TMD factorization bridging large and small x

Vladi Skokov (NC State University)

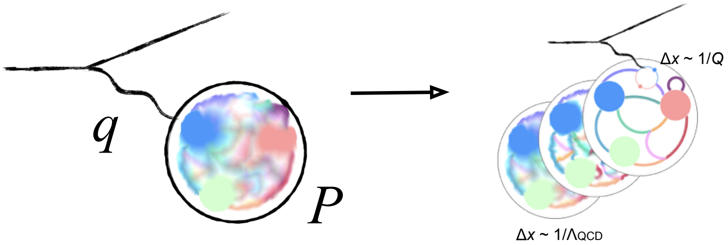
Swagato Mukherjee, V. S., **Andrey Tarasov**, **Shaswat Tiwari**, PRD 109 (2024) 3, 034035; e-Print: 2311.16402

Supported by U.S. DEPARTMENT OF ENERGY

Introduction and motivation

- Initial condition for small-x evolution from lattice QCD
- Small-x evolution (BFKL $\rightarrow$ BK $\rightarrow$ JIMWLK) to obtain various TMD
- TMDPDF from lattice QCD: DGLAP+CSS, no BFKL
- Factorization, IR structure, evolution, nonpertubative TMDPDF are different
- Need for a TMDPDF containing IR of DGLAP and BFKL in the corresponding limits

Factorization

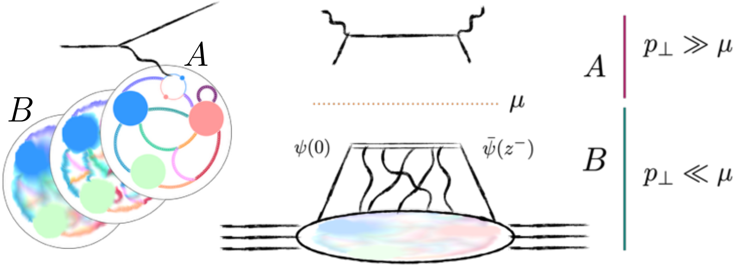


- In interactions, proton displays distinct types of parton dynamics $\rightsquigarrow$ factorization
- Correlation between perturbative and non-perturbative modes $\rightsquigarrow$ computing perturbative mode at high energy scales $\rightsquigarrow$ non-perturbative structure of the proton

$$W^{\mu\nu}(q,P) = \frac{1}{4\pi}\int d^4z e^{iqz}\langle P|j^\mu(z)j^\nu(0)|P\rangle \to W^{\mu\nu}(q,P) = \sum_j \int_x^1 \frac{dz}{z} C_j^{\mu\nu}(Q/\mu, z, x) f_j(z,\mu)$$

Collinear factorization

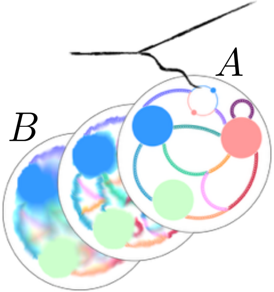
- In DIS, $Q^2 \to \infty$ and fixed x_B. Modes are separated in $k_\perp$.
  Perturbative mode A has $k_\perp \gg \mu$,
  non-perturbative mode B $k_\perp \ll \mu$.

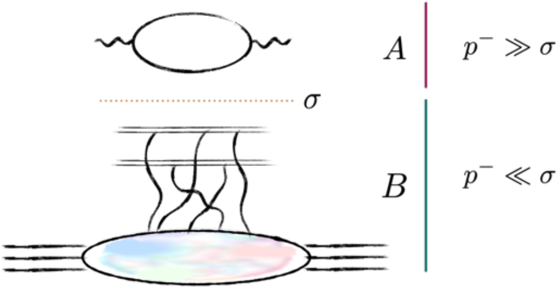


$$W^{\mu\nu}(q,P) = \sum_j \int_x^1 \frac{dz}{z} \underbrace{C_j^{\mu\nu}(Q/\mu, z, x)}_{A} \underbrace{f_j(z,\mu)}_{B}$$

High-energy rapidity factorization

- High energy rapidity factorization, fixed Q^2 and $x_B \to 0$. Modes are separated in long. momentum.
  Perturbative mode A has $p^- \gg \sigma$,
  non-perturbative mode B $p^- \ll \sigma$.





$$W^{\mu\nu} \propto \int \bar{d}^2 p_\perp \underbrace{I^{\mu\nu}(p_\perp)}_{A} \underbrace{\langle P|\mathrm{Tr}\{U(p_\perp)U^\dagger(-p_\perp)\}|P\rangle}_{B}$$

- What is the relation between different factorization schemes?

- How to transition between them?

- How to construct a factorization scheme for a wider kinematic region?

