Longitudinal momentum fraction and longitudinal resolution: CSS in CGC

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The Question

We all know that CGC is the evolution of observables at high energy.

But not of all observables.

Originally: color charge density $j^a(x) = \int dx^- j^a(x, x^-)$. From here eikonal scattering factors $U(x)$ and variety of derivative observables. All such observables are

$$O(\{x_i\}, Y) = \langle O[U(x_i)] \rangle_Y$$

with Y - the evolution interval.

All these observables depend on only one longitudinal variable - Y. But

there are others that do not fall into this category.

These depend on the longitudinal momentum fraction in addition to the overall evolution parameter.

Why is it interesting?

E.g.: gluon TMD.

$$T(k, \mu; x, \zeta) = \langle a^\dagger(x, k) a(x, k) \rangle_\zeta$$

ζ - longitudinal resolution.

There are two longitudinal variables here x and ζ. How do we accommodate such observables in high energy evolution?
They are interesting for variety of reasons.
E.g. recently questions about id'ing the interplay of Sudakov logarithms and energy evolution in dijet production.

The basic observable here is the gluon TMD. But if one wants to discuss different frames (which is done in this context) with different amounts of evolution, one needs to understand how TMD at fixed x is evolved in energy.

So let us ask this question.

JIMWLK

JIMWLK evolution generated by the "Hamiltonian".

$$H_{\rm JIMWLK} = \frac{\alpha_s}{2\pi^2}\int_{\mathbf{x},\mathbf{y},\mathbf{z}} \frac{(\mathbf{x}-\mathbf{z})\cdot(\mathbf{y}-\mathbf{z})}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2} \times \Big[J_L^a(\mathbf{x})\,J_L^a(\mathbf{y}) + J_R^a(\mathbf{x})\,J_R^a(\mathbf{y}) - 2J_L^a(\mathbf{x})\,S^{ab}(\mathbf{z})\,J_R^b(\mathbf{y})\Big].$$

Every gluon in the projectile - an eikonal factor of S. Charges $J_{L(R)}$ - left (right) rotation operator when acting on S.
The Hamiltonian can act either on an observable, or on the projectile (quasi) probability density

$$\frac{d}{dY}\mathcal{W}_P[S] = H_{\rm JIMWLK}[S,J]\mathcal{W}_P[S]$$

where S- an eikonal scattering matrix of a projectile gluon.

What about TMD?

What is TMD? Maybe it is

$$T = \int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \langle b_i(\mathbf{x}) b_i(\mathbf{y}) \rangle = \frac{1}{g^2} \langle \int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} Tr[S^\dagger(\mathbf{x}) \partial_i S(\mathbf{x}) S^\dagger(\mathbf{y}) \partial_i S(\mathbf{y})] \rangle_Y$$

But RHS depends on a single longitudinal variable Y. What is Y? Is it x ($Y = \ln 1/x$)? Or is it resolution ζ?

It is in fact both: B. W. Xiao, F. Yuan and J. Zhou, Nucl. Phys. B 921 (2017), 104. This expression is valid only for TMD in which the longitudinal resolution is equal to the longitudinal momentum fraction

$$T(x = e^{-Y}, \zeta = Y)$$

How do we "decouple" x from ζ?

JIMWLK - recap

Let us recap (schematically) the derivation of JIMWLK

First, diagonalize the soft gluon Hamiltonian in the background of valence color charges, so that the soft vacuum is

$$|0\rangle_{soft} = \Omega[\hat{j}^a_{valence}]|0\rangle$$

Ω - the diagonalizing operator that depends on the *quantum operator* $j_{valence}$.

Then to evolve any JIMWLK - type operator O (i.e. operator that depends on integrated j^a) we consider

$$\langle 0|\Omega^\dagger O(j^a + j^a_{soft})\Omega|0\rangle = \langle 0|\Omega^\dagger R^\dagger O(j^a) R\Omega|0\rangle$$

where

$$\hat{R}^\dagger \hat{j}^a(\mathbf{x}_\perp)\hat{R} = \hat{j}^a(\mathbf{x}_\perp) + \hat{j}^a_{\rm soft}(\mathbf{x}_\perp)$$

Ω evolves the state, R adds the color density of the soft gluons to the valence color charge!

