Quark TMDs at Small-x

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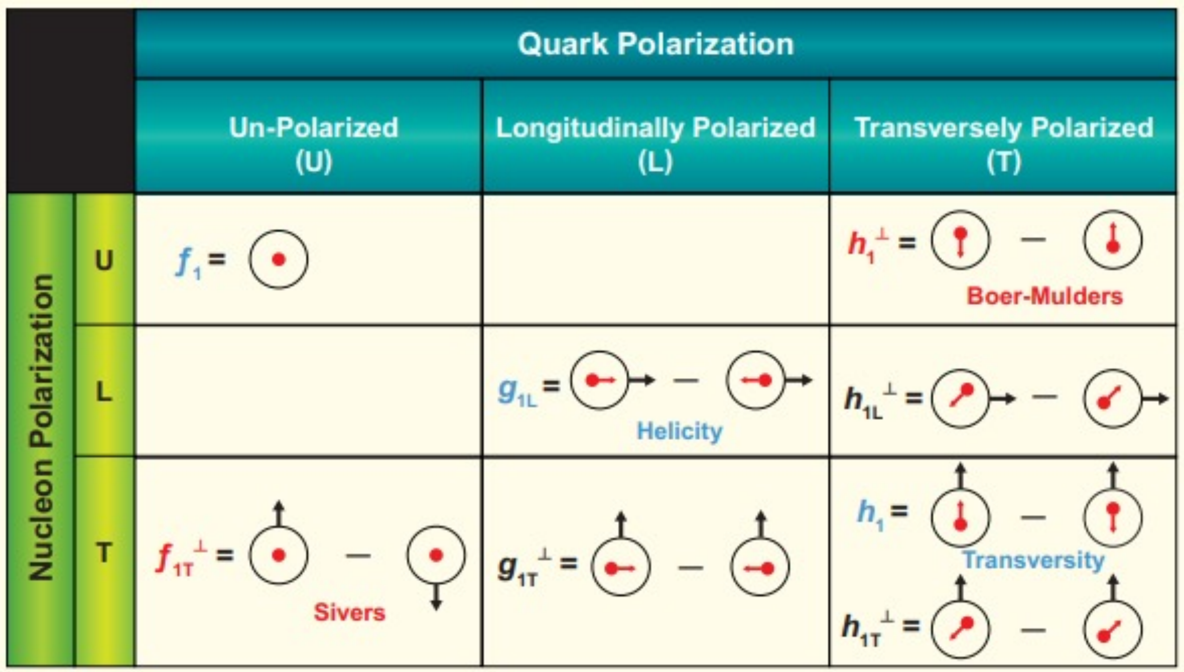
Based largely on work with Yuri V. Kovchegov, Daniel Adamiak, and Yossathorn Tawabutr

Outline

- TMD intro
- Light Cone Operator Treatment (LCOT)
  - From operator definitions to polarized dipole amplitudes
  - Sub-eikonal and sub-sub-eikonal operators
- Evolution and double logs
- LCOT applied to leading-twist quark TMDs at large-N_c with massless quarks
- Summary of results for asymptotic scaling in BFKL regime

TMDs



- The leading-twist quark TMDs give various correlations between the transverse momentum and polarizations of the quarks within a hadron with the polarization of the parent hadron
- Their scale evolution in Q^2 is given by the CSS equations, but the small- x evolution is an ongoing effort

Light Cone Operator Treatment (LCOT)

- Sub-eikonal corrections have been used to develop a framework for studying spin-dependent scattering at small-x over the course of several years
  - Initial calculations for helicity TMDs by Yuri Kovchegov, Daniel Pitonyak and Matthew Sievert in 2016
  - Many advancements and extensions made since with major contributions by Daniel Adamiak, Jeremy Borden, Florian Cougoulic, Ming Li, Brandon Manley, MGS, Andrey Tarasov, Yossathorn Tawabutr
- Started by calculating cross sections, refined to calculate small-x TMDs starting directly from the operator definition
  - We call this formalism the Light Cone Operator Treatment

