Rapidity-only TMD factorization

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Outline

1. Introduction: TMD factorization from rapidity factorization:
   - Drell-Yan hadronic tensors from TMD factorization.
2. Rapidity-only factorization at the tree level:
   - Rapidity factorization from functional integral
   - Classical fields from retarded propagators at $p_\perp^2/p_\parallel^2 \ll 1$.
   - Full list of power corrections for DY hadronic tensor at $\frac{1}{Q^2}$ leading-N_c level
   - Back-of-the envelope estimates of angular asymmetries for Z-boson production.
3. Rapidity-only TMD factorization and evolution:
   - Rapidity factorization beyond the tree level.
   - Rapidity-only cutoff *vs* UV+rapidity regularization
   - Rapidity evolution of TMDs in the Sudakov region.
   - Argument of coupling constant by BLM.
   - Rapidity-only factorization at one loop.
4. Conclusions and outlook

TMD factorization

TMD factorization formula for particle production in hadron-hadron scattering looks like

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp)\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)C(q, k_\perp)$$
$$+ \text{ power corrections } + \text{ ``Y} - \text{terms''}$$

- $\mathcal{D}_{f/A}(x_A, k_\perp)$ is the TMD density of a parton f in hadron A with fraction of momentum x_A and transverse momentum $k_\perp$,
- $\mathcal{D}_{f/B}(x, q_\perp - k_\perp)$ is a similar quantity for hadron B,
- $C_i(q, k)$ are determined by the cross section $\sigma(f\!f \to \mu^+\mu^-)$ of production of DY pair of invariant mass q^2 in the scattering of two partons.

Examples: Drell-Yan process with Q being the mass of DY pair and Higgs production by gluon-gluon fusion

TMD approach is relevant when the transverse momentum $q_\perp \ll Q$

Long-term project: TMD factorization valid at small and moderate x

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A,k_\perp)\mathcal{D}_{f/B}(x_B,q_\perp-k_\perp)C(q,k_\perp)$$
$$+ \text{ power corrections } + \text{ “Y} - \text{terms}''$$

The quantities $\mathcal{D}_{f/A}(x_A,k_\perp)$, $\mathcal{D}_{f/B}(x_B,q_\perp-k_\perp)$, and $C(q,k_\perp)$ are defined with cutoffs. The dependence on the cutoffs cancels in their product order by order in α_s.

At moderate x_A, x_B: CSS/SCET approach. The TMDs $\mathcal{D}_{f/A}(x_A,k_\perp)$ are defined with a combination of UV and rapidity cutoffs.

At $x_A, x_B \ll 1$: k_T-factorization approach. The TMDs are defined with rapidity-only cutoffs.

It is impossible to extend CSS to small x. (Recently: LO BFKL from SCET)

It *is* possible to study TMD factorization at moderate x using small-x methods (rapidity-only factorization etc.) (A. Tarasov, G. Chirilli, I.B, 2015-2023)

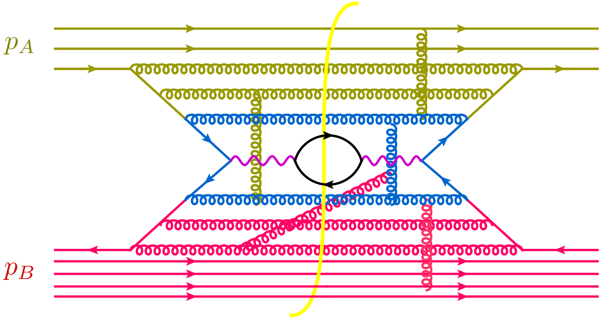
Example: full list of power corrections $\sim \frac{1}{Q^2}$ for DY hadronic tensor, see below.

They are not obtained (yet?) by CSS or SCET

TMD factorization from rapidity factorization (A. Tarasov and I.B.)

Sudakov variables:

$$p \;=\; \alpha p_1 + \beta p_2 + p_\perp, \qquad p_1 \simeq p_A, \;\; p_2 \simeq p_B, \;\; p_1^2 = p_2^2 = 0$$



"Projectile″ fields : $|\beta| < \sigma_p$

"Central" fields

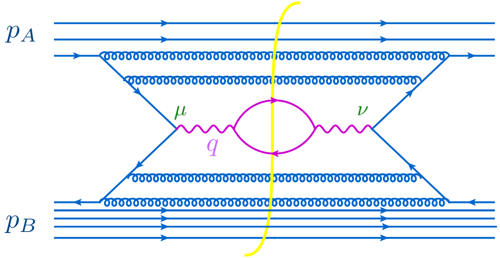
"Target" fields : $|\alpha| < \sigma_t$

The result of the integration over "central" fields in the background of projectile and target fields is a series of TMD operators made from projectile (or target) fields multiplied by powers of $\frac{1}{Q^2}$ $\Rightarrow$ power corrections

Classical example: DY hadronic tensor

DY cross section is given by the product of leptonic tensor and hadronic tensor. The hadronic tensor $W_{\mu\nu}$ is defined as

$$W_{\mu\nu}(p_A,p_B,q) \;=\; \frac{1}{(2\pi)^4}\int d^4x\, e^{-iqx}\langle p_A,p_B|J_\mu(x)J_\nu(0)|p_A,p_B\rangle$$



p_A,p_B = hadron momenta, q = the momentum of DY pair, and J_μ is the electromagnetic or Z-boson current.

There are four tensor structures W_T, W_L, W_Δ, $W_{\Delta\Delta}$

TMD representation for W_i

The hadronic tensor in the Sudakov region $q^2 \equiv Q^2 \gg q_\perp^2$ can be studied by TMD factorization. For example, functions W_T and $W_{\Delta\Delta}$ can be represented as

$$W_i = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}^{(i)}(x_A, k_\perp)\mathcal{D}_{f/B}^{(i)}(x_B, q_\perp - k_\perp)C_i(q, k_\perp) + \text{power corrections} + \text{Y} - \text{terms} \tag{1}$$

There is, however, a problem with Eq. (1) for the functions W_L and W_Δ.

W_T and $W_{\Delta\Delta}$ are determined by leading-twist quark TMDs, but W_Δ and W_L start from terms $\sim \frac{q_\perp}{Q}$ and $\sim \frac{q_\perp^2}{Q^2}$ determined by quark-quark-gluon TMDs.

The power corrections $\sim \frac{q_\perp}{Q}$ were found more than two decades ago but there was no calculation of power corrections $\sim \frac{q_\perp^2}{Q^2}$ until recently.

Part 1: power corrections from tree diagrams in background fields

Goal: TMD factorization formula

TMD factorization formula structure :

$$\langle p'_A, p'_B|J(x_1)J(x_2)|p_A,p_B\rangle \;=\; \sum_{\text{TMD operators}} \int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_i(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\ \langle p'_A|\hat{\mathcal{O}}_i^{\sigma_p}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})|p_B\rangle\langle p'_B|\hat{\mathcal{O}}_i^{\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})|p'_B\rangle$$

$q_\perp^2 \ll Q^2 \Rightarrow$ no dynamics in the transverse space (to be demonstrated below)

$\hat{\mathcal{O}}_i^{\sigma_p}$ - "projectile" TMD operators with $\beta < \sigma_p$ cutoff, e.g

$$\mathcal{O}(z_{1-},z_{1_\perp},z_{2-},z_{2_\perp}) \;\equiv\; \bar\psi(z_{1-},z_{1_\perp})[z_{1-},-\infty]_{z_{1_\perp}}\Gamma[-\infty,z_{2+}]_{z_2}\psi(z_{2+},z_{2_\perp})$$

$\hat{\mathcal{O}}_i^{\sigma_p}$ - "target" TMD operators with $\alpha < \sigma_t$ cutoff, e.g

$$\mathcal{O}(z_{1+},z_{1_\perp},z_{2+},z_{2_\perp}) \;\equiv\; \bar\psi(z_{1+},z_{1_\perp})[z_{1+},-\infty]_{z_{1_\perp}}\Gamma[-\infty,z_{2+}]_{z_{2_\perp}}\psi(z_{2+},z_{2_\perp}).$$

Standard notation for straight-line gauge link

$$[x,y] \;\equiv\; \mathrm{P}e^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)} \text{ – gauge link}$$

