# Physics-induced (functional)RG-flows


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STRUCTURES
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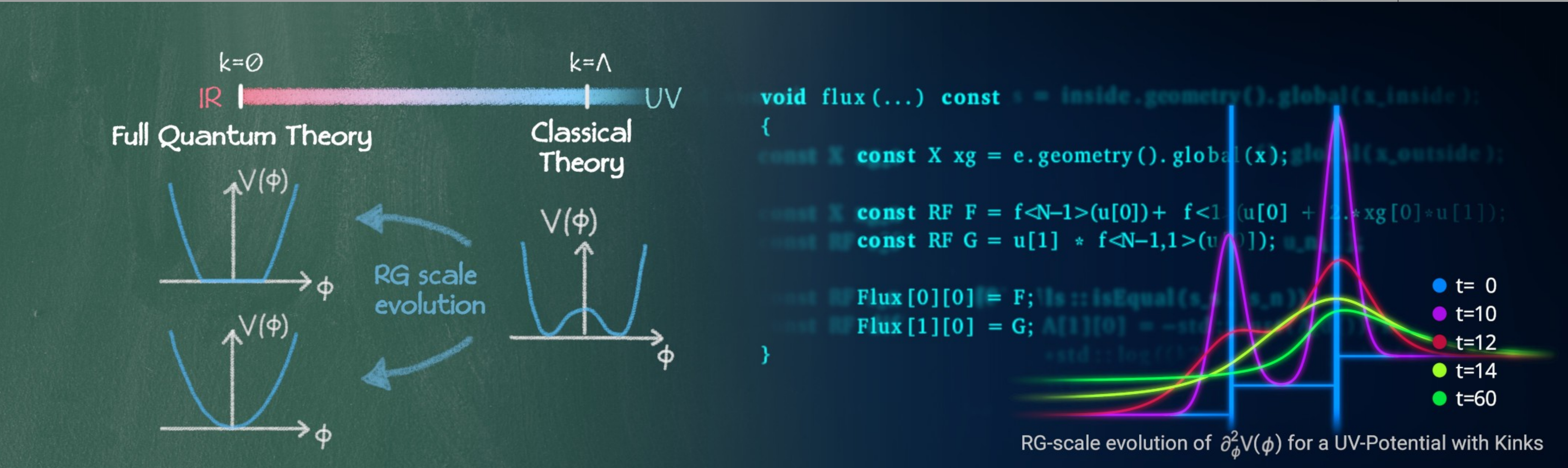
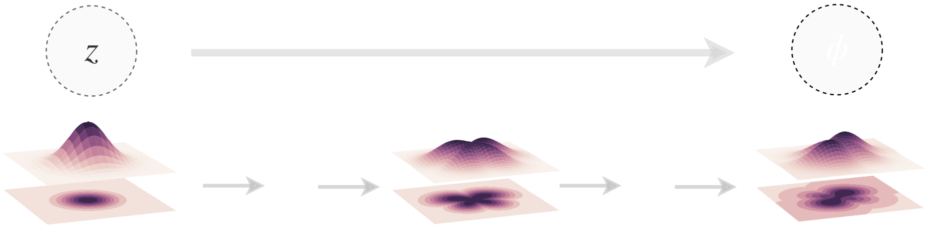
EXCELLENCE




ECT* Workshop: Machine Learning and the Renormalisation Group




## Generative AI for producing lattice configurations

- "Learn" field configuration for some action

  → Invertible field transformation

- Limited lattice size

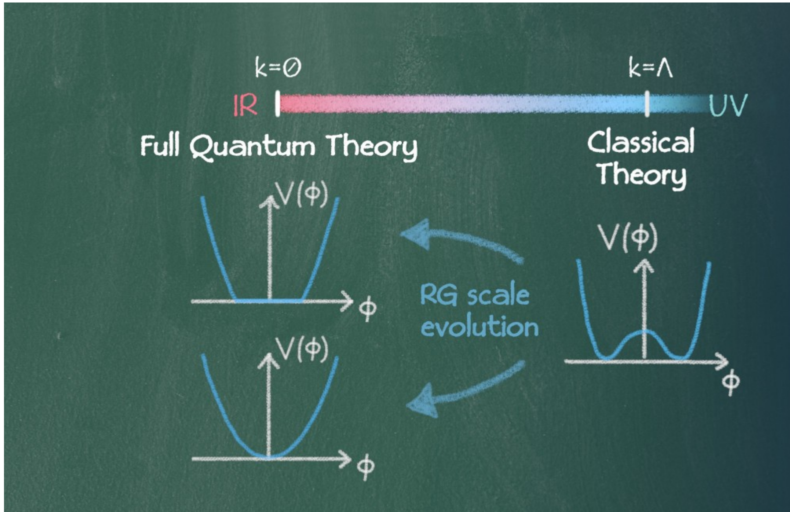
Improve physical predictions by optimising the representation of DoFs

Interface: ML and fRG

## The functional Renormalisation Group

- fRG is a well established tool for physics of phase transitions
- Approximation of degrees of feedom
- Uncharted possibilities in terms of general field transformations

  → Flowing field transformations

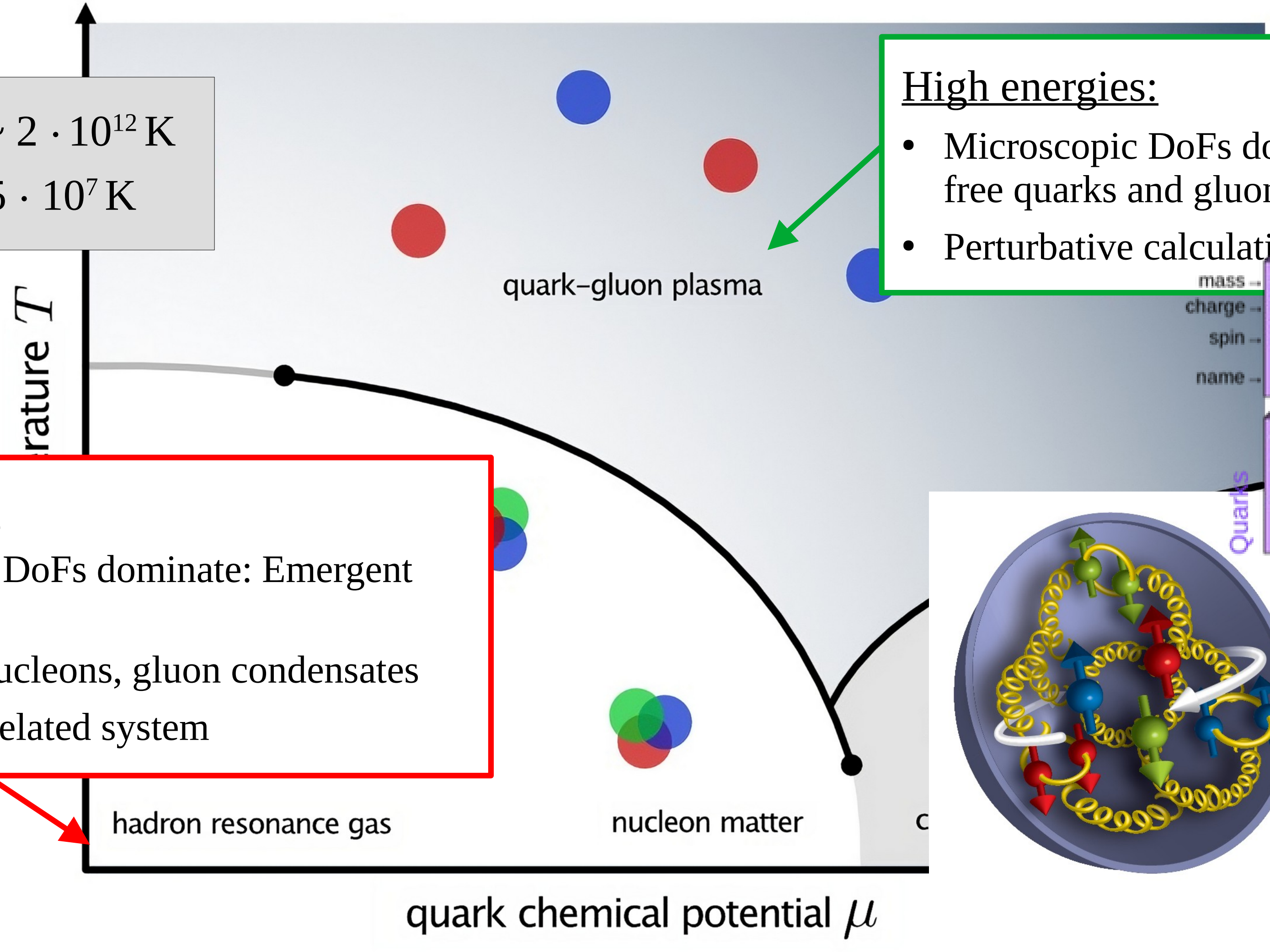


# Example QCD: phase transitions across large scales

$155\ \text{MeV} \sim 2 \cdot 10^{12}\ \text{K}$

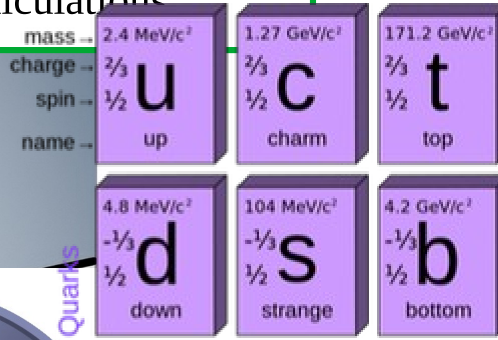
$T_{sun} \sim 1{,}5 \cdot 10^{7}\ \text{K}$

155 MeV




**High energies:**

- Microscopic DoFs dominate: free quarks and gluons
- Perturbative calculations



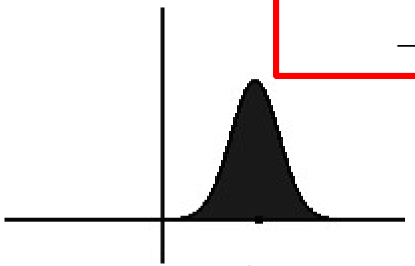

**Low energies:**

- Macroscopic DoFs dominate: Emergent composites
  - Mesons, nucleons, gluon condensates
- Strongly correlated system

# The functional Renormalisation Group: a method of scales

Macroscopic DoFs:

- Gluons decouple: Mass gap
  - → Gluon condensate
- Quark condensates (LEFTs)
  - → Mesons