LCOT for TMDs

- Simplify
  - Rewrite operator definition in small- x limit using shockwave formalism
  - Expand to a given order in eikonality
  - Obtain expression for TMD in terms of 'polarized dipole amplitudes'
- Evolve
  - Calculate small- x gluon/quark emissions in dipole amplitude
  - Take (for example) large- N_c limit to obtain closed equations
- Solve
  - Solve integral equations analytically (if possible) or numerically
  - Plug evolved dipole amplitude back into TMD definition

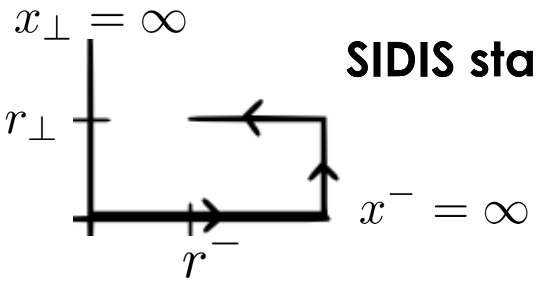
'Staple' Wilson Line becomes a dipole amplitude!

TMDs

- Quark TMDs are defined by the non-local operator product in the hadron state

$$\Phi^{[\Gamma]} = \int \frac{\mathrm{d}r^- \,\mathrm{d}^2 r_\perp}{2\,(2\pi)^3} e^{ik\cdot r} \langle P,S|\bar{\psi}(r)\mathcal{U}[r,0]\Gamma\psi(0)|P,S\rangle$$

$$\mathcal{U}[r,0] = \mathcal{P}\mathrm{exp}\left[ig\int_0^r \mathrm{d}x_\mu\, A^\mu(x)\right]$$



SIDIS staple

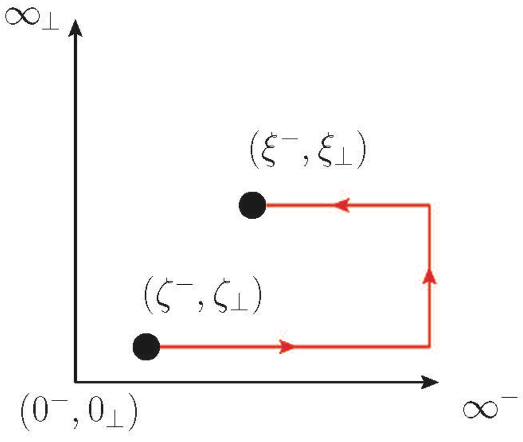
- Linear combinations of different TMDs come from different choices of the Dirac matrix Γ, for example the unintegrated quark density f_1^q and the Sivers function $f_{1T}^{\perp q}$ are given by the taking the matrix to be $\gamma^+/2$

$$f_1^q(x,k_T^2) - \frac{\underline{k}\times\underline{S}_P}{M_P} f_{1\,T}^{\perp\,q}(x,k_T^2) = \int \frac{\mathrm{d}r^- \,\mathrm{d}^2 r_\perp}{2\,(2\pi)^3} e^{ik\cdot r} \langle P,S|\bar{\psi}(r)\mathcal{U}[r,0]\frac{\gamma^+}{2}\psi(0)|P,S\rangle$$

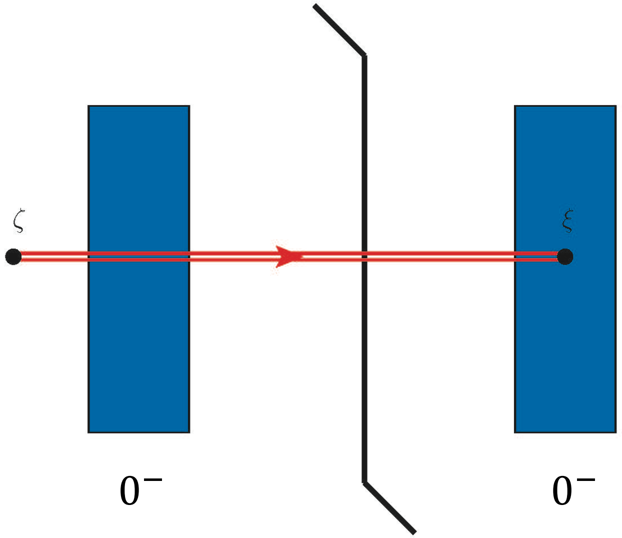
Simplify: Gauge link to dipole amplitude

