Beyond-Eikonal Methods in High-Energy Scattering, ECT* Trento, 20-24 May 2024

Efficient computation of high-order cusp anomalous dimensions in the string-inspired worldline formulation

X.F., A. Tarasov, R. Venugopalan, to appear (hopefully soon)

Xabi Feal

Worldline path integrals

Feynman Phys. Rev. 80 3 440, 1950, Phys. Rev. 84 1 108, 1951
Schwinger Phys. Rev. 82 664, 1951
Fock Physik. Z. Sowjetunion 12, 404, 1937
For a review C. Schubert, Phys. Rept. 355 (2001) 73.

Why and how Feynman, Schwinger and Fock introduced the worldline representation long time ago is explained in the QED foundational papers:

$\langle \Omega | T\left(\Phi(x_f)\bar{\Phi}(x_i)\right) | \Omega \rangle \sim$ amplitude for a particle to go from one point to another as a sum over trajectories of e^{-iS}, with S classical action

One integrates over trajectories that point-like particles follow in space-time $x_\mu(\tau)$ rather than over fields, with their internal d.o.f. (spin and color) exactly described with anti-commuting variables $\psi_\mu(\tau)$ along their paths.

Why these formulations can be relevant now in many practical problems will (try to) be explained in this talk.

Motivations

- **Eikonal approximations, main building block in HEP & NUCL-TH calculations** (there where the interactions of a field with spin/polarization/color with some gauge field need to be defined to all orders in PT)
- **Known problems**: beyond eikonal expansions, kinematic cut-offs, systematics re-exponentiations.
- **Worldline propagators are exact,** they represent the matrix structures of the gauge interactions as path integrals over local spin and color coordinates, opening a window for non-perturbative semi-classical expansions in natural particle variables.
  Halpern, Siegel, Phys. Rev. D16 2486 (1977)
- **PT in the worldline space** does not reproduce the usual Feynman diagrams, but sets of $n!$ topologies at each n^{th} loop order, offering a calculational advantage over conventional field PT.
  Nice discussion Cvitanovic's https://cns.gatech.edu/~predrag/papers/finiteQED.pdf
- **Wide range of applications:**

anomalies and index densities
Álvarez-Gaumé, Witten Nucl. Phys. B234 (1989) 269
Boer, Peeters, Skenderis, Nieuwenhuizen NPB 446 (1995) 211
one-loop effective actions
Strassler Nucl.Phys.B 385 (1992) 145
calculation ß-function in a number of theories
Schmidt, Schubert, Phys. Rev. D 53 (1996) 2150
chiral anomaly and its role in the proton spin puzzle
Tarasov, Venugopalan PRD 105 (2022) 1, 014020

pair production in sQED
Affleck, Álvarez, Manton Nucl.Phys.B 197 (1982) 509
Gould, Rajantie, Phys. Rev. D 96, 076002 (2017).
covariant kinetic theory
Pisarski NATO Sci.Ser.C 511 (1998) 195
Mueller, Venugopalan, PRD 96 (2017) PRD 97 (2018) 5
eikonal & next-to-eikonal IR factorization in QCD
Laenen, Stavenga, White, JHEP 03 054 (2009)
Bonocore, JHEP 02 (2021) 007

Introductory example

Consider the QED vacuum-to-vacuum amplitude $Z = \langle 0 | 0 \rangle$

$$Z = \int \mathscr{D}A \exp\left\{ -\frac{1}{4}\int d^4x F_{\mu\nu}^2 - \frac{1}{2\zeta}\int d^4x\,(\partial_\mu A_\mu)^2 + \ln\det\left(\not{D} + m\right) \right\}$$

Integration over all $A_\mu(x)$ configurations (dynamical gauge field)

1-loop effective action = amplitude for 1-fermion to describe a loop in $A_\mu(x)$

Write the 1-loop fermion determinant in worldline form

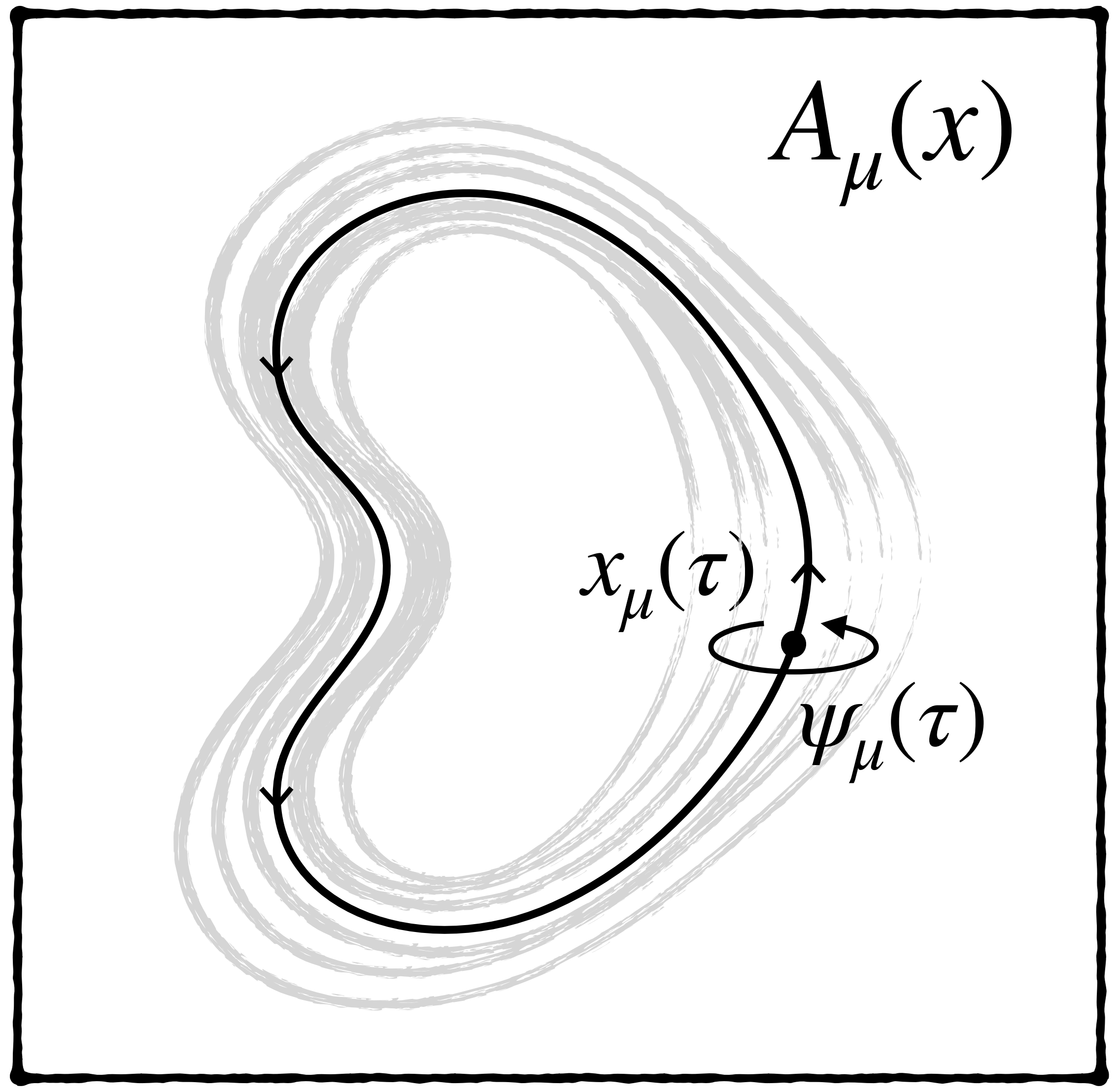
$$\log\det\left(\not{D} + m\right) = \frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{d\varepsilon_0}{\varepsilon_0} e^{-\varepsilon_0} - \frac{1}{2}\int_0^\infty \frac{d\varepsilon_0}{\varepsilon_0} e^{-\varepsilon_0 m^2}$$

$$\times \int_{PBC} \mathscr{D}^4 x \int_{APBC} \mathscr{D}^4 \psi e^{-S_0[x,\psi]} \exp\left\{ ig\int_0^1 d\tau \dot{x}_\mu(\tau) A_\mu\big(x(\tau)\big) - \frac{ig\varepsilon_0}{2}\int_0^1 d\tau \psi_\mu(\tau)\psi_\nu(\tau) F_{\mu\nu}\big(x_\mu(\tau)\big) \right\}$$

Integration over all closed x_μ and ψ_μ worldline contours with weight S_0

(bosonic) Wilson loop term coupling to A_μ

(fermionic) spin tensor term coupling to $F_{\mu\nu}$



$x_\mu(\tau) \equiv$ commuting worldline (τ-dependent 4-position path)

$\psi_\mu(\tau) \equiv$ anti-commuting worldline (τ-dependent spin path)

$$S_0\big[x(\tau), \psi(\tau)\big] = \frac{1}{4\varepsilon_0}\int_0^1 d\tau \dot{x}_\mu^2(\tau) + \frac{1}{4}\int_0^1 d\tau \dot{\psi}_\mu(\tau)\psi_\mu(\tau)$$

Weight of each path

Free action of a (0+1) dimensional spinning particle

Expand in # of virtual fermions and refer to the pure gauge sea of disconnected photon loops to integrate out A_μ

$$\frac{Z}{Z_{MW}} = \frac{1}{Z_{MW}}\int \mathscr{D}A \exp\left\{ -\frac{1}{4}\int d^4x F_{\mu\nu}^2 - \frac{1}{2\zeta}\int d^4x\,(\partial_\mu A_\mu)\right\}\sum_{\ell=0}^{\infty}\frac{1}{\ell\,!}\Big(\ln\det\left(\not{D}+m\right)\Big)^{\ell} \equiv \sum_{\ell=0}^{\infty} Z^{(\ell)}$$

where

$$Z^{(\ell)} = \underbrace{\frac{(-1)^\ell}{\ell\,!}}_{\text{Loop parity and symmetry factor}}\left\langle \exp\left[-\frac{g^2}{8\pi^2}\sum_{i,j=1}^{\ell}\int_0^1 d\tau_i\Big(\underbrace{\dot{x}_\mu^i}_{\text{i-th virtual fermion scalar current}} - i\varepsilon_0^i\underbrace{\sigma_{\mu\rho}\frac{\partial}{\partial x_\rho^i}}_{\text{i-th virtual fermion spin tensor}}\Big)\underbrace{\int_0^1 d\tau_j\Big(\dot{x}_\mu^j - i\varepsilon_0^j\sigma_{\mu\eta}\frac{\partial}{\partial x_\eta^j}\Big)\frac{1}{(x_i-x_j)^2}}_{\text{Dynamical gauge field at point } x_i \text{ induced by j-th virtual fermion}}\right]\right\rangle$$

The notation $\langle\,\star\,\rangle$ means sum/path integrate over all (0+1)-dimensional worldline superpairs $\{x_\mu^i(\tau), \psi_\mu^i(\tau)\}$ representing all possible closed trajectories/precessions of ℓ virtual point-like fermions in space-time/spin:

$$\langle\,\star\,\rangle \equiv \underbrace{\exp\left\{\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{d\varepsilon_0}{\varepsilon_0}e^{-\varepsilon_0}\right\}}_{\text{zero point energy of the QED vacuum (Schwinger proper-time renormalization)}}\prod_{i=1}^{\ell}\left\{\underbrace{\int_0^\infty\frac{d\varepsilon_0^i}{2\varepsilon_0^i}\int_{\mathrm{P}}\mathscr{D}^4x_i\int_{\mathrm{AP}}\mathscr{D}^4\psi_i}_{\text{many-body worldline path integral of } i=1,\ldots,\ell \text{ spin-1/2 particles}}\underbrace{\exp\left\{-\frac{1}{4\varepsilon_0^i}\int_0^1 d\tau\dot{x}_i^2(\tau) - m^2\varepsilon_0^i - \frac{1}{4}\int_0^1 d\tau\psi_\mu^i(\tau)\dot{\psi}_\mu^i(\tau)\right\}}_{\text{worldline free action of the } i=1,\ldots,\ell \text{ spin-1/2 particles}}\right\}\star$$

Feynman Physical Review 80 3 (1950) 440

in velocity. When there are several particles (other than the virtual pairs already included) one use a separate u for each, and writes the amplitude for each set of trajectories as the exponental of $-i$ times

$$\frac{1}{2}\sum_n\int_0^{u_0^{(n)}}\left(\frac{dx_\mu^{(n)}}{du}\right)^2du+\sum_n\int_0^{u_0^{(n)}}\frac{dx_\mu^{(n)}}{du}B_\mu(x_\mu^{(n)}(u))du$$
$$+\frac{e^2}{2}\sum_n\sum_m\int_0^{u_0^{(n)}}\int_0^{u_0^{(m)}}\frac{dx_\nu^{(n)}(u)}{du}\frac{dx_\nu^{(m)}(u')}{du'}$$
$$\times\delta_+((x_\mu^{(n)}(u)-x_\mu^{(m)}(u'))^2)dudu', \quad \text{(11A)}$$

where $x_\mu^{(n)}(u)$ are the coordinates of the trajectory of the nth particle.[22] The solution should depend on the $u_0^{(n)}$ as $\exp(-\frac{1}{2}im^2\sum_n u_0^{(n)})$.