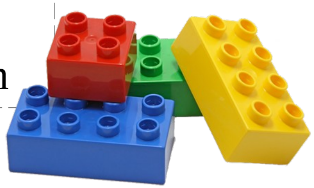


$$\partial_t P[\phi] + \frac{\delta}{\delta\phi(x)}\Big(\Psi[\phi]\, P[\phi]\Big) = 0$$

$$t = \log\left(k/\Lambda\right)$$

$$P[\phi] = e^{-S_{\mathrm{eff}}[\phi]}$$

Modular approach



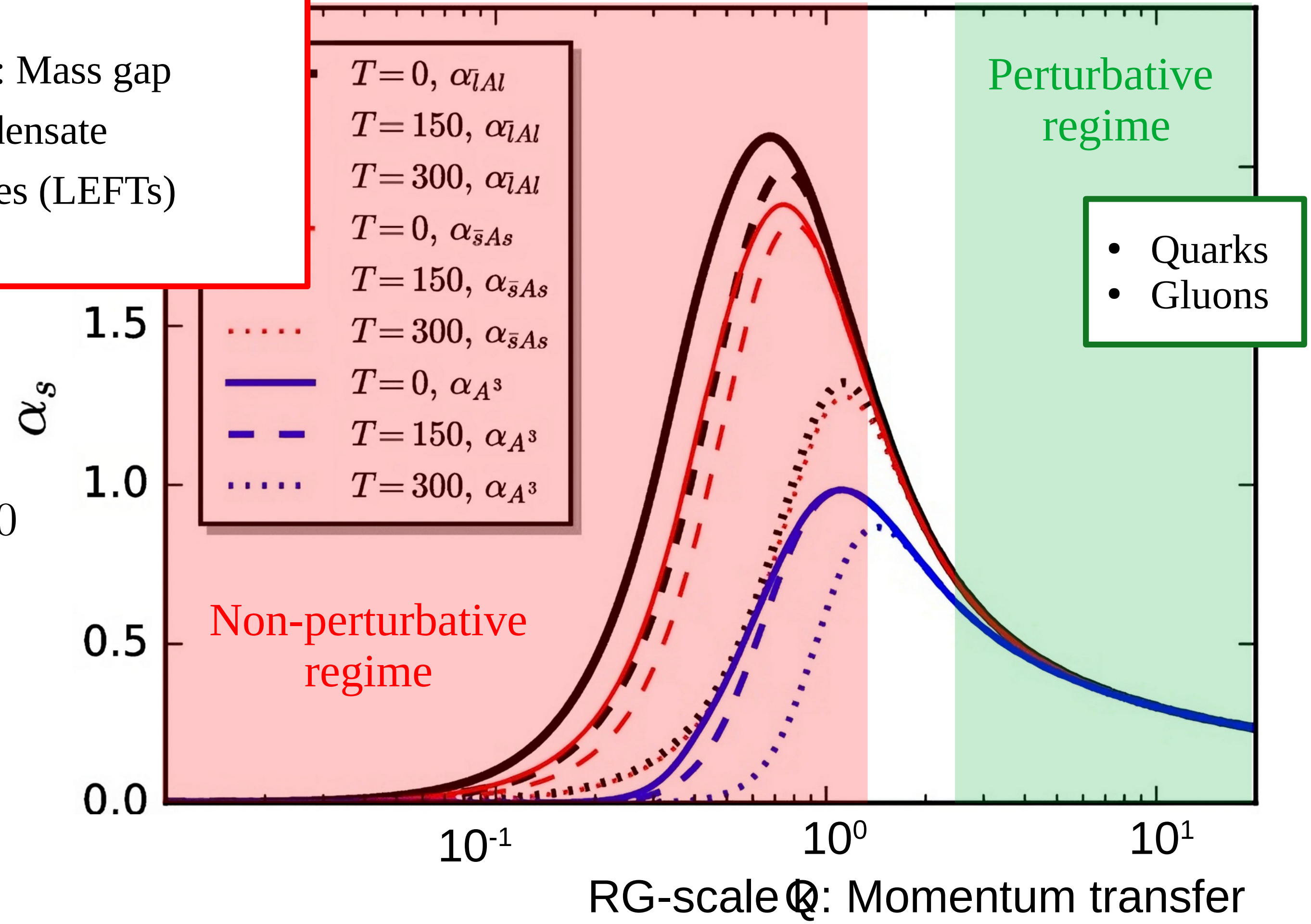



Image: Fu, Pawlowski, Rennecke arXiv:1909.02991

Improve physical predictions by optimising the representation of DoFs
Interface: ML and fRG

Efficient description of QCD <> known dominating DoFs

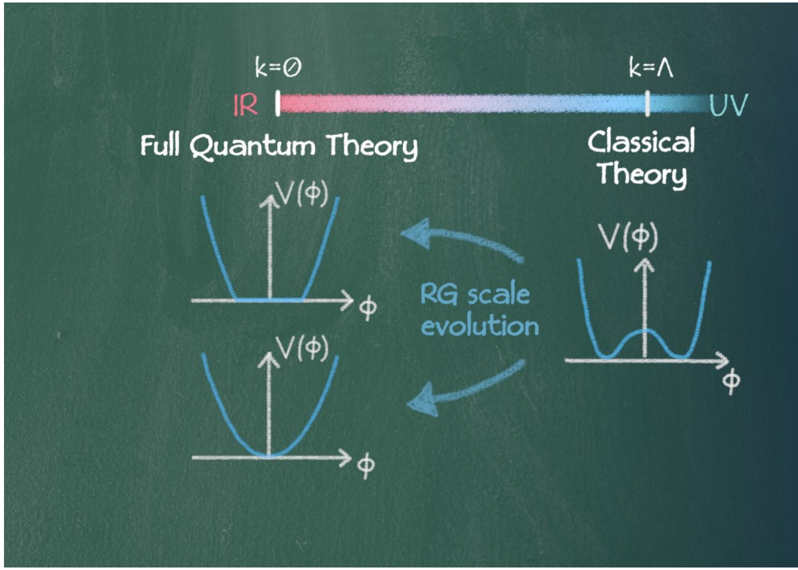
(1) Optimise representation of known emergent physics

What about unknown systems (e.g. Gravity)

(2) Detection of relevant DoFs

Method development:

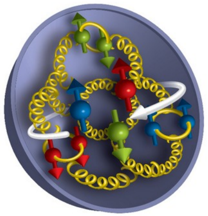
(3) Optimised representation <> efficient computation




$$\partial_t P[\phi] + \frac{\delta}{\delta\phi(x)}\Big(\Psi[\phi]\, P[\phi]\Big) = 0$$

$$P[\phi] = e^{-S_{\mathrm{eff}}[\phi]}$$

# The path integral and probability distributions

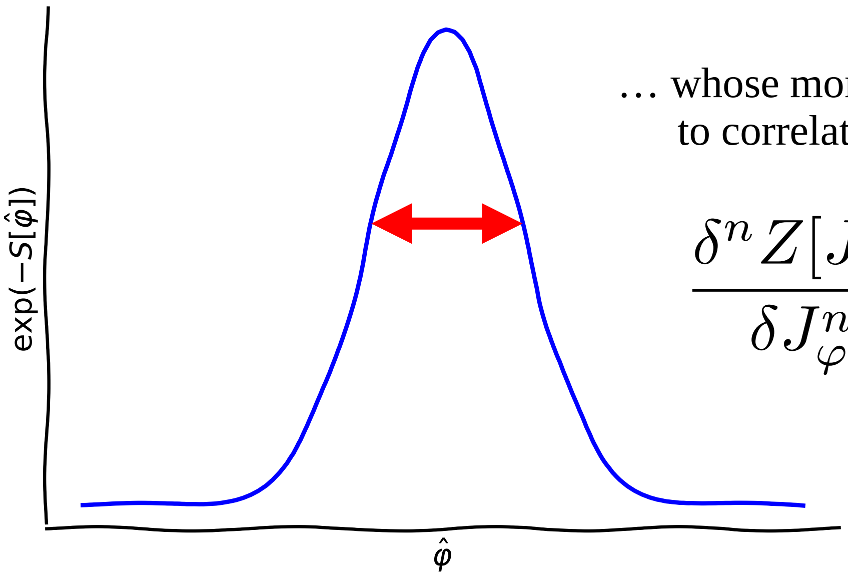


Classical action: $$S[\varphi] = \int_x \left\{ \frac{1}{2}\left(\partial_\mu \varphi\right)^2 + \frac{m_\varphi^2}{2}\varphi^2 + \frac{1}{8}\lambda_\varphi\, \varphi^4 \right\}$$

Generating functional: $$Z[J_\varphi] \simeq \int [d\hat{\varphi}]_{\text{ren}}\, e^{-S[\hat{\varphi}] + \int_{\boldsymbol{x}} J_\varphi \hat{\varphi}}$$

We can also view this as
a probability distribution...

$$d\mu[\hat{\varphi}] = [d\hat{\varphi}]_{\text{ren}}\, e^{-S[\hat{\varphi}]}$$




... whose moments correspond
to correlation functions

$$\frac{\delta^n Z[J_\varphi]}{\delta J_\varphi^n} \simeq \langle \hat{\varphi} \cdots \hat{\varphi} \rangle$$

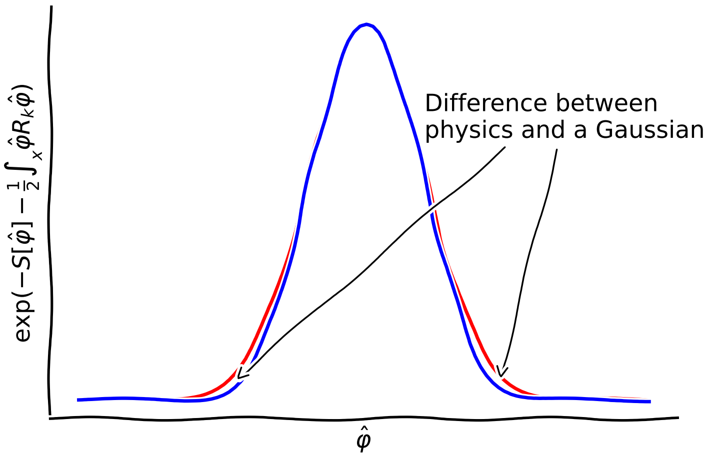
# The functional Renormalisation Group

- Infrared regularised theory ↔ classical theory

$$Z_k[J_\varphi] \simeq \int \boxed{d\mu[\hat{\varphi}]_{p^2 \gtrsim k^2}} \, e^{\int_{\boldsymbol{x}} J_\varphi \hat{\varphi}}$$

