ϕ^4 Theory as a Neural Network Field Theory

Machine Learning and the Renormalization Group Workshop

Jim Halverson

Northeastern University

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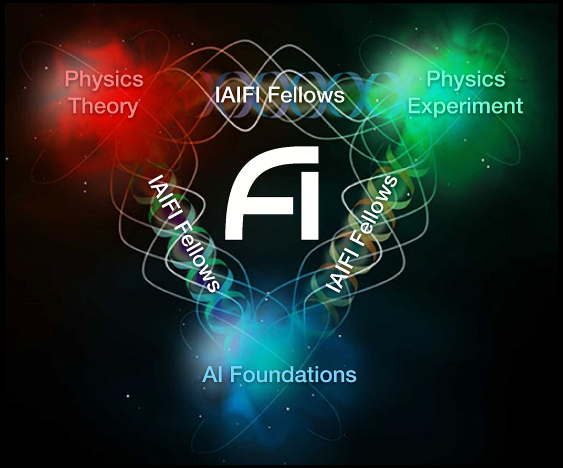

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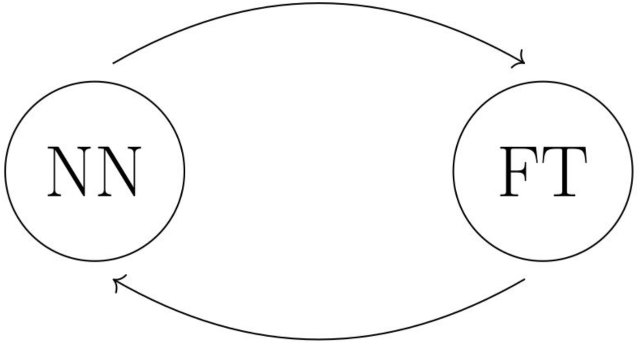


Motivation

The story goes both ways.

NNs provide a different way to do FT.
FT provides a different way to think about NNs, ML.

Neural Network Field Theories: Non-Gaussianity, Actions, and Locality

Mehmet Demirtas[1,2i], James Halverson[1,2ii], Anindita Maiti[1,2,4iii], Matthew D. Schwartz[1,3 iv], and Keegan Stoner[1,2v]

[1] *The NSF AI Institute for Artificial Intelligence and Fundamental Interactions*

[2] *Department of Physics, Northeastern University, Boston, MA 02115 USA*

[3] *Department of Physics, Harvard University, Cambridge, MA 02138 USA*

[4] *School of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138 USA*

most up-to-date account of this story, tackling **interactions, actions, and locality.**

Outline

- **NN-FT Correspondence**
  Why does a NN architecture define a field theory?
  How does it differ from normal descriptions? Is it free or interacting?

- **Interactions:** via Central Limit Theorem Violation
  How can we parametrically control the strength of interactions?

- **Actions:** from Interactions, via a new Feynman diagram approach.
  Given connected correlators, how do we compute the action?

- **Engineering Actions:** Generalities, Locality, and ϕ^4 Theory
  If we deform an action, what happens in the NN-FT? Use to get ϕ^4 Theory?

NN-FT Correspondence

Why does a NN architecture define a field theory?
How does it differ from normal descriptions?
Is it free or interacting?

What is Field Theory?

A Broad Definition:

a *statistical ensemble of fields / functions*, because at a minimum I want fields and their correlators.

(will let us see difference between "normal" field theory and a NN approach)

Tempted to Also Demand Property X?

X = quantum
X = Lagrangian
X = certain symmetries
X = locality
X = . . .

Other communities may not!

This broad definition lets us treat X as an additional property to be added / engineered by restricting the defining data of the theory.

Some we already can do in the NN approach.

A First NN-FT

Consider functions a.k.a. neural networks

$$\phi : \mathbb{R} \to \mathbb{R}$$

that take a specific form, chosen here to be

$$\phi(x) = a\,\sigma(b\,\sigma(c\,x))$$

the ***architecture*** of the neural network,
with parameters a, b, c drawn as

$$a \sim P(a),\ b \sim P(b),\ c \sim P(c)$$

at initialization.
σ is a nonlinear ***activation function***

In practice:

- P(a), P(b), P(c) are often Gaussian
- σ is simple, like tanh, ReLU, sigmoid, etc.
- **name:** width N=1, depth L=1, feedforward network with nonlinearity σ.