Convenient notations

$$[x_+,y_+]_{z_\perp} \;\equiv\; [x_+,0_-,z_\perp;y_+,0_-,z_\perp], \quad [x_-,y_-]_{z_\perp} \;\equiv\; [x_-,0_+,z_\perp;y_-,0_+,z_\perp]$$

Means: “double operator expansion”

Intermediate step: double operator expansion

$$\hat{J}(x_1)\hat{J}(x_2) \;=\; \sum_{I,J}\int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_{IJ}(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\; \hat{\mathcal{O}}_I^{\sigma_p}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\hat{\mathcal{O}}_J^{\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})$$

To find relevant operators and coefficients, it is convenient to consider “matrix” elements of the l.h.s. and r.h.s. in suitable background field

Suitable field $\mathbb{A}$: solution of classical YM equations with boundary condition that at the remote past the field is a sum of projectile and target fields

$$\langle\hat{J}(x_1)\hat{J}(x_2)\rangle_{\mathbb{A}} \;=\; \sum_{I,J}\int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_{IJ}(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\; \langle\hat{\mathcal{O}}_I^{\sigma_p}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\hat{\mathcal{O}}_J^{\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})\rangle_{\mathbb{A}}$$

In the tree approximation

$$\langle\hat{\mathcal{O}}_I^{\sigma_p}\hat{\mathcal{O}}_J^{\sigma_t}\rangle_{\mathbb{A}} \;=\; \hat{\mathcal{O}}_I(\mathbb{A})\hat{\mathcal{O}}_J(\mathbb{A})$$

Classical solution $\mathbb{A}$ and $\psi_{\mathbb{A}}$

Solution of classical YM equations

$$\not{P}\psi_{\mathbb{A}} = 0, \quad \mathscr{D}^\nu \mathscr{F}^a_{\mu\nu} = \sum_f g\bar{\psi}_{\mathbb{A}} t^a \gamma_\mu \psi_{\mathbb{A}}$$

Boundary conditions :

$$\mathbb{A}_\mu(x) \overset{x^+ \to -\infty}{=} \bar{A}_\mu(x^-, x_\perp), \quad \psi_{\mathbb{A}}(x) \overset{x^+ \to -\infty}{=} \psi_a(x^-, x_\perp)$$

$$\mathbb{A}_\mu(x) \overset{x^- \to -\infty}{=} \bar{B}_\mu(x^+, x_\perp), \quad \psi_{\mathbb{A}}(x) \overset{x^- \to -\infty}{=} \psi_b(x^+, x_\perp)$$

The projectile and target fields satisfy YM equations

$$(\not{P} + m_f)\psi_a = 0, \quad D^\nu F^a_{\mu\nu} = g\bar{\psi}_a t^a \gamma_\mu \psi_a$$

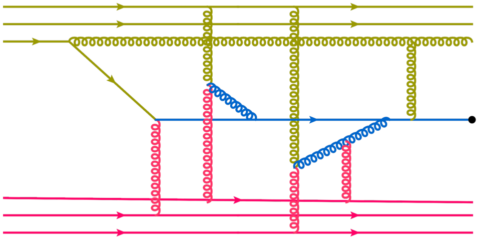
$$(\not{P} + m_f)\psi_b = 0, \quad D^\nu F^a_{\mu\nu} = g\bar{\psi}_b t^a \gamma_\mu \psi_b$$

Method of solution:

- Start with $\psi_A + \psi_B$ and $\bar{A}_\mu + \bar{B}_\mu$ in the gauge $A^+ = 0$, $A^- = 0$
- Correct by computing Feynman diagrams (with retarded propagators) with sources $(\not{P} + m)(\psi_A + \psi_B)$ and $J_\nu = D^\mu F^{\mu\nu}(U + V)$

ψ_C in the tree approximation

It is convenient to choose projectile/target fields as
Projectile fields: $\beta = 0 \Rightarrow A(x^-, x_\perp),\ \psi_A(x^-, x_\perp)$
Target fields: $\alpha = 0 \Rightarrow B(x^+, x_\perp),\ \psi_B(x^-, x_\perp)$



Classical background fields: ψ_C , C_μ
ψ_C = sum of tree diagrams in external $A, \tilde{A}, \psi_A, \tilde{\psi}_A$ and $B, \tilde{B}, \psi_B, \tilde{\psi}_B$ fields with sources
$J_\psi = (\not{P} + m)(\psi_A + \psi_B)$, $\quad J_\nu = D^\mu F^{\mu\nu}(A + B)$
and
$\tilde{J}_\psi = (\not{P} + m)(\tilde{\psi}_A + \tilde{\psi}_B)$, $\quad \tilde{J}_\nu = D^\mu F^{\mu\nu}(\tilde{A} + \tilde{B})$

Classical solution $\equiv \sum$ tree diagrams with retarded propagators

The fields A, ψ and $\tilde{A}, \tilde{\psi}$ do not depend on $x^+ \Rightarrow$
if they coincide at $x^+ = \infty \Rightarrow$ they coincide everywhere.

Similarly,
B, ψ_b and $\tilde{B}, \tilde{\psi}_b$ do not depend on $x^- \Rightarrow$
if they coincide at $x^- = \infty$ they should be equal.

Since $\tilde{A} = A$ and $\tilde{B} = B$ the sources and background fields are the same to the left and to the right of the cut

$\Rightarrow$

ψ_C and C_μ are given by the sum of tree diagrams with *retarded* Green functions

Classical fields in the leading order in $p_\perp^2/p_\parallel^2 \sim q_\perp^2/Q^2$

The solution of such YM equations in general case is yet unsolved problem (goes under the name "glasma" $\Leftrightarrow$ scattering of two "color glass condensates").

Fortunately, for our case of particle production with $\frac{q_\perp}{Q} \ll 1$ we can use this small parameter and construct the approximate solution.

At the tree level transverse momenta are $\sim q_\perp^2$ and longitudinal are $\sim Q^2 \Rightarrow$

$$\psi, A \ = \ \text{series in } \frac{q_\perp}{Q}: \qquad \psi = \psi^{(0)} + \psi^{(1)} + \ldots, \qquad A = A^{(0)} + A^{(1)} + \ldots$$

NB: After the expansion

$$\frac{1}{p^2 + i\epsilon p_0} \ = \ \frac{1}{p_\parallel^2 - p_\perp^2 + i\epsilon p_0} \ = \ \frac{1}{p_\parallel^2} - \frac{1}{p_\parallel^2 + i\epsilon p_0} p_\perp^2 \frac{1}{p_\parallel^2 + i\epsilon p_0} \ + \ldots$$

the dynamics in transverse space is trivial.

Fields are either at the point $x_\perp$ or at the point $0_\perp \Rightarrow$ TMDs

Leading-N_c power corrections

Power corrections are $\sim$ leading twist $\times\left(\frac{q_\perp}{Q} \text{ or } \frac{q_\perp^2}{Q^2}\right) \times \left(1+\frac{1}{N_c}+\frac{1}{N_c^2}\right)$.