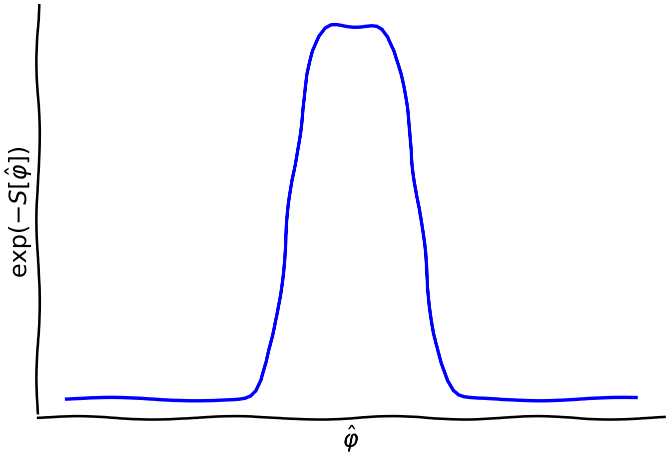
$$d\mu_k[\hat{\varphi}] = d\mu[\hat{\varphi}] \; e^{-\frac{1}{2}\int_x \hat{\varphi} R_k \hat{\varphi}}$$

Artificial mass term




$\Lambda \to k \to 0$

Removal of
the regulator




Successive removal of the artificial mass scale implements the flow from a near Gaussian to the full quantum theory

Normalising Flows

# What is an optimal representation of physical quantities?

$$Z[J_\varphi] \simeq \int d\mu_k[\hat\varphi]\, e^{\int_{\boldsymbol{x}} J_\varphi \hat\varphi}$$

Fundamental field: e.g. Quarks

$$Z[J_\phi] \simeq \int d\mu_k[\hat\varphi]\, e^{\int_{\boldsymbol{x}} J_\phi \hat\phi[\hat\varphi]}$$

Composite field: e.g. Pions
- 2PPI approaches
- Density functional theory

**Field transformations for optimisation:**

$$\hat\varphi \to \hat\phi[\hat\varphi]$$

1) Fundamental fields may not be the physical observable of interest

Ihssen, Pawlowski: arxiv:2305.00816

2) Reduce the amount of cumulants

$$\langle \hat\phi \cdots \hat\phi \rangle_c = 0$$
$$\langle \hat\varphi \cdots \hat\varphi \rangle_c \neq 0$$

3) Decouple degrees of freedom

Ihssen, Pawlowski: in Preparation

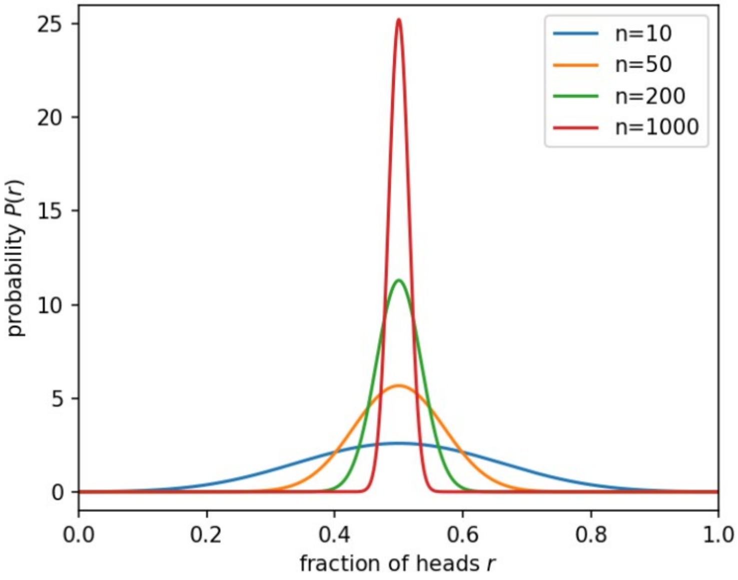
# How do we extract physics?

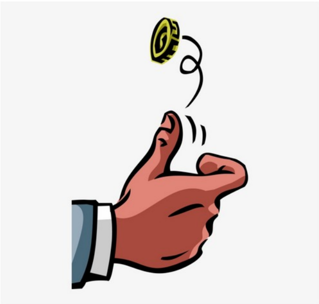
$$Z[J_\varphi] \rightarrow W[J_\varphi] = \log(Z[J_\varphi]) \rightarrow \Gamma[\varphi] = \sup_J \int_x J\varphi - W[J]$$

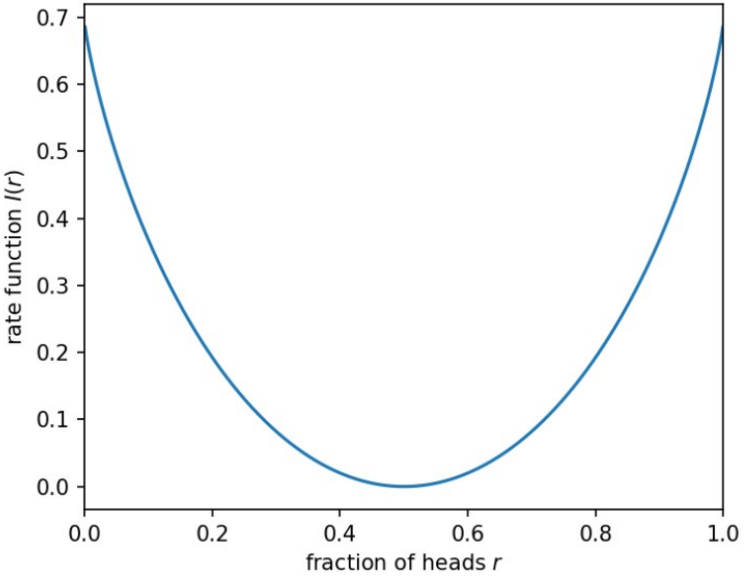
Correlation functions

Cumulants

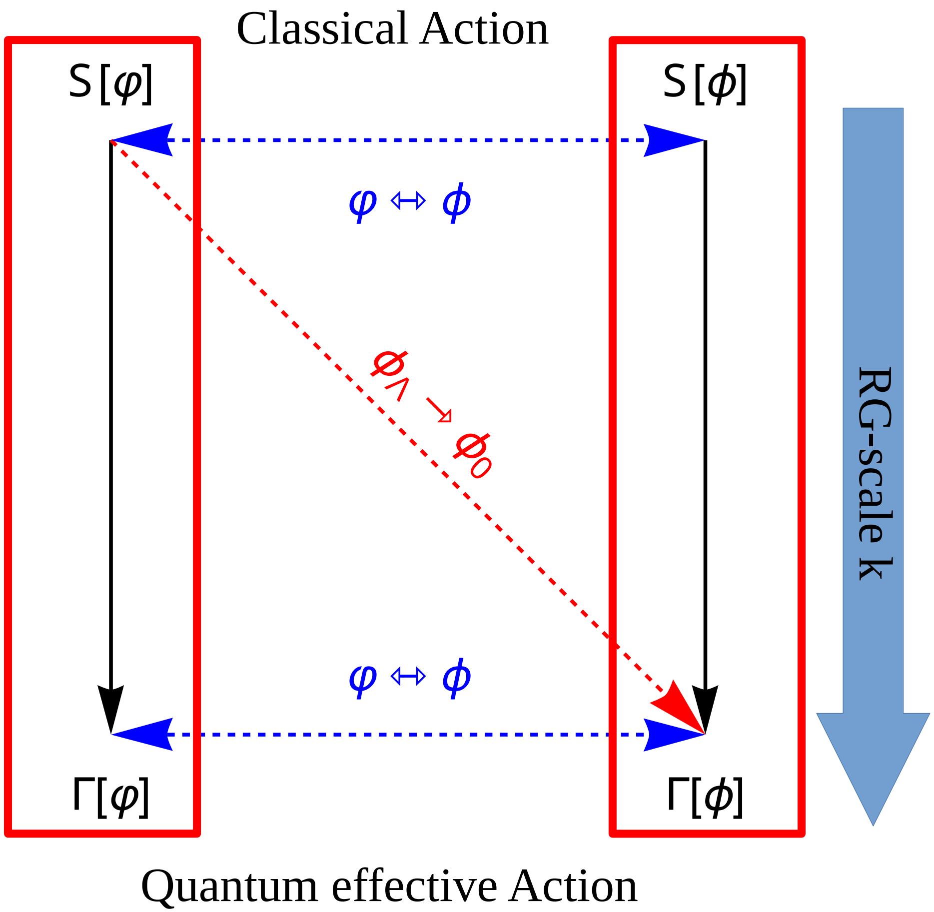
Rate function
Large Deviations Theory





# General field transformations in the fRG




- Infrared regularised theory ↔ classical theory

$$\Gamma_\Lambda[\varphi] = S[\varphi]$$

- Solve RG-flow by integrating over RG-time/RG-scale

Regulator

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2}\, G_k[\varphi]\, \partial_t R_k$$

RG-time

$$t = \log\left(\frac{k}{\Lambda}\right)$$

Mean-field

$$\varphi = \langle \hat{\varphi} \rangle$$

Propagator

- Full quantum effective action $\Gamma[\varphi]$
  Wetterich'92

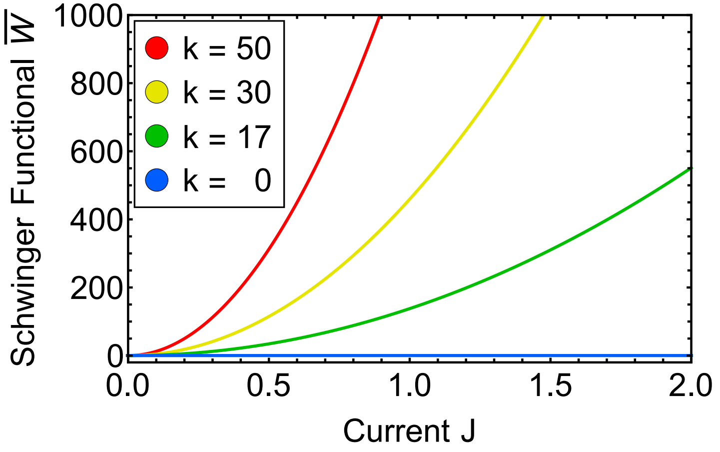
  - Solve PDE for all generated couplings in the effective action

