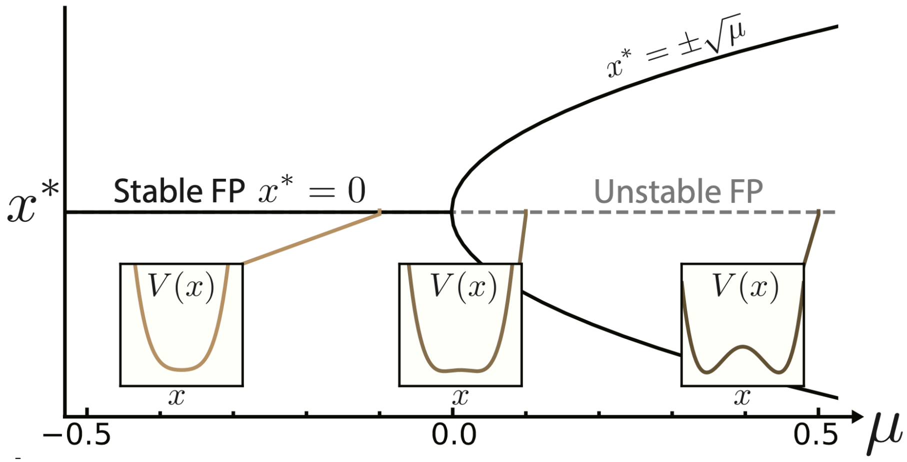
$$p^*_\beta(h|x) = \frac{1}{\mathcal{N}(x)}\, p^*_\beta(h)\, \exp\left[\beta \sum_n \mathrm{e}^{\lambda_n \Delta t} \phi_n(x) f_n(h)\right]$$

Brownian particle in a potential

$$\dot{x}_t = -\partial_x V(x_t) + \sigma \eta_t.$$

$$V(x) = \frac{1}{4}(\mu - x^2)^2$$

Brownian particle in a potential



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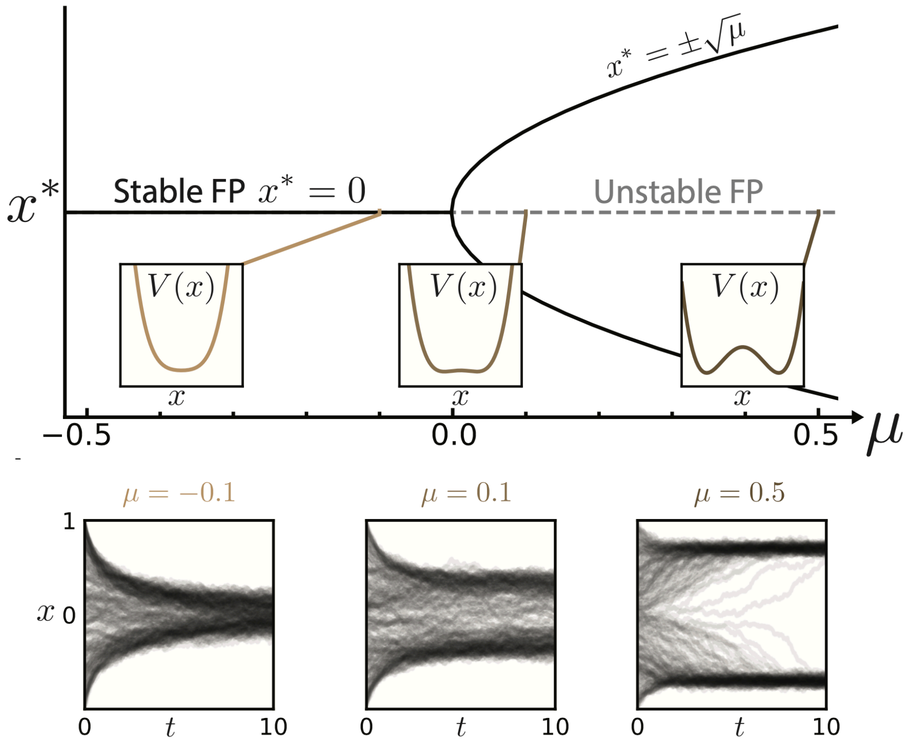

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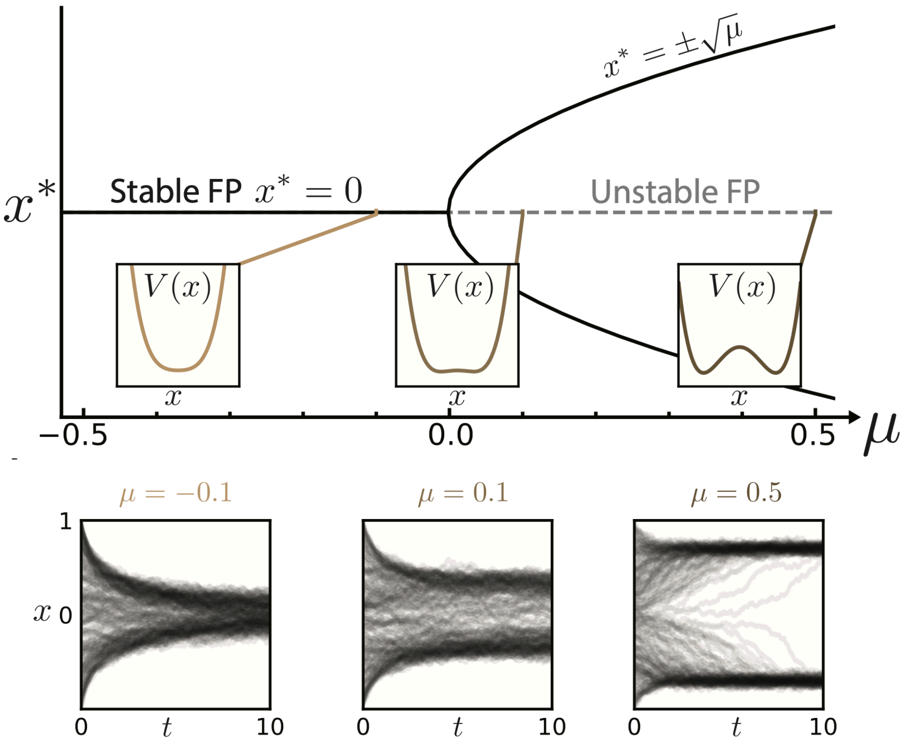