NB: almost all $\bar{q}Gq$ TMDs not suppressed by $\frac{1}{N_c}$ are determined by the $\bar{q}q$ TMDs due to QCD equations of motion

Leading twist: $\varrho \equiv \sqrt{s/2}$

$$\frac{1}{8\pi^3 s}\int dx^- d^2x_\perp \, e^{-i\alpha\varrho x^- + i(k,x)_\perp} \langle A|\hat{\bar{\psi}}(x^-,x_\perp) \not{p}_2 \hat{\psi}(0)|A\rangle = f_1(\alpha,k_\perp^2)$$

Power correction:

$$\frac{1}{8\pi^3 s}\int dx^- dx_\perp \, e^{-i\alpha\varrho x^- + i(k,x)_\perp}$$
$$\times \langle A|\hat{\bar{\psi}}(x^-,x_\perp)\hat{\not{A}}(x^-,x_\perp)\not{p}_2\gamma_i\hat{\psi}(0)|A\rangle$$

$$= k_i f_1(\alpha,k_\perp) - \alpha k_i\big[f_\perp(\alpha,k_\perp) + ig^\perp(\alpha,k_\perp)\big],$$

(Mulders & Tangerman, 1996)

Result for $W_{\mu\nu}$ for unpolarized hadrons

Result:

$$W_{\mu\nu}(q) \; = \; W^1_{\mu\nu}(q) + W^2_{\mu\nu}(q) + W^3_{\mu\nu}(q)$$

The first, gauge-invariant, part is a "gauge completion" of leading-twist result

$$\begin{aligned}
W^1_{\mu\nu}(q) \; &= \; W^{1F}_{\mu\nu}(q) + W^{1H}_{\mu\nu}(q), \\
W^{1F}_{\mu\nu}(q) \; &= \; \sum_f e_f^2 W^{fF}_{\mu\nu}(q), \quad W^{fF}_{\mu\nu}(q) \; = \; \frac{1}{N_c} \int d^2k_\perp F^f(q,k_\perp) \mathcal{W}^F_{\mu\nu}(q,k_\perp), \\
W^{1H}_{\mu\nu}(q) \; &= \; \sum_f e_f^2 W^{fH}_{\mu\nu}(q), \quad W^{fH}_{\mu\nu}(q) \; = \; \frac{1}{N_c} \int d^2k_\perp H^f(q,k_\perp) \mathcal{W}^H_{\mu\nu}(q,k_\perp)
\end{aligned}$$

where F^f and H^f are ($\alpha_q \equiv x_A$, $\beta_q \equiv x_B$)

$$\begin{aligned}
F(q,k_\perp) \; &= \; f_1 \alpha_q, k_\perp) \bar{f}_1 \beta_q, (q-k)_\perp) \; + \; f_1 \leftrightarrow \bar{f}_1 \\
H(q,k_\perp) \; &= \; h_1^\perp(\alpha_q, k_\perp) \bar{h}_1^\perp(\beta_q, (q-k)_\perp) \; + \; h_1^\perp \leftrightarrow \bar{h}_1^\perp
\end{aligned}$$

Gauge-invariant structures $\quad q^\mu W^F_{\mu\nu} = q^\mu W^H_{\mu\nu} = 0$

$$\mathcal{W}^F_{\mu\nu}(q,k_\perp) \;=\; \text{LT} + \text{“gauge} - \text{invariant completion}''$$
$$= \; -g^\perp_{\mu\nu} + \frac{1}{Q^2}(q^\parallel_\mu q^\perp_\nu + q^\parallel_\nu q^\perp_\mu) + \frac{q_\perp^2}{Q^4} q^\parallel_\mu q^\parallel_\nu + \frac{\tilde{q}_\mu \tilde{q}_\nu}{Q^2}[q_\perp^2 - 4(k,q-k)_\perp]$$
$$- \Big[\frac{\tilde{q}_\mu}{Q^2}\Big(g^\perp_{\nu i} - \frac{q^\parallel_\nu q_i}{Q^2}\Big)(q-2k)^i_\perp + \mu \leftrightarrow \nu\Big]$$

$$\tilde{q} \equiv \alpha_q p_1 - \beta_q p_2$$

$$m^2 \mathcal{W}^H_{\mu\nu}(q,k_\perp)$$
$$= \; -\big[k^\perp_\mu (q-k)^\perp_\nu + k^\perp_\nu (q-k)^\perp_\mu + g^\perp_{\mu\nu}(k,q-k)_\perp\big] + 2\frac{\tilde{q}_\mu \tilde{q}_\nu - q^\parallel_\mu q^\parallel_\nu}{Q^4} k_\perp^2 (q-k)^2_\perp$$
$$- \Big(\frac{q^\parallel_\mu}{Q^2}\big[k_\perp^2 (q-k)^\perp_\nu + k^\perp_\nu (q-k)^2_\perp\big] + \frac{\tilde{q}_\mu}{Q^2}\big[k_\perp^2 (q-k)^\perp_\nu - k^\perp_\nu (q-k)^2_\perp\big] + \mu \leftrightarrow \nu\Big)$$
$$- \frac{\tilde{q}_\mu \tilde{q}_\nu + q^\parallel_\mu q^\parallel_\nu}{Q^4}\big[q_\perp^2 - 2(k,q-k)_\perp\big](k,q-k)_\perp - \frac{q^\parallel_\mu \tilde{q}_\nu + \tilde{q}_\mu q^\parallel_\nu}{Q^4}(2k-q,q)_\perp (k,q-k)_\perp$$

Second gauge-invariant part

$$
\begin{aligned}
W^2_{\mu\nu}(q) \;=\; & \frac{2}{N_cQ^2}\int d^2k_\perp \\
\times \Big\{ & \Big[\tilde{q}_\mu(q-k)_\nu + \frac{2}{\beta_q s}\tilde{q}_\mu p_{1\nu}(k,q-k)_\perp + \frac{2}{\alpha_q s}\tilde{q}_\mu p_{2\nu}(q-k)^2_\perp \;+\; \mu\leftrightarrow\nu\Big] \\
& \quad \times \Big(\beta_q\{f_1\bar{f}_\perp + \bar{f}_1 f_\perp\} - \alpha_q\{h\bar{h}_1^\perp + \bar{h}h_1^\perp\}\Big) \\
+ & \Big[\tilde{q}_\mu k_\nu^\perp + \frac{2}{s\beta_q}k_\perp^2\tilde{q}_\mu p_{1\nu} + \frac{2}{s\alpha_q}(k,q-k)_\perp\tilde{q}_\mu p_{2\nu} \;+\; \mu\leftrightarrow\nu\Big] \\
& \quad \times \Big(-\alpha_q\{f_\perp\bar{f}_1 + \bar{f}_\perp f_1\} + \beta_q\{h_1^\perp\bar{h} + \bar{h}_1^\perp h\}\Big) \\
+ & \frac{4\tilde{q}_\mu\tilde{q}_\nu}{Q^2}\Big[m^2\Big(\alpha_q^2\{f_3\bar{f}_1 + \bar{f}_3 f_1\} + \beta_q^2\{f_1\bar{f}_3 + \bar{f}_1 f_3\} + \alpha_q\beta_q\big[\{e\bar{e} + \bar{e}e\} + \{h\bar{h} + \bar{h}h\}\big]\Big) \\
& \quad + (k,q-k)_\perp\Big(-\alpha_q\beta_q\big[\{f_\perp\bar{f}_\perp + \bar{f}_\perp f_\perp\} + \{g_\perp\bar{g}_\perp + \bar{g}_\perp g_\perp\}\big] \\
& \quad + \beta_q^2\{h_1^\perp\bar{h}_3^\perp + \bar{h}_1^\perp h_3^\perp\} + \alpha_q^2\{h_3^\perp\bar{h}_1^\perp + \bar{h}_3^\perp h_1^\perp\}\Big)\Big]
\end{aligned}
$$

$f_\perp, f_3, h, h_3^\perp, g_\perp, e$ are the quark-antiquark TMDs of a non-leading twist.

Third part

$$W^3_{\mu\nu}(q) \ = \ \frac{2}{N_c Q^2}\int d^2k_\perp$$
$$\times \Big\{g^\perp_{\mu\nu} \times \big[\text{a bunch of quark–antiquark and quark–antiquark–gluon TMDs}\big]$$
$$+ \big[k^\perp_\mu(q-k)^\perp_\nu + k^\perp_\nu(q-k)^\perp_\mu + g^\perp_{\mu\nu}(k,q-k)_\perp\big] \times \text{same}\Big\}$$

Similarly to LT contribution, is not EM gauge invariant $\Rightarrow$ needs "gauge completion" by $\frac{1}{Q^3}$ and $\frac{1}{Q^4}$ power corrections, for example

$$g^\perp_{\mu\nu} \to \mathcal{W}^F_{\mu\nu}, \quad k^\perp_\mu(q-k)^\perp_\nu + k^\perp_\nu(q-k)^\perp_\mu + g^\perp_{\mu\nu}(k,q-k)_\perp \to \mathcal{W}^H_{\mu\nu}$$