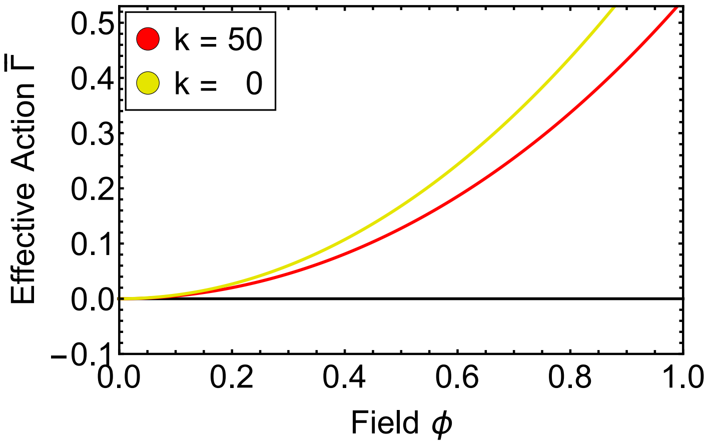
$$\partial_t W_k[J] = -\frac{1}{2}\partial_t R_k \left(W_k^{(2)}[J] + (W_k^{(1)}[J])^2\right)$$

- Algebraic equation with a linear diffusion term
- High degree of redundance

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\,\frac{1}{\Gamma_k^{(2)} + R_k}\,\partial_t R_k$$

- Non algebraic convection-diffusion equation
- Convex functional
- Very condensed information

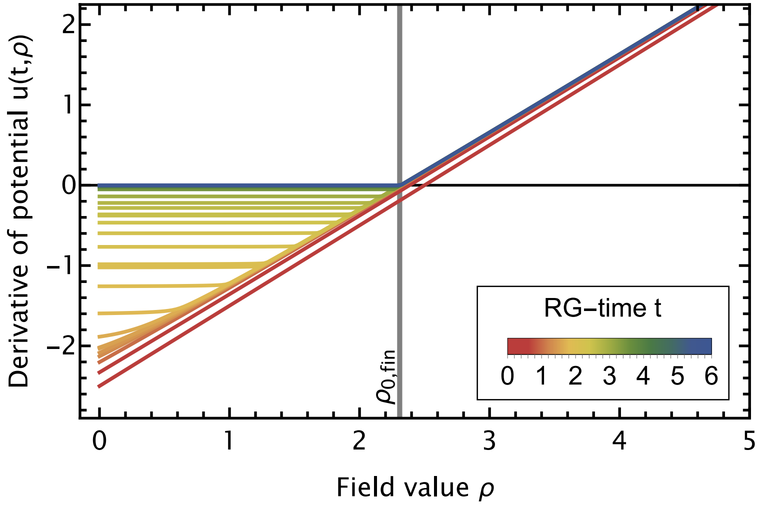


# An example: the local potential approximation

$$\Gamma_k[\phi] = \int_x \left[-\frac{1}{2}\phi(x)\,\partial_\mu^2\phi(x) + V_{\text{eff},k}\right]$$

$$\partial_t V(\phi) = A_d \frac{k^{2+d}}{k^2 + V^{(2)}(\phi)} + \text{Pions}$$

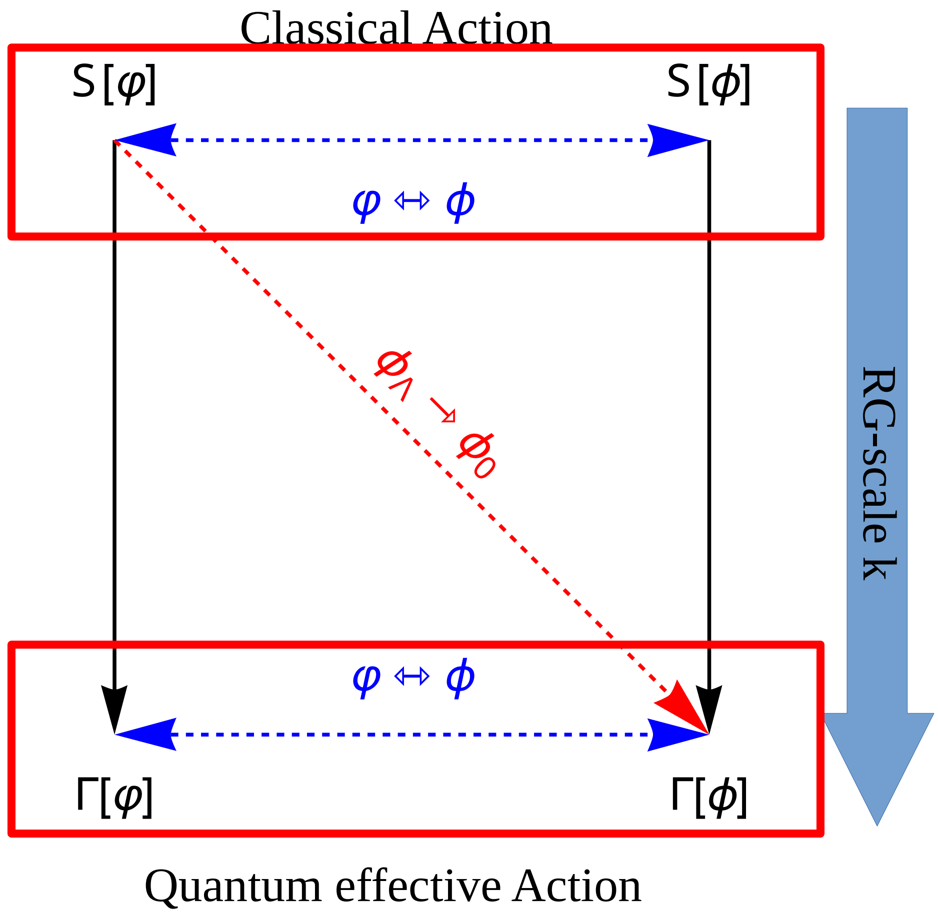



Ihssen arXiv2302.04736

- LPA: solve a PDE for the effective potential only
  - $V_{\text{eff}}$ is a function of constant background $\phi(x) = \phi$
- Wide variety of numerical developments
  - Convexity restoration
  - Time-stepping
  - Shock development

Grossi arXiv1903.09503, Grossi arXiv2102.01602, Koenigstein arXiv2108.02504, Ihssen arXiv2309.07335

# General field transformations in the fRG




- Explicit field transformations (also possible with the RG)

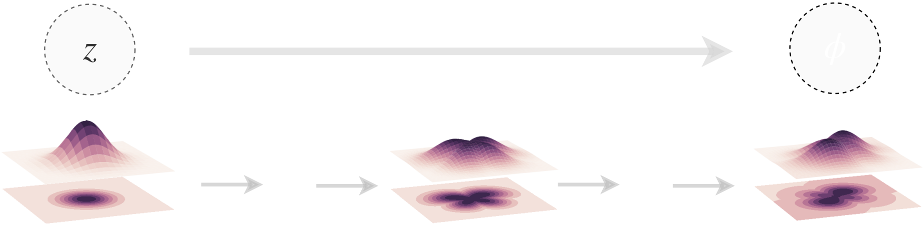
$$Z[J_\varphi] \simeq \int [d\hat{\varphi}]\, e^{-S[\hat{\varphi}]+\int_{\boldsymbol{x}} J_\varphi \hat{\varphi}}$$

$$Z[J_\phi] \simeq \int [d\phi]\ \det\left|\frac{\delta\hat{\varphi}}{\delta\hat{\phi}}\right| e^{-S[\hat{\varphi}[\hat{\phi}]]+\int_{\boldsymbol{x}} J_\phi \hat{\phi}}$$

- Or a normalising flow for the full quantum theory

$$\langle\hat{\phi}\cdots\hat{\phi}\rangle_c = 0$$
$$\langle\hat{\varphi}\cdots\hat{\varphi}\rangle_c \neq 0$$

where a free theory is mapped on an interacting one
Albergo et al. arXiV:2101.08176




## Generative AI for producing lattice configurations

- "Learn" field configuration for some action
  - → Invertible field transformation
- Limited lattice size

# Normalising flows and the effective action

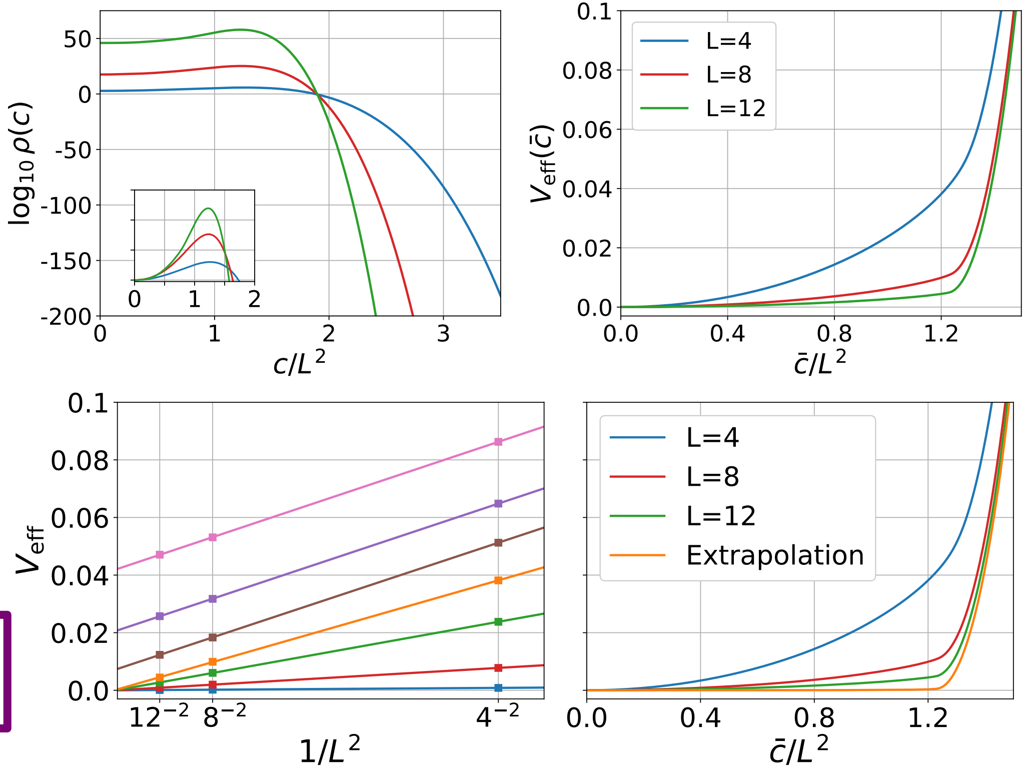
- Resolution of the effective action requires a cheap sampling method

  Attanasio, Bauer, Pawlowski, Temmen in Prep.