$$V_x[b^-, a^-] = \mathcal{P} \exp \left[ig \int_{a^-}^{b^-} \mathrm{d}s^- \, A^+(s^-, x_\perp) \right]$$

$$V_x = V_x[\infty, -\infty]$$



Insert a complete set of states

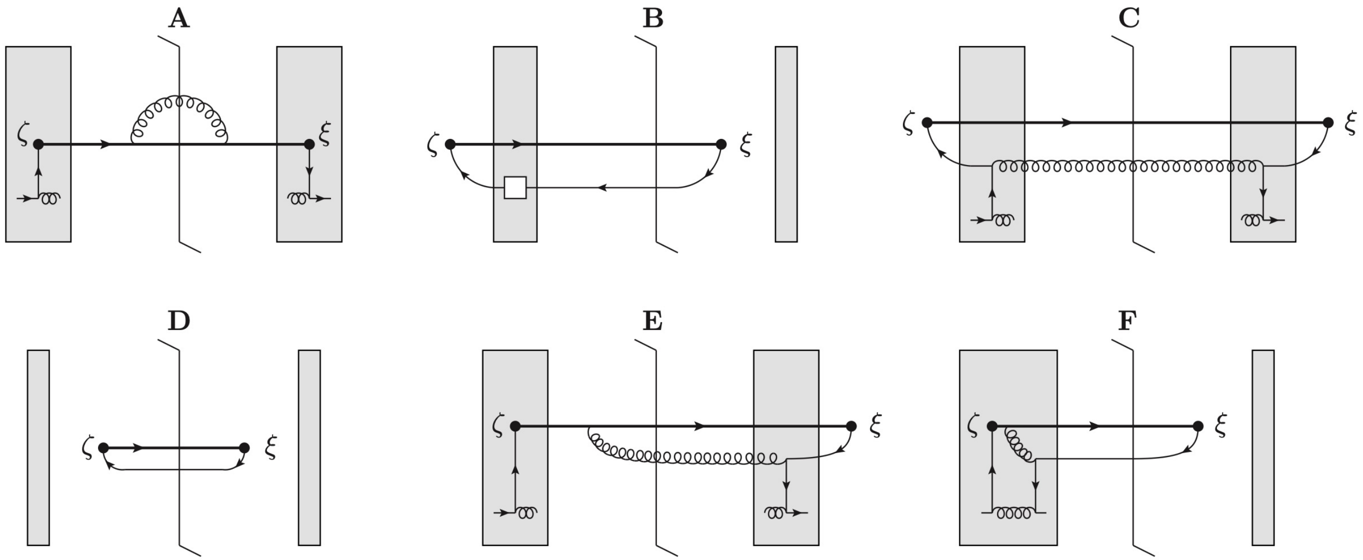


$$\langle P, S | \, \bar{\psi}(\xi) U[\xi, \zeta] \psi(\zeta) \, | P, S \rangle$$

$$\sum_X \langle P, S | \, \mathrm{T} \bar{\psi}(\xi) V_{\underline{\xi}}[\xi^-, \infty] \, | X \rangle \, \langle X | \, \mathrm{T} V_{\underline{\zeta}}[\infty, \zeta^-] \psi(\zeta) \, | P, S \rangle$$