Not only theoretical but also an important phenomenological questions for EIC

Transverse-momentum dependent factorization

- TMD factorization is applied to analysis of a final state with the transverse momentum $q_\perp$, such that $q_\perp^2 \ll Q^2$

- Successfully applied to Drell-Yan, W/Z boson production, SIDIS etc

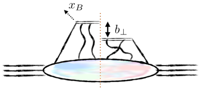
- Color singlet state from unpolarized hadron-hadron scattering:

$$\frac{d\sigma}{dQdyd^2q_\perp} = \sum_{ij} \underbrace{H_{ij}(Q,\mu)}_{\text{hard function}} \int d^2b_\perp e^{iq_\perp b_\perp} f_i(x_a, b_\perp, \mu, \zeta_a) \underbrace{f_j(x_b, b_\perp, \mu, \zeta_b)}_{\text{TMDPDF}} + O\left(\frac{q_\perp^2}{Q^2}\right)$$

Transverse-momentum dependent factorization: collinear limit

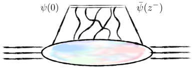
- TMD distribution

$$\underbrace{\Phi_{ij}^{\text{TMD}}(x,\mathbf{b}) = \int \frac{dz}{2\pi} e^{-ixz(pn)} \langle P|\bar{q}_j(zn+\mathbf{b})[zn+\mathbf{b},\pm\infty n+\mathbf{b}][\pm\infty n,0]q_i(0)|P\rangle}_{\rightsquigarrow f(x_B,b_\perp,\mu,\xi)}$$



- collinear distribution

$$\underbrace{\Phi_{ij}^{\text{col.}}(x) = \int \frac{dz}{2\pi} e^{-ixz(pn)} \langle P|\bar{q}_j(zn)[zn,\pm\infty n][\pm\infty n,0]q_i(0)|P\rangle}_{\rightsquigarrow f(x_B,\mu)}$$



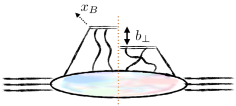
$$\text{TMDPDF}(x,b_\perp,\mu,\zeta) = C_1(\mu,\zeta) \otimes \underbrace{\text{PDF}(x,\mu)}_{\textbf{Twist 2}} + b_\perp^2\, C_2(\mu,\zeta) \otimes \underbrace{\text{PDF}_3(x,\mu) + \ldots\ldots\ldots\ldots}_{\textbf{Higher twist content}}$$

- Only for small $b_\perp$
- TMD region of interest is for large $b_\perp$

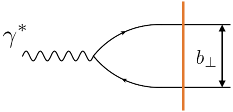
- Usual practice is to introduce non-perturbative function for extrapolation

Transverse-momentum dependent factorization: dipole limit

- TMD distribution



- Small x dipole amplitude

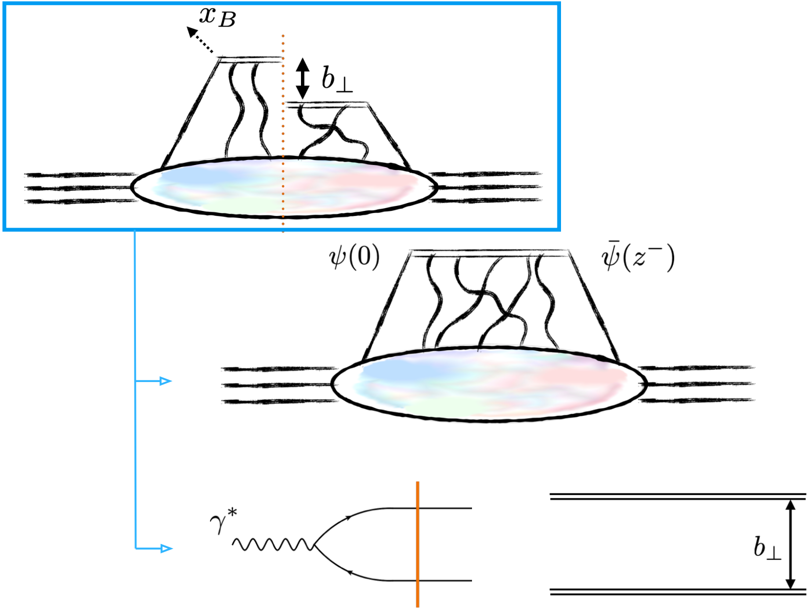


} Wilson lines separated by $b_\perp$

F. Dominguez, C. Marquet, B.W. Xiao, and F. Yuan (2011),
... ,
G. Beuf's talk on Monday

$$\text{TMDPDF}(x_B, b_\perp, \mu, \zeta) = \underbrace{\tilde{C}_1 \otimes \text{WW}(b_\perp, \zeta)}_{\textbf{BFKL logs}} + x \underbrace{\tilde{C}_2 \otimes D_2(b_\perp, \zeta) + \ldots\ldots.}_{\textbf{Sub-eikonal corrections}}$$

A factorization scheme describing a smooth transition is needed.