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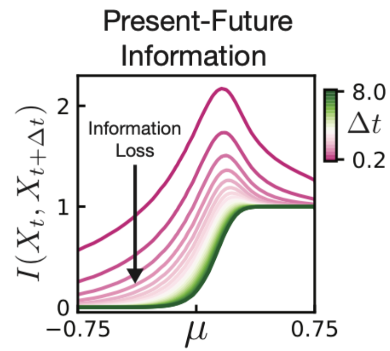

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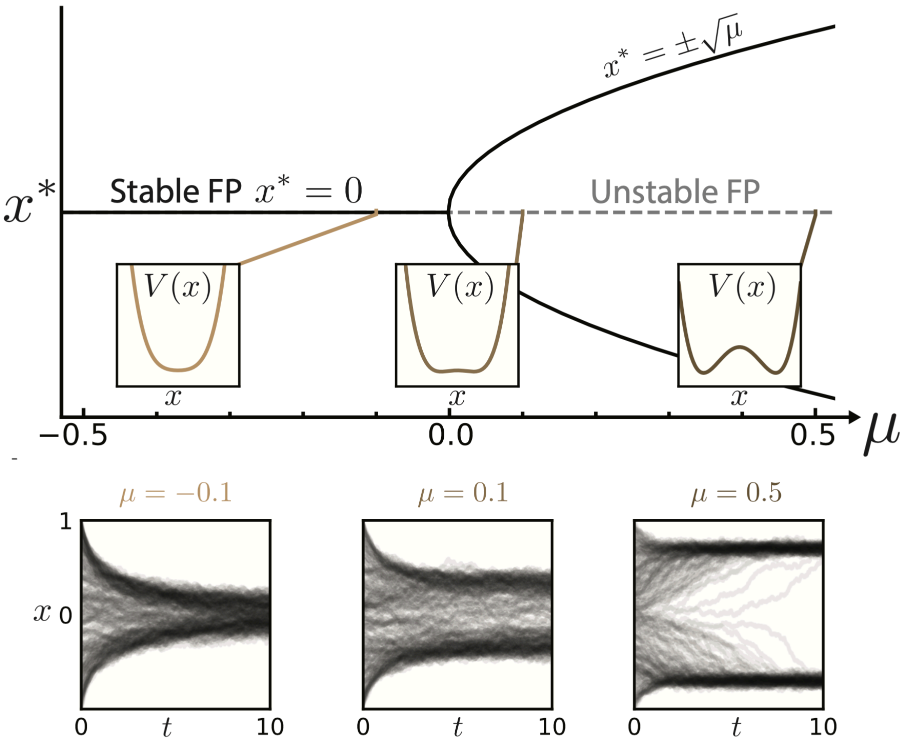

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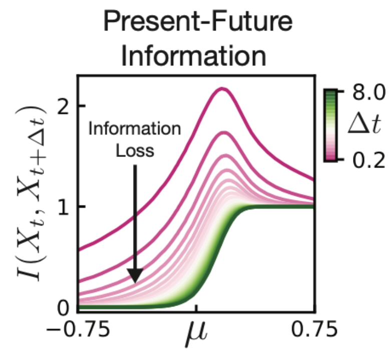