Key insight: redraw a,b,c many times at init, defines ensemble of (a,b,c) tuples,
and therefore an ensemble of random functions.

i.e., a field theory.

Key difference: a ***neural network field theory (NN-FT)*** is a field theory defined by a NN architecture and parameter density.

Fields and Parameter-Space / Function-Space Duality

Broad FT definition gets me a partition function

$$Z[J] = \mathbb{E}[e^{\int d^dx J(x)\phi(x)}]$$

from which the correlation functions

$$G^{(n)}(x_1, \ldots, x_n) := \mathbb{E}[\phi(x_1)\ldots\phi(x_n)]$$

may be computed from Z[J] by J-derivatives.
Connected corr. from J-derivs of W[J] := ln Z[J].

Key Difference: ***statistics-agnostic*** here,
i.e. we have not defined how to compute $\mathbb{E}[\cdot]$,
which allows us to consider different methods.

Function-Space: usual QFT1 story, via the Feynman Path Integral

$$Z[J] = \int D\phi \, e^{-S[\phi]+\int d^dx J(x)\phi(x)}$$

Parameter-Space: immediate for any NN-FT

$$Z[J] = \int d\theta \, P(\theta) \, e^{\int d^dx J(x)\phi(x)}$$

Note: yield different ways to compute correlators!
If you know param- and function-space desc of thy, two ways to compute same correlators.

NN-FT: Definition and Discussion

Definition: a ***neural network field theory*** (NN-FT) is a field theory defined by a NN architecture and a density for NN parameters.

- param. space description always available, giving another way to do computations.

  e.g. correlation functions. [Williams]

  e.g. symmetries of full theory
  [J.H., Maiti, Stoner]
  [J.H.]

- sampling can be much easier, could make this very useful for lattice.

- not necessarily a **quantum** field theory, need NN-FT w/ OS axioms [J.H.].

Free Theory Limits: NNGP Correspondence

90's: [Neal]

2010's: generality
[Google team] [Yang]

Key Fact:
Many architectures have a discrete parameter N s.t. as N → ∞, the NN is drawn from a Gaussian Process (GP) by the Central Limit Theorem (CLT).
Neural Network Gaussian Process Correspondence.

Physics Interpretation: in the limit, the associated NN-FT is a generalized **free field theory**,

$$S[\phi] = \int d^d x\, d^d y\, \phi(x)\, G^{(2)}(x,y)^{-1}\, \phi(y)$$

In practice, since NN-FT always have param description, 2-pt function computed in param space and GP action is *inferred*.

Example: $\phi : \mathbb{R}^d \to \mathbb{R}$ width N, depth 1.

Architecture: $$\phi(x) = \sum_{i=1}^{N} \sum_{j=1}^{d} a_i\, \sigma\left(b_{ij} x_j\right)$$

Parameter Density:

$$a \sim \mathcal{N}(0, \sigma^2/N), \;\; b \sim \mathcal{N}(0, \sigma^2/d)$$

Result: $$G_c^{(2k)}(x_1, \ldots, x_{2k}) \propto \frac{1}{N^{k-1}}$$

- Gaussian b/c higher connected correlators vanish.
- Two-point function (k=1) know analytically sometimes, e.g. for Erf, Tanh, ReLU non-linearities.

Interactions

How can we control the strength of interactions
via parametrically violating CLT assumptions?

Central Limit Theorem from Generating Functions

Connected Correlation Functions:

$$G_c^{(n)}(x_1,\ldots,x_n) := \left(\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}W[J]\right)\Bigg|_{J=0}$$

$$W[J] := \ln Z[J]$$

$$\text{Gaussian if } G_c^{(n)} = 0 \text{ for } n > 2$$

connected correlators = cumulants, in statistics.

W[J] the cumulant generating functional.

Central Limit Theorem: Single Variable Ex. (0D FT)

$$\phi = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} X_i \qquad \kappa_r^{\phi} := \left(\frac{\partial}{\partial J}\right)^r W_\phi[J]$$

$$W_\phi[J] := \log\,\mathbb{E}[e^{J\phi}] = \log\,\mathbb{E}[e^{J\sum_i X_i/\sqrt{N}}]$$

when vars ***independent*** $\quad Z_{X_1+\cdots+X_N}[J] = \prod_i^N Z_{X_i}[J]$

$$W_{X_1+\cdots+X_N}[J] = W_{X_1}[J] + \cdots + W_{X_N}[J]$$

$$\kappa_r^{X_1+\cdots+X_N} = \kappa_r^{X_1} + \cdots + \kappa_r^{X_N}$$

with identicalness and $\kappa_r^{cX} = c^r\,\kappa_r^X$ gives

$$\kappa_r^{\phi} = \frac{\kappa_r^{X_i}}{N^{r/2-1}}$$ establishing Gaussianity.