Background field

- Consider a matrix element of an arbitrary operator

$$\langle P_1|\mathcal{V}(\hat{C})|P_2\rangle = \int \mathcal{D}C\ \Psi^*_{P_1}(\mathbf{C}(t_f))\mathcal{V}(C)\Psi_{P_2}(\mathbf{C}(t_i))e^{iS_{QCD}(C)}$$

- Decompose fields into different components and integrate over them separately

$$C_\mu \to \underbrace{A_\mu}_{\text{quantum fields}} + \underbrace{B_\mu}_{\text{background fields}}$$

- In most generic bg. field method, the separation is arbitrary; here the separation scale(s) = factorization scale(s) σ
- Integrating quantum modes

$$\langle P_1|\mathcal{V}(\hat{C})|P_2\rangle = \int \mathcal{D}B\ \Psi^*_{P_1}(\mathbf{B}(t_f))\tilde{\mathcal{V}}(B,\sigma)\ \underbrace{\Psi_{P_2}(\mathbf{B}(t_i))}_{\text{func. of b.g. field}}\ e^{iS_{QCD}(B)}$$

with

$$\boxed{\tilde{\mathcal{V}}(B,\sigma) = \int \mathcal{D}A\ \mathcal{V}(A+B)e^{iS_{bQCD}(A,B)}}$$

$$S_{bQCD}(A,B) = S_{QCD}(A+B) - S_{QCD}(B)$$

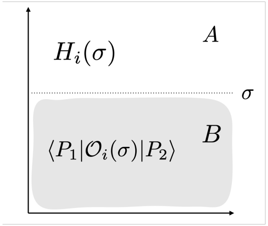
Background field functional and factorization

- Perturbative integration over quantum fields

$$\tilde{\mathcal{V}}(B,\sigma) = \int \mathcal{D}A\ \mathcal{V}(A+B) e^{iS_{bQCD}(A,B)} \rightarrow \sum_i H_i(\sigma) \otimes \mathcal{O}_i(B,\sigma)$$

- Thus, the matrix element

$$\langle P_1|\mathcal{V}(\hat{C})|P_2\rangle = \int \mathcal{D}B\ \Psi^*_{P_1}(\mathbf{B}(t_f))\tilde{\mathcal{V}}(B,\sigma)\Psi_{P_2}(\mathbf{B}(t_i)) e^{iS_{QCD}(B)} \rightarrow \boxed{\sum_i H_i(\sigma) \otimes \langle P_1|\mathcal{O}_i(\sigma)|P_2\rangle}$$



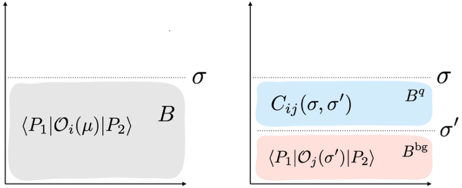
- $\langle P_1|\mathcal{O}_i(\sigma)|P_2\rangle$ defines distribution functions

Background field method and QCD evolution

- Matrix elements depend on factorization scale(s)

- To study this dependence/obtain evolution equations, decompose $B \to B^q + B^{bg}$ and integrate over B^q:

$$\langle P_1|\mathcal{O}_i(\sigma)|P_2\rangle = \sum_j C_{ij}(\sigma,\sigma') \otimes \langle P_1|\mathcal{O}_j(\sigma')|P_2\rangle$$



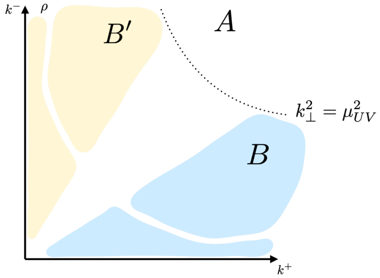
- Evolution equations follow from $\partial_\sigma C_{ij}$

Factorization

- Specific decomposition of modes $C_\mu \to A_\mu + B_\mu$
- Transverse scale $\mu_{\rm UV}$ separates A from b.g. field modes
- Integrating over A:

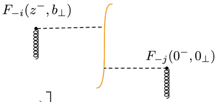
$$\langle P_1|\mathcal{V}(\hat{C})|P_2\rangle = \sum_i H_i(\mu_{UV}^2) \otimes \langle P_1|\mathcal{O}_i(\mu_{UV}^2)|P_2\rangle$$

- The matrix element can be parameterized in terms of TMD distributions



E.g. for gluons

$$\mathcal{B}_{ij}(x_B, b_\perp) = \int_{-\infty}^{\infty} dz^- e^{-ix_B P^+ z^-} \langle P|F^m_{-i}(z^-, b_\perp)[z^-, \infty]_b^{ma}\}[\infty, 0^-]_0^{an} F^n_{-j}(0^-, 0_\perp)|P\rangle$$



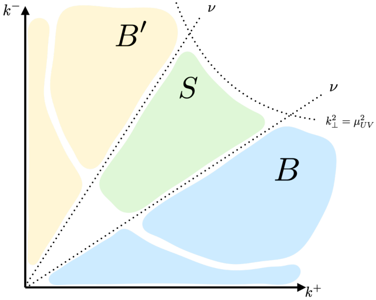
$$[x^-, y^-]_{z_\perp} = \mathcal{P} \exp\left[ig \int_{y^-}^{x^-} dz^- B_-(z^-, z_\perp)\right]$$

Factorization: soft function

- B.g. modes of two hadrons are separated by rapidity
- Possible intersection between the modes is resolved by introducing ν and the soft factor

$$f_{ij}(x_B, b_\perp) = \sqrt{S(b_\perp)}\mathcal{B}_{ij}(x_b, b_\perp)$$

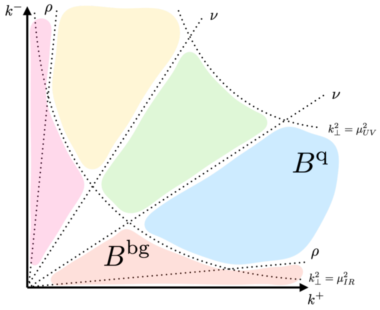
- The pair (ν, $\mu_{\rm UV}$) define TMDPDF