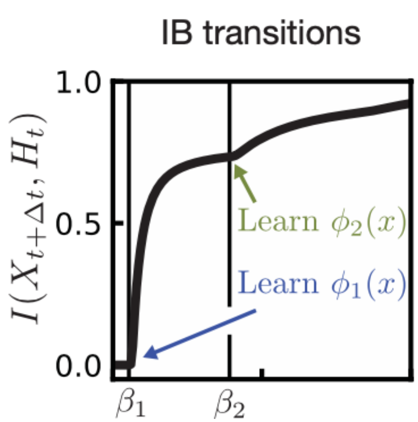

- IB learns the transfer operator eigenmodes

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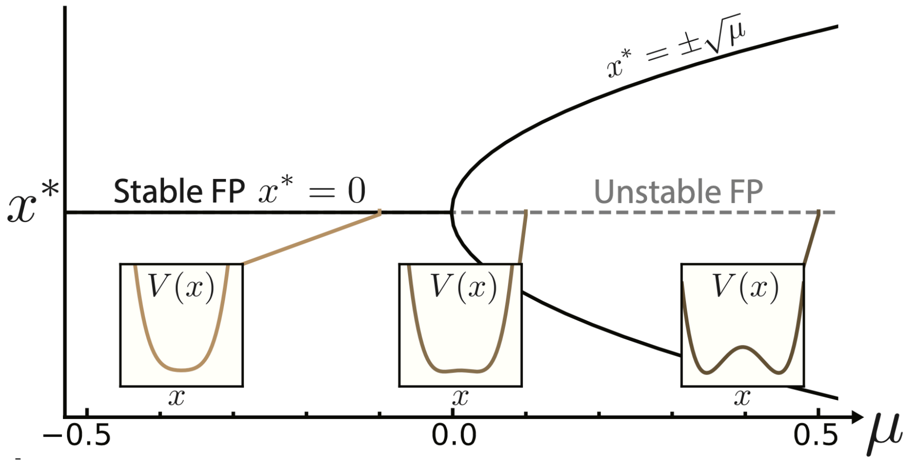

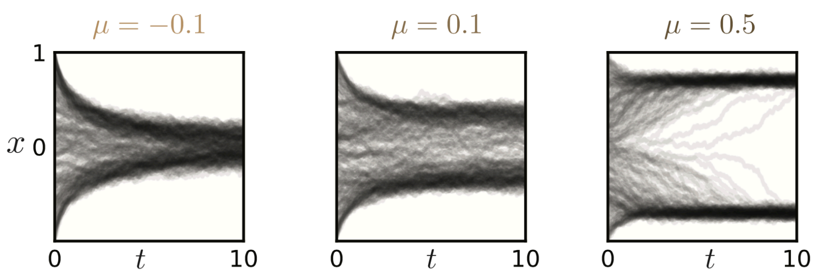

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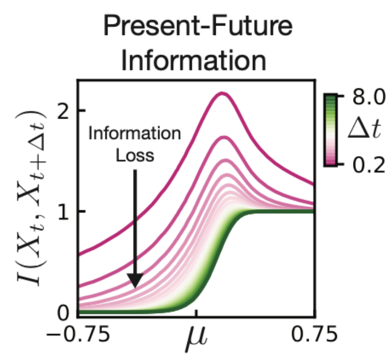

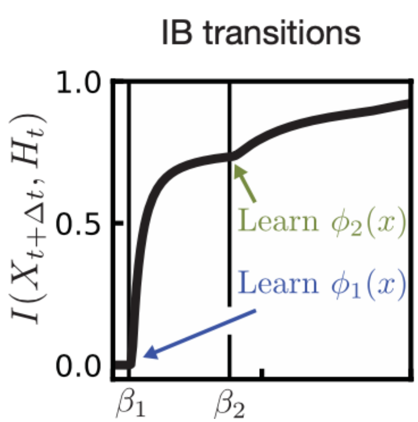

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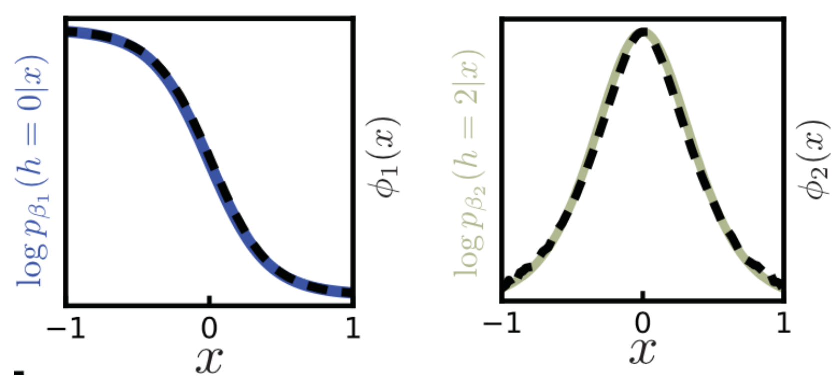

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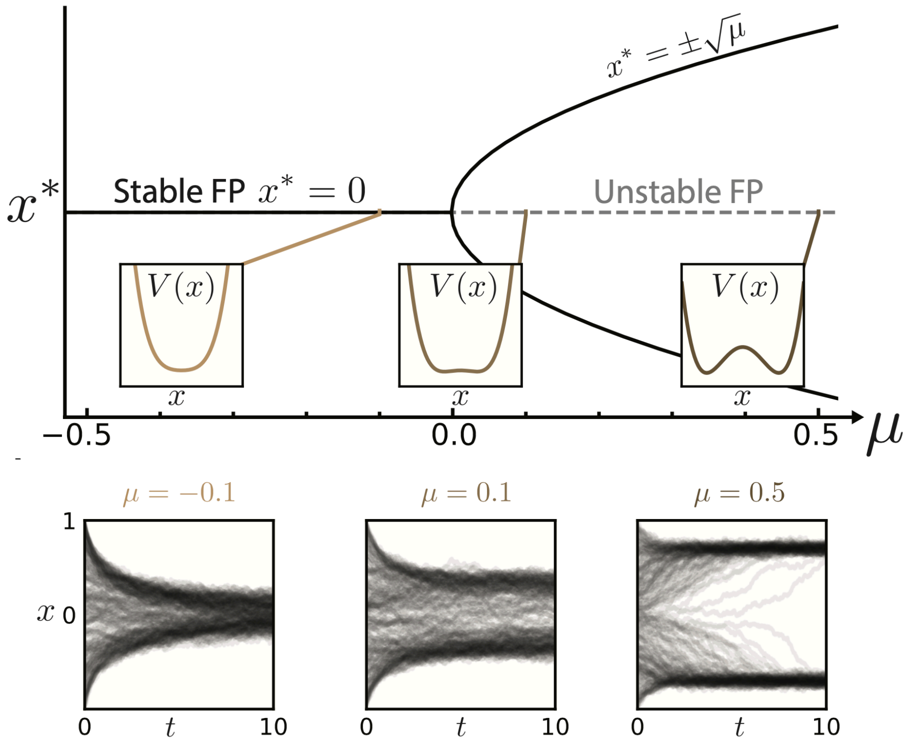