Feynman Physical Review 84 1 (1950) 108

I have expended considerable effort to obtain an equally simple word description of the quantum mechanics of the Dirac equation. Very many modes of description have been found, but none are thoroughly satisfactory. For example, that of Eq. (32-a) is incomplete, even aside from the geometrical mysteries involved in the superposition of hypercomplex numbers. For in (32-a) the

Semi-classical expansion: classical motion of a spinning particle describing a loop.

$$\mathscr{L} = -p^\mu \dot{x}_\mu - \pi\dot{\epsilon} + \frac{i}{4}\psi^\lambda \dot{\psi}_\lambda + H(p, x, \psi, \epsilon), \quad \tau \in [0,1] \qquad H = -\epsilon\left[m^2 - \left(p_\mu + gA_\mu\right)^2 - \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}\right]$$

Take $\delta S = \delta \int_0^1 d\tau \mathscr{L} = 0$ and choose $\tau = s$ proper time

$$\underbrace{\frac{d}{ds}\left(m_R \frac{dx^\rho}{ds}\right) = g\frac{dx_\mu}{ds}F^{\rho\mu} - \frac{g}{4m_R}\sigma_{\mu\nu}\frac{\partial F^{\mu\nu}}{\partial x_\rho}}_{\text{Motion of a classical virtual fermion in } A_\mu} \qquad \underbrace{\frac{d\psi^\rho}{ds} = -\frac{g}{m_R}\psi_\nu F^{\rho\nu}}_{\text{Spin precession}} \qquad \underbrace{\sigma_{\mu\nu} = \frac{i}{2}[\psi_\mu, \psi_\nu]}_{\text{Spin tensor}}$$

- Homogeneous A_μ -> **Bargman-Michel-Telegdi (BMT) equations** in covariant form.
- Real particles -> **Berezin-Marinov action**, i.e. BMT eqs. with energy-momentum and helicity-momentum constraint.
- Dynamical A_μ -> many-body theory of classical spinning charges in pairwise/non-local interaction through classical Lorentz forces.
- Non-Abelian $A_\mu \rightarrow$ **Wong equations.**
- Gravity $\rightarrow$ **Papapetrou-Mathisson-Dixon equations.**

Berezin, Marinov Ann. Phys. 104 (1977) 336
Barducci, Casalbuoni, Lusanna, Nuov. Cim. A 25 (377) 1976
Wong, Nuov. Cim. A 65 (1970) 689
Papapetrou, Proc. Roy. Soc. A 209 (1951) 248
Mathisson, Act. Phys. Pol. 6 (1937) 163
Dixon, Proc. Roy. Soc. A 314 (1970) 499

Open worldline propagator (for a real spin-1/2 particle in an Abelian background)

$$D_F^A(x_f, x_i)\gamma_5 = \frac{1}{N_5}\exp\left[\bar{\gamma}_\lambda \frac{\partial}{\partial\theta_\lambda}\right]\int_0^\infty d\varepsilon_0 \int d\chi_0 \underbrace{\int \mathscr{D}^4 x \int \mathscr{D}^5\psi}\, \exp\left[-S[x,\dot{x},\psi,\dot{\psi}]\right]\Bigg|_{\theta=0}$$

Integration over all **open** x_μ and ψ_μ worldline contours with weight S

$$\underbrace{S[x,\dot{x},\psi,\dot{\psi}]} = \frac{1}{4}\psi_\lambda(1)\psi_\lambda(0) + \int_0^1 d\tau \mathscr{L}$$

QED worldline action

$$x_\mu(1) = x_\mu^f\,,\ x_\mu(0) = x_\mu^i\,,\ \mu = 1{,}2{,}3{,}4.$$

$$\psi_\lambda(1) = -\psi_\lambda(0) + 2\theta_\lambda\,,\ \lambda = 1{,}2{,}3{,}4{,}5.$$

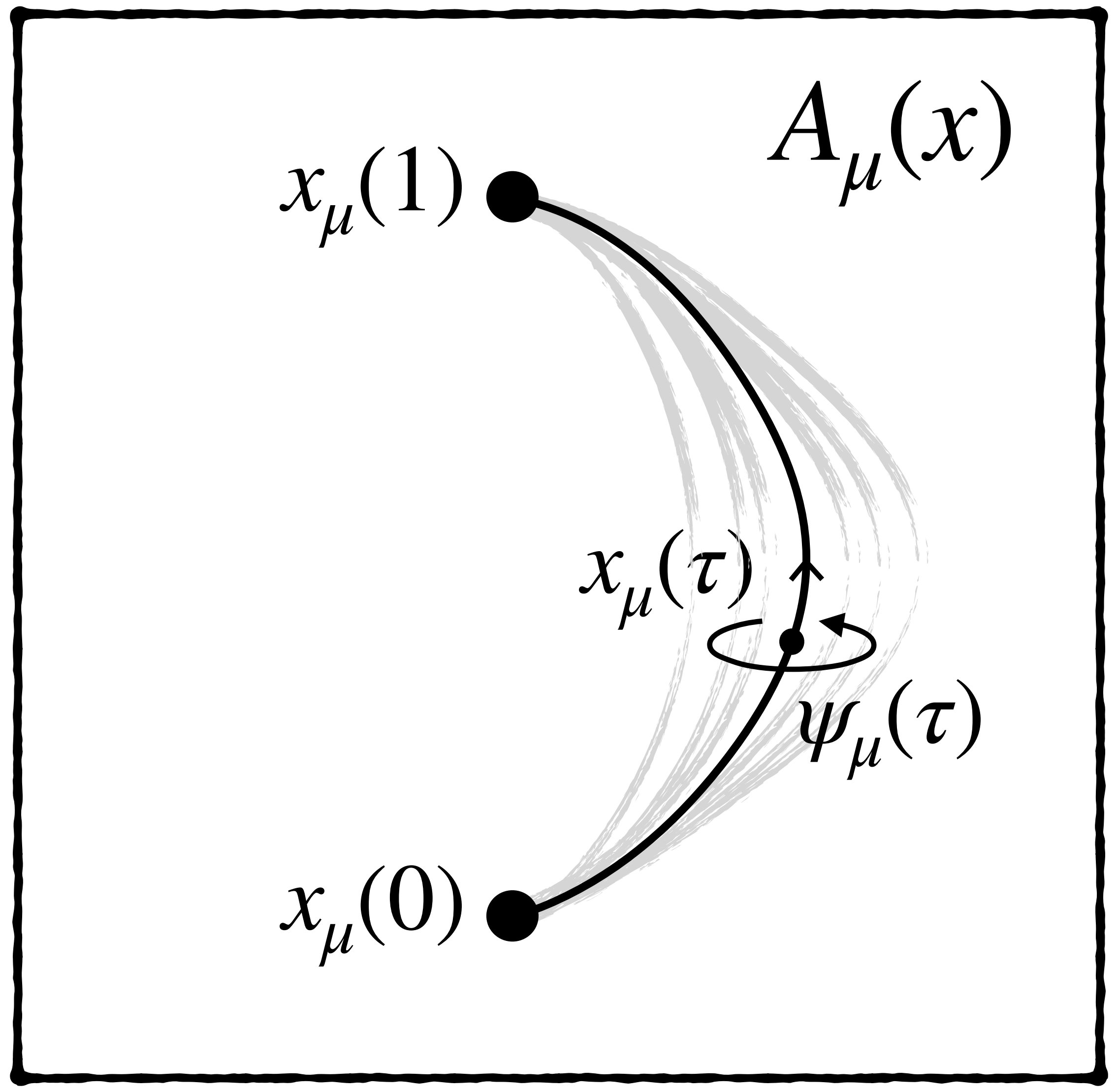
$$\underbrace{\mathscr{L}} = \varepsilon_0 m^2 + \frac{1}{4\varepsilon_0}\dot{x}_\mu^2 + \frac{1}{4}\psi_\lambda\dot{\psi}_\lambda - \chi_0\left[m\psi_5 + \frac{i}{2\varepsilon_0}\dot{x}_\mu\psi_\mu\right]$$

Worldline Lagrangian

$$\underbrace{-ig\dot{x}_\mu A_\mu(x)} + \underbrace{i\frac{g\varepsilon_0}{2}\psi_\mu\psi_\nu F_{\mu\nu}(x)}$$

(bosonic) Wilson line term coupling to A_μ

(fermionic) spin tensor term coupling to $F_{\mu\nu}$



S-matrix formalism for all-order computations:

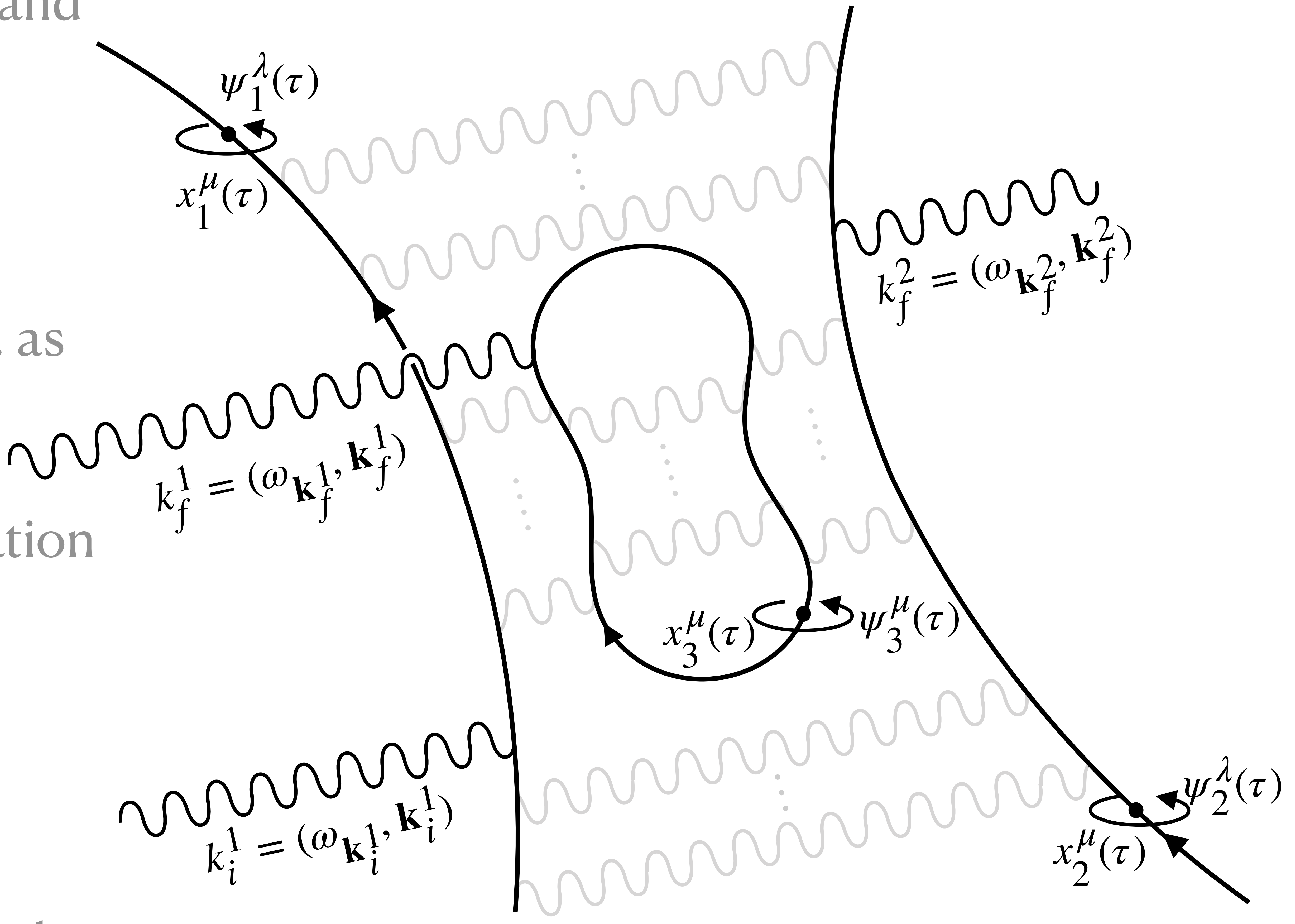
Feal, Tarasov, Venugopalan, PRD 106 (2022) 056009, PRD 107 (2023) 096021

- Generalization of Wilson loop/lines with spin precession and dynamical fields Lorentz invariantly exponentiated.
- Explicit gauge invariance (dynamical fields).
- Non-local and many-body generalization of the BMT eqs. as classical limit

  Bargmann, L. Michel, and V. L. Telegdi, PRL 2 (1959) 435
- Soft-theorems / Ward identities / soft Abelian exponentiation follow naturally.
- First all-order proof of IR safety of the Faddeev-Kulish S-matrix.