Evolution

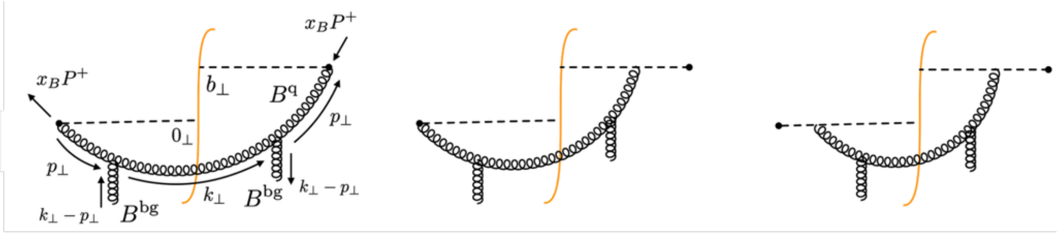
- To study dependence on ν and $\mu_{\rm UV}$ introduce IR scales $\sigma' = (\rho, \mu_{\rm IR})$
- How to separate different field modes?
- Rigid cut-offs are possible but awkward to implement
- We use renormalization-inspired approach. Divergent integrals are regulated with scales that we assign the meaning of cut-offs
- Dim. reg. for transverse integrals
- For rapidity integrals:

$$\int_0^\infty \frac{dk^-}{k^-} \to \nu^\eta \int_0^\infty \frac{dk^-}{k^-} |k^+|^{-\eta}$$



J. Chiu, A. Jain, D. Neill, I. Rothstein, 1202.0814

Real emissions



a b c

Non-zero transverse momentum from B^{bg}; c.f. SCET

$$\mathcal{B}_{ij}^{\mathrm{q}(1)+\mathrm{bg;real}}(x_B, b_\perp) = -4\alpha_s N_c \int \bar{d}^2 p_\perp e^{ip_\perp b_\perp} \int_0^1 \frac{dz}{z(1-z)}$$

$$\times \int \bar{d}^2 k_\perp \Big[\underbrace{\mathcal{R}^a_{ij;lm}(z, p_\perp, k_\perp) + \mathcal{R}^b_{ij;lm}(z, p_\perp, k_\perp)}_{\text{finite due to b.g.}k_\perp} + \underbrace{\mathcal{R}^c_{ij;lm}(k_\perp)}_{\text{divergent in } k^- \ \& \ k_\perp} \Big] \int d^2 z_\perp e^{i(k-p)_\perp z_\perp} \mathcal{B}^{\mathrm{bg}}_{lm}\left(\frac{x_B}{z}, z_\perp\right)$$

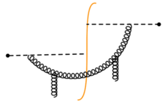
Real emission kernels, $\mathcal{R}$

$$\mathcal{R}^a_{ij;lm}(z,p_\perp,k_\perp) \equiv (1-z)^2\left(\frac{1}{2}\frac{(p+k)_l}{k_\perp^2}\frac{zk_\perp^2\delta_i^k - 2(1-z)k^k k_i}{zk_\perp^2+(1-z)p_\perp^2} - \frac{\delta_l^k p_i + g_{li}k^k}{zk_\perp^2+(1-z)p_\perp^2}\right)$$
$$\times\left(\frac{1}{2}\frac{(p+k)_m}{k_\perp^2}\frac{zk_\perp^2 g_{kj} - 2(1-z)k_k k_j}{zk_\perp^2+(1-z)p_\perp^2} - \frac{g_{mk}p_j + g_{mj}k_k}{zk_\perp^2+(1-z)p_\perp^2}\right)$$

$$\mathcal{R}^b_{ij;lm}(z,p_\perp,k_\perp) \equiv (1-z)\frac{g_{il}}{k_\perp^2}\left(\frac{(p+k)_m}{2}\frac{zk_j+2(1-z)k_j}{zk_\perp^2+(1-z)p_\perp^2} - \frac{k_m p_j - g_{mj}k_\perp^2}{zk_\perp^2+(1-z)p_\perp^2}\right)$$
$$+(1-z)\left(\frac{(p+k)_l}{2}\frac{zk_i+2(1-z)k_i}{zk_\perp^2+(1-z)p_\perp^2} - \frac{k_l p_i - g_{li}k_\perp^2}{zk_\perp^2+(1-z)p_\perp^2}\right)\frac{g_{mj}}{k_\perp^2}$$

$$\mathcal{R}^c_{ij;lm}(k_\perp) \equiv -\frac{g_{il}g_{mj}}{k_\perp^2}$$

Real emissions: divergent term



c

$$\rightarrow \quad 4\alpha_s N_c \boxed{\int \frac{d\!\!{}^-{}^2 k_\perp}{k_\perp^2} e^{ik_\perp b_\perp}} \left(\int_0^1 dz \left[\frac{1}{(1-z)_+} + \frac{1}{z} \right] \mathcal{B}^{\rm bg}_{ij}\left(\frac{x_B}{z}, b_\perp\right) + \boxed{\int_0^1 \frac{dz}{1-z} \mathcal{B}^{\rm bg}_{ij}(x_B, b_\perp)} \right)$$

$$z \equiv \frac{x_B}{x_B + k_\perp^2/2P^+ k^-}$$

- IR transverse momentum divergence: $\boxed{\dfrac{1}{\epsilon_{\rm IR}} + L_b^{\mu \rm IR}}$ with $L_b^\mu \equiv \ln\left(\dfrac{b_\perp^2 \mu^2}{4e^{-2\gamma_E}}\right)$