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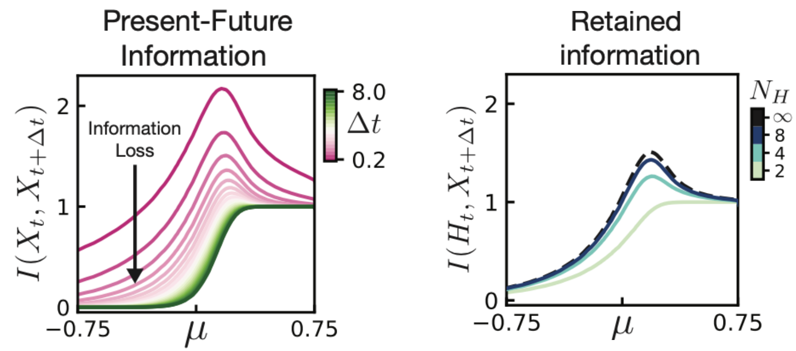

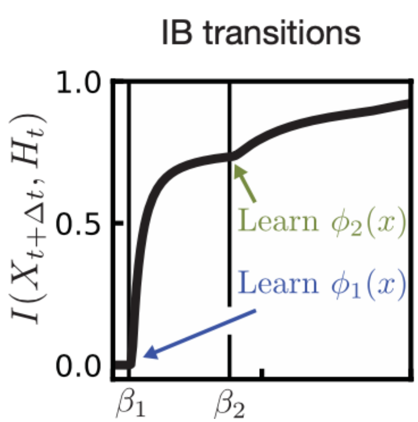

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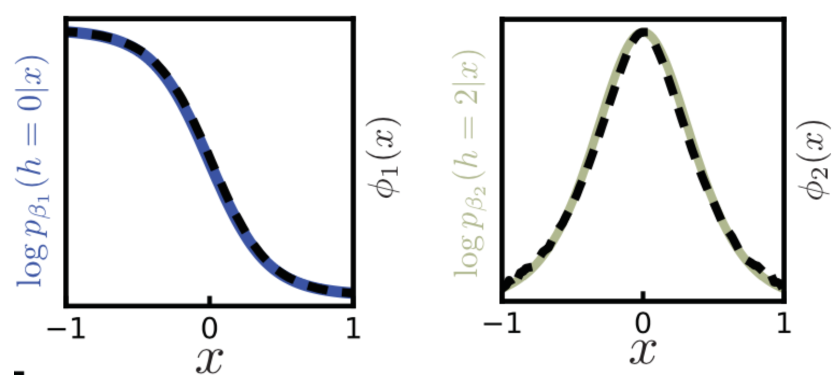
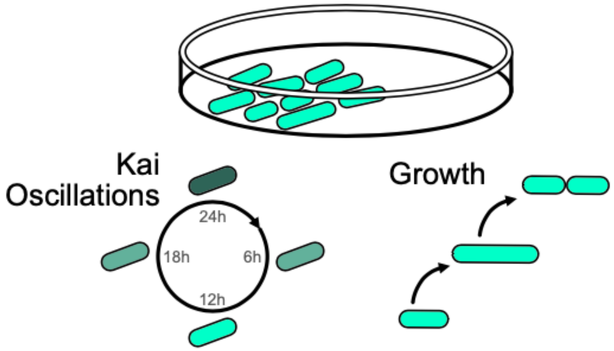


Cyanobacteria experiments



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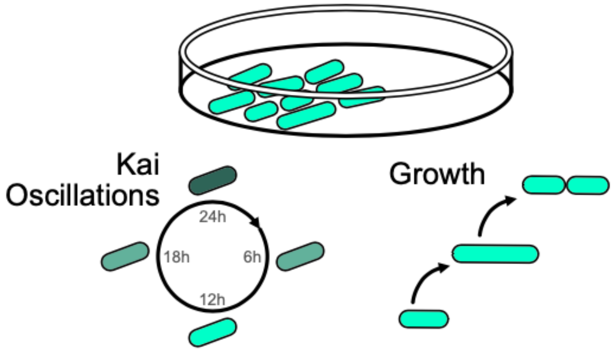