  Kulish, Faddeev Theor. Math. Phys. 4 (1970) 745
- Multi-loop generalization of the (Abelian) Bern-Kosower rules & compact form universal expression to compute any given order in PT (examples in this talk).

  Bern-Kosower NPB 379 (1992) 451 , Strassler NPB 385 (1992) 145



Practical application: QED cusp anomalous dimension

Amplitude for a spin-1/2 particle to go from state i at t_i to f at t_f in background $B_\mu(x)$:

$$\mathcal{S}_{fi}[B] = \frac{1}{Z}\int dA_2 dA_1 d\Psi_2 d\Psi_1 \langle f; t_f \,|\, A_2, \Psi_2; t_f\rangle \int_{A_1}^{A_2} \mathscr{D}A \int_{\Psi_1}^{\Psi_2} \mathscr{D}\bar{\Psi}\mathscr{D}\Psi e^{-iS[A,B,\bar{\Psi},\Psi]} \big\langle A_1, \Psi_1; t_i \,|\, i; t_i\rangle\,,$$

here $Z = \langle 0\,|\,0\rangle$, $D_\mu = \partial_\mu - igA_\mu - igB_\mu$, and

$$S[A,B,\bar{\Psi},\Psi] = \int_{t_i}^{t_f} dt \int d^3\mathbf{x}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A_\mu)^2 - \bar{\Psi}(i\not{D} - m)\Psi\right]$$

Expand in # of interactions with $B_\mu(x)$

$$\mathcal{S}_{fi}[B] = \mathcal{S}_{fi}^{[0]}[B] + \mathscr{M}_{fi}^{[1]}[B] + \mathscr{M}_{fi}^{[2]}[B] + \cdots$$

For $B_\mu(x)$ with finite support and $t_{f,i}$ large enough

$$\mathscr{M}_{fi}^{[1]}[B] = ig\tilde{B}_\mu(p_f - p_i)\Gamma^\mu(p_f, p_i)$$

with the **QED 1-point vertex function all PT orders**

$$\Gamma^\mu(p_f,p_i) = \frac{(2\pi)^3}{V}\left[\frac{E_f}{m}\frac{E_i}{m}\right]^{1/2}\left[\frac{|t_f|}{E_f}\frac{|t_i|}{E_i}\right]^{3/2} e^{-i\frac{\pi}{2}+i|t_f|\frac{m^2}{E_f}+i|t_i|\frac{m^2}{E_i}} u^\dagger_{\beta_f}(p_f,s_f) j^\mu_{\beta_f\beta_i}\left(\frac{t_f p_f}{E_f}, \frac{t_i p_i}{E_i}\right) u_{\beta_i}(p_i,s_i)$$

and the QED 1-point vertex function defined to all-orders in PT as

$$j_{\mu;\beta_f\beta_i}(y_f,y_i) = \frac{1}{Z[0,0]}\int \mathscr{D}A \exp\left\{-\frac{1}{4}\int d^4x F_{\mu\nu}^2 - \frac{1}{2\xi}\int d^4x(\partial_\mu A_\mu) + \log\det(\not{D}+m)\right\}$$

$$\times\left\{\underbrace{\left[D_F^A(y_f,0)\right]_{\beta_f\alpha}\left[\gamma_\mu\right]_{\alpha\beta}\left[D_F^A(0,y_i)\right]_{\beta\alpha_i}\left[\gamma_0\right]_{\alpha_i\beta_i}}_{\text{Direct scattering of the external spin-1/2 particle off } B_\mu(x)} - \underbrace{\mathrm{Tr}\left[\gamma_\mu D_F^A(0,0)\right]\left[D_F^A(y_f,y_i)\gamma_0\right]_{\beta_f\beta_i}}_{\text{Topologies in which the external particle interacts with a virtual fermion polarized by } B_\mu(x)}\right\}.$$

$$y_\mu^{f,i} = \frac{p_\mu^{f,i} t_{f,i}}{E_{f,i}}$$

- Step 1: express $\log\det(D+m)$ - encoding virtual loop diagrams to all PT orders - and the external particle dressed fermion propagators $D_F^A(x_f,x_i)$ as **closed** and **open** propagators for a point-like spin-1/2 particle.

- Step 2: integrate the S-matrix over all $A_\mu(x)$ configurations to get an **exact path-integral definition of the QED vertex function to all orders in PT in terms of point-like particle variables.**

- Step 3: evaluate the path integrals either using **(a) semi-classical expansions** or **(b) PT.**

- Our main focus today will be how to **use (b) to evaluate the QED cusp anomalous dimension.**

IR limit of $\Gamma_\mu(p_f, p_i)$**:** assume $B_\mu(x)$ **hard** and **real**, and $A_\mu(x)$ soft compared with hard $p_\mu^{f,i}$, i.e.:

$$\underbrace{D_F^A(x_f,x_i)}_{\text{Dressed spin-1/2 propagator}} \simeq \underbrace{D_F^0(x_f,x_i)}_{\text{Free spin-1/2 propagator}}\underbrace{U^A(x_f,x_i)}_{\text{Wilson line}}\,, \quad U_A(x_f,x_i) = \exp\left[ig\int_{t_i}^{t_f} dt\, \dot{x}_\mu^{cl} A_\mu(x^{cl})\right], \quad \underbrace{x_\mu^{cl}(t) = \frac{x_\mu^f - x_\mu^i}{t_f - t_i}(t - t_i) + x_\mu^i}_{\text{Saddle point of the worldline path integral in the free theory}}$$

One gets for the QED 1-point vertex function

$$\Gamma^\mu(p_f,p_i) = \frac{1}{V}\left[\frac{E_f}{m}\frac{E_i}{m}\right]^{3/2}\frac{1}{2m}\bar{u}(s_f,p_f)\left[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu\right]u(s_i,p_i)\,\Phi[y^{cl}]$$

where $\Phi[y_{cl}(\tau)]$ returns the **self-energy of a classical charge** g moving along $y_{cl}(\tau)$ to all orders in PT

$$\Phi[y^{cl}] = \mathcal{N}\int \underbrace{\mathcal{D}A\exp\left[-\frac{1}{4}\int d^dxF_{\mu\nu}^2 - \frac{1}{2\zeta}\int d^dx(\partial_\mu A_\mu)^2\right.}_{\text{Sum over all gauge field configurations}} + \underbrace{\log\det(\not{D}+m)}_{\text{Arbitrary \# fermion loop insertions}} + \underbrace{\left.ig\mu^{\frac{4-d}{2}}\int_{t_i}^{t_f} d\tau\,\dot{y}_\mu^{cl}A_\mu\big(y_{cl}(t)\big)\right]}_{\text{Cusped Wilson line in } A_\mu}$$

Expand now in power series the fermion loop determinant

$$\Phi\left[x(\tau)\right] = \mathcal{N}\int \mathcal{D}A \exp\left[-\frac{1}{4}\int d^d x F_{\mu\nu}^2 - \frac{1}{2\zeta}\int d^d x(\partial_\mu A_\mu)^2 + ig\mu^{\frac{4-d}{2}}\int_{\tau_i}^{\tau_f} d\tau \dot{x}_\mu A_\mu\left(x(\tau)\right)\right]$$

$$\times \sum_{\ell=0}^{\infty}\frac{1}{\ell\,!}\left\{\ln\det\left(\not{D}+m\right)\right\}^{\ell} \equiv \sum_{\ell=0}^{\infty}\Phi^{(\ell)}\left[x(\tau)\right]$$

The ℓ-th term returns the self-energy of $x(\tau)$ with a fixed number ℓ of virtual fermions loops

$$\Phi^{(\ell)}\left[x(\tau)\right] = \frac{(-1)^\ell}{\ell\,!}\left[\frac{1}{2}\int_0^\infty \frac{d\varepsilon_0}{\varepsilon_0} e^{-\varepsilon_0 m^2}\int \mathcal{D}^4 x \mathcal{D}^4 \psi e^{-S_0[x,\psi]}\right]^\ell \mathcal{N}\int \mathcal{D}A \exp\left[-\frac{1}{4}\int d^d x F_{\mu\nu}^2 - \frac{1}{2\zeta}\int d^d x(\partial_\mu A_\mu)^2\right.$$

Integration over ℓ closed loop x_μ^i and ψ_μ^i contours

Integration over all A_μ gauge field configurations

$$\left. + ig\mu^{\frac{4-d}{2}}\int_{\tau_i}^{\tau_f} d\tau \dot{x}_\mu A_\mu\left(x(\tau)\right) + ig\mu^{\frac{4-d}{2}}\sum_{i=1}^{\ell}\int_0^1 d\tau \dot{x}_\mu^i(\tau) A_\mu\left(x^i(\tau)\right) - ig\mu^{\frac{4-d}{2}}\sum_{i=1}^{\ell}\frac{\varepsilon_0^i}{2}\int_0^1 d\tau \psi_\mu^i(\tau)\psi_\nu^i(\tau) F_{\mu\nu}\left(x_\mu^i(\tau)\right)\right]$$

(bosonic) Wilson line

ℓ (bosonic) worldline loops

ℓ (fermionic) worldline loops

The integral in A_μ is quadratic:

$$\Phi^{(\ell)}[x(\tau)] = \mathcal{N}' \frac{(-1)^\ell}{\ell\,!} \left\langle \exp\left[\frac{1}{2}\sum_{ij=0}^{\ell}\int \frac{d^dk}{(2\pi)^d} i\tilde{J}^i_\mu(-k)\tilde{D}_{\mu\nu}(k) i\tilde{J}^j_\nu(+k)\right]\right\rangle$$

$D_{\mu\nu}(k) \equiv$ photon propagator

Many-body path integrals for a theory of electromagnetic currents in non-local interaction:

$$\tilde{J}^0_\mu(k) = g\mu^{\frac{4-d}{2}} \int_{-\infty}^{+\infty} d\tau_0 \dot{x}_\mu(\tau_0) e^{ik\cdot x(\tau_0)}$$

eikonal

Charged current induced by the external particle

$$\tilde{J}^i_\mu(k) = g\mu^{\frac{4-d}{2}} \int_0^1 d\tau_i \left\{ \dot{x}^i_\mu(\tau_i) + i\epsilon^i_0 k_\nu \psi^i_\mu(\tau_i)\psi^i_\nu(\tau_i) \right\} e^{ik\cdot x_i(\tau_i)}$$