  For the last term, $\frac{1}{\epsilon_{\rm IR}}$ cancels in combination with virtual term leading to $\frac{1}{\epsilon_{\rm UV}} + L_b^{\mu \rm UV}$

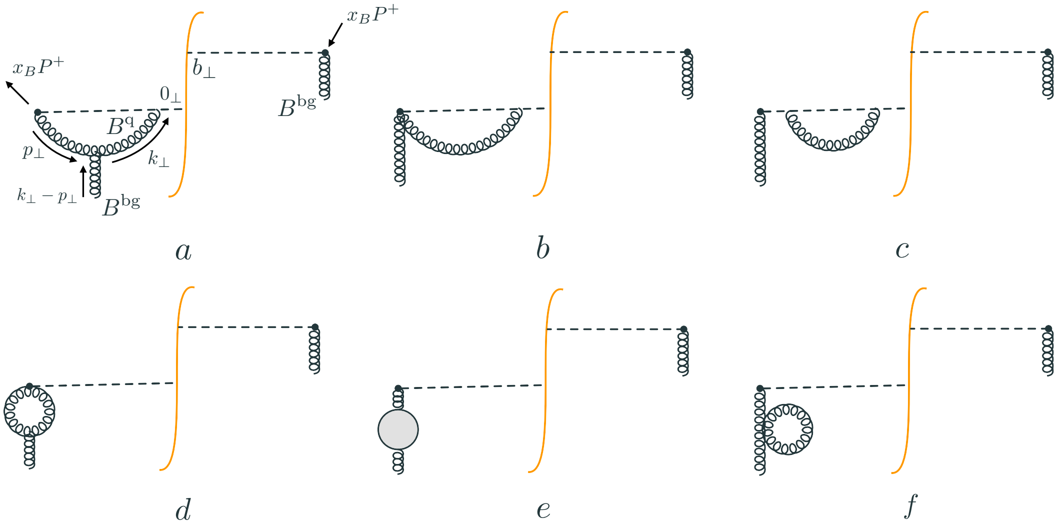
- UV rapidity divergence $\boxed{\dfrac{1}{\eta} + \ln\left(\dfrac{\nu}{x_B P^+}\right)}$

$$\mathcal{B}^{\rm real}_{ij}(x_B, b_\perp) = -\frac{4\alpha_s N_c}{(2\pi)^4} \int d^2 p_\perp e^{ip_\perp b_\perp} \int_0^1 \frac{dz}{z(1-z)} \int d^2 k_\perp \left[\mathcal{R}^a_{ij;lm}(z, p_\perp, k_\perp) + \mathcal{R}^b_{ij;lm}(z, p_\perp, k_\perp) \right]$$

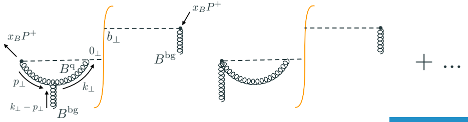
$$\times \int d^2 z_\perp e^{-i(p_\perp - k_\perp) z_\perp} \mathcal{B}_{lm}\left(\frac{x_B}{z}, z_\perp\right) - \frac{\alpha_s N_c}{\pi} \left(\frac{1}{\epsilon_{IR}} + L_b^{\mu IR} \right) \int_0^1 dz \left[\frac{1}{(1-z)_+} + \frac{1}{z} \right] \mathcal{B}_{ij}\left(\frac{x_B}{z}, b_\perp\right)$$

$$+ \frac{\alpha_s N_c}{\pi} \left(\frac{1}{\epsilon_{UV}} + L_b^{\mu UV} \right) \left(\frac{1}{\eta} + \ln\left(\frac{\nu}{x_B P^+}\right) \right) \mathcal{B}_{ij}(x_B, b_\perp)$$