- Field transformation between theories with different UV-cutoffs

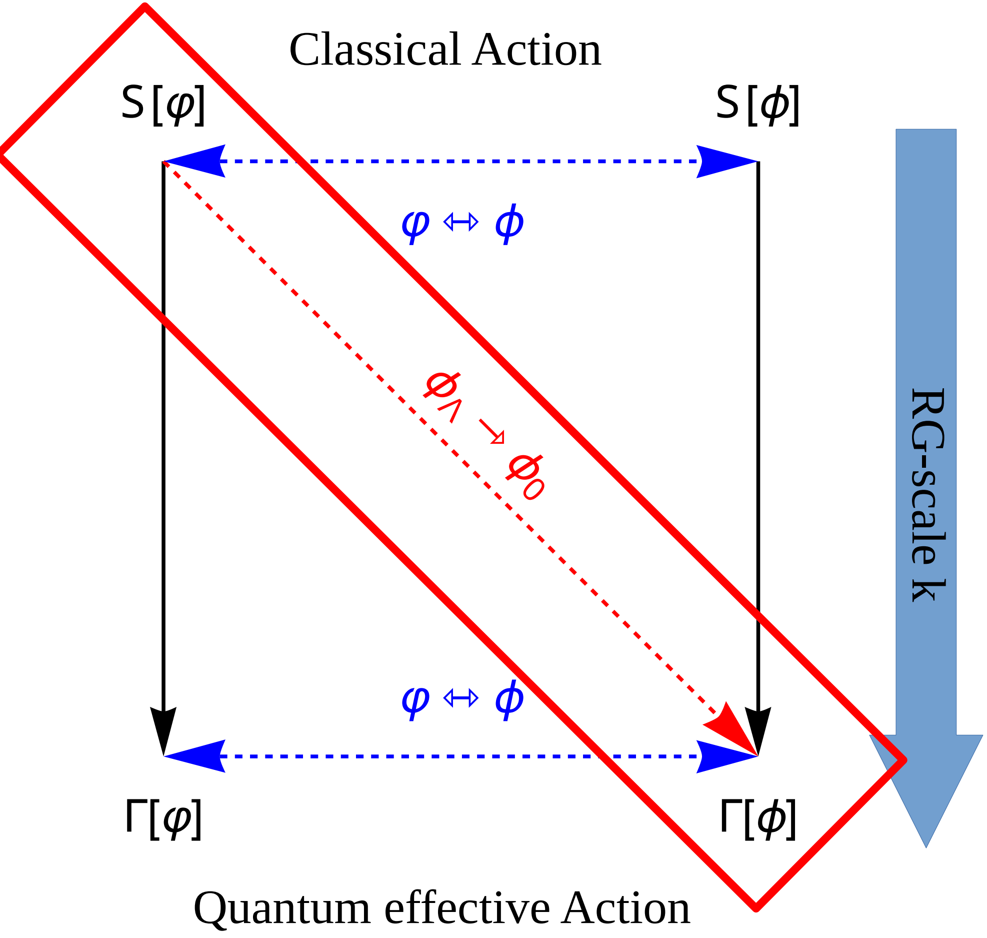
  Bauer, Kapust, Pawlowski, Temmen in Prep.

RG Flows between different lattice sizes

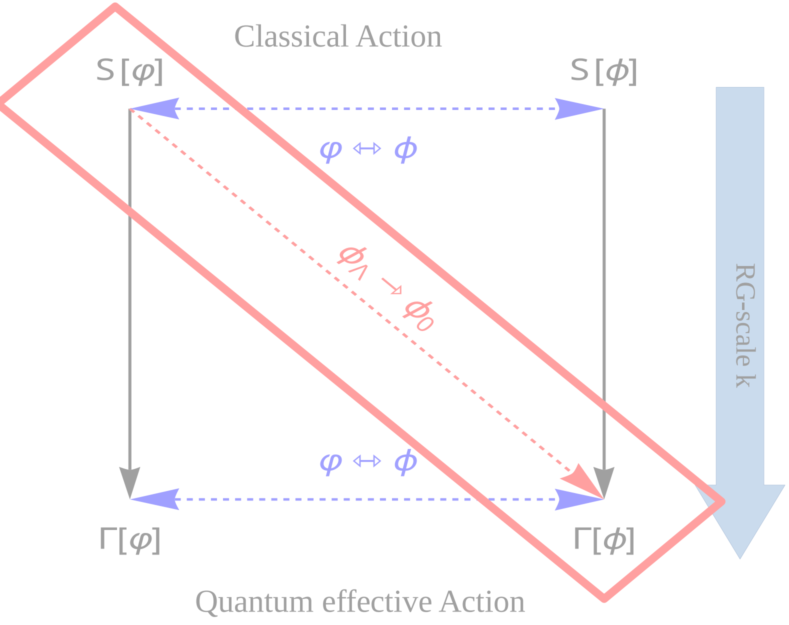



Marc Bauer, Renzo Kapust

# General field transformations in the fRG



# General field transformations in the fRG




- Generalised functional Flows

RG-time — 1PI gen. funct. — Propagator — Regulator

$$t = \log\left(\frac{k}{\Lambda}\right)$$

$$\left(\partial_t + \int_x \dot\phi \frac{\delta}{\delta\phi}\right)\Gamma_k[\phi] = \frac{1}{2}\left[G[\phi]\left(\partial_t + 2\frac{\delta\dot\phi}{\delta\phi}\right) R_k\right]$$

Pawlowski arXiv0512261

- RG-scale dependent composite

$$\dot\phi[\phi] = \langle \partial_t \hat\phi_k \rangle[\phi]$$

- 1PI formulation of general transformations of the path integral Wegner '74

- RG-kernels and optimal transport
  Cotler, Rexchikov arXiv2202.11737

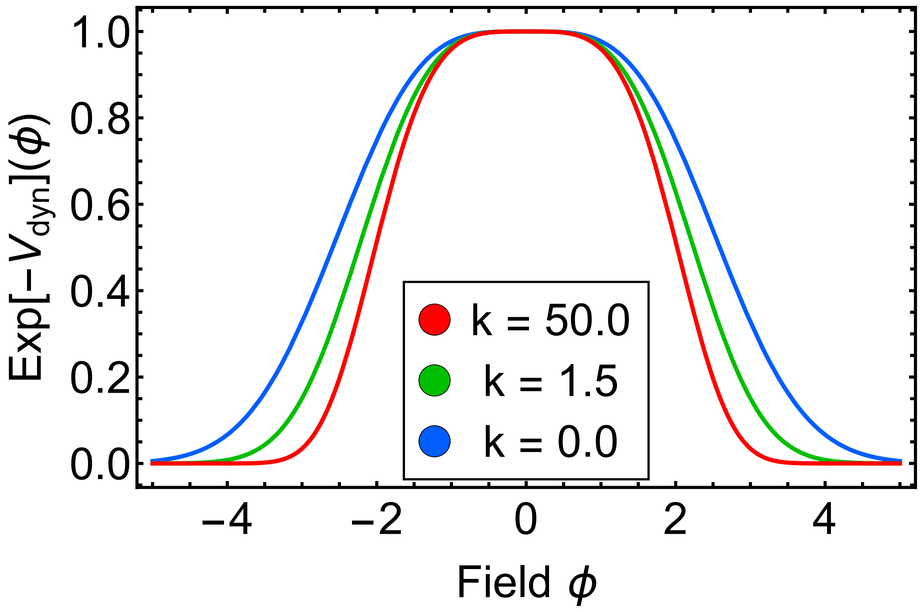
$$\partial_t P[\phi] = \frac{\delta}{\delta\phi}\Psi P[\phi]\,, \quad P[\phi] = e^{-S[\phi]}$$

# Side Note: Wegner's generalised flows

$$\partial_t P[\phi] = \frac{\delta}{\delta\phi}\Psi P[\phi]\,, \qquad P[\phi] = e^{-S[\phi]}$$

- General transformations, which leave the path integral unchanged Wegner '74
- RG-Kernel $\Psi[\phi]$ , applications to optimal transport, ML
  Cotler arXiv2202.11737
- Complex functional flows
  Ihssen arXiv220710057

$$\Psi = \dot{\phi}$$



# (1) Physically motivated applications

"Absorption of functions"
Baldazzi arXiv2105.11482, Braun arXiv0810.1727, arXiv1412.1045, Rennecke arXiv1504.03585, Fu arXiv1909.02991

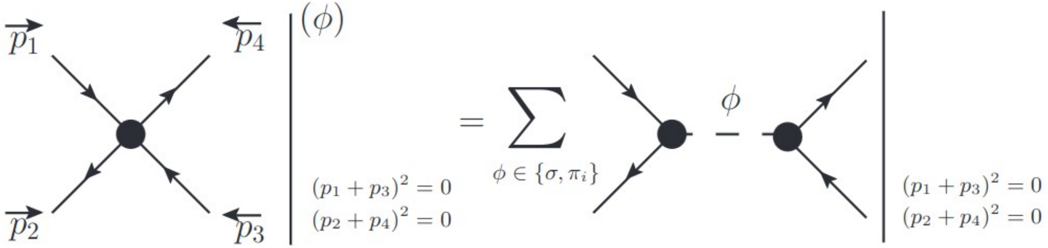
Absorb flows of correlation functions into the field

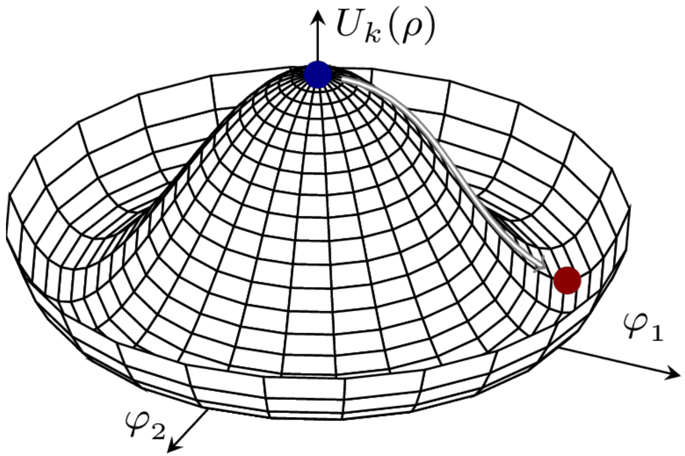
$$\phi_k(\varphi, k) \;\; \to \partial_t \Gamma^{(n)} \equiv 0$$