JIMWLK - recap continued

All said and done this leads to the Hamiltonian (a little long, but bear with me)

$$H_{RFT} = -\frac{1}{4\pi}\left[(\tilde{b}_L^\alpha - b_R^\alpha)^\dagger (N_\perp^\dagger N_\perp)_{\alpha\beta}(\tilde{b}_L^\beta - b_R^\beta) + h.c.\right]$$

with

$$\partial_i b_i^a(\mathbf{x}_\perp) = j^a(\mathbf{x}_\perp)\,,$$
$$\partial_i b_j^a(\mathbf{x}_\perp) - \partial_j b_i^a(\mathbf{x}_\perp) - gf^{abc} b_i^b(\mathbf{x}_\perp) b_j^c(\mathbf{x}_\perp) = 0\,.$$

$$N_\perp = [1 - l - L] = \left[\delta_{ij} - \partial_i \frac{1}{\partial^2}\partial_j - D_i \frac{1}{D^2} D_j\right]^{ab}(\mathbf{x}_\perp, \mathbf{y}_\perp)\,.$$

The two "classical fields" depend on the right and left charges :

$$b_L^\beta \equiv b^\beta[j_L];\quad b_R^\beta \equiv b^\beta[j_R]$$

JIMWLK - recap continued still

$$
\begin{aligned}
j^a_L =& j^b\left[\frac{\tau}{2}\coth\frac{\tau}{2}+\frac{\tau}{2}\right]^{ba} \\
=& j^a+\frac{1}{2}j^b\left(gT^e\frac{\delta}{\delta j^e}\right)_{ba}+\frac{1}{12}j^b\left(gT^e\frac{\delta}{\delta j^e}\right)^2_{ba}-\frac{1}{720}j^b\left(gT^e\frac{\delta}{\delta j^e}\right)^4_{ba}+\dots \\
j^a_R =& j^b\left[\frac{\tau}{2}\coth\frac{\tau}{2}-\frac{\tau}{2}\right]^{ba} \\
=& j^a-\frac{1}{2}j^b\left(gT^e\frac{\delta}{\delta j^e}\right)_{ba}+\frac{1}{12}j^b\left(gT^e\frac{\delta}{\delta j^e}\right)^2_{ba}-\frac{1}{720}j^b\left(gT^e\frac{\delta}{\delta j^e}\right)^4_{ba}+\dots
\end{aligned}
$$

with

$$\tau = gT^e\frac{\delta}{\delta j^e}$$

Why two charges? This is necessary to represent quantum operators in terms of classical phase space variables- "Wigner-Weyl" representation of quantum mechanics: A. Kovner and M. Li, JHEP 05 (2020) 036 (mostly Ming Li)

JIMWLK - recap continued a little bit longer

Important note: H_{RFT} contains

$$\tilde{b}_L^{\alpha} = b_L^{\beta} \mathcal{R}_p^{\beta\alpha}, \qquad \mathcal{R} = e^{\tau}; \qquad \tau = gT^e \frac{\delta}{\delta j^e}$$

JIMWLK is the "dilute target" limit of H_{RFT}, i.e. H_{RFT} expanded to leading order in τ. AT order τ^2 this expansion yields the usual JIMWLK Hamiltonian.

Very important: this expansion involves expanding j_L and j_R around j, and **also** the explicit factor $\mathcal{R}$ in the expression for $\tilde{b}$.

From JIMWLK to "CSS"

So what is the evolution of an observable that depends on x and not just on Y?

It's straightforward: when evolving past $Y_c = \ln 1/x$ the color charge of soft gluons should not be added to the existing j. Apart from that, the evolution of the wave function, and therefore the operator Ω is the same as in JIMWLK.

In H_{RFT} addition of j_{soft} is enforced by the factor $\mathcal{R}$ in the definition of $\tilde{b}$. That's all.

All we have to do is to drop the tilde:

$$\tilde{b} \to b$$

$$H'_{RFT} = -\frac{1}{4\pi}\left[(b_L^\alpha - b_R^\alpha)^\dagger (N_\perp^\dagger N_\perp)_{\alpha\beta}(b_L^\beta - b_R^\beta) + h.c.\right]$$

The "CSS" Hamiltonian

Expand H'_{RFT} in τ and we get

$$H_{CSS} = \frac{g^2}{2\pi}\int_{x,y}\left[j(\mathbf{x})T^a\frac{\delta}{\delta j(\mathbf{x})}\right]\left[\frac{1}{\partial^2}\right](\mathbf{x},\mathbf{y})\left[j(\mathbf{y})T^a\frac{\delta}{\delta j(\mathbf{y})}\right]$$

It's simple: emission of an gluon during the evolution rotates the color charge density: $j(\mathbf{x})T^a\frac{\delta}{\delta j(\mathbf{x})}$ is the operator of such rotation. For "CSS" all that matters is how j changes due to this rotation, and not what the emitted gluon itself does.