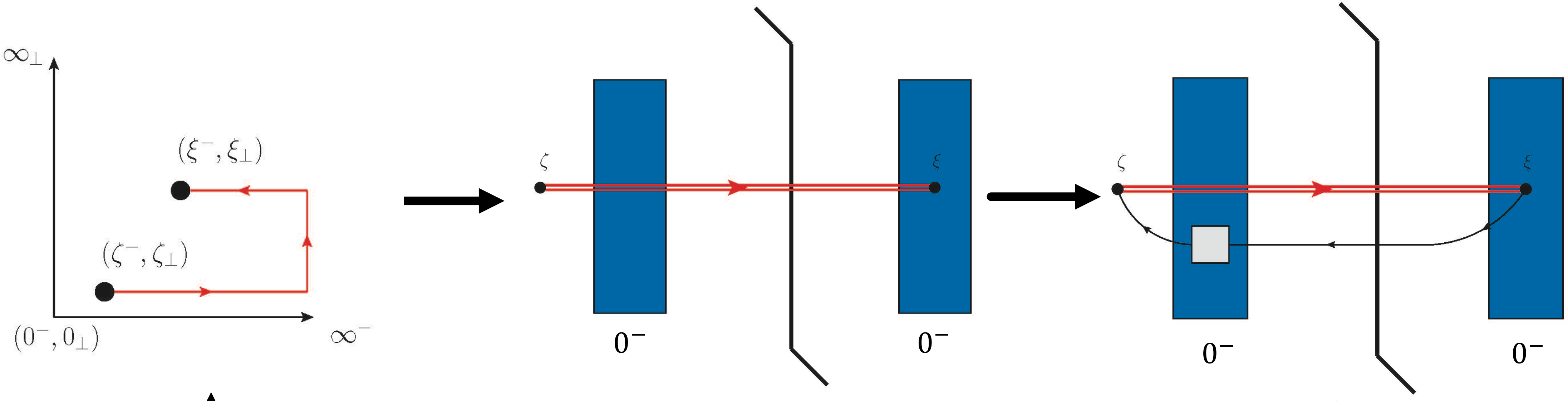
Simplify: Gauge link to dipole amplitude

- We consider the order α_s corrections to the correlator



- A fairly general analysis (Kovchegov and Sievert 2019) shows that only the diagrams in class B give the leading spin-dependent contribution, with the white box denoting a sub-eikonal interaction/operator

Simplify: Gauge link to dipole amplitude



$\langle P, S|\, \bar{\psi}(\xi) U[\xi, \zeta] \psi(\zeta)\, |P, S\rangle$

$\sum_X \langle P, S|\, \mathrm{T}\bar{\psi}(\xi) V_{\underline{\xi}}[\xi^-, \infty]\, |X\rangle \langle X|\, \mathrm{T} V_{\underline{\zeta}}[\infty, \zeta^-] \psi(\zeta)\, |P, S\rangle$

Leading order correction

Simplify: Shock wave picture

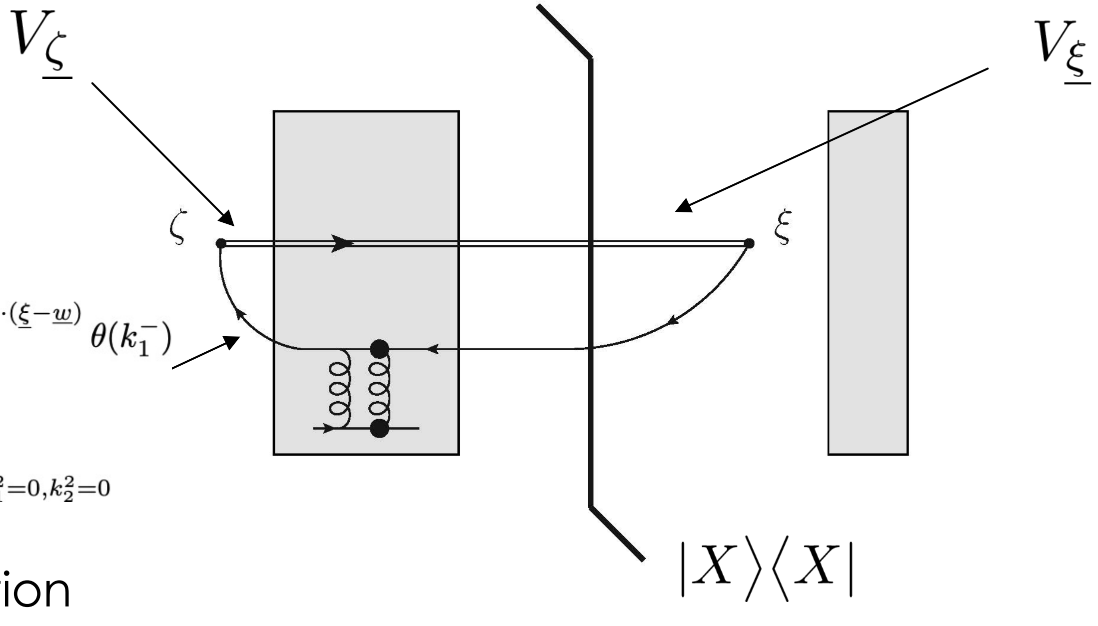
- By inserting a complete set of states, one can write the operator product as a sum over cut diagrams for the scattering of a quark on the shockwave of a target hadron