Virtual emissions



Virtual emissions



- collinear matching and SCET calculations: these diagrams = zero in dim. reg. due to negligible transverse momentum transferred from b.g. field

$$\mathcal{B}_{ij}^{\mathrm{q}(1)+\mathrm{bg;virt}}(x_B, b_\perp) = -2\alpha_s N_c \boxed{\int_0^1 \frac{dz}{z}} \int đ^2 p_\perp e^{ip_\perp b_\perp} \int đ^2 k_\perp e^{-ik_\perp b_\perp} \mathcal{V}_{ij;lm}(z, p_\perp - k_\perp, k_\perp)$$

$$\times \int d^2 z_\perp e^{i(k-p)_\perp z_\perp} \mathcal{B}_{lm}^{\mathrm{bg}}(x_B, z_\perp) - \boxed{4\alpha_s N_c \int_0^1 \frac{dz}{1-z} \int \frac{đ^2 k_\perp}{k_\perp^2} \mathcal{B}_{ij}^{\mathrm{bg}}(x_B, b_\perp)}$$

$$\mathcal{B}_{ij}^{\mathrm{q}(1)+\mathrm{bg;virt}}(x_B, b_\perp) = -\frac{\alpha_s N_c}{2\pi} \left(\frac{1}{\epsilon_{\mathrm{IR}}^2} + \frac{1}{\epsilon_{\mathrm{IR}}} \left(\frac{1}{\xi} + \ln(\frac{\rho}{x_B P^+}) \right) - \frac{\pi^2}{12} \right)$$

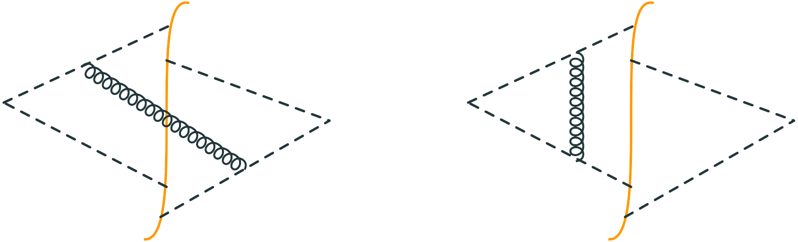
$$\times \int d^2 z_\perp \int đ^2 p_\perp e^{ip_\perp (b-z)_\perp} \left(\frac{\mu_{\mathrm{IR}}^2}{p_\perp^2} \right)^{\epsilon_{\mathrm{IR}}} \frac{g_{il} p_j p_m + p_i p_l g_{mj}}{p_\perp^2} \mathcal{B}_{lm}^{\mathrm{bg}}(x_B, z_\perp)$$

$$+ \frac{\alpha_s N_c}{2\pi} \left(\frac{1}{\epsilon_{\mathrm{UV}}} \frac{\beta_0}{2N_c} + \frac{67}{18} - \frac{5N_f}{9N_c} \right) \int d^2 z_\perp \int đ^2 p_\perp e^{ip_\perp (b-z)_\perp} \left(\frac{\mu_{\mathrm{UV}}^2}{p_\perp^2} \right)^{\epsilon_{\mathrm{UV}}} \mathcal{B}_{ij}^{\mathrm{bg}}(x_B, z_\perp)$$

Virtual emission kernel

$$\mathcal{V}_{ij;lm}(z, l_\perp, k_\perp) \equiv \frac{g_{il}(2l_j k_m - l_m k_j) + (2l_i k_l - l_l k_i) g_{mj}}{k_\perp^2 (z k_\perp^2 + (1-z)(l+k)_\perp^2)}$$

Soft factor



$$\mathcal{S}(b_\perp) = \frac{1}{N_c^2 - 1}\langle 0|\mathrm{Tr}[S_{\bar{n}}^\dagger(b_\perp)S_n(b_\perp)S_n^\dagger(0_\perp)S_{\bar{n}}(0_\perp)]|0\rangle$$

- Regulating UV singularities at g^2 order

$$\mathcal{S}^{(1)}(b_\perp) = \frac{\alpha_s N_c}{2\pi}\left(\frac{2}{\epsilon_{\mathrm{UV}}^2} + 4(\frac{1}{\epsilon_{\mathrm{UV}}} + L_b)\left(-\frac{1}{\eta} + \ln\frac{\mu}{\nu}\right) - L_b^2 - \frac{\pi^2}{6}\right)$$

- Full matrix element

$$f_{ij}(x_B, b_\perp) = \sqrt{\mathcal{S}(b_\perp)}\mathcal{B}_{ij}(x_B, b_\perp)$$

Renormalization

- UV rapidity singularity $(1/\eta)$ cancels in combination of soft and beam functions
- $1/\epsilon_{\mathrm{UV}}$ is to be removed by the universal UV renormalization factor

$$Z_{UV} = 1 - \frac{\alpha_s N_c}{2\pi}\left[\frac{1}{\epsilon_{UV}^2} + \frac{1}{\epsilon_{UV}}\ln\left(\frac{\mu_{UV}^2}{(x_B P^+)^2}\right)\right]$$

- Infrared singularities $1/\rho$ & $1/\epsilon_{\mathrm{IR}}$ are absorbed into matrix elements of b.g. fields