"Geometric transformations" Lamprecht '07
Flow from a Cartesian to a polar basis

$$\phi^t = (\rho, \theta)$$

$$\varphi = \sqrt{2\rho}\, e^{\theta^a t^a}\, (1,\, 0,\, \dots,\, 0)^t$$



# (1) An expansion about the ground state

Ihssen, Pawlowski: arxiv:2305.00816

O(N) model: $\varphi^t = (\varphi_1, \dots, \varphi_N)$ vs. $\phi^t = (\phi_1, \dots, \phi_N)$

$$\rho_\varphi = \frac{\varphi^2}{2} \qquad \rho = \frac{\phi^2}{2}$$

$$\Gamma^{(2)}_{\varphi_i \varphi_i}[\varphi](p) = Z_\varphi(\rho_\varphi, p)\Big(p^2 + m^2_{\varphi_i}(\rho_\varphi)\Big)$$

Field dependent wave function renormalisation and its derivatives

*__Take away message:__*

- Expansion about classical dispersion
- Technical simplification with improved truncation

$$\Gamma^{(2)}_{\phi_i \phi_i}[\phi](p) = \left[p^2 + m^2_{\phi_i}(\rho)\right]$$

$$\Gamma_k[\phi] = \int_x \left[\frac{1}{2} Z_{\phi,k} (\partial_\mu \phi)^2 + V_k(\rho) - c_\sigma \sigma\right]$$

Expand about ground state using the flowing Fields:

$$\dot{\phi}_k(\phi, k) \rightarrow Z_{\phi,k}(\phi, p) \equiv 1$$

$$\partial_t Z_\phi(\phi, p) \equiv 0$$

# (2) Detect relevant DoFs

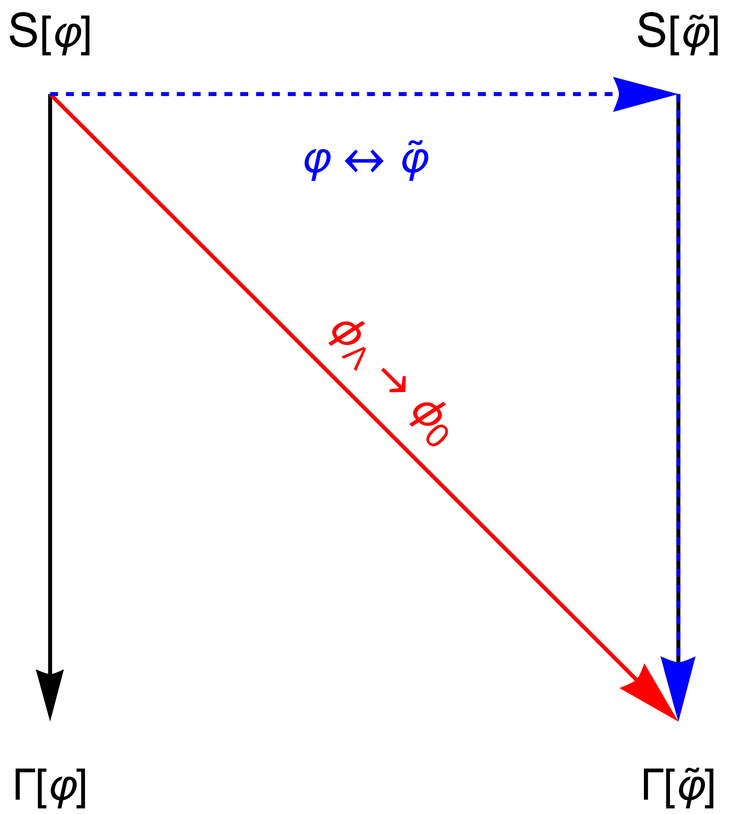
Can we find trivialising maps?

$$S[\varphi] = \frac{1}{2}\int_x K(\varphi)(\partial_\mu \varphi)^2 \quad \Rightarrow \quad \frac{\partial \tilde{\varphi}}{\partial \varphi} = K^{1/2}(\varphi)$$

$$\tilde{S}[\tilde{\varphi}] = \int_x \frac{1}{2}(\partial_\mu \tilde{\varphi})^2 + \frac{\epsilon^{-d}}{2} \ln K[\varphi(\tilde{\varphi})]$$

$$S[\varphi] = \int_x \left\{ \frac{1}{2}\left(\partial_\mu \varphi(x)\right)^2 + \frac{1}{2} m^2 \varphi(x)^2 \right\} \quad \Rightarrow \quad \varphi = \phi + \frac{\alpha}{3}\phi^3$$

$$S[\phi] = \int_x \left\{ \frac{1}{2}\left(1+\alpha\phi^2\right)^2 (\partial_\mu \phi)^2 + \frac{1}{2} m^2 (\phi + \frac{\alpha}{3}\phi^3)^2 - \ln\left(1+\alpha\phi^2\right) \right\}$$



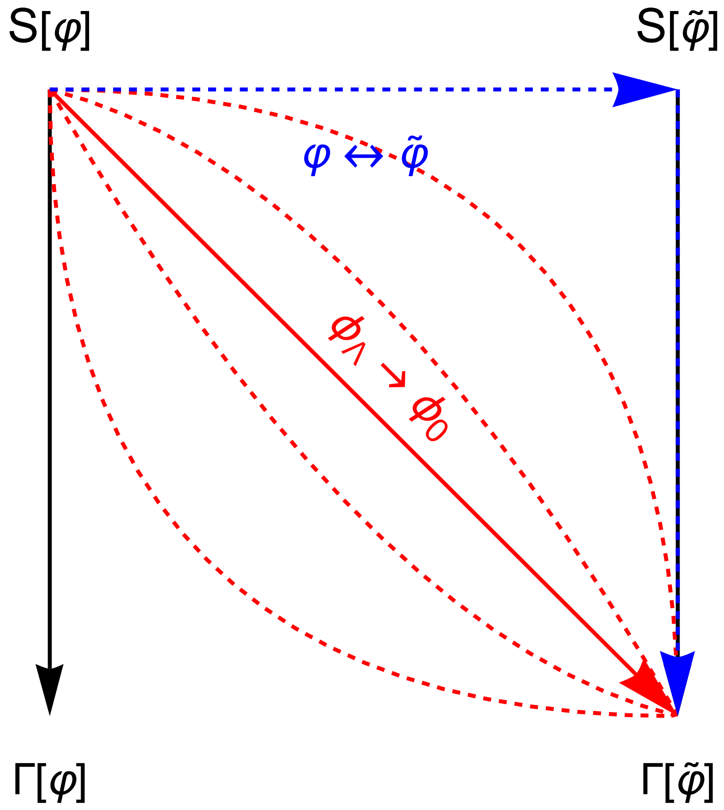
# (2) Detect relevant DoFs

Can we find trivialising maps?

- Absorb Kinetical into Potential   Wetterich arXiV:2402.04679
- Restore a free theory   Defenu, Ihssen Pawlowski in Prep.

Optimal transport

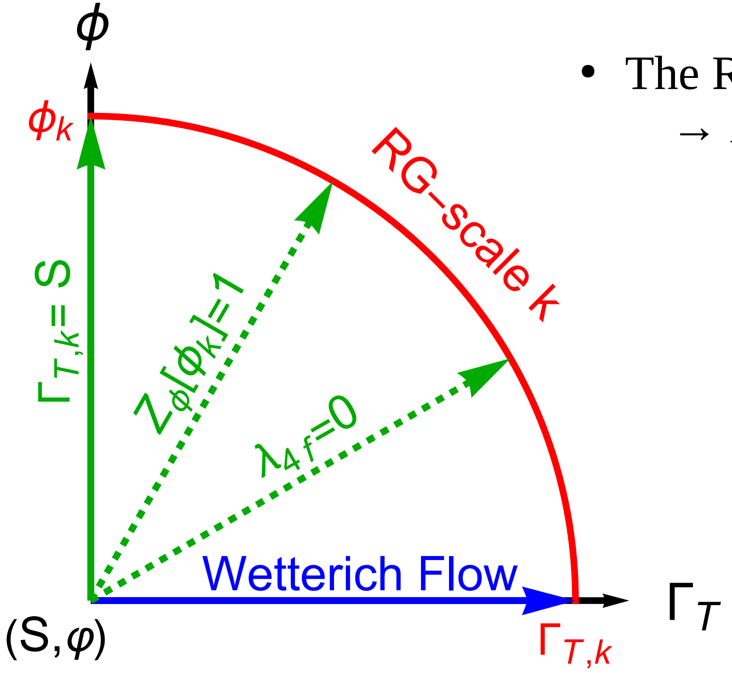
- Reduce RG-flow by adjusting coordinates



# (3) Computational Simplifications

- Let's shift our perspective: Target Actions $\partial_t \Gamma[\phi] \overset{!}{=} \partial_t \Gamma_T[\phi]$

$$\int_{\boldsymbol{x}} \dot{\phi}[\phi]\, \Gamma_T^{(1)}[\phi] - \mathrm{Tr}\Big[G_T[\phi]\, \dot{\phi}^{(1)}[\phi]\, R_k\Big] = \Bigg[\frac{1}{2}\mathrm{Tr}\Big[G_T[\phi]\, \partial_t R_k\Big]\Bigg] - \partial_t \Gamma_T[\phi]$$




- The RG-flow is stored in the pair $\big(\dot{\phi}[\phi]\,,\,\Gamma_T\big)$
  → *Physics-induced flows*

$$\partial_t \lambda_\psi \overset{!}{=} 0 \longrightarrow \dot{\phi}_k[\phi]$$

$$\partial_t Z_\phi[\phi](p) \overset{!}{=} 0 \longrightarrow \dot{\phi}_k[\phi]$$

$$\partial_t \Gamma_T[\phi] \overset{!}{=} 0 \longrightarrow \dot{\phi}_k[\phi]$$

......