$$f_1^q(x,k_T^2) - \frac{\underline{k} \times \underline{S}_P}{M_P} f_{1T}^{\perp q}(x,k_T^2) = \frac{2p_1^+}{2(2\pi)^3} \sum_X \int \mathrm{d}\xi^- \, \mathrm{d}^2\xi_\perp \, \mathrm{d}\zeta^- \, \mathrm{d}^2\zeta_\perp \, e^{ik\cdot(\zeta-\xi)} \left[\frac{\gamma^+}{2}\right]_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) V_{\underline{\xi}}[\xi^-,\infty] |X\rangle\langle X| V_{\underline{\zeta}}[\infty,\zeta^-] \psi_\beta(\zeta) \right\rangle$$

Quark propagator through shock wave background

$$\overline{\bar{\psi}^i_\alpha(\xi)\,\psi^j_\beta}(\zeta) = \int d^2w \, \frac{d^2k_1 \, dk_1^-}{(2\pi)^3} \frac{d^2k_2}{(2\pi)^2} \, e^{i\frac{\underline{k}_1^2}{2k_1^-}\zeta^- - i\frac{\underline{k}_2^2}{2k_1^-}\xi^- + i\underline{k}_1\cdot(\underline{w}-\underline{\zeta}) + i\underline{k}_2\cdot(\underline{\xi}-\underline{w})} \, \theta(k_1^-)$$
$$\times \left\{ \left[\frac{\not{k}_1}{2k_1^-}\right] \left[\left(\hat{V}^\dagger_{\underline{w}}\right)^{ji}\right] \left[\frac{\not{k}_2}{2k_1^-}\right] \right\}_{\beta\alpha} \Bigg|_{k_2^-=k_1^-,\,k_1^2=0,\,k_2^2=0}$$

Wilson line with sub-eikonal insertion

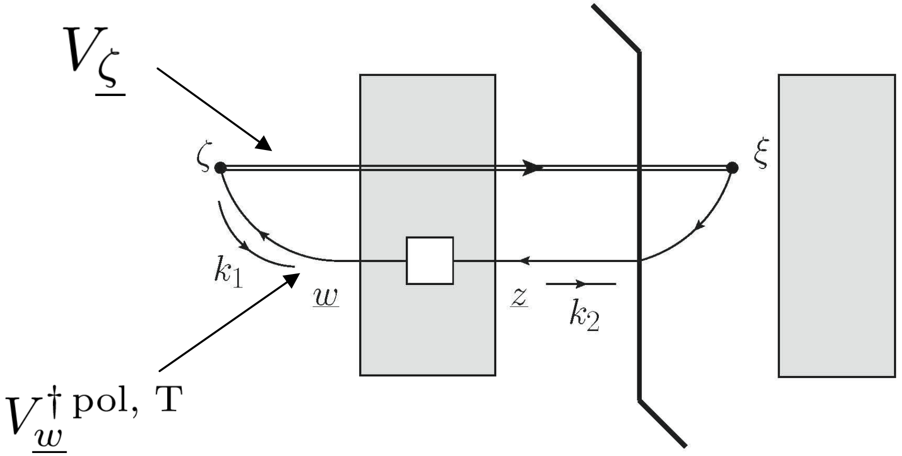


Simplify: Shock wave picture

- Writing the antiquark propagator as a polarized Wilson line lets us write the operator product in terms of a polarized dipole amplitude