Funny that: dilute limit of H'_{RFT} is identical to the dense limit, i.e. expanding to leading order in j gives the same result as expanding to leading order in τ.

Why is this CSS?

Why do we call it CSS?

Consider a dilute projectile.

Suppose we are interested in the TMD at $\zeta > \ln 1/x$. To calculate it we start with initial condition at $Y_0 = \ln 1/x$

$$T(\mathbf{k}; x, \zeta = \ln 1/x) = \int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \langle b_i(\mathbf{x}) b_i(\mathbf{y}) \rangle$$

$$b_i^a(\mathbf{x}) = g \int_{\mathbf{u}} \frac{\partial_i}{\partial^2}(\mathbf{x}-\mathbf{u}) j^a(\mathbf{u})$$

and evolve with H_{CSS} from $Y_0 = \ln 1/x$ to $Y = \zeta$. Acting on the

observable with H_{CSS} yields a homogeneous equation:

$$\frac{\partial T(\mathbf{k}, x, \zeta)}{\partial \zeta} = -\frac{\alpha_s N_c}{\pi} \ln \frac{\Lambda^2}{\mathbf{k}^2} T(\mathbf{k}, x, \zeta)$$

Λ^2 - UV cutoff

UV cutoff?

TMD as defined above has not transverse resolution scale, or rather - the transverse resolution scale is UV cutoff.

This is easily rectified. Define

$$T(\mathbf{k},\mu^2;x,\zeta)=\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\langle b_i^\mu(\mathbf{x})b_i^\mu(\mathbf{y})\rangle$$

$$b_i^\mu(\mathbf{x})\equiv\int_z b_i(\mathbf{x}-\mathbf{z})\gamma(\mathbf{z});\qquad \gamma(0)=1;\ \ \text{and}\ \ \gamma(\mathbf{z})\to 0;\quad \mathbf{z}^2>1/\mu^2$$

A goof choice is e.g.: $\gamma(\mathbf{z})=e^{-\mathbf{z}^2\mu^2}$

Repeat the exercise:

$$\frac{\partial T(\mathbf{k},\mu^2;x,\zeta)}{\partial\zeta}=-\frac{\alpha_s N_c}{\pi}\ln\frac{\mu^2}{\mathbf{k}^2}T(\mathbf{k},\mu^2;x,\zeta)$$

Now this is (one of the) CSS.

Dense projectile

In the dense case our favorite variables are usually the eikonal factors $S(x)$. What does H_{CSS} do to those?

The action is not trivial: simple rotation of j^a is a rather complicated nonlinear transformation of S is given by j

$$\partial_i b_i = \frac{i}{g}\partial_i\left(S^\dagger \partial_i S\right) = gj$$

One can use this to express $\delta/\delta j$ in terms of $\mathcal{J}_{R(L)}$ - the right and left rotation operators of the Wilson line.

The final result

$$H_{CSS} = g^2 \mathcal{J}_R \frac{1}{\partial D}\left(D\partial - \partial D\right)\frac{1}{\partial^2}\left(\partial D - D\partial\right)\frac{1}{D\partial}\mathcal{J}_R$$

In JIMWLK expressions like $1/\partial D$ appeared in the intermediate stages, but cancelled in the final result. In CSS they don't.

So we still have a challenge to understand how to deal with those.

Conclusions

So one can evolve observables at fixed x to high energy. But not with H_{JIMWLK} - rather with H_{CSS}.

$$\frac{d}{d\zeta} < O(x,\zeta) >= \int DjO[j]H_{CSS}[j,\frac{\partial}{\partial j}]W_{CSS}[j]$$

H_{CSS} can act either on O or on W_{CSS}. The initial condition for the evolution is taken at $\zeta_0 = \ln 1/x$:

$$O(x,\zeta_0) = O_{\zeta_0}[S]; \qquad W_{CSS}[j,\zeta_0] = W_{JIMWLK}[j,\zeta_0]$$

H_{CSS} is universal in the same sense as JIMWLK- it is applicable not only to TMD, but to any observable that depends on gluonic degrees of freedom at fixed x.

j is simple, S is complicated. Hopefully for some interesting observables the awkward denominators cancel. Have to understand it better.