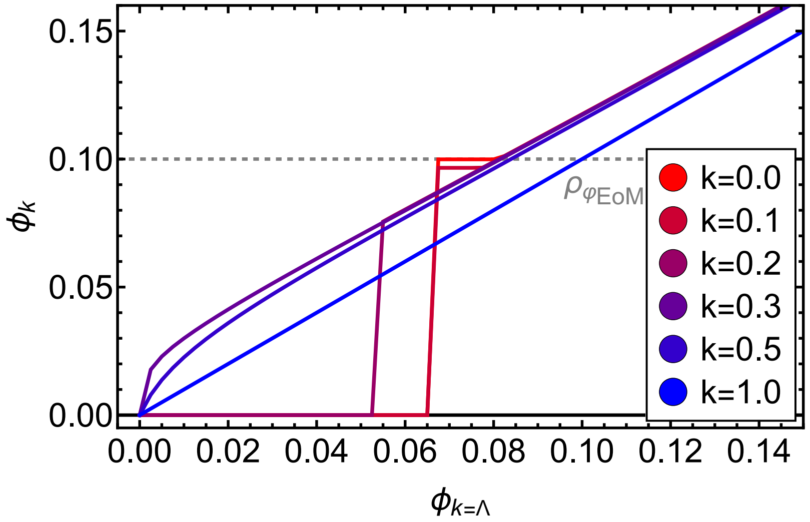
# Target action space: for constant mean field $\phi(x) = \phi$

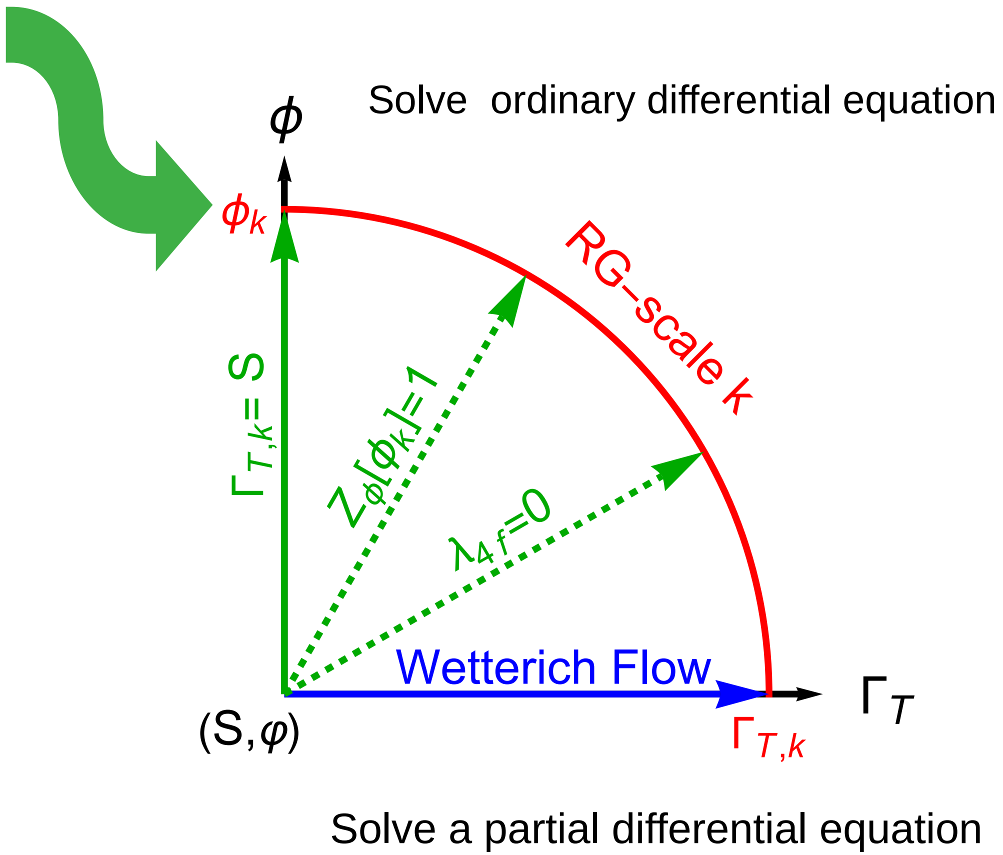
Solve an ordinary differential equation for $\dot{\phi}$

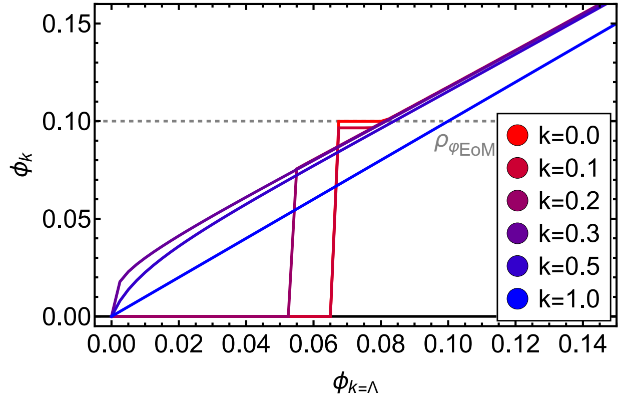
$$\Gamma_T[\phi] = \frac{1}{2}\int_x (\partial_\mu \phi)^2 + \int_x V_{\rm cl}(\phi)$$

→ RG-flow is stored in the map

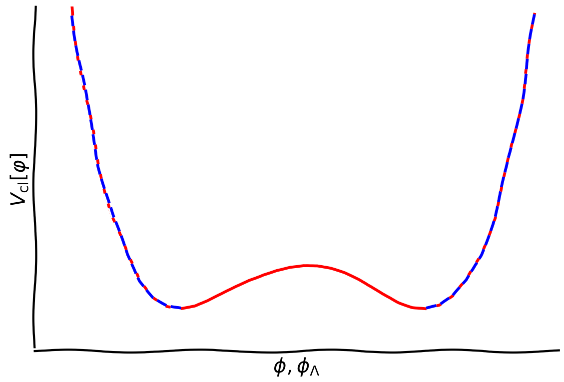
$$\phi[\phi_\Lambda] = \phi_\Lambda + \int_\Lambda^0 \frac{dk}{k}\, \dot{\phi}[\phi]$$



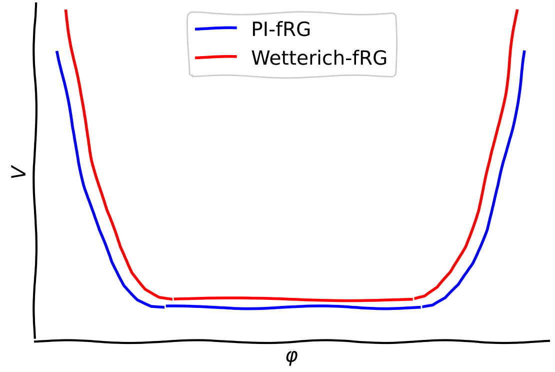


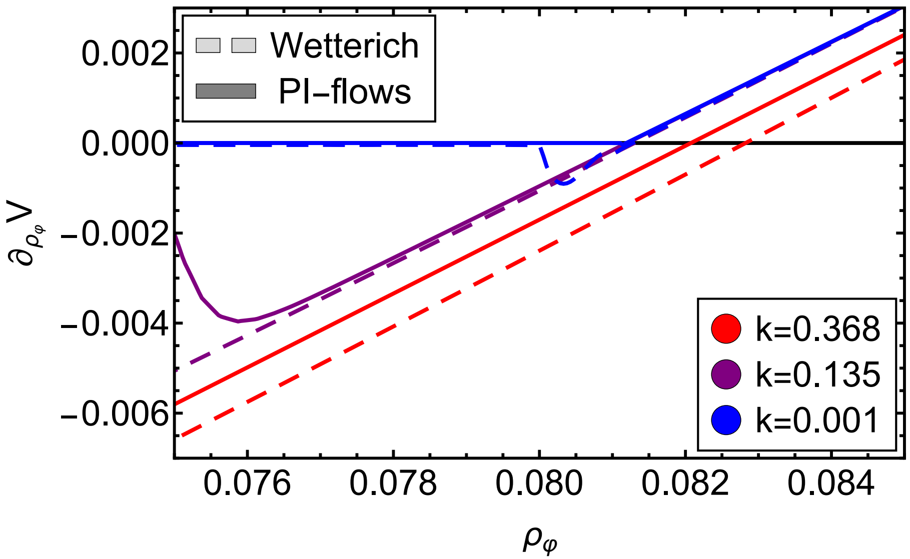

$$V(\phi) = V_{\mathrm{cl}}(\phi_k)$$




$$V_\phi(\varphi) = V(\phi(\varphi))$$

$$\phi_\Lambda = \varphi$$




No coincidence with Wetterich RG in general

Some DoF still available

$$\partial_t V_T \to \partial_t V_T + \dot{\mathcal{C}}_k$$

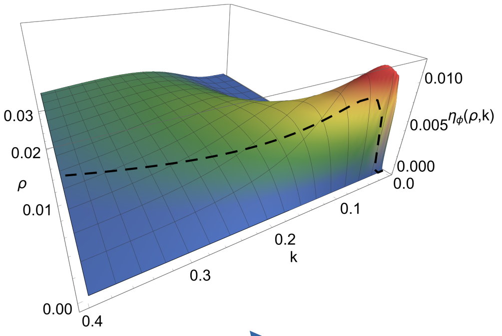
Optimisation of *physics content* of field definitions: Physics-Induced flows

PDE to ODE: Classical target action

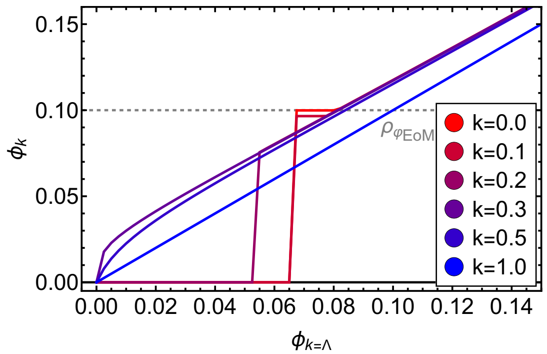
e.g. ground-state-expansion

Tremendous potential for computational simplifications

Implementation of *feed-down-flows*




$$\Gamma_T[\phi[\varphi]] \neq \Gamma[\varphi]$$

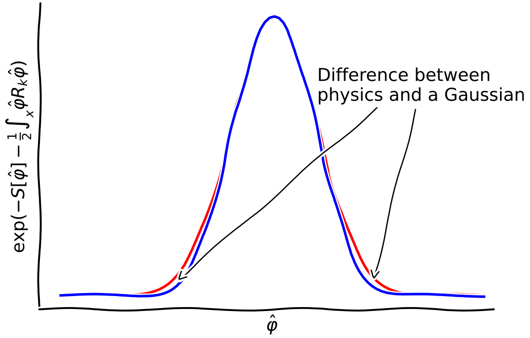



Optimise the *physics content*

Can we extract physical observables from correlation functions of the composite?

Reconstruction of physics: What is the role of the composite?