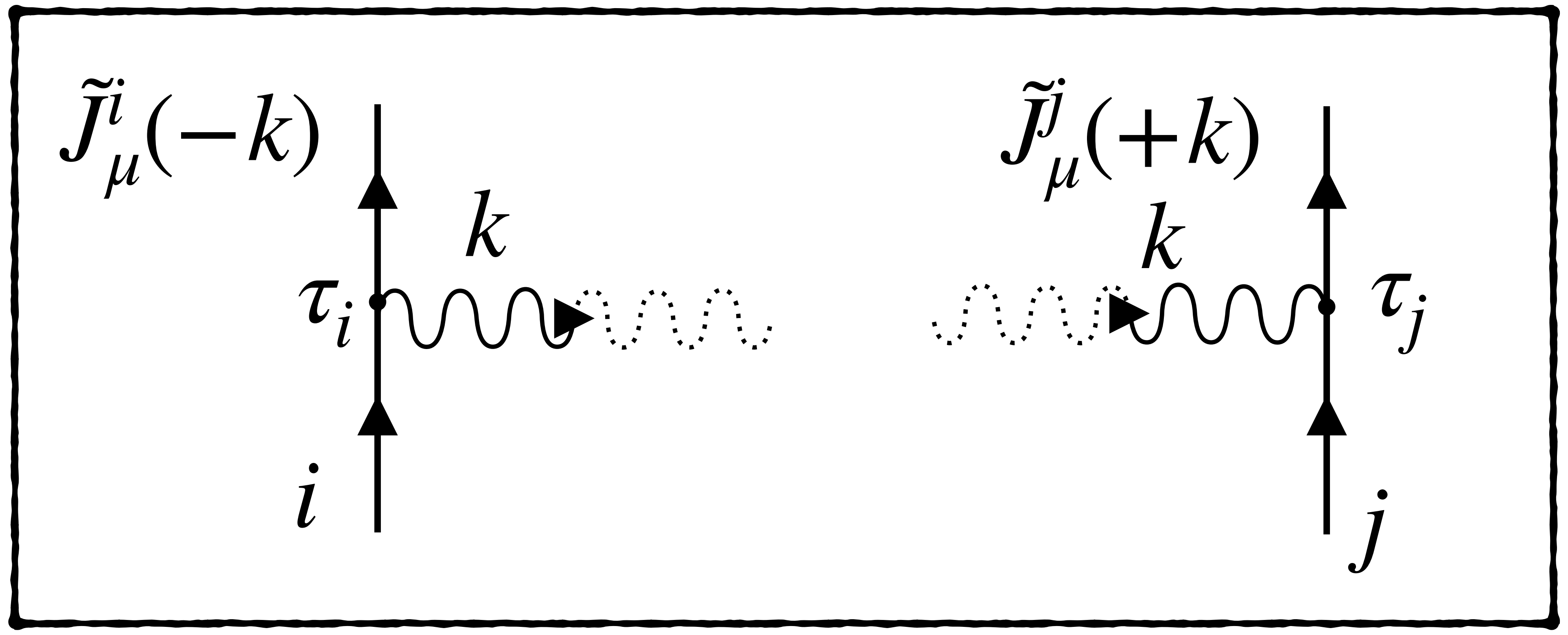
exact kinematics = all eikonal orders

Charged current induced by the i-th virtual fermion polarized from the vacuum

$$\left\langle \mathcal{O}\left[\{J^i_\mu\}\right]\right\rangle \equiv \prod_{i=1}^{\ell}\left[\int_0^\infty \frac{d\varepsilon^i_0}{2\varepsilon^i_0} e^{-\varepsilon^i_0 m^2}\int_{\mathrm{P}} \mathscr{D}^4 x_i \int_{\mathrm{A}} \mathscr{D}^4\psi_i e^{-S_0[x_i,\psi_i]}\right]$$

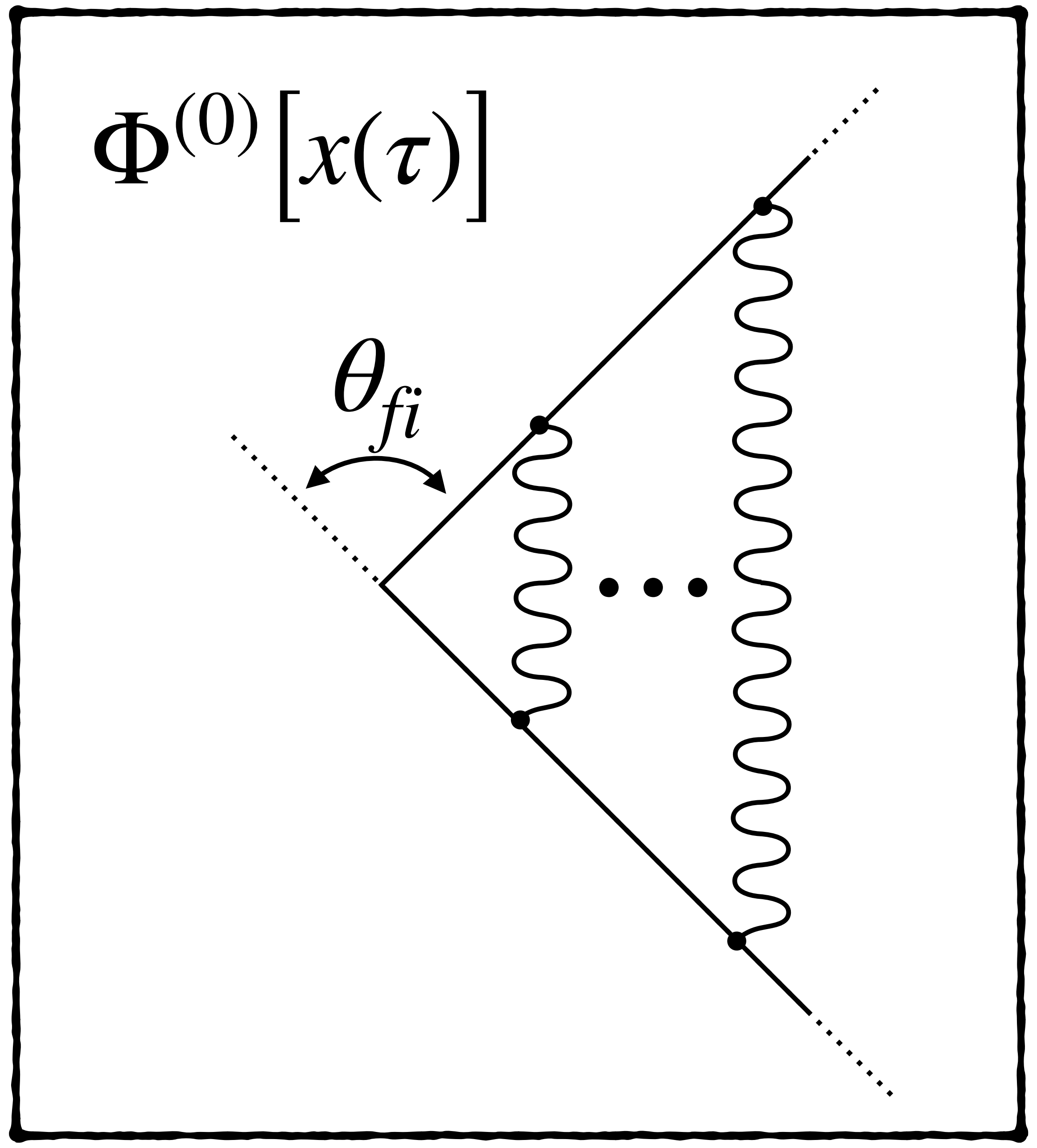
$$\times\, \mathcal{O}\left[\{J^i_\mu\}\right]$$

ℓ path integrals summing over all possible virtual worldline fermion contours



Each term in the loop expansion contains (infinite) UV's and IR's poles

$$\Phi\left[x(\tau)\right] = \underbrace{\Phi^{(0)}\left[x(\tau)\right]}_{\text{0-fermion loops}} + \underbrace{\Phi^{(1)}\left[x(\tau)\right]}_{\text{1-fermion loops}} + \underbrace{\Phi^{(2)}\left[x(\tau)\right]}_{\text{2-fermion loops}} + \cdots$$



For instance, the quenched term ($\ell = 0$) is given by

$$\Phi^{(0)}\left[x(\tau)\right] = \mathcal{N}'\exp\left\{-\frac{g^2\mu^{4-d}}{8\pi^{\frac{d}{2}}}\Gamma\left(\frac{d-2}{2}\right)\left(\frac{1+\zeta}{2}\right)\int_0^1 d\tau\int_0^1 d\tau'\frac{\dot{x}_\mu(\tau)\dot{x}_\mu(\tau')}{\left[\left(x_\mu(\tau)-x_\mu(\tau')\right)^2\right]^{\frac{d}{2}-1}}\right\}$$

with the integral having UVs from $\tau' \to \tau$ regions, and IRs from $\{\tau, \tau'\} \to \pm\infty$ regions,

$$\Phi^{(0)}\left[x(\tau)\right] = \mathcal{N}'\exp\left\{-\frac{\alpha}{\pi}\left[\frac{1}{2\epsilon_{\text{UV}}} + \ln\frac{\mu}{\lambda_{\text{IR}}}\right]\left(\theta_{fi}\coth\theta_{fi} - 1\right)\right\}$$

$\Phi\left[x(\tau)\right]$ needs to be renormalized, and will pick an anomalous dimension depending only on γ_{fi}

To compute the anomalous dimension consider now the perturbation theory for $\Phi\left[x(\tau)\right]$

$$\Phi\left[x(\tau)\right] = \left\{ \sum_{\ell,n=0}^{\infty} Z_{(n)}^{(\ell)} \right\}^{-1} \sum_{\ell,n=0}^{\infty} \Phi_{(n)}^{(\ell)}\left[x(\tau)\right]$$

$\ell = 1$
Bern-Kosower NPB 379 (1992) 451
Strassler NPB 385 (1992) 145

where we power expanded to get

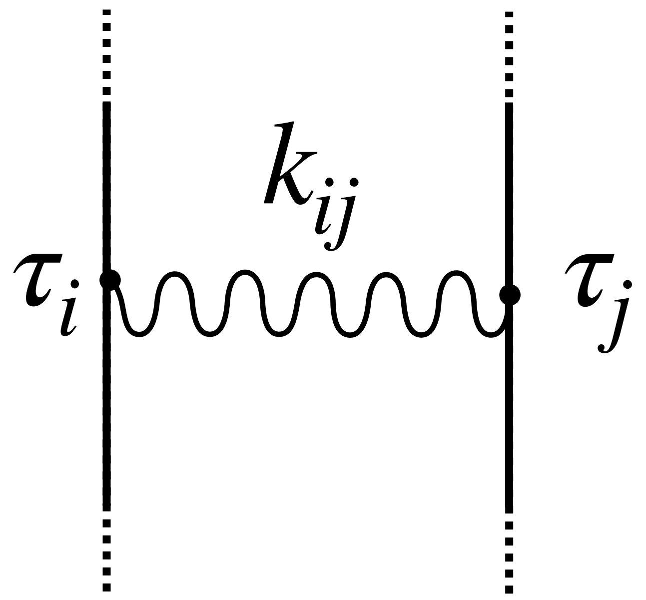
$$\Phi_{(n)}^{(\ell)}\left[x(\tau)\right] = \frac{(-1)^{\ell}}{\ell\,!} \sum_{\sum n_{ij}=n} \left\langle \prod_{i,j=0}^{\ell} \frac{S_{ij}^{n_{ij}}}{n_{ij}\,!} \right\rangle, \qquad Z_{(n)}^{(\ell)}\left[x(\tau)\right] = \frac{(-1)^{\ell}}{\ell\,!} \sum_{\sum n_{ij}=n} \left\langle \prod_{i,j=1}^{\ell} \frac{S_{ij}^{n_{ij}}}{n_{ij}\,!} \right\rangle,$$

Diagrams including the external particle

Vacuum-to-vacuum diagrams removing disconnected fermion loops

$\ell \equiv$ total # fermion loops, $n \equiv$ total # photon lines, $n_{ij} \equiv$ # photon lines connecting i and j