Third part

$$W^3_{\mu\nu}(q) \;=\; \frac{2}{N_c Q^2}\int d^2k_\perp$$
$$\times \Big\{ g^\perp_{\mu\nu} \times \big[\text{a bunch of quark–antiquark and quark–antiquark–gluon TMDs}\big]$$
$$+ \big[k^\perp_\mu (q-k)^\perp_\nu + k^\perp_\nu (q-k)^\perp_\mu + g^\perp_{\mu\nu}(k, q-k)_\perp\big] \times \text{same}\Big\}$$

Similarly to LT contribution, is not EM gauge invariant ⇒ needs "gauge completion" by $\frac{1}{Q^3}$ and $\frac{1}{Q^4}$ power corrections, for example

$$g^\perp_{\mu\nu} \to \mathcal{W}^F_{\mu\nu}, \quad k^\perp_\mu (q-k)^\perp_\nu + k^\perp_\nu (q-k)^\perp_\mu + g^\perp_{\mu\nu}(k, q-k)_\perp \to \mathcal{W}^H_{\mu\nu}$$

NB: basis of operators for $\frac{1}{Q^2}$ corrections ($\varrho \equiv \sqrt{s/2}$)

$$EOM: \quad \begin{aligned} \not{A}_\perp(x)\psi(x) &= -i\not{\partial}_\perp\psi(x) - i\frac{1}{\varrho}\not{p}_1\partial_+\psi(x) - i\frac{1}{\varrho}\not{p}_2 D_-\psi(x) \\ \not{B}_\perp(x)\psi(x) &= -i\not{\partial}_\perp\psi(x) - i\frac{1}{\varrho}\not{p}_2\partial_-\psi(x) - i\frac{1}{\varrho}\not{p}_1 D_+\psi(x) \end{aligned}$$

$-i\frac{1}{\varrho}\not{p}_1 D_\pm\psi(x)$ = my choice, $\not{A}_\perp(x)\psi(x)$ = Vladimirov, Scimemi *et al*

Application: angular coefficients of Z-boson production

In CMS and ATLAS experiments $s = 8$ TeV, $Q = 80 - 100$ GeV and $Q_\perp$ varies from 0 to 120 GeV.

Our analysis is valid at $Q_\perp = 10 - 30$ GeV and $Y \simeq 0$ ($x_A \sim x_B \sim 0.1$) so that power corrections are small but sizable.

Angular distribution of DY leptons in the Collins-Soper frame ($c_\phi \equiv \cos\phi$, $s_\phi \equiv \sin\phi$ etc.)

$$\frac{d\sigma}{dQ^2 dy d\Omega_l} = \frac{3}{16\pi}\frac{d\sigma}{dQ^2 dy}\Big[(1+c_\theta^2) + \frac{A_0}{2}(1-3c_\theta^2) + A_1 s_{2\theta}c_\phi + \frac{A_2}{2}s_\theta^2 c_{2\phi} + A_3 s_\theta c_\phi + A_4 c_\theta + A_5 s_\theta^2 s_{2\phi} + A_6 s_{2\theta}s_\phi + A_7 s_\theta s_\phi\Big]$$

Back-of-the envelope estimation: take only f_1 contribution at large N_c, use "factorization hypothesis" for TMD $f_1(x,k_\perp) \simeq f(x)g(k_\perp)$ and calculate integrals over $k_\perp$ in the leading log approximation using $f_1(x,k_\perp^2) \simeq \frac{f(x)}{k_\perp^2}$

Comparison of A_0 with LHC results

Logarithmic estimate of A_0 (m_z -Z-boson mass, m - proton mass)

$$A_0 \;=\; \frac{Q_\perp^2}{m_z^2}\,\frac{1+2\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}}{1+\frac{Q_\perp^2}{m_z^2}\,\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}} \qquad (*)$$

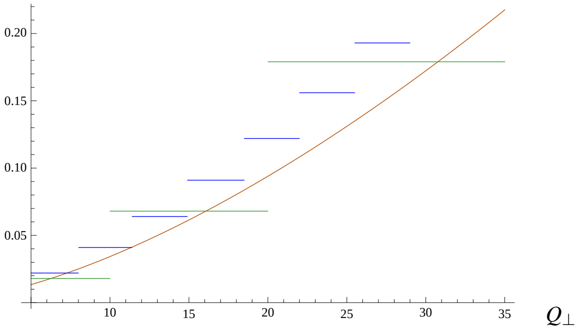


Figure: Comparison of prediction $(*)$ with lines depicting angular coefficient A_0 in bins of $Q_\perp$ and $Y<1$ from CMS (arXiv:1504.03512) and ATLAS (arXiv1606.00689)

Comparison of A_2 with LHC results

Logarithmic estimate of A_2

$$A_2 = \frac{Q_\perp^2}{m_z^2} \frac{1}{1 + \frac{Q_\perp^2}{m_z^2} \frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}} \qquad (**)$$

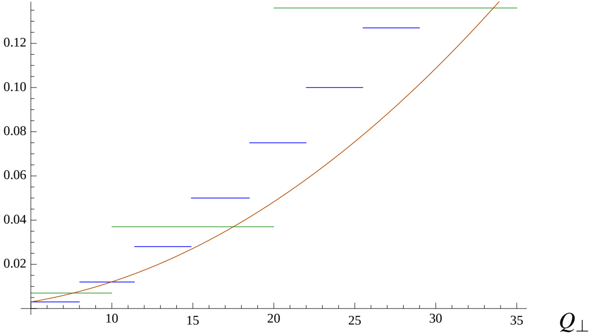


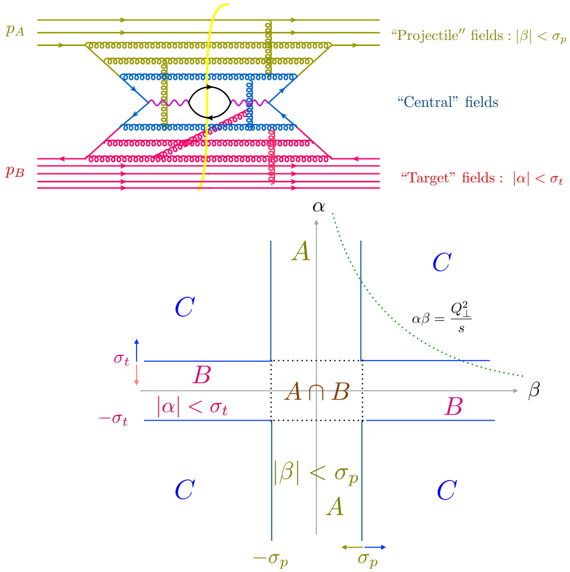
Figure: Comparison of prediction $(**)$ with lines depicting angular coefficient A_2 in bins of $Q_\perp$ and $Y < 1$ from CMS (arXiv:1504.03512) and ATLAS (arXiv1606.00689)

Conclusions to Part 1

1. Power corrections $\sim \frac{1}{Q^2}$ for DY hadronic tensor $\Rightarrow$ "gauge-invariant completion" of the LT result.
2. Bookkeeping: full list of $\frac{1}{Q^2}$ power corrections
3. Back-of-the-envelope estimates of angular distributions for DY Z-boson production are in good agreement with LHC data.

Part 2: logarithms and evolution

Rapidity-only cutoffs and matching of logs



Matching: $\ln \sigma_p$ in the projectile TMDs and $\ln \sigma_t$ in the target TMDs should cancel with $\ln \sigma_p$ and $\ln \sigma_t$ in the coefficient functions.