# Successive removal of the regulator function

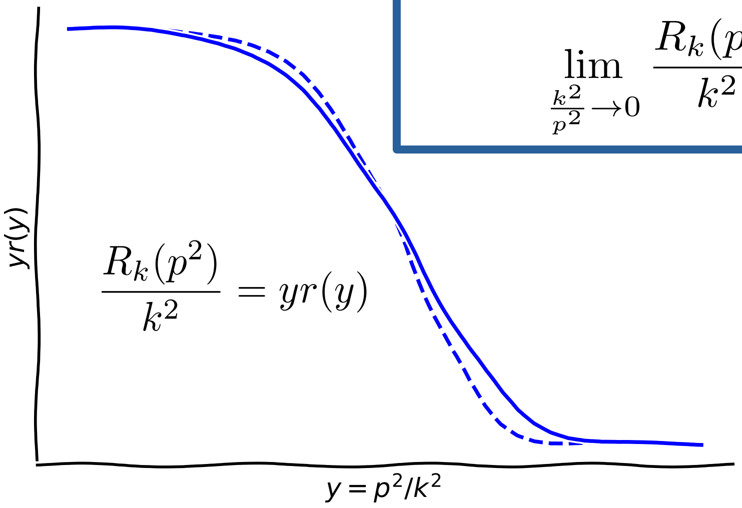


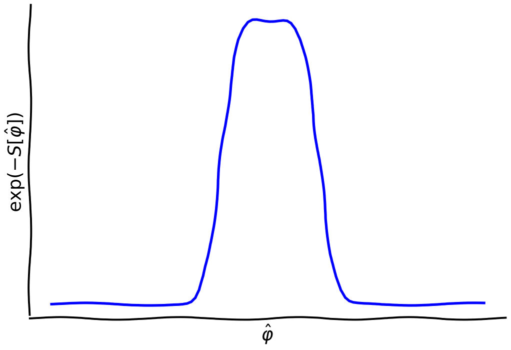

- IR regularisation

$$\lim_{\frac{p^2}{k^2}\to 0} \frac{R_k(p^2)}{k^2} \geq 0$$

- Physical limit

$$\lim_{\frac{k^2}{p^2}\to 0} \frac{R_k(p^2)}{k^2} = 0$$



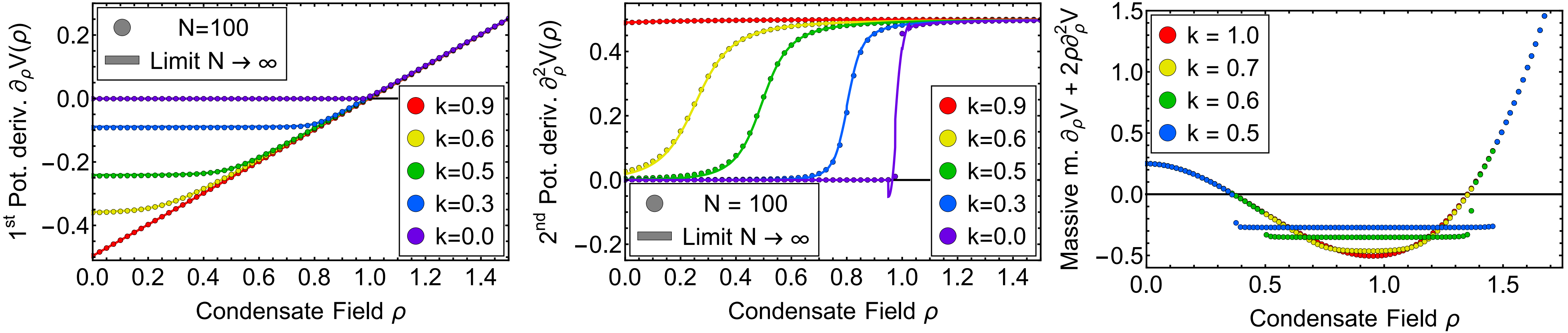




<u>Regulator choice</u>

→ Irrelevant in an ideal representation of the generating functionals

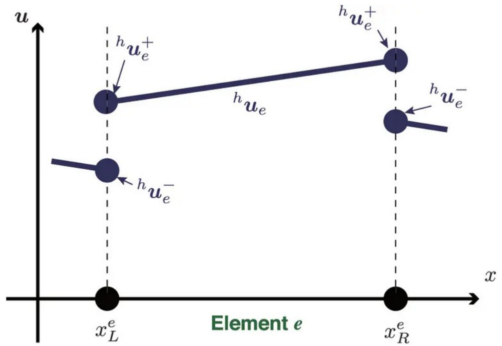
→ Optimisation problem depending on truncation

# Solving RG-flows with Discontinuous Galerkin:




$$\partial_t V = -A_d k^{d+2}\left(\frac{(N_f^2-1)}{k^2+\partial_\rho V}+\frac{1}{k^2+\partial_\rho V+2\rho\partial_\rho^2 V}\right)$$

Convexity restoration + Time-stepping




$$\partial_t u = -\partial_\rho\left[A_d k^{d+2}\left(\frac{(N_f^2-1)}{k^2+u}+\frac{1}{k^2+u+2\rho\partial_\rho u}\right)\right]$$

Convection Diffusive contributions

- This application: $Z_\phi(\rho, p) \approx Z_\phi(\rho)$ (1st order deriv. exp.)
- Task: Solve two equations

1) $\partial_t Z_\phi = 0$ : ODE, determines $\eta_\phi(\rho)$

2) $\partial_t V_k = \ldots$ : PDE, integrate $k \to k - \Delta k$

$$\eta_\phi = 4A_d\bar{\rho}(\bar{V}'')^2\mathcal{B}\mathcal{B}_{(2,2)}\left(1 - \frac{\eta_\phi + 2\rho\eta_\phi'}{d+1}\right)$$

1-loop

Boundary conditions?

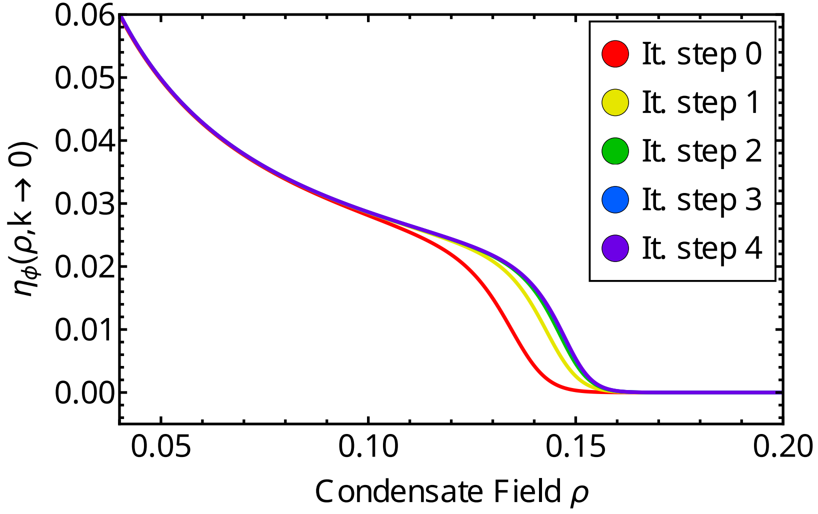
Here: iterative procedure

Parametrisation:

$$\dot{\phi} = -\frac{1}{2}\eta_\phi(\rho)\,\phi$$

And accordingly:

$$\eta_\phi(\rho) = -\frac{\partial_t Z_\varphi(\rho)}{Z_\varphi(\rho)}$$



# A field dependent anomalous dimension

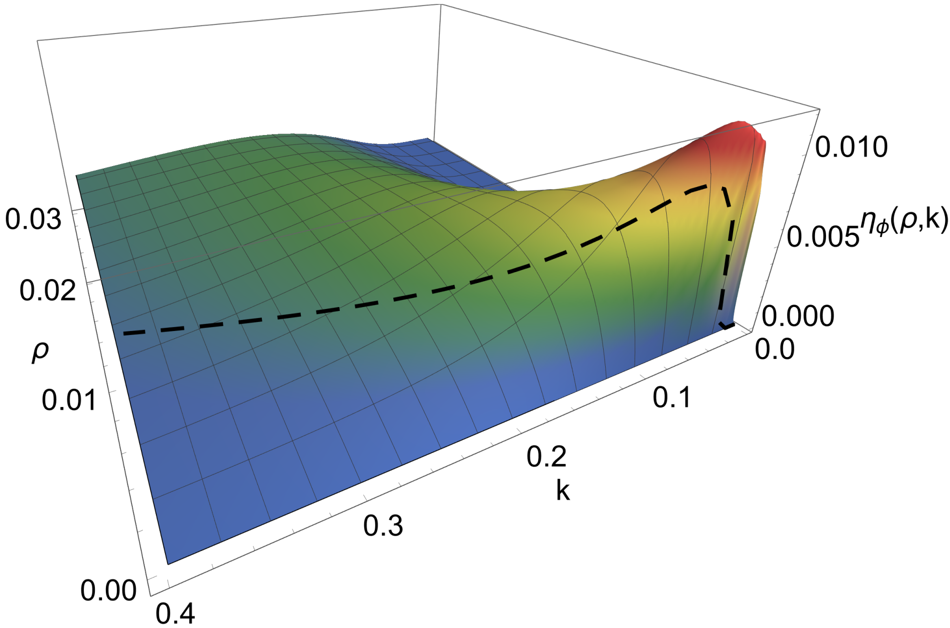
Parametrisation:

$$\dot{\phi} = -\frac{1}{2}\eta_\phi(\rho)\,\phi$$

And accordingly:

$$\eta_\phi(\rho) = -\frac{\partial_t Z_\varphi(\rho)}{Z_\varphi(\rho)}$$

Broken phase,
Low temperature




Symmetric phase,
High temperature

- Application: $Z_\phi(\rho, p) \approx Z_\phi(\rho)$ (1st order deriv. exp.)
- Task: Solve two equations

1) $\partial_t Z_\phi = 0$ : ODE, determines $\eta_\phi(\rho)$

2) $\partial_t V_k = \ldots$ : PDE, integrate $k \to k - \Delta k$

Parametrisation:

$$\dot{\phi} = -\frac{1}{2}\eta_\phi(\rho)\,\phi$$

And accordingly:

$$\eta_\phi(\rho) = -\frac{\partial_t Z_\varphi(\rho)}{Z_\varphi(\rho)}$$

*Take away message:*

- Reminder: standard 1st order derivative expansion is a system of 2 coupled PDEs

  → Technical simplification

- At the same time, the approximation is better

  → More momentum dependences