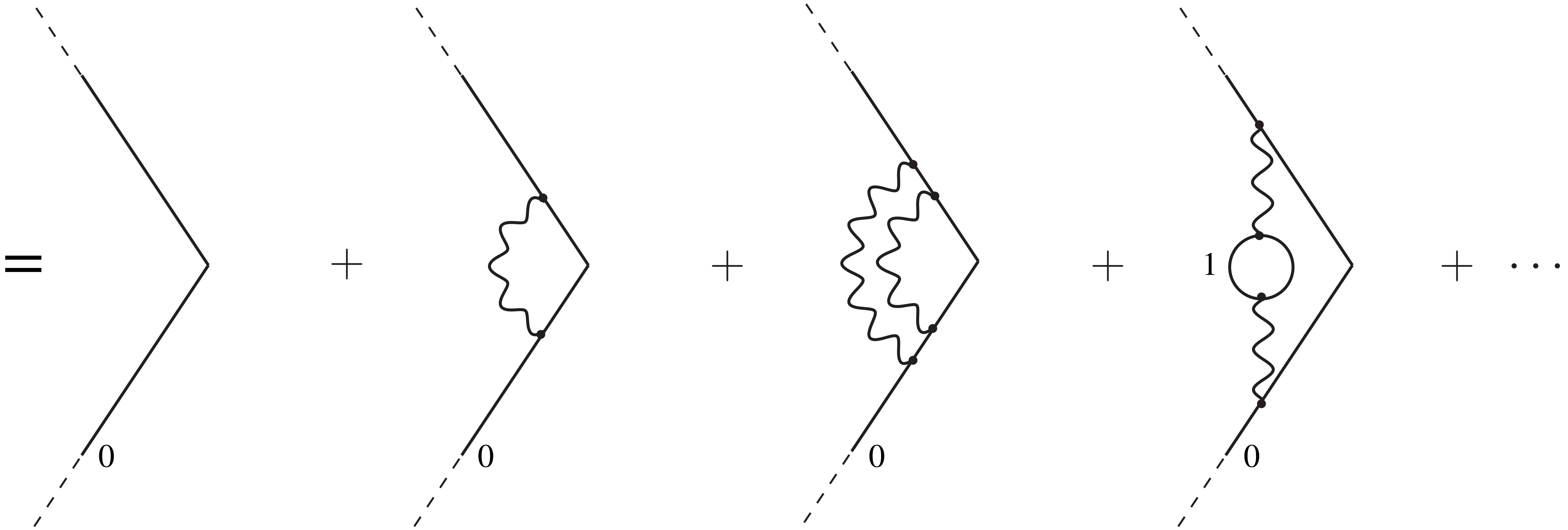
$$S_{ij} \equiv \frac{1}{2} \int \frac{d^d k_{ij}}{(2\pi)^d} i\tilde{J}_{\mu}^{i}(-k_{ij}) \tilde{D}_{\mu\nu}(k_{ij}) i\tilde{J}_{\nu}^{j}(+k_{ij}) \equiv$$



Photon subgraph connecting i and j

To 2-loops α^2

$$\Phi\left[x(\tau)\right] = 1 + S_{00} + \frac{1}{2}S_{00}^2 - 2\left\langle S_{01}^2 \right\rangle + \cdots$$

$\Phi\left[x(\tau)\right] =$ 

Recall that:

$$\langle S_{01}^2 \rangle = \langle S_{01} S_{10} \rangle \equiv \frac{1}{2^2} \int \frac{d^d k_{10}}{(2\pi)^d} \int \frac{d^d k_{01}}{(2\pi)^d} i\tilde{J}^0_\mu(-k_{01}) \tilde{D}_{\mu\nu}(k_{01}) \left\langle i\tilde{J}^1_\nu(+k_{01}) i\tilde{J}^1_\rho(-k_{10}) \right\rangle \tilde{D}_{\rho\sigma}(k_{10}) i\tilde{J}^0_\sigma(+k_{10})$$

eikonal: external particle scalar current insertion

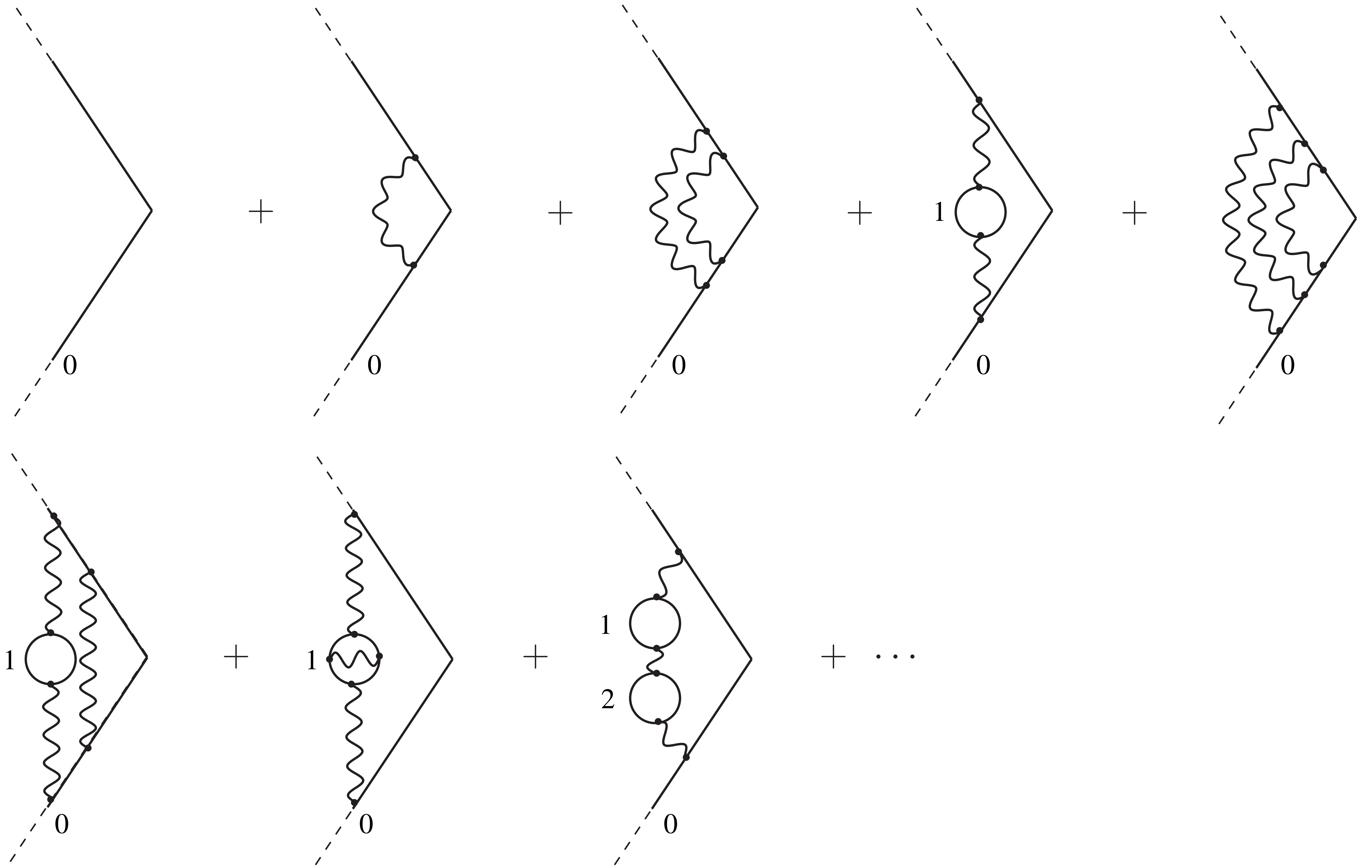
exact = all eikonal orders: path integral virtual fermion loop with 2 fermionic current insertions

eikonal: real/external particle scalar current insertion

To three-loops α^3

$$\Phi\left[x(\tau)\right] = 1 + S_{00} + \frac{1}{2}S_{00}^2 - 2\left\langle S_{01}^2 \right\rangle + \frac{1}{6}S_{00}^3 - 2S_{00}\left\langle S_{01}^2 \right\rangle - 2\left\langle S_{01}^2 S_{11} \right\rangle + 4\left\langle S_{01}S_{12}S_{20} \right\rangle + \cdots$$

$$\Phi\left[x(\tau)\right] =$$

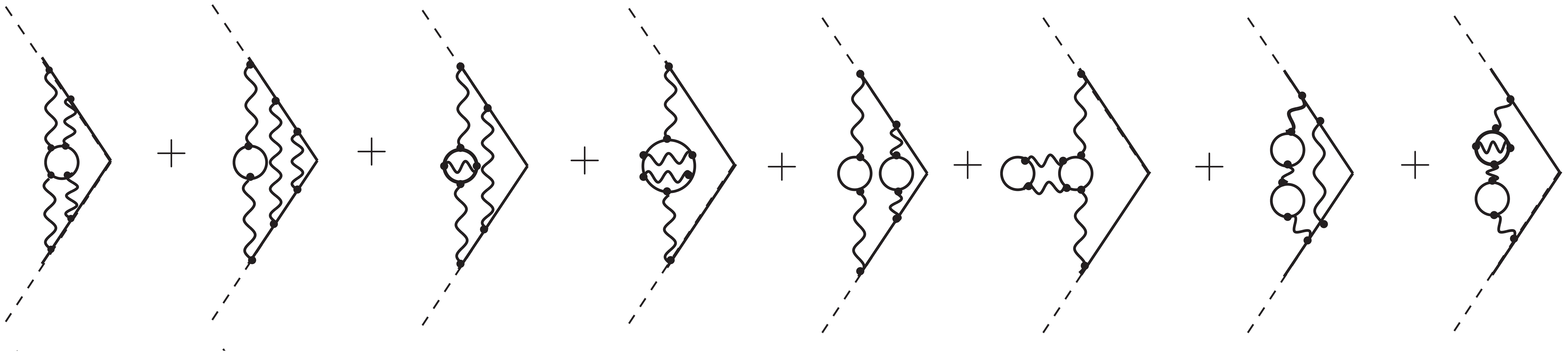


To four-loops α^4

$$\Phi\left[x(\tau)\right] = 1 + S_{00} + \frac{1}{2}S_{00}^2 - 2\left\langle S_{01}^2\right\rangle + \frac{1}{6}S_{00}^3 - 2S_{00}\left\langle S_{01}^2\right\rangle - 2\left\langle S_{01}^2 S_{11}\right\rangle + 4\left\langle S_{01}S_{12}S_{20}\right\rangle$$

$$-\frac{2}{3}\left\langle S_{01}^4\right\rangle - \frac{1}{2}S_{00}^2\langle S_{01}^2\rangle - S_{00}\left\langle S_{01}^2 S_{11}\right\rangle - \frac{1}{2}\left\langle S_{01}^2 S_{11}^2\right\rangle + \frac{3}{2}\left\langle S_{01}^2\right\rangle\left\langle S_{02}^2\right\rangle + \frac{3}{2}\left\langle S_{01}^2 S_{12}^2\right\rangle + \frac{3}{2}\left\langle S_{02}^2 S_{12}^2\right\rangle$$

$$+4S_{00}\left\langle S_{01}S_{02}S_{12}\right\rangle + 4\left\langle S_{01}S_{02}S_{11}S_{12}\right\rangle + 4\left\langle S_{01}S_{02}S_{12}S_{22}\right\rangle + \cdots$$



Disclaimer: I am not a big fan of perturbative calculations

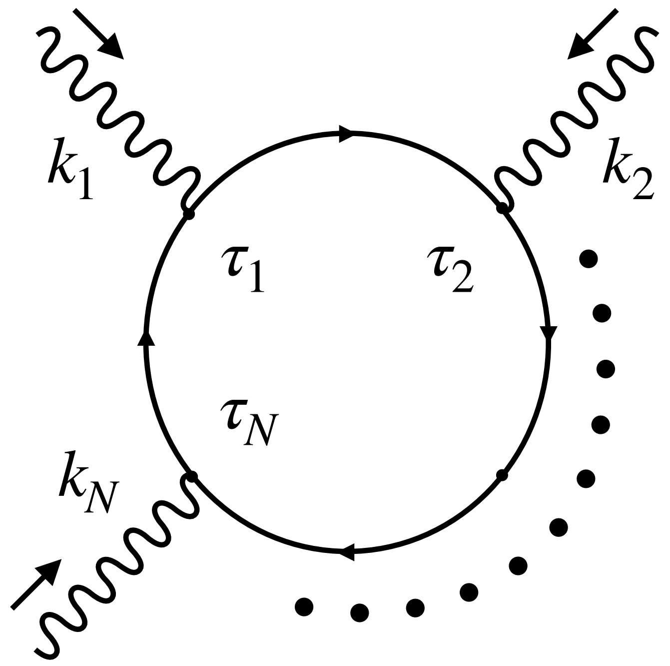
Multi-loop ℓ-fermion N'-photon amplitudes reduced to compute products of 1-fermion N-photon.