$A \cap B$, $k_\perp \sim m_\perp$: Glauber gluons
$A \cap B$, $k_\perp \ll m_\perp$: soft gluons

$A \cap B$ gluons $\equiv$ soft/Glauber (sG) gluons cancel out

Rapidity-only cutoff

Typical diagram in the background field $\Psi(\beta_B, p_{B_\perp}) \ = \ \varrho\int dz^+ dz_\perp \ \Psi(z^+, z_\perp) e^{i\varrho\beta_B z^+ - i(p_B, z)_\perp}$

$$\langle [x^+, -\infty]_x \Gamma\psi(y^+, y_\perp)\rangle_\Psi \ = \ g^2 c_F \int d\!\!\!/ \beta_B d\!\!\!/ p_{B_\perp} e^{-ip_B y}\Gamma\Psi(\beta_B, p_{B_\perp})$$

$$\times \int_0^\infty d\!\!\!/ \alpha \int \frac{d\!\!\!/ p_\perp}{p_\perp^2} \frac{\beta_B s e^{-i\frac{p_\perp^2}{\alpha s}\varrho\Delta^+ + i(p,\Delta)}}{\alpha\beta_B s + (p - p_B)_\perp^2 + i\epsilon} \qquad \leftarrow \text{divergent as } \alpha \to \infty$$

$$\langle [x^+, -\infty]_x \Gamma\psi(y^+, y_\perp, -\delta^-)\rangle_\Psi \ = \ g^2 c_F \int d\!\!\!/ \beta_B d\!\!\!/ p_{B_\perp} e^{-ip_B y}\Gamma\Psi(\beta_B, p_{B_\perp}) \qquad \sigma \equiv \frac{1}{\varrho\delta^-}$$

$$\times \int_0^\infty d\!\!\!/ \alpha \int \frac{d\!\!\!/ p_\perp}{p_\perp^2} \frac{\beta_B s e^{-i\frac{p_\perp^2}{\alpha s}\varrho\Delta^+ + i(p, x-y)_\perp}}{\alpha\beta_B s + (p - p_B)_\perp^2 + i\epsilon} e^{-i\frac{\alpha}{\sigma}} \qquad \leftarrow \text{convergent as } \alpha \to \infty$$

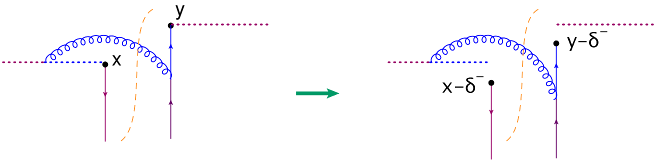


Figure: Point-splitting visualization of “smooth” rapidity-only cutoff.

Rapidity-only cutoff *vs* UV+rapidity regularization

Typical divergent integral ($\varepsilon = \frac{d}{2} - 2$, $đ^n p \equiv \frac{d^n p}{(2\pi)^n}$)

$$-i\mu^{-2\varepsilon}\int đ\alpha đ\beta đp_\perp \frac{1}{\beta - i\epsilon}\frac{1}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \mu^{-2\varepsilon}\int\frac{đp_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^{\beta_B}\frac{đ\beta}{\beta_B}\frac{\beta_B - \beta}{\beta - i\epsilon} = -\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta - i\epsilon}$$

Regularization with $A^-(z^+) \to A^-(z^+)e^{\pm\delta z^+}$

$$-\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta - i\delta} \simeq \frac{1}{8\pi^2}\Big(-\frac{1}{\varepsilon} + \ln\mu^2\frac{x_\perp^2}{4} + \gamma_E\Big)\Big(\ln\frac{\beta_B}{-i\delta} - 1\Big)$$

Rapidity-only cutoff

$$-i\int đ\alpha đ\beta đp_\perp \frac{1}{\beta - i\epsilon}\frac{e^{-i\frac{\alpha}{\sigma}}}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \int\frac{đp_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^\infty đ\alpha\frac{\beta_B s}{\alpha\beta_B s + p_\perp^2}e^{-i\frac{\alpha}{\sigma}} = \frac{1}{16\pi^2}\ln^2\Big(-i\beta_B\sigma s\frac{x_\perp^2}{4}e^{\gamma_E}\Big)$$

Rapidity evolution of TMDs

Quark TMD operator

$$\mathcal{O}(z_{1+}, z_{1_\perp}, z_{2+}, z_{2_\perp}) \equiv \bar{\psi}(z_{1+}, z_{1_\perp})[z_{1+}, -\infty]_{z_1}\Gamma[-\infty, z_{2+}]_{z_2}\psi(z_{2+}, z_{2_\perp})$$

Sudakov regime: $Q^2 \gg Q_\perp^2 \Leftrightarrow z_{12+}z_{12-} \ll z_{12_\perp}^2$

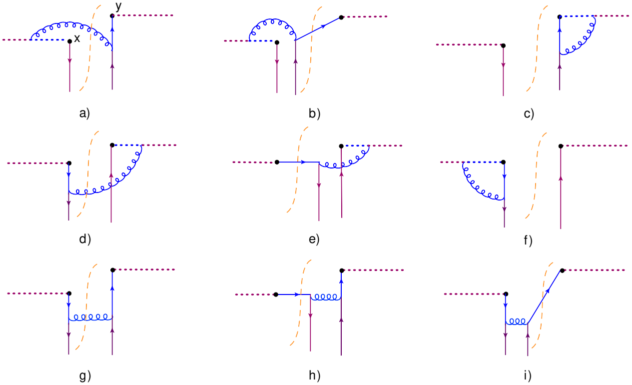


Figure: Diagrams for leading-order rapidity evolution of quark TMD in the Sudakov regime.

Conformally invariant evolution at the LO (G.A. Chirilli & I.B.,2019)

Evolution equation ($\lambda \equiv \sigma(x-y)_\perp^2 \frac{s}{4}$)

$$\lambda\frac{d}{d\lambda}\mathcal{O}(x_+,y_+;\lambda)$$
$$= \Big[\int_{x_+}^\infty dx'_+ \frac{1}{x'_+-y_+}e^{i\frac{\lambda\sqrt{2/s}}{x'_+-y_+}}\mathcal{O}(x_+,y_+;\lambda) - \int_{y_+}^\infty dy'_+ \frac{\mathcal{O}(x_+,y_+;\lambda)-\mathcal{O}(x_+t,y'_+;\lambda)}{y'_+-y_+}$$
$$+\int_{y_+}^\infty dy'_+ \frac{1}{y'_+-x_+}e^{i\frac{\lambda\sqrt{2/s}}{y'_+-x_+}}\mathcal{O}(x_+,y_+;\lambda) - \int_{x_+}^\infty dx'_+ \frac{\mathcal{O}(x_+,y_+;\lambda)-\mathcal{O}(x'_+,y_+;\lambda)}{x'_+-x_+}\Big]$$

If we use rapidity cutoff at $\sigma = \frac{8\varsigma}{|x-y|_\perp\sqrt{s}} \quad \Leftrightarrow \lambda = \varsigma|x-y|\sqrt{s}$,
the solution

$$\mathcal{O}(x_+,y_+;\sigma) = e^{-\frac{\tilde\alpha_s}{2}\left(\ln^2\frac{2(x-y)_\perp^2\varsigma^2}{x_+y_+} - \ln^2\frac{2(x-y)_\perp^2\varsigma_0^2}{x_+y_+}\right)} e^{4\tilde\alpha_s\psi(1)\ln\frac{\varsigma}{\varsigma_0}} \int dx'_+dy'_+\ \mathcal{O}(x'_+,y'_+;\sigma_0)$$
$$\times (x_+y_+)^{-\tilde\alpha_s\ln\frac{\varsigma}{\varsigma_0}}\left[\frac{i\Gamma\left(1-\tilde\alpha_s\ln\frac{\varsigma}{\varsigma_0}\right)}{(x_+-x'_++i\epsilon)^{1-\tilde\alpha_s\ln\frac{\varsigma}{\varsigma_0}}} - \frac{i\Gamma\left(1-\tilde\alpha_s\ln\frac{\varsigma}{\varsigma_0}\right)}{(x_+-x'_+-i\epsilon)^{1-\tilde\alpha_s\ln\frac{\varsigma}{\varsigma_0}}}\right]$$
$$\times\left[\frac{i\Gamma\left(1-\tilde\alpha_s\ln\frac{\varsigma}{\varsigma_0}\right)}{(y_+-y'_++i\epsilon)^{1-\tilde\alpha_s\ln\frac{\varsigma}{\varsigma_0}}} - \frac{i\Gamma\left(1-\tilde\alpha_s\ln\frac{\varsigma}{\varsigma_0}\right)}{(y_+-y'_+-i\epsilon)^{1-\tilde\alpha_s\ln\frac{\varsigma}{\varsigma_0}}}\right]$$

is obviously invariant under the inversion $x_+\to\frac{x_+}{x_+^2}$, $y_+\to\frac{y_+}{y_+^2}$.