CLT followed from i.i.d. assumption and $N \to \infty$

We control interactions (i.e., non-Gaussianities) in NN-FT
by violating these assumptions.

The CLT derivation already shows how
finite-N generates non-Gaussianities,

But independence breaking can also generate them.

In our paper we present examples,

e.g. 1/N-corrections to connected 4-pt function
in ReLU-net, Gauss-net, Cos-net.

e.g. independence-breaking corrections in ReLU-net.

Further details are in backup slides.

Actions from Interactions: A New Feynman Diagram Approach

Given connected correlators, how do we compute the action?

Edgeworth Expansion: Expand Around Gaussian Density

Single Variable Example:

$$P[\phi] = e^{-S[\phi]} = \int dJ e^{W[J]-J\phi}$$

$$W[J] = \sum_{r=1}^{\infty} \frac{\kappa_r}{r!} J^r$$

$$P_\phi = \exp\left[\sum_{r=3}^{\infty} \frac{\kappa_r}{r!}(-\partial_\phi)^r\right] \int dJ e^{\kappa_2 \frac{J^2}{2} + \kappa_1 J - J\phi}$$
$$= \exp\left[\sum_{r=3}^{\infty} \frac{\kappa_r}{r!}(-\partial_\phi)^r\right] e^{-\frac{(\phi-\kappa_1)^2}{2\kappa_2}},$$

see book "Tensor Methods in Statistics"

Field Theory Case:

$$e^{-S[\phi]} = \frac{1}{Z}\exp\Big(\sum_{r=3}^{\infty}\frac{(-1)^r}{r!}\int\prod_{i=1}^{r} d^d x_i G_c^{(r)}(x_1,\cdots,x_r)\frac{\delta}{\delta\phi(x_1)}\cdots\frac{\delta}{\delta\phi(x_r)}\Big)e^{-S_{GP}[\phi]}$$

$$S_{GP}[\phi] = \frac{1}{2}\int d^d x_1 d^d x_2\, \phi(x_1) G_c^{(2)}(x_1,x_2)^{-1}\phi(x_2)$$

formal expression for the action,
but what do to with it?

Is there a better way to understand / compute?

An Unexpected Source-Field Duality

Edgeworth Mirrors a Canonical FT Calculation:

$$Z[J] = \int D\phi \, e^{-S_{\text{free}}[\phi] - S_{\text{int}} + \int d^d x J(x)\phi(x)}$$

$$S_{\text{int}} = \sum_{r=3}^{\infty} \int \prod_{i=1}^{r} d^d x_i \, g_r(x_1, \ldots, x_r) \, \phi(x_1) \ldots \phi(x_r)$$

can easily rewrite Z[J] as

$$e^{W[J]} = Z[J] = c' \exp\Big(\sum_{r=3}^{\infty} (-1)^r \int \prod_{i=1}^{r} d^d x_i \, g_r(x_1, \cdots, x_r) \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_r)}\Big) e^{-S_0[J]}$$

$$S_0[J] = \int dx_1 dx_2 J(x_1) \Delta(x_1, x_2) J(x_2)$$

compare to Edgeworth

$$e^{-S[\phi]} = \frac{1}{Z} \exp\Big(\sum_{r=3}^{\infty} \frac{(-1)^r}{r!} \int \prod_{i=1}^{r} d^d x_i G_c^{(r)}(x_1, \cdots, x_r) \frac{\delta}{\delta \phi(x_1)} \cdots \frac{\delta}{\delta \phi(x_r)}\Big) e^{-S_{GP}[\phi]}$$

A Duality:

	Field Picture	Source Picture
Field	$\phi(x)$	$J(x)$
CGF	$W[J] = \log(Z[J])$	$S[\phi] = -\log(P[\phi])$
Cumulant	$G_c^{(r)}(x_1, \ldots, x_r)$	$g_r(x_1, \ldots, x_r)$

we see the equations are relative by change of vars.