Remarkably, in the worldline the 1-fermion N-photon amplitude can be evaluated for general N.

$$\left\langle i\tilde{J}_{\mu_1}(k_1)\cdots\tilde{J}_{\mu_N}(k_N)\right\rangle = (2\pi)^d\delta(k_1+\cdots+k_N)2\frac{g^N}{N!}\frac{1}{(4\pi)^{d/2}}\int_0^\infty\frac{d\varepsilon_0}{\varepsilon_0^{1+d/2}}e^{-\varepsilon_0 m^2}\left\{\prod_{i=1}^N\int_0^1 d\tau_i\right\}\exp\left\{\frac{1}{2}\varepsilon_0\sum_{i,j=1}^N k_i\cdot k_j G_B^{ij}\right\}I_{\mu_1\ldots\mu_N}$$

with

$$I_{\mu_1\cdots\mu_N} = \sum_{2N_a+N_b+2N_c=N}\frac{(-1)^{N_c}}{N_a!N_b!N_c!N_c!}\epsilon_{i_1j_1\ldots i_{N_a}j_{N_a}k_1\ldots k_{N_b}p_1q_1\ldots p_{N_c}q_{N_c}}\epsilon_{i_1j_1\ldots i_{N_a}j_{N_a}l_1\ldots l_{N_b}r_1s_1\ldots r_{n_c}s_{N_c}}$$



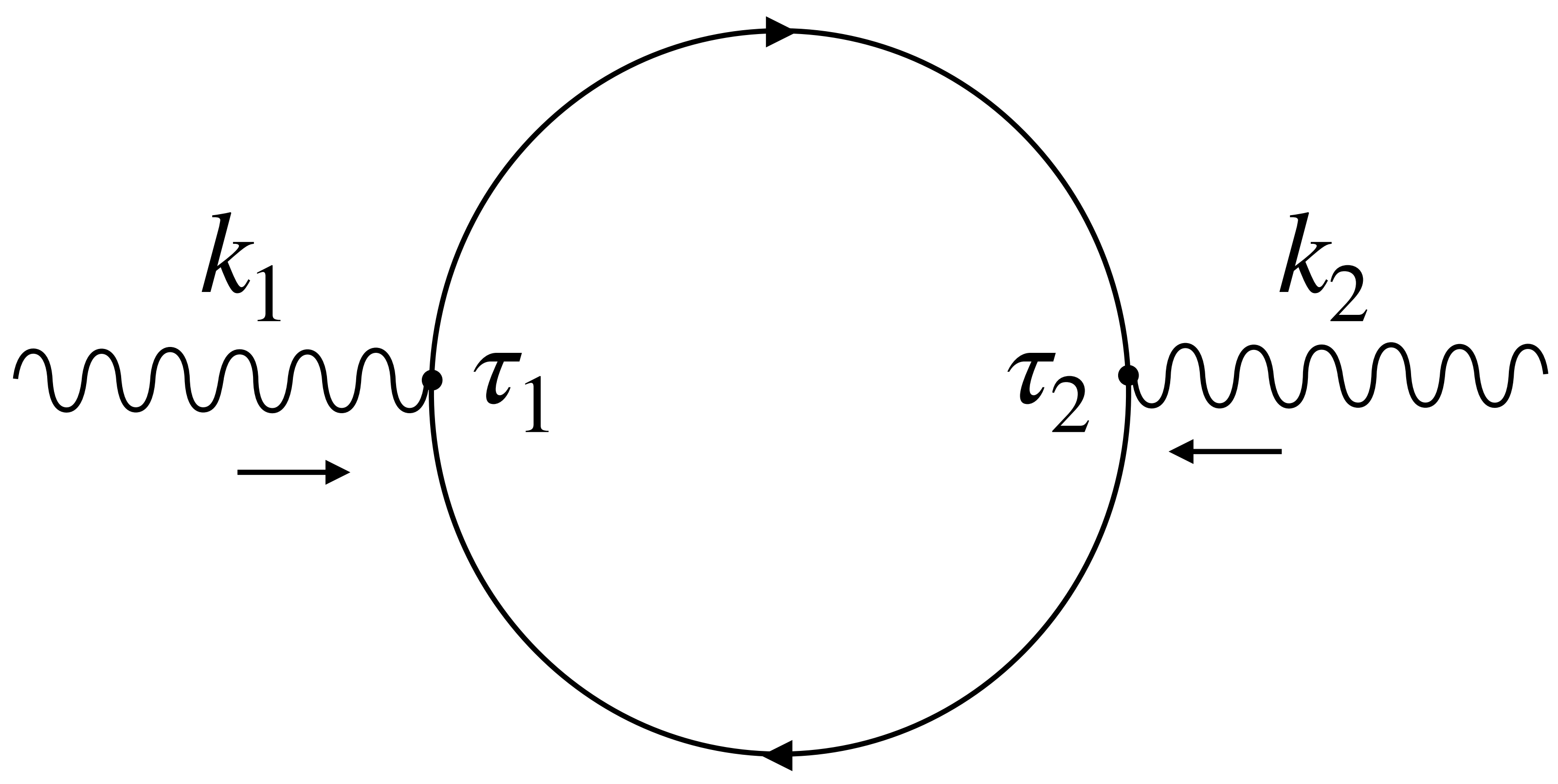
$$\times\prod_{\alpha_a=1}^{N_a}\prod_{\alpha_b=1}^{N_b}\prod_{\alpha_c=1}^{N_c}\prod_{\alpha_d=1}^{N_d}A_{i_{\alpha_a}j_{\alpha_a}}B_{k_{\alpha_b}l_{\alpha_b}}C_{p_{\alpha_c}q_{\alpha_c}}D_{r_{\alpha_d}s_{\alpha_d}}$$

$$A_{ij}=\frac{\varepsilon_0}{2}\eta_{\mu_i\mu_j}\ddot{G}^B_{ij}\,,\quad B_{ij}=\varepsilon_0\sum_{\alpha=1}^N k^\alpha_{\mu_i}\left(\delta_{ij}\dot{G}^B_{i\alpha}+\delta_{\alpha j}G^F_{\alpha i}\right),\quad C_{ij}=\frac{1}{2}\eta_{\mu_i\mu_j}G^F_{ij}\,,\quad D_{ij}=\frac{\varepsilon_0^2}{2}k^i\cdot k^j G^F_{ij}$$

$$G^B_{ij}=|\tau_i-\tau_j|(1-|\tau_i-\tau_j|)\,,\quad G^F_{ij}=\mathrm{sign}(\tau_i-\tau_j)$$

Universal compact form expression to compute any order in PT in terms of **unordered time integrals** of a tensor which is just a **polynomial function of time variables**

Example:



$$I_{\mu_1\mu_2} = A_{12} + A_{21} + \left(B_{11}B_{22} - B_{12}B_{21}\right) - \left(C_{12} - C_{21}\right)\left(D_{12} - D_{21}\right)$$

$$= \varepsilon_0 \ddot{G}^B_{12} + \varepsilon_0^2 k^2_{\mu_1} k^1_{\mu_2} \dot{G}^B_{12} \dot{G}^B_{21} - \varepsilon_0^2 \left(\eta_{\mu_1\mu_2} k_1 \cdot k_2 - k^2_{\mu_1} k^1_{\mu_2}\right)$$

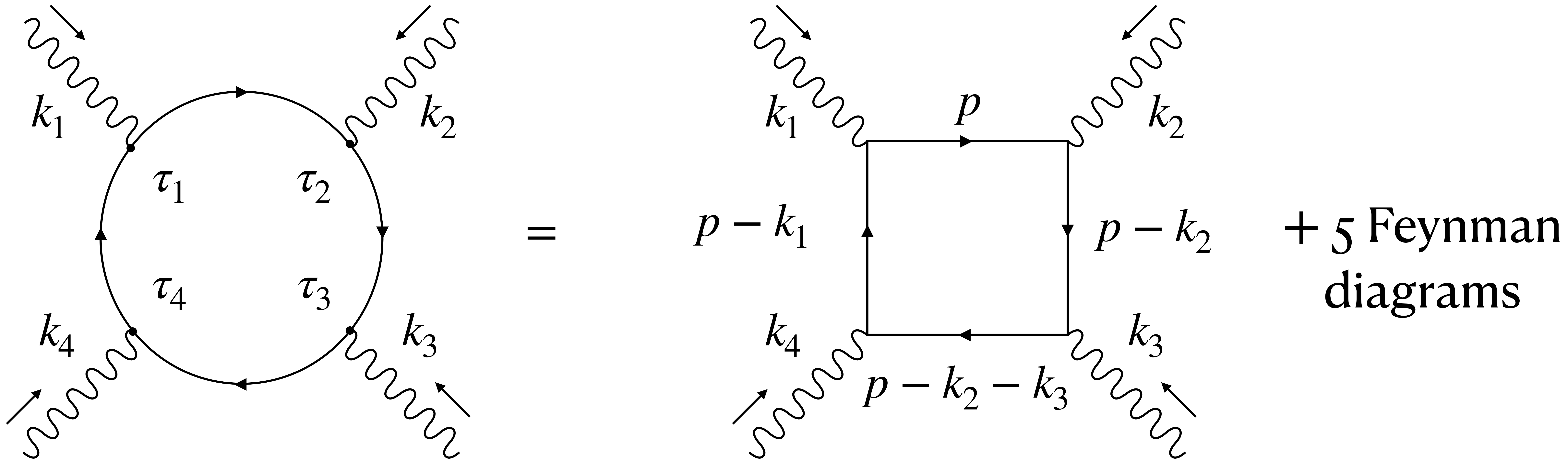
Easily integrating over proper-time ε_0:

$$\Pi_{\mu_1\mu_2}(k_1, k_2) = (2\pi)^d \delta^d(k_1 + k_2) \frac{8g^2}{(4\pi)^{d/2}} \left(\eta_{\mu_1\mu_2} k_1^2 - k^1_{\mu_1} k^1_{\mu_2}\right) \Gamma\left[\frac{4-d}{2}\right] \int_0^1 d\tau \tau(1-\tau) \frac{1}{\left(m^2 + k^2\tau(1-\tau)\right)^{\frac{4-d}{2}}}$$

This is just the standard textbook result after:

computing Dirac traces, introducing Feynman parameters, Wick rotating,
dimensional regularization, manipulation tensor structures,
performing loop momentum integral (and be careful with the symmetry factors)

Not only a convenient bookkeeping formula for the automatic generation of diagrams, it also avoids the factorial explosion of Feynman diagrams at higher-orders in PT