Argument of coupling constant by BLM (G.A. Chirilli & I.B.,2022)

A problem with leading-order rapidity evolution: what is the argument of coupling constant?

In CSS approach - no problem, argument is defined by renormgroup

With rapidity-only evolution (BFKL, BK and the like) - argument of α_s may be obtained from the NLO calculations. BLM approach: calculate the small part of the NLO result, namely quark loop contribution to gluon propagator, and promote $-\frac{2}{3}n_f$ to the full $b = \frac{11}{3}N_c - \frac{2}{3}n_f$.

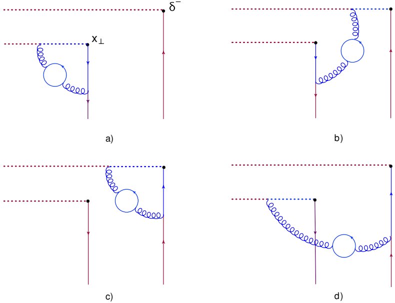
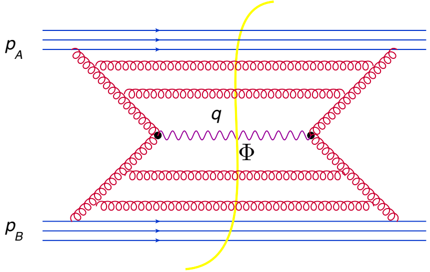


Figure: Quark loop correction to quark TMD evolution

Result: BLM optimal scale is logarithmically halfway between transverse momentum ($b_\perp^{-1/2}$) and energy ($\sigma\beta_B s$) of TMD both for quarks and gluons

Coefficient function for TMD factorization at one loop

Particle production by gluon fusion



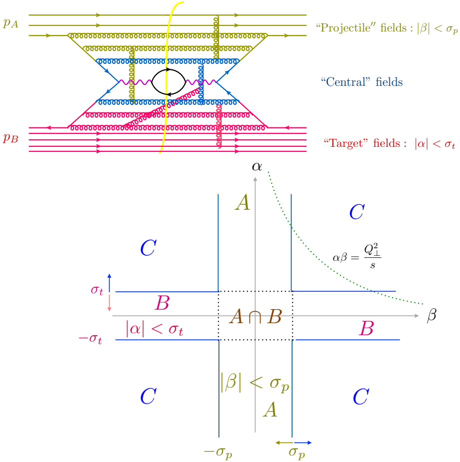
$$s \gtrsim Q^2 \quad \gg \quad Q_\perp^2 \gtrsim m^2$$
$$q^2 \equiv Q^2 = M_\Phi^2, \quad Q_\perp^2 \equiv q_\perp^2$$

Goal: one-loop TMD factorization formula for hadronic tensor.

Result of calculations:

$$W(p_A, p_B; q) \;=\; \int db_\perp \; e^{i(q,b)_\perp} \mathcal{D}_{g/A}(x_A, b_\perp; \sigma_a) \mathcal{D}_{g/B}(x_B, b_\perp; \sigma_b)$$
$$\times \;\exp\Big\{ \frac{\alpha_s N_c}{2\pi} \Big[\ln^2 \frac{b_\perp^2 s \sigma_p \sigma_t}{4} - 2\big(\ln \frac{x_A}{\sigma_t} + \gamma \big)\big(\ln \frac{x_B}{\sigma_p} + \gamma \big) + \frac{\pi^2}{2} \Big] \Big\}$$
$$+ \text{ NLO terms} \sim O\big(\alpha_s^2\big) \;+\; \text{power corrections}$$

Reminder: rapidity factorization of functional integral



Matching: $\ln \sigma_p$ in the projectile TMDs and $\ln \sigma_t$ in the target TMDs should cancel with $\ln \sigma_p$ and $\ln \sigma_t$ in the coefficient functions.

$A \cap B$, $k_\perp \sim m_\perp$: Glauber gluons
$A \cap B$, $k_\perp \ll m_\perp$: soft gluons

$A \cap B$ gluons $\equiv$ soft/Glauber (sG) gluons cancel out

Formal rescaling: $s = \zeta s_0,\ \zeta \to \infty$, $Q_\perp^2$-fixed

$$\text{Rapidity cutoffs}: \ \alpha_a \gg \sigma_t \gg \frac{Q_\perp^2}{\beta_b s} \sim \zeta^{-1},\ \ \beta_b \gg \sigma_p \gg \frac{Q_\perp^2}{\alpha_a s} \sim \zeta^{-1},\ \ \frac{\sigma_p \sigma_t s}{Q_\perp^2} \sim \zeta^{-1/2}$$

Coefficient function in the functional-integral language

After integration over central fields

$$\frac{1}{16}(N_c^2-1)\langle p'_A,p'_B|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle$$
$$= \int d\mathscr{A}\,d\psi_{\mathscr{A}}\ \Psi^*_{p'_A}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p'_B}(t_i)\Psi_{p_B}(t_i)\Big[\mathcal{O}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp}$$
$$+\int dz_1^-dz_{1_\perp}dz_2^-dz_{2_\perp}dw_1^+dw_{1_\perp}dw_2^+dw_{2_\perp}\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,z_{i_\perp},w_i^+,w_{i_\perp};\sigma_p,\sigma_t)$$
$$\times\ \mathcal{O}^{\sigma_p}_{ij}(z_2^-,z_{2_\perp};z_1^-,z_{1_\perp})\mathcal{O}^{ij;\sigma_t}(z_2^+,z_{2_\perp};z_1^+,z_{1_\perp})\ +\ldots\Big]$$

where $\mathscr{A}=A\cup B$, $\Psi_{p_A},\ \Psi_{p_B}$ are proton wave functionals, and

$$\mathcal{O}^{\sigma_p}_{ij}(x^-,x_\perp;y^-,y_\perp)\ \equiv\ g^2F_{+i}(x^-,x_\perp)[x^-,-\infty^-]_{x_\perp}[-\infty^-,y^-]_{y_\perp}F_{+j}(y^-,y_\perp)$$
$$\mathcal{O}^{\sigma_t}_{ij}(x^+,x_\perp;y^+,y_\perp)\ \equiv\ g^2F_{-i}(x^+,x_\perp)[x^+,-\infty^+]_{x_\perp}[-\infty^+,y^+]_{y_\perp}F_{-j}(y^+,y_\perp)$$

are gluon TMD operators

Coefficient function in the functional-integral language

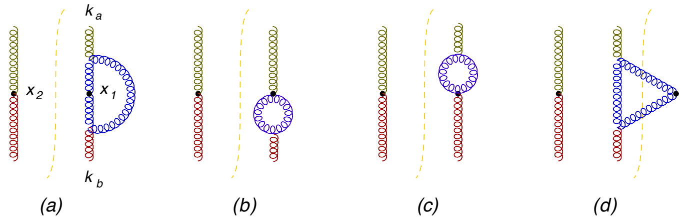
Calculation of coefficient function $\mathfrak{C}_1$ in the background field $\mathbb{A} = \bar{A} + \bar{B} + \bar{C}$

$$\int dz_2^- dz_{2_\perp} dz_1^- dz_{1_\perp} dw_1^+ dw_{1_\perp} dw_2^+ dw_{2_\perp} \frac{\alpha_s N_c}{2\pi} \mathfrak{C}_1(x_1, x_2; z_i^-, z_{i_\perp}, w_i^+, w_{i_\perp}; \sigma_p, \sigma_t)$$
$$\times F^{-i,a}(z_2^+, z_{2_\perp}) F^{-j,a}(z_1^+, z_{1_\perp}) F^{+i,a}(z_2^-, z_{2_\perp}) F^{+j,a}(z_1^-, z_{1_\perp})$$
$$= \frac{N_c^2 - 1}{16} g^4 \langle \tilde{F}^a_{\mu\nu} \tilde{F}^{a\mu\nu}(x_2) F^b_{\lambda\rho} F^{b\lambda\rho}(x_1) \rangle_{\mathbb{A} = \bar{A} + \bar{B}}$$
$$- \langle \hat{\mathcal{O}}^{ij,\sigma_p}(x_2^-, x_{2_\perp}; x_1^-, x_{1_\perp}) \hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp}) \rangle_{\mathbb{A} = \bar{A} + \bar{B}}$$