$$f_1^q(x,k_T^2) - \frac{\underline{k} \times \underline{S}_P}{M_P} f_{1\,T}^{\perp\,q}(x,k_T^2) = -\frac{2p_1^+}{2(2\pi)^3} \int d^2\zeta_\perp \, d^2w_\perp \, \frac{d^2k_{1\perp}\, dk_1^-}{(2\pi)^3} e^{i(\underline{k}_1+\underline{k})\cdot(\underline{w}-\underline{\zeta})}$$

$$\theta(k_1^-) \frac{1}{(xp_1^+k_1^- + \underline{k}_1^2)(xp_1^+k_1^- + \underline{k}^2)} \sum_{\chi_1,\chi_2} \bar{v}_{\chi_2}(k_2)\frac{\gamma^+}{2} v_{\chi_1}(k_1) \left\langle \mathrm{T}\, V_{\underline{\zeta}}^{ij}\, \bar{v}_{\chi_1}(k_1) V_{\underline{w}}^{\dagger\,\mathrm{pol,\,T}ji} v_{\chi_2}(k_2) \right\rangle$$

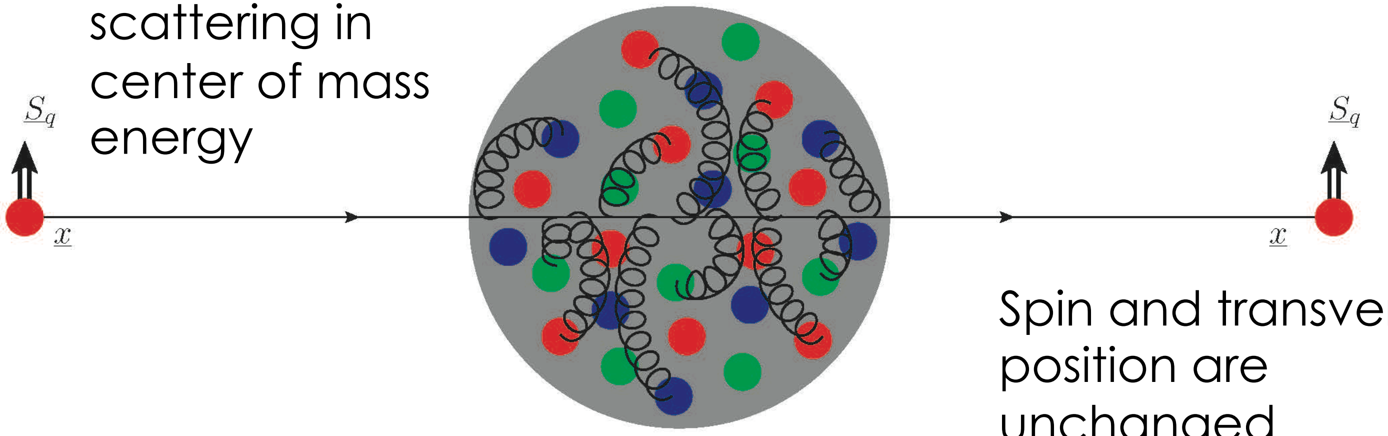


$$\supset \mathrm{tr}\left[V_{\underline{\zeta}} V_{\underline{w}}^{\mathrm{pol,\,T}\dagger}\right]$$

Spin at small-x

- The eikonal approximation only sees the projectile's color charge/representation