Normal FT: compute coeff fns of J-expansion of W[J] via connected Feyn diags, with g-vertices.

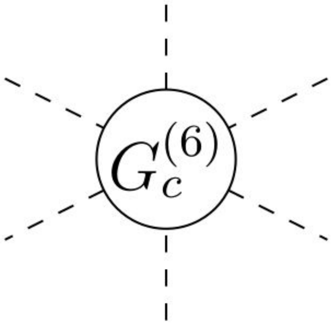
New: compute coeff fns of ϕ-expansion of S[ϕ] via connected Feyn diags, with G-vertices.

Compute couplings as Feyn diags w/ corr-vertices!

Actions from Interactions

Feynman diagrams for couplings g_r:

vertices are connected correlators, e.g.



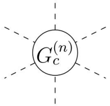
method: to compute g_r, write down all connected diagrams with G-vertices, apply Feynman Rules.

Feynman Rules:

1. Internal points associated to vertices are unlabelled, for diagrammatic simplicity. Propagators therefore connect to internal points in all possible ways.
2. For each propagator between z_i and y_j, where $z_i = x_i$ (internal) or $z_i = y_i$ (external),

$$z_i \text{ - - - - } y_j \;=\; G_c^{(2)}(z_i, y_j)^{-1}. \tag{3.22}$$

3. For each vertex,



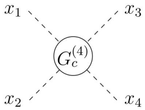
$$= (-1)^n \int d^d y_1 \cdots d^d y_n \, G_c^{(n)}(y_1, \cdots, y_n). \tag{3.23}$$

4. Divide by symmetry factor and $(-1)^r$ factor of coupling $g_r(x_1, \cdots, x_r)$.

Example:

$$g_4(x_1, \ldots, x_4) = \frac{1}{4!}\Big[\int dy_1 dy_2 dy_3 dy_4 \, G_c^{(4)}(y_1, y_2, y_3, y_4)\, G_c^{(2)}(y_1, x_1)^{-1} G_c^{(2)}(y_2, x_2)^{-1}$$
$$G_c^{(2)}(y_3, x_3)^{-1} G_c^{(2)}(y_4, x_4)^{-1} + \text{Comb.}\Big] + \ldots$$

$=$



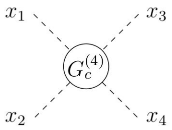
$+ \ldots,$

NN-FT: Actions for 1/N and Independence Breaking

Leading order in 1/N:

$$S = S_{\mathrm{GP}} + \int d^d x_1 \ldots d^d x_4 \;\; g_4(x_1, \ldots, x_4)\, \phi(x_1) \ldots \phi(x_4) + O\left(\frac{1}{N^2}\right)$$

$$g_4(x_1, \ldots, x_4) = \frac{1}{4!}\left[\int dy_1 dy_2 dy_3 dy_4\, G_c^{(4)}(y_1, y_2, y_3, y_4)\, G_c^{(2)}(y_1, x_1)^{-1} G_c^{(2)}(y_2, x_2)^{-1} G_c^{(2)}(y_3, x_3)^{-1} G_c^{(2)}(y_4, x_4)^{-1} \; + \text{Comb.}\right] + O\left(\frac{1}{N^2}\right),$$

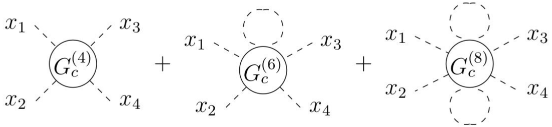
$$= $$


$$+ \, O\left(\frac{1}{N^2}\right).$$

Leading order in independence breaking:

$$S = S_{\mathrm{GP}} + \sum_{r=4}^{\infty} \int d^d x_1 \ldots d^d x_r \, g_r(x_1, \ldots, x_r)\, \phi(x_1) \ldots \phi(x_r) + O(\alpha^2)$$

$$g_4(x_1, x_2, x_3, x_4) = \sum_{n=2}^{\infty} \frac{(-1)^{2n-4}}{(2n)!}\Big[\int dy_1 \cdots dy_{2n}\, G_c^{(2n)}(y_1, \cdots, y_{2n})\, G_c^{(2)}(y_1, x_1)^{-1} G_c^{(2)}(y_2, x_2)^{-1}\, G_c^{(2)}(y_3, x_3)^{-1}\, G_c^{(2)}(y_4, x_4)^{-1} \prod_{m=5}^{2n-1} G_c^{(2)}(y_m, y_{m+1})^{-1} + \text{Comb.}\Big] + O(\alpha^2)$$