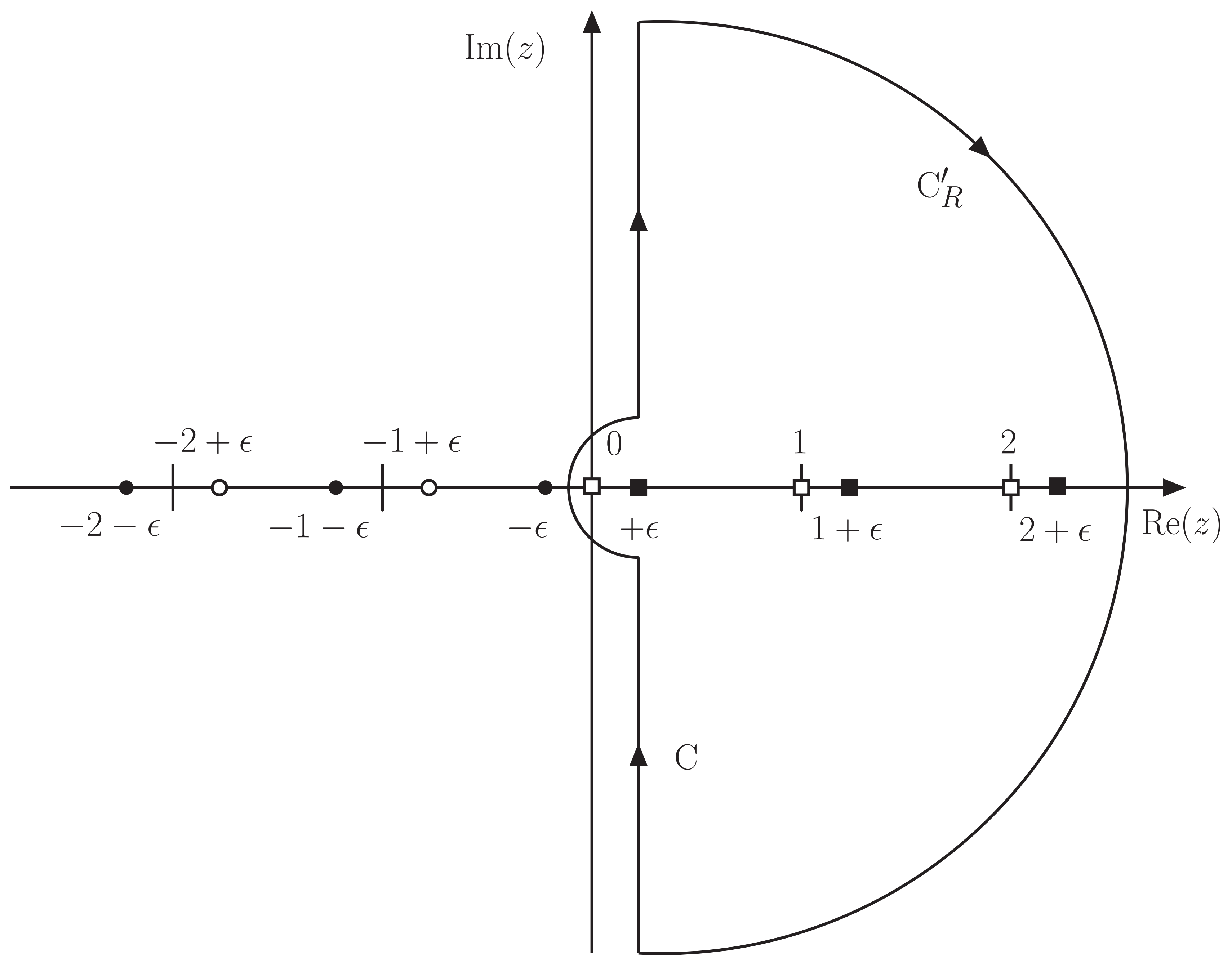
4-photon amplitude in PT in the worldline

(4-1)! Feynman diagrams in conventional field PT

Basic building block: N-photon amplitudes encoding at once $N!$ Feynman diagrams, conjectured cancellations rendering asymptotic the g-2 series? Cvitanovic, Kinoshita PRD 10 (1974) 4007

At 2-loops Mellin-Barnes (MB) allows to easily integrate over all worldline and Schwinger parameters:

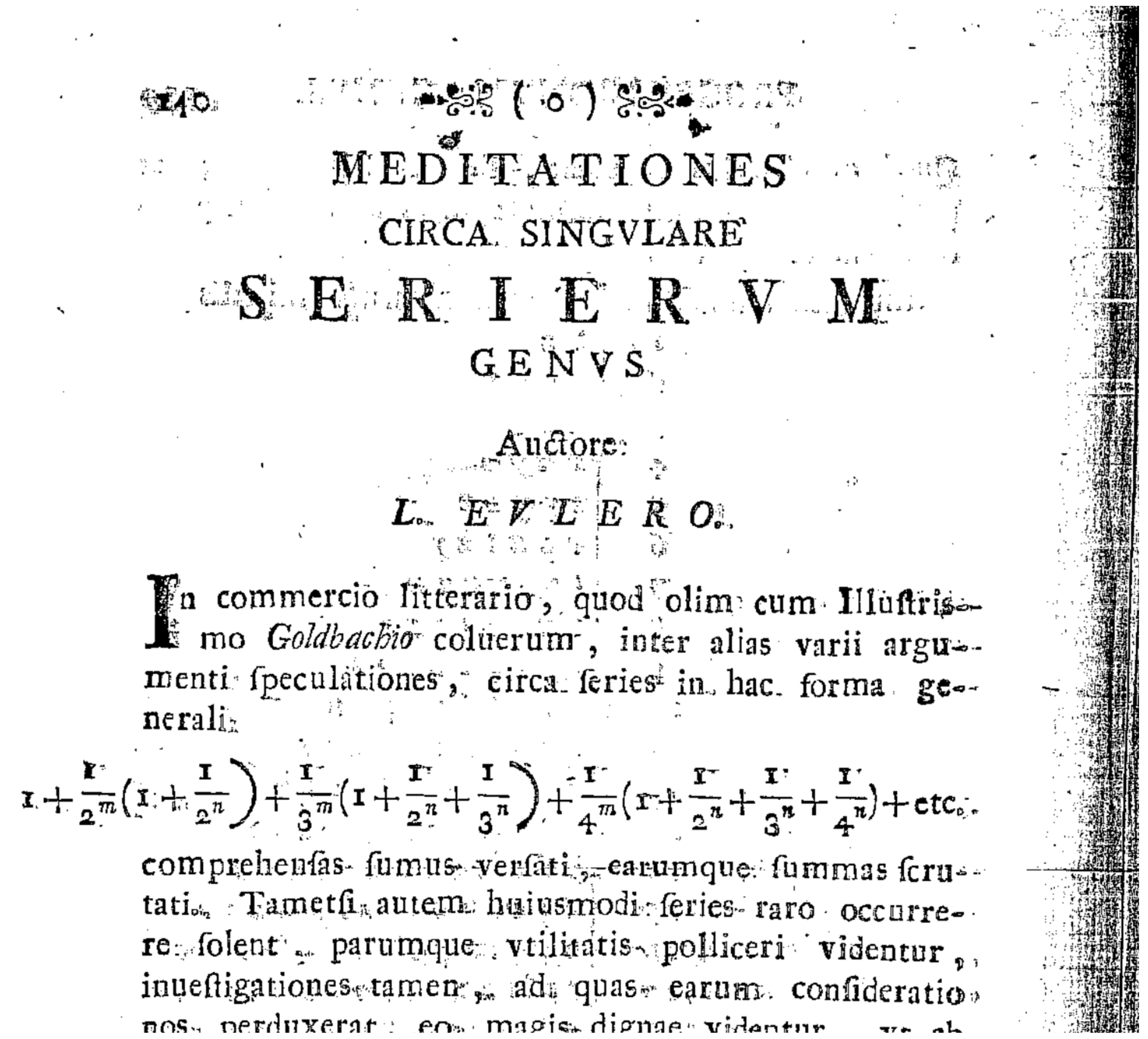
$$-2\left\langle S_{01}S_{10}\right\rangle_R = -\frac{16g^4}{(4\pi)^4}\left\{\frac{1}{12\epsilon^2}-\frac{1}{6\epsilon}\left(\gamma_e+\frac{5}{6}+\log\frac{m^2}{\bar{\mu}^2}\right)+\frac{1}{6}\left(\gamma_e+\frac{5}{6}+\log\frac{m^2}{\bar{\mu}^2}\right)^2+\frac{31+3\pi^2}{216}\right\}f(\theta)$$



Mellin-Barnes contour for 2-loop worldline diagram

At 3-loops MB & worldline amplitudes lead to infinite sums:

$$\begin{aligned}
4\left\langle S_{01}S_{12}S_{20}\right\rangle \propto\; & \left[\frac{\bar{\mu}^6}{m^4\lambda^2}\right]^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\sum_{n_2=0}^{\infty}\frac{1}{n_2!}\left[-\frac{\lambda^2}{m^2}\right]^{n_2}\sum_{n_3=0}^{\infty}\frac{1}{n_3!}\left[-\frac{\lambda^2}{m^2}\right]^{n_3}\Gamma(\epsilon+n_2)\Gamma^2(2+n_2)\\
&\times\Gamma^{-1}(4+2n_2)\Gamma(\epsilon+n_3)\Gamma^2(2+n_3)\Gamma^{-1}(4+2n_3)\Gamma(\epsilon-n_2-n_3)\Gamma(1-\epsilon+n_2+n_3)\\
&+\left[\frac{\bar{\mu}^6}{m^6}\right]^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\sum_{n_2=0}^{\infty}\frac{(-1)^{n_2}}{n_2!}\sum_{n_3=0}^{\infty}\frac{1}{n_3!}\left[-\frac{\lambda^2}{m^2}\right]^{n_3}\Gamma(\epsilon+n_2)\Gamma^2(2+n_2)\Gamma^{-1}(4+2n_2)\\
&\times\Gamma(-\epsilon+n_2-n_3)\Gamma(2\epsilon-n_2+n_3)\Gamma^2(2+\epsilon-n_2+n_3)\Gamma^{-1}(4+2\epsilon-2n_2+2n_3)\\
&+\left[\frac{\bar{\mu}^6}{m^6}\right]^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\sum_{n_2=0}^{\infty}\left[-\frac{\lambda^2}{m^2}\right]^{n_2}\sum_{n_3=0}^{\infty}\frac{(-1)^{n_3}}{n_3!}\Gamma(\epsilon+n_3)\Gamma^2(2+n_3)\Gamma^{-1}(4+2n_3)\\
&\times\Gamma(-\epsilon+n_3-n_2)\Gamma^2(2+\epsilon+n_2-n_3)\Gamma^{-1}(4+2\epsilon+2n_2-2n_3)\Gamma(2\epsilon+n_2-n_3)\\
&+\left[\frac{\bar{\mu}^6}{m^6}\right]^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\sum_{n_2=0}^{\infty}\left[-\frac{\lambda^2}{m^2}\right]^{n_2}\sum_{n_3=0}^{\infty}\frac{(-1)^{n_3}}{n_3!}\Gamma(-2\epsilon-n_2-n_3)\Gamma(3\epsilon+n_2+n_3)\\
&\times\Gamma^2(2+2\epsilon+n_2+n_3)\Gamma^{-1}(4+4\epsilon+2n_2+2n_3)\Gamma(\epsilon+n_3)\Gamma^{-1}(4-2\epsilon-2n_3)\Gamma^2(2-\epsilon-n_3)\\
&+\left[\frac{\bar{\mu}^6}{m^6}\right]^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\sum_{n_2=0}^{\infty}\left[-\frac{\lambda^2}{m^2}\right]^{n_2}\sum_{n_3=0}^{\infty}2\frac{(2n_3)!}{(n_3!)^2}\Gamma(-2-\epsilon-n_2-n_3)\Gamma(2\epsilon+2+n_2+n_3)\\
&\times\Gamma^2(4+\epsilon+n_2+n_3)\Gamma^{-1}(8+2\epsilon+2n_2+2n_3)\Gamma(2+n_3)\Gamma(\epsilon-2-n_3)\,.
\end{aligned}$$


240 (o)

MEDITATIONES
CIRCA SINGVLARE
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Auctore
L. EVLERO.

In commercio litterario, quod olim cum Illuſtriſſimo *Goldbachio* colueram, inter alias varii argumenti ſpeculationes, circa ſeries in hac forma generali

$$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\text{etc.}$$

comprehenſas ſumus verſati, earumque ſummas ſcrutati. Tametſi autem huiusmodi ſeries raro occurrere ſolent, parumque vtilitatis polliceri videntur, inueſtigationes tamen, ad quas earum conſideratio nos perduxerat, eo magis dignae videntur

a) UV and IR poles can be systematically extracted using the re-exponentiation of the Euler gamma functions Laurent and Taylor series expansions in terms of Riemann ζ-functions and harmonic series.

b) Multiple nested sums of products of Euler gamma functions and harmonic series can be performed after reading Euler's original paper.