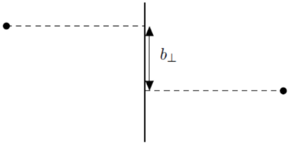
Full result

The full result contains CSS and parts of DGLAP and BFKL

$$
\begin{aligned}
f_{ij}(x_B, b_\perp, \mu_{\rm UV}^2, \zeta) &= f_{ij}(x_B, b_\perp, \mu_{\rm IR}^2, \rho) - 4\alpha_s N_c \int đ^2 p_\perp e^{ip_\perp b_\perp} \int_0^1 \frac{dz}{z(1-z)} \int đ^2 k_\perp \Big[\mathcal{R}^a_{ij;lm}(z, p_\perp, k_\perp) \\
&+ \mathcal{R}^b_{ij;lm}(z, p_\perp, k_\perp)\Big] \int d^2 z_\perp e^{-i(p_\perp - k_\perp)z_\perp} f_{lm}\Big(\frac{x_B}{z}, z_\perp, \mu_{\rm IR}^2, \rho\Big) + \boxed{\frac{\alpha_s N_c}{2\pi}\Big(-\frac{1}{2}(L_b^{\mu_{\rm UV}})^2 + L_b^{\mu_{\rm UV}} \ln\frac{\mu_{\rm UV}^2}{\zeta^2} - \frac{\pi^2}{12}\Big)} \\
&\times f_{ij}(x_B, b_\perp, \mu_{\rm IR}^2, \rho) - \boxed{\frac{\alpha_s N_c}{\pi} L_b^{\mu_{\rm IR}} \int_0^1 dz \Big[\frac{1}{(1-z)_+} + \frac{1}{z}\Big]} f_{ij}\Big(\frac{x_B}{z}, b_\perp, \mu_{\rm IR}^2, \rho\Big) \\
&- \frac{\alpha_s N_c}{2\pi} \int d^2 z_\perp \int đ^2 p_\perp e^{ip_\perp (b-z)_\perp} \Big(\frac{1}{2}\ln^2\frac{\mu_{\rm IR}^2}{p_\perp^2} + \boxed{\ln\frac{\mu_{\rm IR}^2}{p_\perp^2}\ln\frac{\rho}{\zeta}} - \frac{\pi^2}{12}\Big) \frac{g_{il}p_j p_m + p_i p_l g_{mj}}{p_\perp^2} f_{lm}(x_B, z_\perp, \mu_{\rm IR}^2, \rho) \\
&+ \frac{\alpha_s N_c}{2\pi} \int d^2 z_\perp \int đ^2 p_\perp e^{ip_\perp (b-z)_\perp} \Big(\frac{\beta_0}{2N_c}\ln\frac{\mu_{\rm UV}^2}{p_\perp^2} + \frac{67}{18} - \frac{5N_f}{9N_c}\Big) f_{ij}(x_B, z_\perp, \mu_{\rm IR}^2, \rho) + O(\alpha_s^2)\,.
\end{aligned}
$$

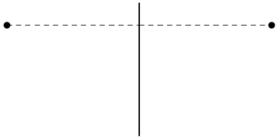
$$
\zeta = x_B P^+, \quad L_b^\mu = \ln\frac{b_\perp^2 \mu^2}{4e^{-2\gamma_E}}
$$

Collinear matching



$$b_\perp \to 0$$

$$\longrightarrow$$



$$\mathcal{B}^{\rm bg}_{ij}(x_B, p_\perp) \to (2\pi)^2 \delta^2(p_\perp) \mathcal{B}^{\rm bg}_{ij}(x_B)$$

additional factorization condition $k_{\rm b.g.} \ll k_{\rm q}$

- Convenient to decompose

$$f_{ij}(x_B, b_\perp) = x_B P^+ \Big[-\frac{g_{ij}}{2} f_1(x_B, b_\perp) + \Big(\frac{g_{ij}}{2} + \frac{b_i b_j}{b_\perp^2}\Big) h_1(x_B, b_\perp) \Big]$$

$$f_1(x_B, b_\perp, \mu^2_{\rm UV}, \zeta) = f_1(x_B, 0_\perp, \mu^2_{\rm IR})$$

$$- \boxed{\frac{\alpha_s N_c}{\pi} L_b^{\mu_{\rm IR}} \int_0^1 \frac{dz}{z} P_{gg}(z) f_1(\frac{x_B}{z}, 0_\perp, \mu^2_{\rm IR})} + \boxed{\frac{\alpha_s N_c}{2\pi} \Big(-\frac{1}{2} (L_b^{\mu_{\rm UV}})^2 + L_b^{\mu_{\rm UV}} \ln \frac{\mu^2_{\rm UV}}{\zeta^2} - \frac{\pi^2}{12} \Big) f_1(x_B, 0_\perp, \mu^2_{\rm IR})}$$

DGLAP $P_{gg}(z) = \frac{1}{(1-z)_+} + \frac{1}{z} - 2 + z - z^2$

CSS

- Collinear singularity in previously finite diagrams
- Reduces to standard collinear matching result/SCET (expansion in leading collinear twists)

Small-x limit

- Full results contains all subeikonal orders; expanding to leading order of eikonal expansion

$$\mathcal{B}_{ij}^{\rm bg}(x_B,p_\perp) = \int_{-\infty}^{\infty} dz^- e^{-ix_B P^+ z^-} \int d^2b_\perp e^{-ib_\perp p_\perp} \langle P|F_{-i}^m(z^-,b_\perp)[z^-,\infty]_b^{ma}[\infty,0^-]_0^{an} F_{-j}^n(0^-,0_\perp)|P\rangle^{\rm bg}$$

$$\to \int d^2b_\perp e^{-ib_\perp p_\perp}\langle p|\mathrm{Tr}(U_b\partial_i U_b^\dagger)(U_0\partial_j U_0^\dagger)|p\rangle^{\rm bg} \rightsquigarrow \lim_{x_B\to 0} f_{ij}(x_B,p_\perp) = \frac{p_ip_j}{p_\perp^2}\mathcal{H}_1(p_\perp)$$