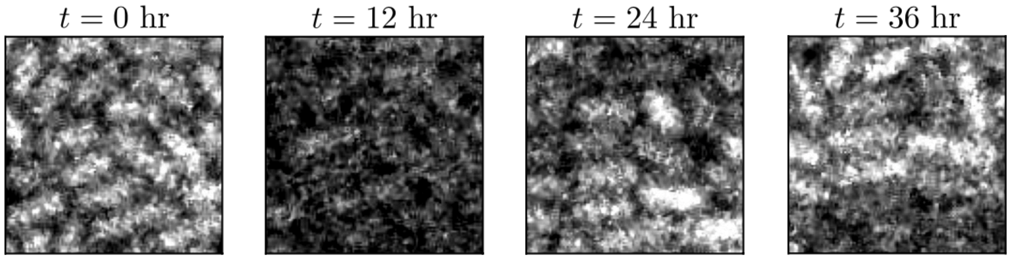

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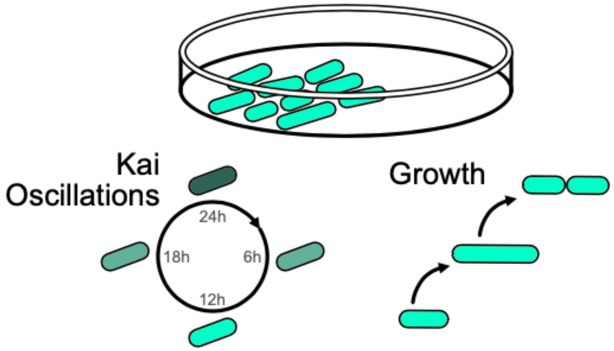

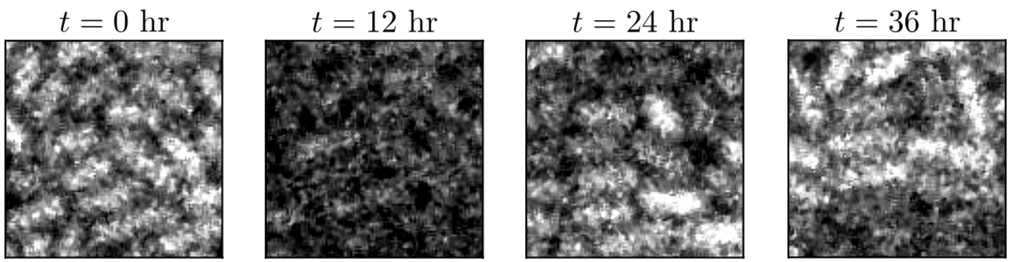

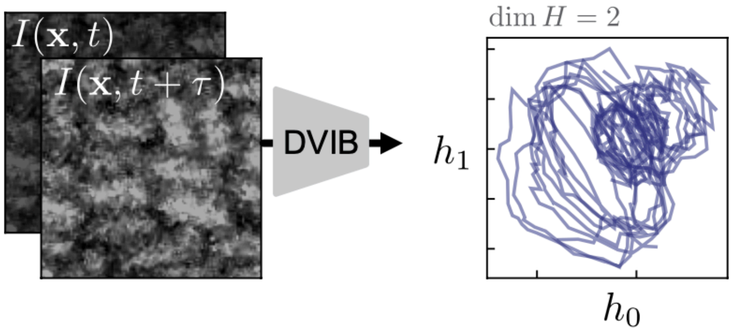

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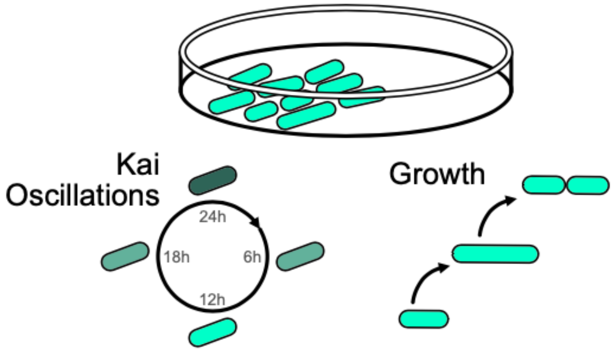
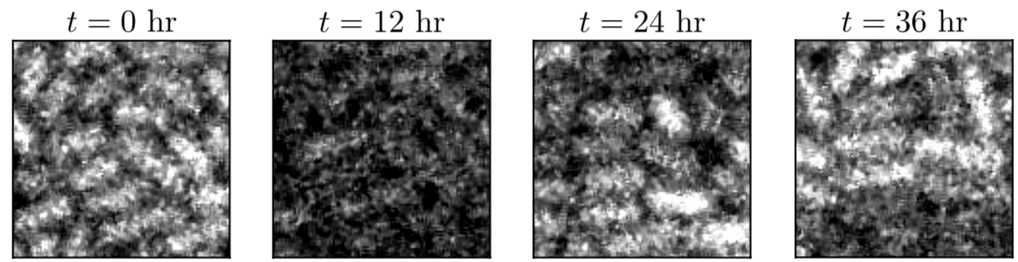
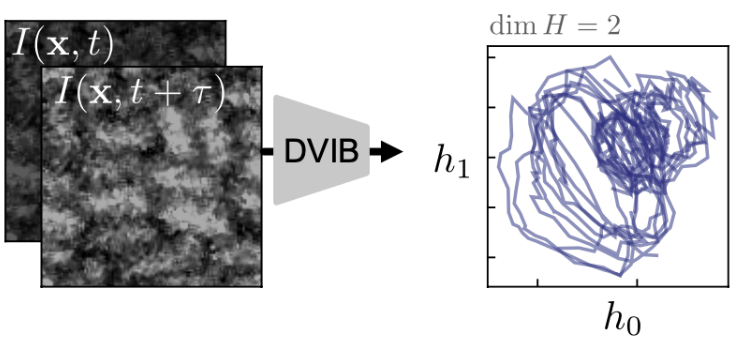
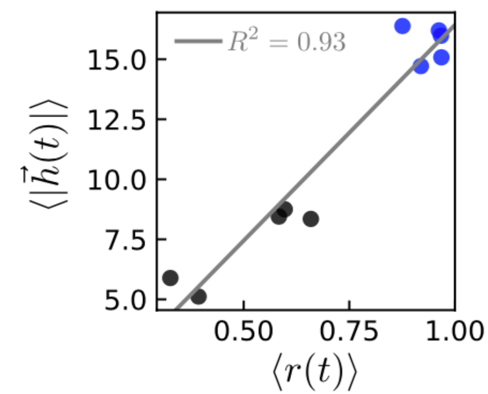