$$=(-1)^{2n-4}\Bigg($$


$$+ \cdots\Bigg) + O(\alpha^2)$$

Concrete Example: Cos-Net Action

single-layer feedforward network, width N, judiciously chosen parameter densities

$$S_{\text{Cos}}[\phi] = \frac{2\sigma_{W_0}}{\sigma_{W_1}^2\sqrt{d}} \int d^d x\, \phi(x)\, e^{-\frac{\sigma_{W_0}^2 \nabla_x^2}{2d}} \phi(x) \; - \int d^d x_1 \cdots d^d x_4 \Bigg[\frac{4\sqrt{6}\pi^{3/2}\sigma_{W_0}^4}{N d^2 \sigma_{W_1}^4} \sum_{\mathcal{P}(abcd)} e^{-\frac{\sigma_{W_0}^2 \nabla_{r_{abcd}}^2}{6d}}$$

$$- \frac{8\pi\sigma_{W_0}^4}{N d^2 \sigma_{W_1}^4} \sum_{\mathcal{P}(ab,cd)} e^{-\frac{\sigma_{W_0}^2 (\nabla_{r_{ab}}^2 + \nabla_{r_{cd}}^2)}{2d}} \Bigg] \phi(x_1) \cdots \phi(x_4) + O(1/N^2)$$

Engineering Actions:
Generalities, Locality, and ϕ^4 Theory

If we deform an action, what happens in the NN-FT?
Can we use it to engineer ϕ^4 Theory?

Theory Deformations in Function- and Parameter-Space

Begin with a fixed theory. Here, a Gaussian theory

$$Z_G[J] = \mathbb{E}_G[e^{\int d^d x\, J(x)\phi(x)}]$$

deform it by an operator insertion

$$Z[J] = \mathbb{E}_G[e^{-\lambda \int d^d x_1 \ldots d^d x_r\, \mathcal{O}_\phi(x_1,\ldots,x_r)} e^{\int d^d x\, J(x)\phi(x)}]$$

In function space, this deforms the action

$$Z[J] = \int D\phi\, e^{-S[\phi] + \int d^d x J(x)\phi(x)}$$

$$S_G[\phi] + \lambda \int d^d x_1 \ldots d^d x_r\, \mathcal{O}_\phi(x_1, \ldots, x_r)$$

But because we know NN ϕ as func. of params,

$$Z[J] = \int d\theta P_G(\theta)\; e^{-\lambda \int d^d x_1 \ldots d^d x_r\, \mathcal{O}_{\phi_\theta}(x_1,\ldots,x_r)} e^{\int d^d x J(x)\phi_\theta(x)}$$

$$P(\theta) := P_G(\theta)\; e^{-\lambda \int d^d x_1 \ldots d^d x_r\, \mathcal{O}_{\phi_\theta}(x_1,\ldots,x_r)}$$

therefore op. insertion deforms param. density in NN-FT.
Deforming an NNGP by operator insertion gives interactions from independence breaking!

Deforming a Free NN-FT

Putting it more explicitly in NNGP-form, consider a NN

$$\phi_\theta(x) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} a_i h_i(x)$$

that has a density

$$P_G(a) P_G(\theta_h)$$

yielding a GP in the N → ∞ limit.

Deforming the theory by an operator insertion

$$Z[J] = \int da\, d\theta_h\ P_G(a) P_G(\theta_h)\ e^{-\lambda \int d^d x_1 \ldots d^d x_r\, \mathcal{O}_{\phi_{a,\theta_h}}(x_1,\ldots,x_r)} e^{\int d^d x J(x) \phi_\theta(x)}$$

gives a new theory

$$Z[J] = \int da\, d\theta_h\ P(a, \theta_h)\, e^{\int d^d x J(x) \phi_\theta(x)}$$

$$P(a, \theta_h) = P_G(a) P_G(\theta_h)\, e^{-\lambda \int d^d x_1 \ldots d^d x_r\, \mathcal{O}_{\phi_{a,\theta_h}}(x_1,\ldots,x_r)}$$

with a non-trivial joint parameter density that breaks independence.