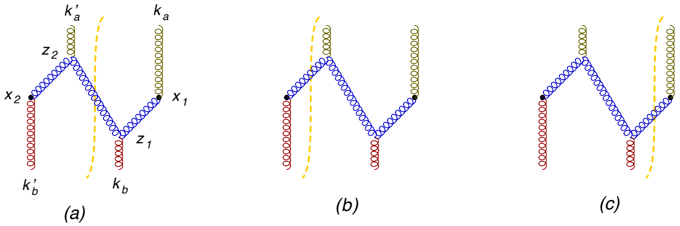
(for the purpose of calculating leading-twist coefficient function the "correction field" C can be neglected)

Diagrams for $\langle \tilde{F}^a_{\mu\nu}\tilde{F}^{a\mu\nu}(x_2)F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}}$ in background fields

"Virtual" diagrams



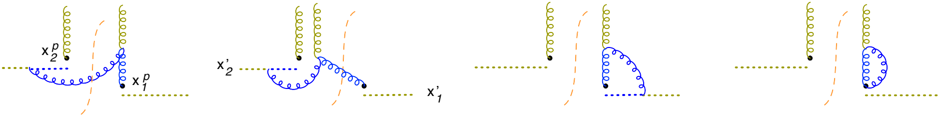
"Real" diagrams



Diagrams for subtracted TMD matrix elements

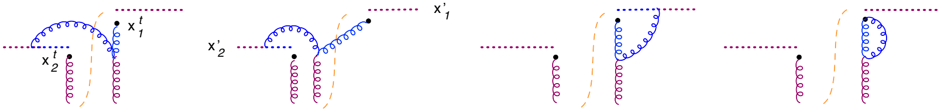
"Projectile" TMD matrix elements.

The $e^{-i\frac{\beta}{\sigma_p}}$ regularization is depicted by point splitting: positions of $F's$ are separated from the beginnings of gauge links. (Violations of gauge invariance are power corrections).



(a) (b) (c) (d)

"Target" TMD matrix elements. The $e^{-i\frac{\alpha}{\sigma_t}}$ regularization is depicted by point splitting.



(a) (b) (c) (d)

$$\begin{aligned}
&\mathcal{W}(x_1,x_2) \;-\; \mathcal{W}^{\rm eik}(x_1,x_2) \\
&= \int \bar{d}\alpha'_a \bar{d}k'_{a_\perp} \bar{d}\beta'_b \bar{d}k_{b_\perp} \bar{d}\alpha_a \bar{d}k'_{a_\perp} \bar{d}\beta_b \bar{d}k'_{b_\perp}\, e^{-i\alpha'_a\varrho x_2^- - i\alpha_a\varrho x_1^-} e^{-i\beta'_b\varrho x_2^+ - i\beta_b\varrho x_1^+} \\
&\quad\times e^{-i(k_a+k_b,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp} F^{+,b}_{i}(\alpha'_a,k'_{a_\perp}) F^{-i,a}(\beta'_b,k'_{b_\perp}) F^{+,b}_{j}(\alpha_a,k_{a_\perp}) F^{-j,a}(\beta_b,k_{b_\perp}) \\
&\quad\times g^2[I - I^{\sigma_p,\sigma_t}_{\rm eik}](\alpha_a,\alpha'_a,\beta_b,\beta'_b,k_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_1,x_2)
\end{aligned}$$

with

$$\begin{aligned}
&[I - I^{\sigma_p,\sigma_t}_{\rm eik}](\alpha'_a,\alpha_a,\beta'_b,\beta_b,k'_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_2,x_1) \\
&\quad= -\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}} \ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}} + \ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4} \\
&\qquad - \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2
\end{aligned}$$

where $(-i\alpha_a) \equiv -i(\alpha_a + i\epsilon)$ etc. Power corrections $\sim \zeta^{-1}$ and $\sim \zeta^{-1/2}$ are neglected.

(Intermediate) Result

$$\begin{aligned}
&\mathcal{W}(x_1,x_2) \; - \; \mathcal{W}^{\rm eik}(x_1,x_2) \\
&= \int đ\alpha'_a đk'_{a_\perp} đ\beta'_b đk_{b_\perp} đ\alpha_a đk'_{a_\perp} đ\beta_b đk'_{b_\perp} e^{-i\alpha'_a \varrho x_2^- - i\alpha_a \varrho x_1^-} e^{-i\beta'_b \varrho x_2^+ - i\beta_b \varrho x_1^+} \\
&\quad \times e^{-i(k_a+k_b,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp} F_i^{+,b}(\alpha'_a, k'_{a_\perp}) F^{-i,a}(\beta'_b, k'_{b_\perp}) F_j^{+,b}(\alpha_a, k_{a_\perp}) F^{-j,a}(\beta_b, k_{b_\perp}) \\
&\quad \times g^2[I - I_{\rm eik}^{\sigma_p,\sigma_t}](\alpha_a, \alpha'_a, \beta_b, \beta'_b, k_{a_\perp}, k'_{a_\perp}, k_{b_\perp}, k'_{b_\perp}, x_1, x_2)
\end{aligned}$$

with

$$\begin{aligned}
&[I - I_{\rm eik}^{\sigma_p,\sigma_t}](\alpha'_a, \alpha_a, \beta'_b, \beta_b, k'_{a_\perp}, k'_{a_\perp}, k_{b_\perp}, k'_{b_\perp}, x_2, x_1) \\
&\qquad = -\ln \frac{(-i\alpha'_a) k'^2_{a_\perp}}{(-i\alpha_a) k'^2_{a_\perp}} \ln \frac{(-i\beta'_b) k'^2_{b_\perp}}{(-i\beta_b) k^2_{b_\perp}} + \ln^2 \frac{x^2_{12_\perp} s\sigma_p\sigma_t}{4} \\
&\qquad\quad - \ln \frac{(-i\alpha'_a) e^\gamma}{\sigma_t} \ln \frac{(-i\beta'_b) e^\gamma}{\sigma_p} - \ln \frac{(-i\alpha_a) e^\gamma}{\sigma_t} \ln \frac{(-i\beta_b) e^\gamma}{\sigma_p} + \pi^2
\end{aligned}$$

where $(-i\alpha_a) \equiv -i(\alpha_a + i\epsilon)$ etc. Power corrections $\sim \zeta^{-1}$ and $\sim \zeta^{-1/2}$ are neglected.

This formula is not yet the final result for the coefficient function. The coefficient function was defined as a result of integration over C-fields with $\alpha > \sigma_t$ and $\beta > \sigma_p$. Since we did not impose these restrictions while calculating the loop integrals, we need to subtract sG contributions (with $\alpha < \sigma_t, \beta < \sigma_p$) to these integrals.

Result for the coefficient function

Result of sG subtraction:

term $-\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}}\ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}}$ disappears ⇒ no dynamics in the transverse plane

$$\begin{aligned}
&\mathcal{W}(x_1,x_2) - \mathcal{W}^{\rm eik}(x_1,x_2) - \mathcal{W}^{\rm sG}(x_1,x_2)\\
&= \int đ\alpha'_a đ\beta'_b đ\alpha_a đ\beta_b e^{-i\alpha'_a\varrho x_2^- - i\alpha_a\varrho x_1^-} e^{-i\beta'_b\varrho x_2^+ - i\beta_b\varrho x_1^+}\\
&\quad\times F_i^{+,b}(\alpha'_a,x_{2_\perp})F^{-i,a}(\beta'_b,x_{2_\perp})F_j^{+,b}(\alpha_a,x_{1_\perp})F^{-j,a}(\beta_b,x_{1_\perp})\\
&\quad\times g^2\mathfrak{C}_1(\alpha'_a,\alpha_a,\beta'_b,\beta_b;x_1,x_2)
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{C}_1(\alpha'_a,\alpha_a,\beta'_b,\beta_b;x_2,x_1) &= I - I_{\rm eik}^{\sigma_p,\sigma_t} - I_{\rm sG}^{\sigma_p,\sigma_t}\\
&= \ln^2\frac{x_{12_\perp}^2 s\sigma_p\sigma_t}{4} - \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2
\end{aligned}$$

The coefficient function in the coordinate space is made of $(+)$ - prescriptions since

$$\int d\alpha\, e^{i\alpha z}\Big[\ln\Big(-i\frac{\alpha}{\sigma}+\epsilon\Big) = \frac{\theta(-z)}{z} + \delta(z)\int_0^{1/\sigma}\frac{dz'}{z'}$$

Result for the coefficient function

Our formula

$$
\begin{aligned}
&\frac{1}{16}(N_c^2-1)\langle p_A',p_B'|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle\\
&=\int\mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p_A'}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p_B'}(t_i)\Psi_{p_B}(t_i)\Big[\mathcal{O}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})\\
&\quad+\int dz_1^-dz_2^-dw_1^+dw_2^+\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)\\
&\quad\times\mathcal{O}^{\sigma_p}_{ij}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})\ +\ O(\alpha_s^2)\Big]
\end{aligned}
$$

is not yet TMD formula since $\mathscr{A}=A\cup B$ and soft/Glauber gluons $sG=A\cap B$ connect "projectile" and "target" gluons.