- Dynamics in latent space reveals populations differing by **synchronisation**





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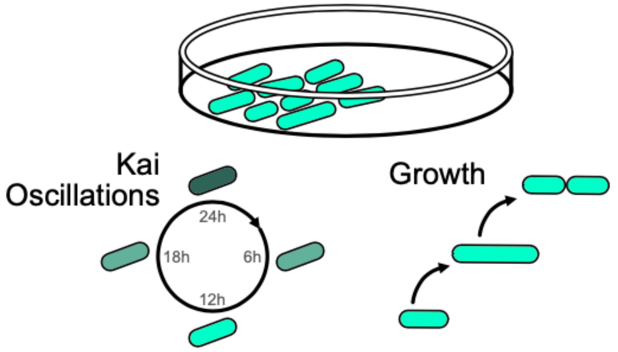

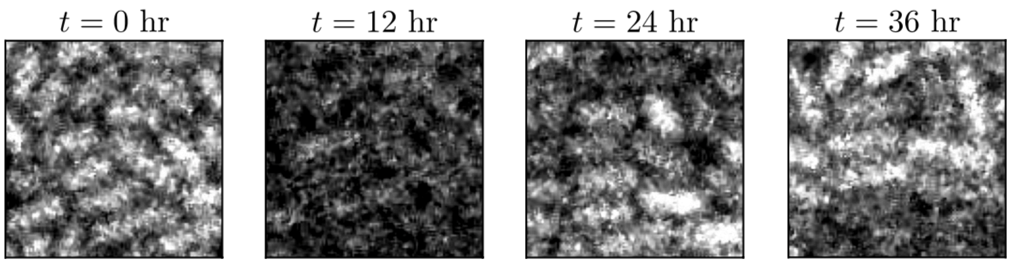

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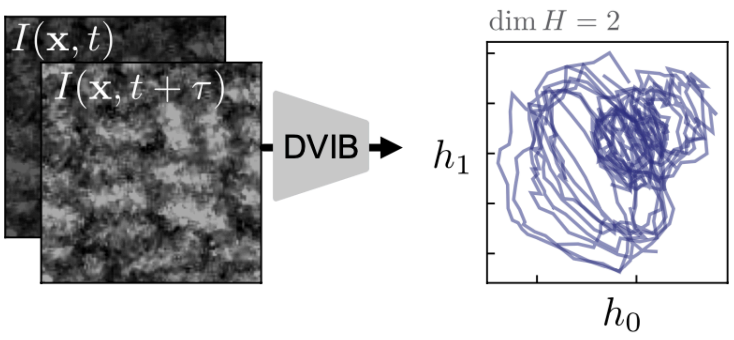

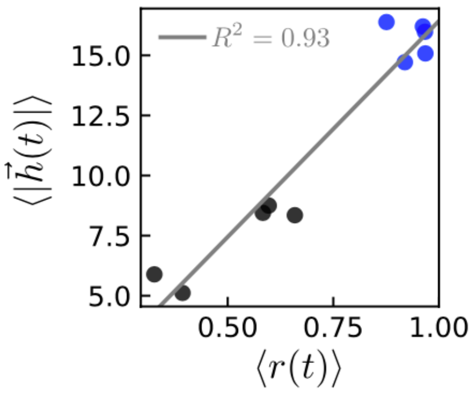

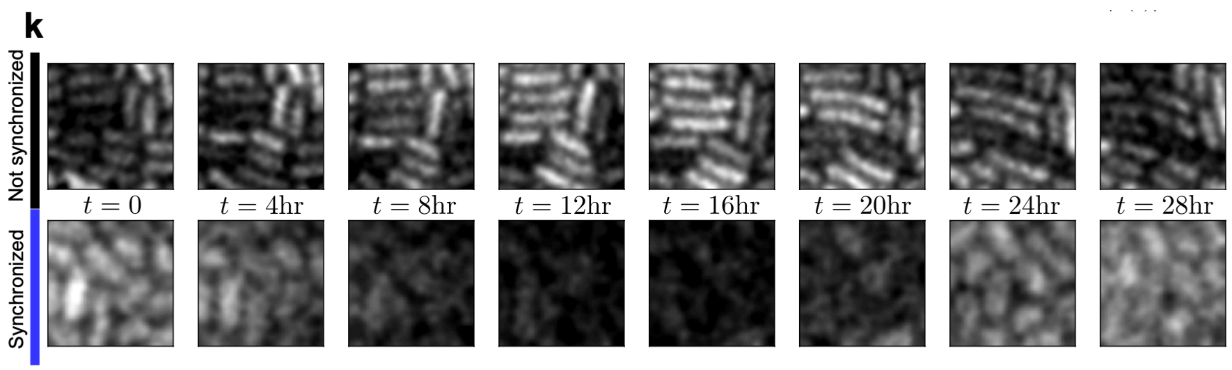

Outlook

- Applications to 3D stat.mech. models
- Automating discovering the algebraic properties satisfied by the operators
- Dynamical graphs
- Application to experimental data: soft-matter,