ϕ^4 Theory as a NN-FT

Engineer the NNGP: [J.H.]

$$\phi_{a,b,c}(x) = \sqrt{\frac{2\,\mathrm{vol}(B^d_\Lambda)}{\sigma_a^2(2\pi)^d}} \sum_{i,j} \frac{a_i\,\cos(b_{ij}x_j + c_i)}{\sqrt{\mathbf{b}_i^2 + m^2}}$$

$$P_G(a) = \prod_i e^{-\frac{N}{2\sigma_a^2} a_i a_i}$$

$$P_G(b) = \prod_i P_G(\mathbf{b}_i) \ \text{ with } \ P_G(\mathbf{b}_i) = \mathrm{Unif}(B^d_\Lambda)$$

$$P_G(c) = \prod_i P_G(c_i) \ \text{ with } \ P_G(c_i) = \mathrm{Unif}([-\pi, \pi])$$

where i = 1, ..., N. in N $\to \infty$ limit get NNGP with

$$G^{(2)}(p) = \frac{1}{p^2 + m^2}$$

Introduce the Operator Insertion:

$$e^{-\frac{\lambda}{4!}\int d^dx\, \phi_{a,b,c}(x)^4}$$

Absorb into Param. Density Deformation:

$$P(a,b,c) = P_G(a)P_G(b)P_G(c)\ e^{-\frac{\lambda}{4!}\int d^dx\, \phi_{a,b,c}(x)^4}$$

Write the Partition Function:

$$Z[J] = \int da\, db\, dc\ \ P(a,b,c)\ e^{\int d^dx J(x)\, \phi_{a,b,c}(x)}$$

this is ϕ^4 theory as an infinite width NN-FT!

interactions are from *independence breaking*.

Conclusions

- **NN-FT Correspondence**

  Why does a NN architecture define a field theory?

  It defines an ensemble of functions, via different initializations.

  How does it differ from normal descriptions? Is it free or interacting?

  Naturally defined in parameter space, not via an action.

  Very often a free $N \to \infty$ limit, via central limit theorem (CLT).

- **Interacting Theories via Central Limit Theorem Violation**

  How can we parametrically control the strength of interactions?

  By parametrically violating assumptions of CLT,

  e.g. infinite N and statistical independence.

Conclusions

- **Actions from Interactions: A New Feynman Diagram Approach**
  Given connected correlators, how do we compute the action?
  Edgeworth expansion gives field density $P[\phi]$ in terms of connected correlators G_c.
  Can compute couplings in $S[\phi] = -\ln P[\phi]$ by Feynman diags with G_c vertices.

- **Engineering Actions: Generalities, Locality, and ϕ^4 Theory**
  If we deform an action, what happens in the NN-FT?
  It deforms the parameter density, generally to something non-independent.

  Use to get ϕ^4 Theory?
  Engineer free scalar theory, deform via our method of deforming parameter densities.

To appear in 2024:

NN-CFT, Grassmann NN-FT, and super NN-FT.

Thanks!

Questions?

Feel free to get in touch:

e-mail: jhh@neu.edu
Twitter: @jhhalverson
web: www.jhhalverson.com

Interactions from Independence Breaking

In CLT derivation, this independence identity

$$W_{X_1+\cdots+X_N}[J] = W_{X_1}[J] + \cdots + W_{X_N}[J]$$

was crucial to the derivation.

What if we break independence parametrically?
by having a non-trivial joint distribution on X_i

$$p(X;\alpha) \qquad \text{such that} \quad p(X;\alpha=0) = \prod_i p(X_i)$$

i.e. independence holds only for α = 0,
where α is a hyperparameter.

Observation:
if independence is broken, W[J] no longer splits!

$$W_\phi[J] = \log\left[\prod_j \mathbb{E}_{p(X,\alpha=0)}\left[e^{JX_j/\sqrt{N}}\right] + \sum_{k=1}^{\infty}\frac{\alpha^k}{k!}\mathbb{E}_{p(X,\alpha=0)}\left[e^{J\sum_i X_i/\sqrt{N}}\,\mathcal{P}_k|_{\alpha=0}\right]\right]$$

computing cumulants / connected correlators
from this yields non-Gaussianities.

interactions from breaking of independence!
concrete example in paper.