It is well known that Glauber gluons cancel and soft gluons form soft factors.

With rapidity-only cutoffs, soft factors are power corrections ⇒ one-loop TMD formula

$$
\begin{aligned}
&\frac{1}{16}(N_c^2-1)\langle p_A',p_B'|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle\\
&=\ \langle p_A'|\hat{\mathcal{O}}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})|p_A\rangle\langle p_B'|\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})|p_B\rangle\\
&\quad+\int dz_1^-dz_2^-dw_1^+dw_2^+\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)\\
&\quad\times\ \langle p_A'|\hat{\mathcal{O}}^{\sigma_p}_{ij}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})|p_A\rangle\langle p_B'|\hat{\mathcal{O}}^{ij;\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})|p_B\rangle
\end{aligned}
$$

Matching of coefficient function and TMDs

The solution of TMD evolution equations compatible with this first-order result is

$$\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) \;=\; e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$

$\Rightarrow$ hadronic tensor is

$$W(\alpha'_a, \alpha_a, \beta'_b, \beta_b, x_{1_\perp}, x_{2_\perp}) \;=\; \int đ\alpha'_a đ\alpha_a đ\beta'_b đ\beta_b \; e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$
$$\times \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})|p_B\rangle \;+\; \ldots$$

Reminder

$$\mathfrak{C}_1(\alpha'_a, \alpha_a, \beta'_b, \beta_b; x_1, x_2; \sigma_p, \sigma_t)$$
$$= \;\ln^2\frac{x^2_{12_\perp} s\sigma_p\sigma_t}{4} - \;\ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2$$

Forward case ($\equiv$ particle production by gluon fusion)

$$
\begin{aligned}
W(p_A,p_B;q) &= \int db_\perp\, e^{i(q,b)_\perp} W(p_A,p_B;\alpha_q,\beta_q,b_\perp),\\
W(p_A,p_B;\alpha_q,\beta_q,b_\perp) &= \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\sigma_p}(\alpha_q,b_\perp;p_A)\,\mathcal{G}^{ij;\sigma_t}(\beta_q,b_\perp;p_B)\\
&\times \exp\Big\{\frac{\alpha_s N_c}{2\pi}\Big[\ln^2\frac{b_\perp^2 s\sigma_p\sigma_t}{4} - 2\Big(\ln\frac{\alpha_q}{\sigma_t}+\gamma\Big)\Big(\ln\frac{\beta_q}{\sigma_p}+\gamma\Big)+\frac{\pi^2}{2}\Big]\Big\}\\
&+ \text{NLO terms} \sim O\big(\alpha_s^2\big) \;+\; \text{power corrections} \qquad (2)
\end{aligned}
$$

where $\mathcal{G}_{ij}^{\sigma_p}$, $\mathcal{G}_{ij}^{\sigma_t}$ are gluon TMDs:

$$
\begin{aligned}
\langle p_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(z^-,0^-,b_\perp)|p_A\rangle &= -g^2\varrho^2\int_0^1 du\, u\mathcal{G}_{ij}^{\sigma_p}(u,b_\perp)\cos u\varrho z^-,\\
\langle p_B|\hat{\mathcal{O}}_{ij}^{\sigma_t}(z^-,0^-,b_\perp)|p_B\rangle &= -g^2\varrho^2\int_0^1 du\, u\mathcal{G}_{ij}^{\sigma_t}(u,b_\perp)\cos u\varrho z^-.
\end{aligned}
$$

Matching of coefficient function and TMDs

The r.h.s. of this evolution formula (2) does not depend on cutoffs σ_p and σ_t as long as $\sigma_p \geq \tilde{\sigma}_p = \frac{4b_\perp^{-2}}{\alpha_q s}$ and $\sigma_t \geq \tilde{\sigma}_t \equiv \frac{4b_\perp^{-2}}{\beta_q s}$. Thus, the result of double-log Sudakov evolution reads

$$W(p_A, p_B; \alpha_q, \beta_q, b_\perp) \;=\; \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\tilde{\sigma}_p}(\alpha_q, b_\perp; p_A) \mathcal{G}^{ij;\tilde{\sigma}_t}(\beta_q, b_\perp; p_B)$$
$$\times\; \exp\Big\{-\frac{\alpha_s N_c}{2\pi}\Big[\Big(\ln\frac{Q^2 b_\perp^2}{4} + 2\gamma\Big)^2 - 2\gamma^2 - \frac{\pi^2}{2}\Big]\Big\} +\; O\big(\alpha_s^2\big)\text{ terms } +\text{ power corrections}$$

This result is universal for moderate x and small-x hadronic tensor. The difference lies in the continuation of the evolution beyond Sudakov region.

Double-log Sudakov evolution should stop at $\beta_B \sigma_0 s \simeq b_\perp^{-2}$. After that:

- If $\beta_B \equiv x_B \sim 1$ - DGLAP-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\rm fin} = \frac{m_N^2}{s}$: summation of $\big(\alpha_s \ln \frac{b_\perp^{-2}}{m_N^2}\big)^n$
- If $\beta_B \equiv x_B \ll 1$ - BFKL-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\rm fin} = \frac{b_\perp^{-2}}{s}$: summation of $\big(\alpha_s \ln x_B\big)^n$

Conclusions

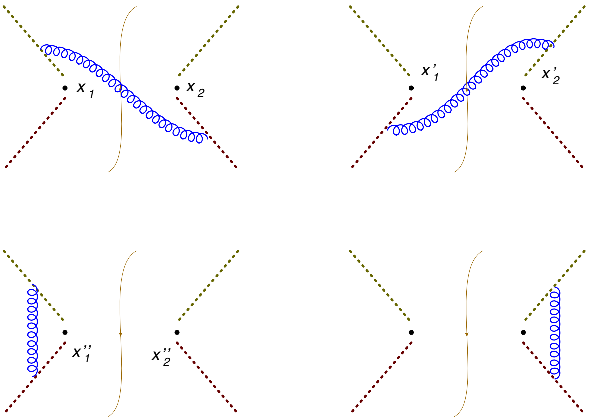
1. Conclusion to part 2: rapidity-only TMD factorization works!
   - The rapidity evolution of quark and gluon TMDs is conformally invariant in the leading order (with proper rapidity-only cutoff).
   - Rapidity-only evolution with BLM prescription for running coupling gives the same universal formula for Sudakov double logs at both small and moderate x for both quark and gluon TMDs.
   - Rapidity factorization at the one-loop level gives Sudakov-type double logs for both small and intermediate x_B

2. Outlook
   - Matching to DGLAP and BFKL/BK evolutions
   - Conformal invariance of rapidity-only factorization

Thank you for attention!

Backup slide: soft factor

Leading-order diagrams



Result of calculation: $\frac{1}{4\pi^2}\mathrm{Li}_2\Big(-\frac{x^2_{12\perp}}{2\delta^+\delta^-}\Big) \sim O\Big(\frac{\Delta^2_\perp}{2\delta^+\delta^-}\Big) \sim O\Big(\frac{\sigma_p\sigma_t s}{Q^2_\perp}\Big) \sim O(\zeta^{-1/2})$

Soft factor with rapidity-only regularization does not have perturbative contributions which can mix with the TMD evolution