Operator content of theories on different lattices, equilibrium or not, can be extracted using compression theory tools from raw data alone.

$$\Lambda_{\mathrm{S1}} \cdot \mathcal{V}(\mathbf{r}) = (-1)^{x+y} N_{|}(\mathbf{r}),$$
$$\Lambda_{\mathrm{S2}} \cdot \mathcal{V}(\mathbf{r}) = (-1)^{x+y+1} N_{-}(\mathbf{r})$$

$$N_{|}(\mathbf{r}) = \frac{1}{4} + \frac{(-1)^{x+y+1}}{2\pi}\partial_x\varphi(\mathbf{r}) + (-1)^y \sin\varphi(\mathbf{r})$$
$$N_{-}(\mathbf{r}) = \frac{1}{4} + \frac{(-1)^{x+y}}{2\pi}\partial_y\varphi(\mathbf{r}) + (-1)^x \cos\varphi(\mathbf{r})$$

Papanikolaou et al. PRB 76, 134514 (2007)

$$\mathcal{H}_1 \sim \sum_{\mathbf{r}\in\mathcal{V}} \Lambda_{\mathrm{S1}} \cdot \mathcal{V}(\mathbf{r}) = \sum_{\mathbf{r}\in\mathcal{V}} \left[\frac{(-1)^{x+y}}{4} + \frac{\partial_x\varphi(\mathbf{r})}{2\pi} + (-1)^x \sin\varphi(\mathbf{r})\right] = \sum_{\mathbf{r}\in\mathcal{V}} \left[\frac{\partial_x\varphi(\mathbf{r})}{2\pi} + (-1)^x \sin\varphi(\mathbf{r})\right]$$
$$\mathcal{H}_2 \sim \sum_{\mathbf{r}\in\mathcal{V}} \Lambda_{\mathrm{S2}} \cdot \mathcal{V}(\mathbf{r}) = \sum_{\mathbf{r}\in\mathcal{V}} \left[\frac{(-1)^{x+y}}{4} + \frac{\partial_y\varphi(\mathbf{r})}{2\pi} + (-1)^y \cos\varphi(\mathbf{r})\right] = \sum_{\mathbf{r}\in\mathcal{V}} \left[\frac{\partial_y\varphi(\mathbf{r})}{2\pi} + (-1)^y \cos\varphi(\mathbf{r})\right]$$

$$\mathcal{H} \propto \tau \circ \nabla \langle\varphi(\mathbf{r})\rangle_{\mathbf{r}\in\mathcal{V}}$$

- The staggered filters:

$$\Lambda_{\mathrm{S1}} \cdot \mathcal{V}(\mathbf{r}) = (-1)^{x+y} N_{|}(\mathbf{r}),$$
$$\Lambda_{\mathrm{S2}} \cdot \mathcal{V}(\mathbf{r}) = (-1)^{x+y+1} N_{-}(\mathbf{r})$$

$$N_{|}(\mathbf{r}) = \frac{1}{4} + \frac{(-1)^{x+y+1}}{2\pi} \partial_x \varphi(\mathbf{r}) + (-1)^y \sin \varphi(\mathbf{r})$$
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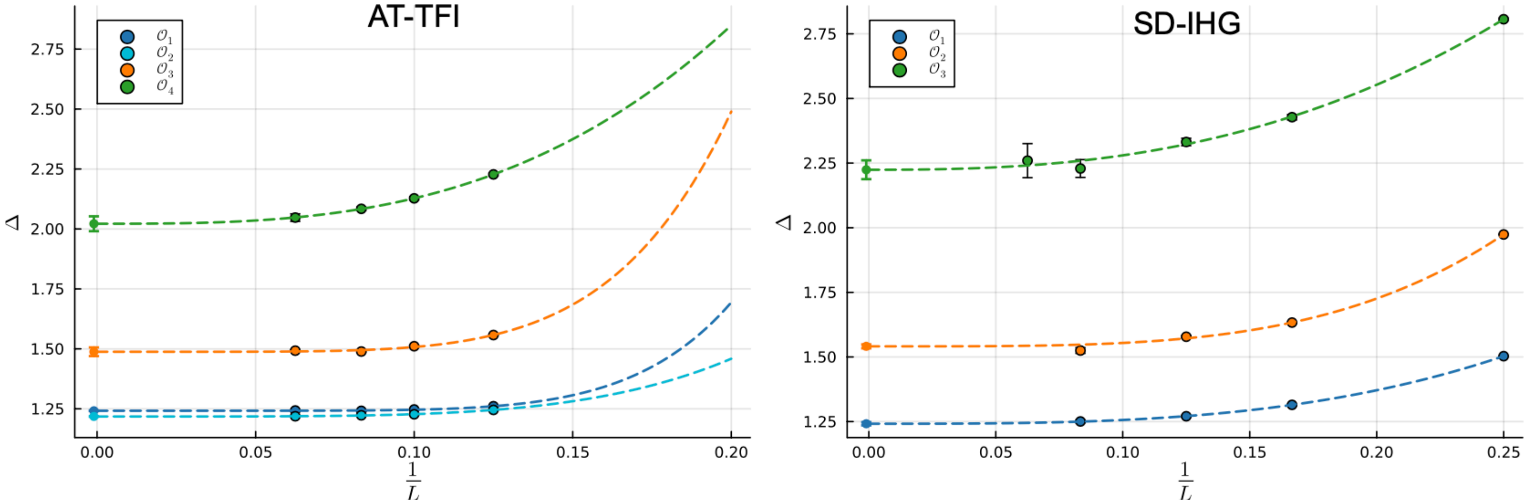
- Expanding sin/cos and averaging we obtain:

$$\mathcal{H} \propto \tau \circ \nabla \langle \varphi(\mathbf{r}) \rangle_{\mathbf{r}\in\mathcal{V}}$$

Critical exponents for the Self-dual Ising-Higgs Gauge theory and the Ashkin-Teller model



Filters as scaling operators

https://github.com/RSMI-NE/RSMI-NE

Phys.Rev.Lett 127, 240603 (2021)

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- They can used as operators in correlation functions

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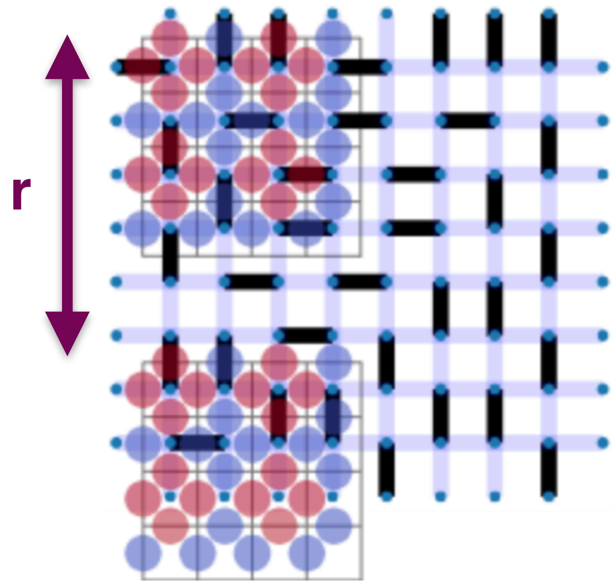

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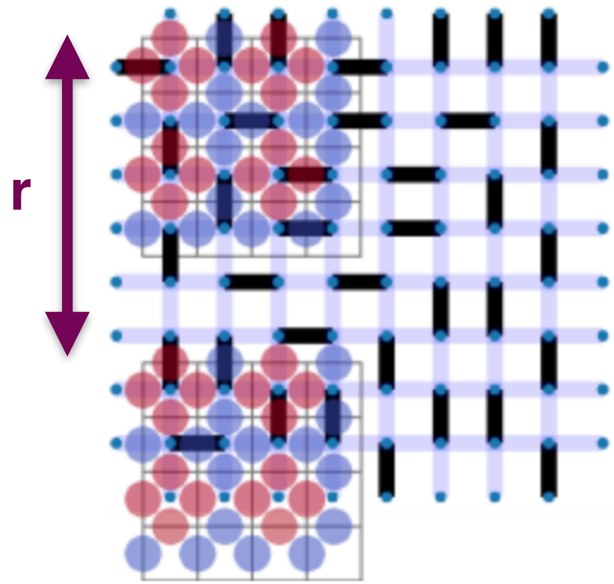

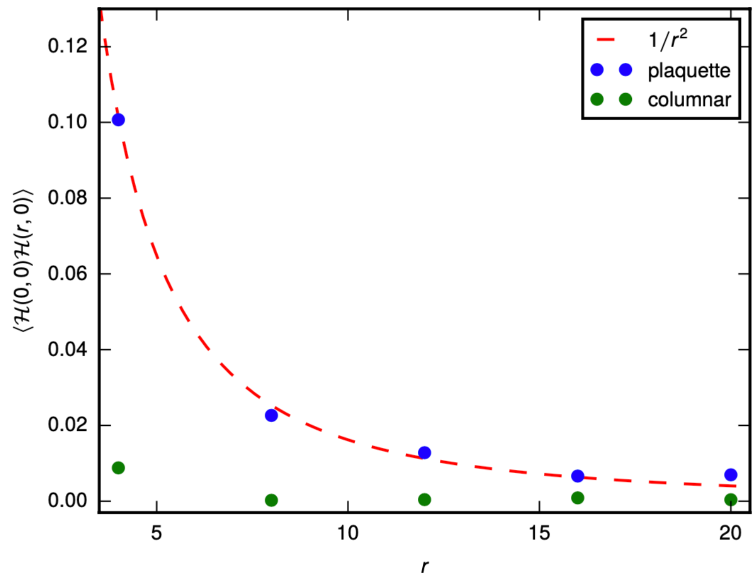

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