- Additional factorization condition: $k_{\rm q}^- \gg k_{\rm b.g.}^-$
- Single log-terms lead to BFKL (all collinear twists)

$$K_{\rm BFKL}(p_\perp,k_\perp) = -\frac{\alpha_s N_c}{\pi}\int d^2b_\perp e^{ik_\perp b_\perp} L_b^{\mu_{\rm IR}} + \frac{\alpha_s N_c}{\pi}(2\pi)^2\delta^2(k_\perp)\ln\frac{\mu_{\rm IR}^2}{p_\perp^2}$$

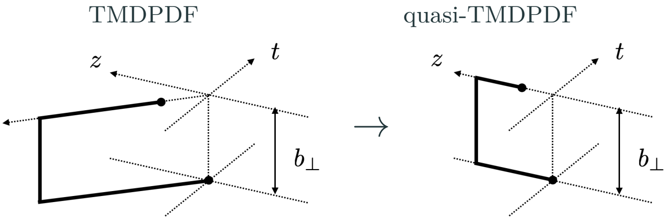
$$f_1(x_B,p_\perp,\mu_{\rm UV}^2,\zeta) = \mathcal{H}_1(p_\perp,\mu_{\rm IR}^2,\rho) + \boxed{\ln\frac{\rho}{\zeta}\int d\!\!\bar{}^2k_\perp K_{\rm BFKL}(p_\perp,k_\perp)\mathcal{H}_1(p_\perp-k_\perp,\mu_{\rm IR}^2,\rho)}$$

$$+\frac{\alpha_s N_c}{2\pi}\int d^2b_\perp\Big(-\frac{1}{2}(L_b^{\mu_{\rm UV}})^2 + L_b^{\mu_{\rm UV}}\ln\frac{\mu_{\rm UV}^2}{\zeta^2} - \frac{\pi^2}{12}\Big)\int d\!\!\bar{}^2k_\perp e^{ik_\perp b_\perp}\mathcal{H}_1(p_\perp-k_\perp,\mu_{\rm IR}^2,\rho)$$

$$-4\alpha_s N_c\int_0^1\frac{dz}{z(1-z)}\int d\!\!\bar{}^2k_\perp\Big[\mathcal{R}^a_{ii;lm}(z,p_\perp,k_\perp)+\mathcal{R}^b_{ii;lm}(z,p_\perp,k_\perp)\Big]\frac{(p-k)_l(p-k)_m}{(p-k)_\perp^2}\mathcal{H}_1^{(0)}(p_\perp-k_\perp,\mu_{\rm IR}^2,\rho)$$

$$+\frac{\alpha_s N_c}{\pi}\Big(\frac{1}{2}\ln^2\frac{\mu_{\rm IR}^2}{p_\perp^2}-\frac{\pi^2}{12}\Big)\mathcal{H}_1(p_\perp,\mu_{\rm IR}^2,\rho) + \frac{\alpha_s N_c}{2\pi}\Big(\frac{\beta_0}{2N_c}\ln\frac{\mu_{\rm UV}^2}{p_\perp^2}+\frac{67}{18}-\frac{5N_f}{9N_c}\Big)\mathcal{H}_1(p_\perp,\mu_{\rm IR}^2,\rho)+\ldots$$

Outlook: lattice QCD

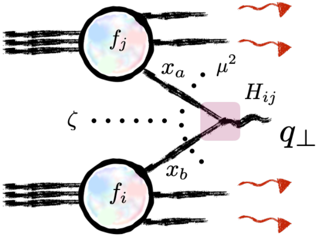


- Perturbatively derivable matching coefficients:

$$f^{\text{quasi}}(x, b_\perp, \mu, \zeta, xP^z) = C(xP^z, \mu) f^{\text{phys.}}(x, b_\perp, \mu, \zeta)$$

- Applied to standard CSS TMDPDF/SCET
- Our aspiration: extension to our scheme $\rightarrow$ initial conditions for small x evolution

Outlook: phenomenology



- TMD factorization

$$\frac{d\sigma}{dQdyd^2q_\perp} = \sum_{ij} H_{ij}(Q,\mu) \int d^2b_\perp e^{iq_\perp b_\perp} f_i(x_a, b_\perp, \mu, \zeta_a) f_j(x_b, b_\perp, \mu, \zeta_b) + O(\frac{q_\perp^2}{Q^2})$$

- Our all twist TMDPDF (all twists) instead of CSS TMDPDFs (leading twists)

Summary

gluon TMDPDF($x_B, b_\perp$) operator

NLO(2-gluon) background ↓ Introducing IR and UV cutoffs

TMDPDF($x_B, b_\perp, \sigma$) valid for all x_B

$x_B \to 0$

$b_\perp \to 0$

Small-x factorization/BFKL

Large-x factorization/CSS