Leading order scattering in center of mass energy

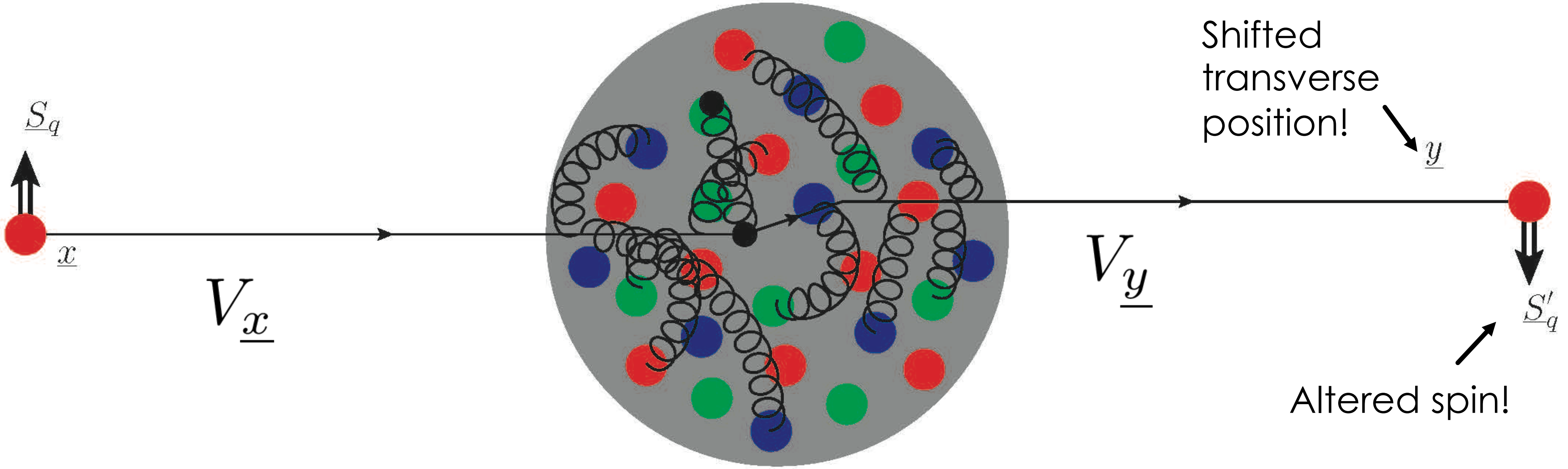


Spin and transverse position are unchanged

- Spin-dependence can only enter in the target background fields – eikonal Sivers function!