Let me focus on the 2-loop correction

$$\Phi\big[x(\tau)\big] = 1 - \frac{\alpha}{\pi}\left\{\frac{1}{2\epsilon_{\rm UV}} + \frac{1}{2}\left(\log\frac{\bar\mu^2}{\lambda_{\rm IR}^2} - \gamma_e\right)\right\} f(\theta_{fi}) + \frac{\alpha^2}{\pi^2}\left\{\frac{1}{8\epsilon_{\rm UV}^2} + \frac{1}{4\epsilon_{\rm UV}}\left(\log\frac{\bar\mu^2}{\lambda_{\rm IR}^2} - \gamma_e\right) + \frac{1}{4}\left(\log\frac{\bar\mu^2}{\lambda_{\rm IR}^2} - \gamma_e\right)^2 + \frac{\pi^2}{6}\right\} f^2(\theta_{fi})$$

$$-\frac{\alpha^2}{\pi^2}\left\{\frac{1}{12\epsilon_{\rm UV}^2} - \frac{1}{6\epsilon_{\rm UV}}\left(\gamma_e + \log\frac{m_f^2}{\bar\mu^2} + \frac{5}{6}\right) + \frac{1}{6}\left(\gamma_e + \log\frac{m_f^2}{\bar\mu^2} + \frac{5}{6}\right)^2 + \frac{31\pi^2+31}{216}\right\} f(\theta_{fi})$$

It can be made UV finite by multiplicative renormalization:

$$\Phi_R\big[x(\tau)\big] = \lim_{\epsilon\to 0}\left\{\mathscr{Z}\Phi\big[x(\tau)\big]\right\}\Big|_{g=g(g_R,\mu,\epsilon)}$$

$\mathscr{Z}$ ≡ renormalization constant, removes UVs
g_R ≡ renormalized coupling
$d = 4 - 2\epsilon$
m ≡ bare fermion mass

RGE $(\Phi_R = \mathcal{Z}\Phi)$

$$\underbrace{\mu\frac{d\Phi}{d\mu} = 0}_{\text{scale indep.}} \;\rightarrow\; \underbrace{\mu\frac{d\Phi_R}{d\mu}}_{\text{RGE}} = \frac{\mu}{\mathcal{Z}}\frac{d\mathcal{Z}}{d\mu} = \frac{d\log\mathcal{Z}}{d\log\mu} \equiv \underbrace{-\Gamma(\alpha_R)}_{\text{cusp anomalous dimension}} \;\rightarrow\; \frac{\Phi_R(\mu_1)}{\Phi_R(\mu_0)} = \exp\left\{-\int_{\alpha_R(\mu_0)}^{\alpha_R(\mu_1)}\frac{d\alpha_R}{\beta(\alpha_R)}\Gamma(\alpha_R)\right\}$$

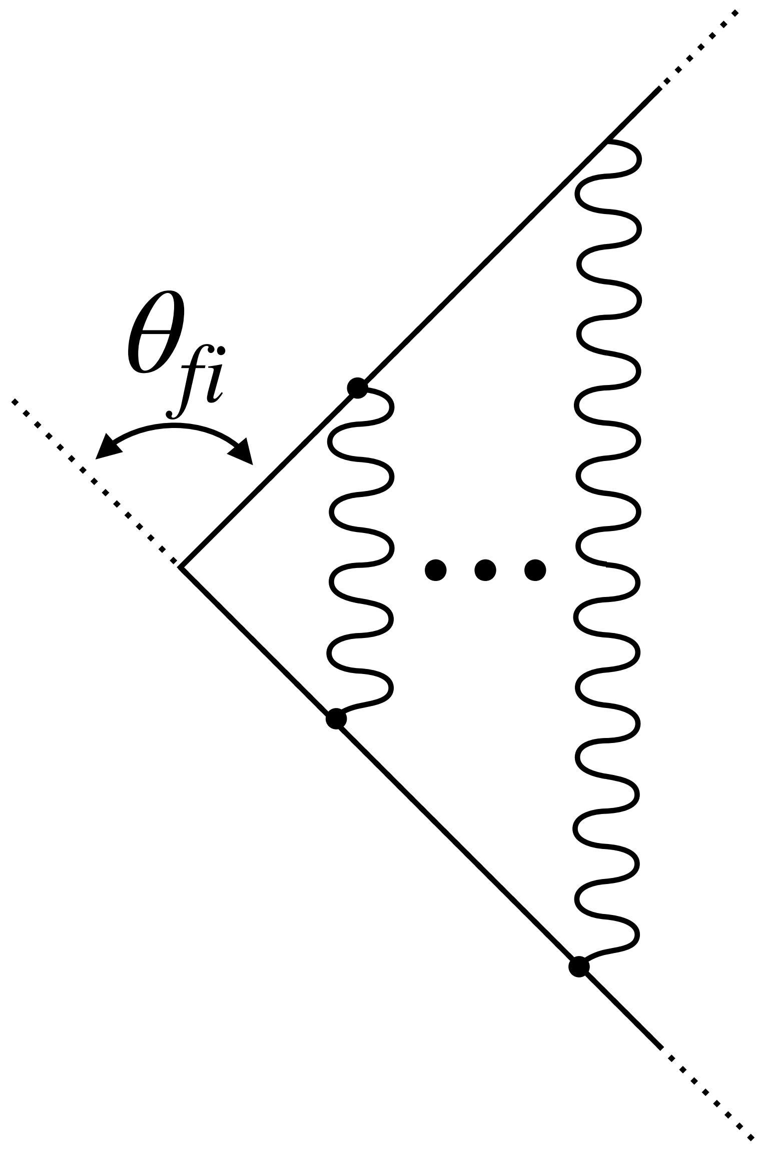
Polyakov, NPB164 (1979) 171
Brandt, Neri, Sato, PRD 24 (1981) 879
Korchemsky and Radyushkin NPB 283 (1987) 342

Example: Renormalization to one-loop

$$\Phi\left[x(\tau)\right] = 1 - \frac{\alpha}{\pi}\left\{\frac{1}{2\epsilon_{\text{UV}}} + \frac{1}{2}\left(\log\frac{\bar{\mu}^2}{\lambda_{\text{IR}}^2} - \gamma_e\right)\right\}f(\theta_{fi}) \quad \text{then} \quad \mathcal{Z} = 1 + \frac{\alpha}{\pi}\frac{1}{2\epsilon_{\text{UV}}}f(\theta_{fi}) + \text{UV finite}$$

This gives

$$\Gamma(\alpha_R) = -\frac{\mu}{\mathcal{Z}}\frac{d\mathcal{Z}}{d\mu} = -\frac{1}{\mathcal{Z}}\frac{f(\theta_{fi})}{\pi}\frac{1}{2\epsilon}\mu\frac{d\alpha_R}{d\mu} = -\frac{f(\theta_{fi})}{\pi}\frac{1}{1+\cdots}\frac{1}{2\epsilon}\left(-2\alpha_R\epsilon + \cdots\right) = \frac{\alpha_R}{\pi}f(\theta_{fi})$$



namely (running coupling) re-summation of virtual IR divergences with $N_f = 0$

$$\frac{\Phi_R(\mu_1)}{\Phi_R(\mu_0)} = \exp\left\{-\left(\theta_{fi}\coth\theta_{fi} - 1\right)\int_{\mu_0}^{\mu_1}\frac{d\mu}{\mu}\frac{\alpha_R}{\pi}\right\}$$

Example: the procedure can be continued to higher-loops, to two-loops and $N_f \neq 0$

$$\Gamma(\alpha_R) = \left\{ \frac{\alpha_R}{\pi} - \frac{\alpha_R^2}{\pi^2}\frac{5N_f}{9} + \cdots \right\} \left(\theta_{fi} \coth \theta_{fi} - 1 \right)$$

Relation to BFKL:
Caron-Huot, Gardi, Reichel, Vernazza JHEP 03 (2018) 198

Work in progress: Outlining a general strategy to automatize this calculation using the general expression for the N-photon amplitude

$$\Phi\left[x(\tau)\right] = 1 + \mathrm{S}_{00} + \frac{1}{2}\mathrm{S}_{00}^2 - 2\left\langle \mathrm{S}_{01}^2 \right\rangle + \frac{1}{6}\mathrm{S}_{00}^3 - 2\mathrm{S}_{00}\left\langle \mathrm{S}_{01}^2 \right\rangle - 2\left\langle \mathrm{S}_{01}^2\mathrm{S}_{11} \right\rangle + 4\left\langle \mathrm{S}_{01}\mathrm{S}_{12}\mathrm{S}_{20} \right\rangle$$

$$-\frac{2}{3}\left\langle \mathrm{S}_{01}^4 \right\rangle - \frac{1}{2}\mathrm{S}_{00}^2\left\langle \mathrm{S}_{01}^2 \right\rangle - \mathrm{S}_{00}\left\langle \mathrm{S}_{01}^2\mathrm{S}_{11} \right\rangle - \frac{1}{2}\underbrace{\left\langle \mathrm{S}_{01}^2\mathrm{S}_{11}^2 \right\rangle}_{5!=120 \text{ Feynman diagrams}} + \frac{3}{2}\left\langle \mathrm{S}_{01}^2 \right\rangle\left\langle \mathrm{S}_{02}^2 \right\rangle + \frac{3}{2}\left\langle \mathrm{S}_{01}^2\mathrm{S}_{12}^2 \right\rangle + \frac{3}{2}\left\langle \mathrm{S}_{02}^2\mathrm{S}_{12}^2 \right\rangle$$

$$+4\mathrm{S}_{00}\left\langle \mathrm{S}_{01}\mathrm{S}_{02}\mathrm{S}_{12} \right\rangle + 4\left\langle \mathrm{S}_{01}\mathrm{S}_{02}\mathrm{S}_{11}\mathrm{S}_{12} \right\rangle + 4\left\langle \mathrm{S}_{01}\mathrm{S}_{02}\mathrm{S}_{12}\mathrm{S}_{22} \right\rangle + \cdots$$

QCD 2-loops Korchemsky, Radyushkin NPB 283 (1987) 342
QCD 3-loops Grozin, Henn, Korchemsky, Marquard JHEP 01 (2016) 140
QCD & N=4 SYM 4-loops Henn, Korchemsky, Mistlberger JHEP 04 (2020) 018
QED 4-loops Brüser, Dlapa, Henn, Yan, PRL 126 (2021) 021601

Future directions: QCD

$$\mathrm{TrP}\exp\left[i\int_0^T M(t)\right] = \int \mathscr{D}\phi \int \mathscr{D}\bar{\lambda}\mathscr{D}\lambda \exp\left[i\phi\left(\lambda^\dagger\lambda + \frac{n}{2} - 1\right)\right]\exp\left[i\int_0^T d\tau\left(i\lambda^\dagger\dot{\lambda}d\tau + \lambda^\dagger M\lambda\right)\right]$$

D'Hoker, Gagné, NPB 467 272 (1996)

$M(\tau) \equiv n \times n$ hermitian matrix

$\lambda^\dagger, \lambda \equiv$ Grassmann valued eigenvalues fermionic creation/anihilation operators

$\mathscr{D}\phi \exp() \equiv$ constrain P.I. to the finite dimensional representation of symm. group

- Successfully used for covariant kinetic theory and Keldysh-Schwinger problems
  Mueller, Venugopalan, Phys.Rev.D 96 (2017) Phys.Rev.D 97 (2018) 5
- Can we extend this result to open boundary/scattering problems (S-matrix)?
- Can we extend it to Non-Abelian dynamical fields?
- Can we easily obtain soft theorems and the systematics of the multipole soft expansion in QCD?
- Can implement it for high-order perturbative computations?

Brief summary:

The extension of this QED program to (try to) reformulate QCD amplitudes along the same lines presents clear advantages both for **perturbative** and **non-perturbative** calculations:

On one hand, worldlines allow for the exact exponentiation of color and spin d.o.f. allowing to investigate non-perturbative effects without the cumbersome problem of having to re-exponentiate path-ordered Wilsonian (or Hamiltonian) operators, opening a window to strong coupling semi-classical expansions in particle variables.

Halpern, Siegel, Phys. Rev. D16 2486 (1977)

On the other hand, perturbative expansions of worldline amplitudes with a full implementation of colored fermions offer a clear path for much more efficient perturbative QCD computations.

D'Hoker, Gagné, NPB 467 272 (1996)

Thanks