Interactions in NN-FT

consider a field built out of N neurons as:

$$\phi(x) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} h_i(x)$$

Connected Correlators from 1/N-Corrections:

$$G_c^{(r)}(x_1, \cdots, x_r) = \frac{G_{c,h_i}^{(r)}(x_1, \cdots, x_r)}{N^{r/2-1}}$$

simple generalization of sing-var case to continuum.
G4c is leading non-Gaussianity in 1/N.

Connected Correlators from Indep. Breaking:
[J.H.], this systematic from [Demirtas et. al.]
neurons that are not independent

$$P(h; \vec{\alpha})$$

$$P(h; \vec{\alpha} = \vec{0}) = \prod_i P(h_i)$$

interactions arise in alpha-dependence of
connected correlators, from W[J] below.

$$W_\phi[J] = \log\left[e^{W_{\phi,\vec{\alpha}=0}[J]} + \sum_{r=1}^{\infty} \sum_{s_1,\cdots,s_r=1}^{q} \frac{\alpha_{s_1}\cdots\alpha_{s_r}}{r!} \prod_{i=1}^{N} \mathbb{E}_{P_i(h_i)}\left[e^{\frac{1}{\sqrt{N}}\int d^dx\, h_i(x)J(x)} \cdot \mathcal{P}_{r,\{s_1,\cdots,s_r\}}\Big|_{\vec{\alpha}=0}\right]\right]$$

$$\mathcal{P}_{r,\{s_1,\cdots,s_r\}} := \frac{1}{P(h|\vec{\alpha})}\partial_{\alpha_{s_1}}\cdots\partial_{\alpha_{s_r}} P(h|\vec{\alpha})$$

A Connected 4-pt Example

Cos-net at Finite-N with Independence Breaking:

definition of this NN-FT

$$\phi(x) = W^1_j \cos(W^0_{jk} x_k + b^0_j)$$

$$W^1_j \sim \mathcal{N}(0, \sigma^2_{W_1}/N)$$

$$b^0_j \sim \text{Unif}[-\pi, \pi]$$

$$\mathcal{P}(W^0) = c \exp\left[-\sum_{i,j} \left(\frac{d}{2\sigma^2_{W_0}} (W^0_{ij})^2 + \frac{\alpha_{\text{IB}}}{N^2} \sum_{i_1,j_1,i_2,j_2} (W^0_{i_1 j_1})^2 (W^0_{i_2 j_2})^2 \right) \right]$$

one may computed the connected 4-pt function directly in parameter space. The result at leading order in the independence breaking parameter is:

$$\begin{aligned} G^{(4)}_{c,\text{Cos}}(x_1,x_2,x_3,x_4) = \frac{\sigma^4_{W_1}}{8N} \sum_{\mathcal{P}(abcd)} \Bigg[\Bigg(& -2e^{-\frac{\sigma^2_{W_0}\left((\Delta x_{ab})^2 + (\Delta x_{cd})^2\right)}{2d}} + 3e^{-\frac{\sigma^2_{W_0}(\Delta x_{ab}+\Delta x_{cd})^2}{2d}} \Bigg) \\ + \frac{\alpha_{\text{IB}}\sigma^4_{W_0}}{d^4 N} \Bigg(& -6d^2 e^{-\frac{\sigma^2_{W_0}\left((\Delta x_{ab})^2 + (\Delta x_{cd})^2\right)}{2d}} + 3d^2 e^{-\frac{\sigma^2_{W_0}(\Delta x_{ab}+\Delta x_{cd})^2}{2d}} + 3d\sigma^2_{W_0}(\Delta x_{ab} + \Delta x_{cd})^2 \\ & e^{-\frac{\sigma^2_{W_0}(\Delta x_{ab}+\Delta x_{cd})^2}{2d}} - 2d\sigma^2_{W_0} \left((\Delta x_{ab})^2 + (\Delta x_{cd})^2\right) e^{-\frac{\sigma^2_{W_0}\left((\Delta x_{ab})^2 + (\Delta x_{cd})^2\right)}{2d}} - 2\sigma^4_{W_0}(\Delta x_{ab})^2(\Delta x_{cd})^2 \\ & e^{-\frac{\sigma^2_{W_0}\left((\Delta x_{ab})^2 + (\Delta x_{cd})^2\right)}{2d}} \Bigg) \Bigg], \end{aligned} \tag{B.7}$$

which recovers the result of [J.H.] in the $\alpha_{\text{IB}} \to 0$ limit, which restores independence.