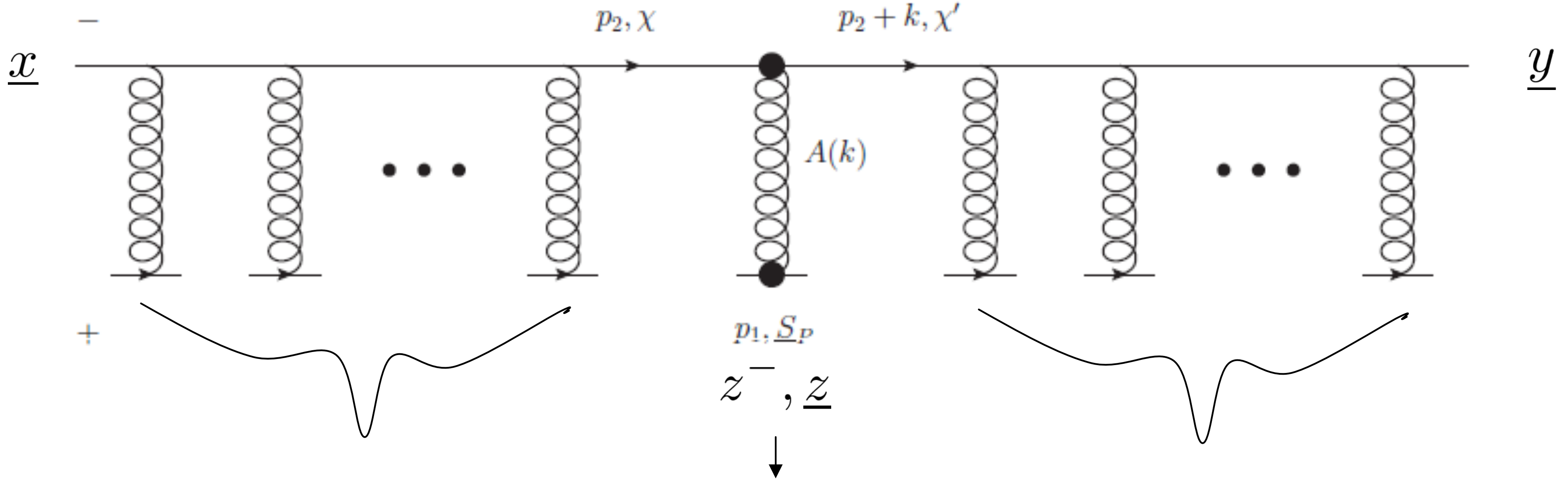
Spin at small-x

- Relaxing the eikonal approximation allows not only for momentum kicks, but also for the transfer of spin information!



Polarized Wilson Line

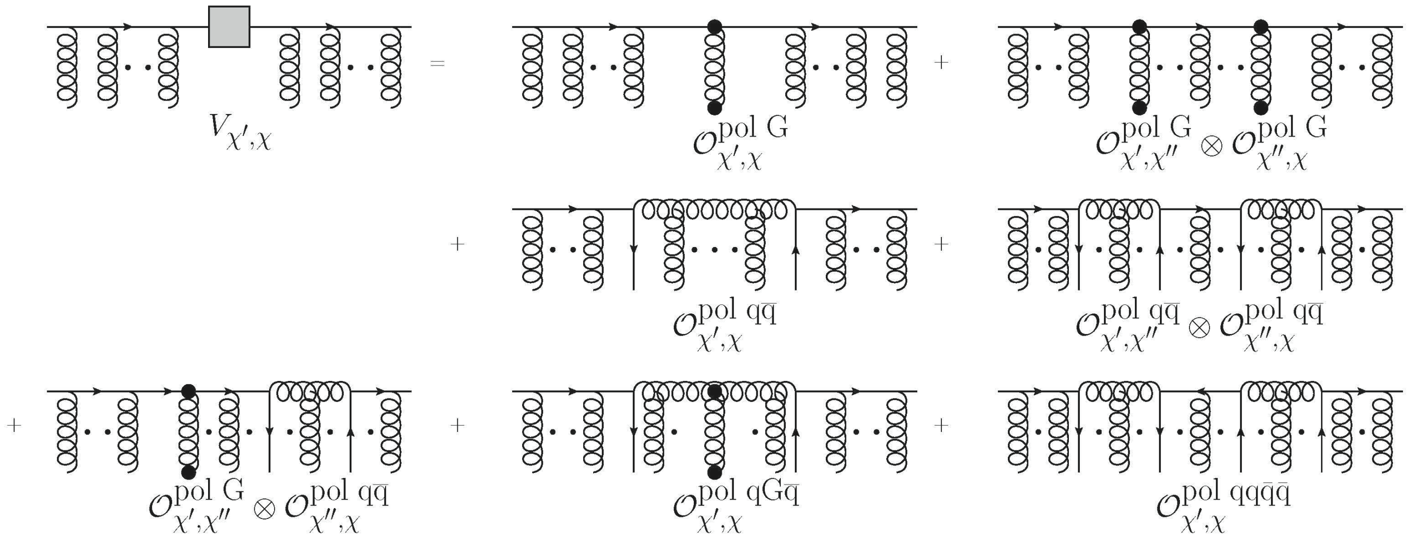
- Dependence on the spin of the quarks in the dipole requires the insertion of sub-eikonal operators in the Wilson lines
- Add all possible operator insertions integrated over x^- positions along Wilson lines



$$\int \mathrm{d}z^- \, \mathrm{d}^2 z_\perp \quad V_{\underline{x}} \quad \mathcal{O}^{\mathrm{pol\ G}}_{\chi,\chi'} \quad V_{\underline{y}}$$

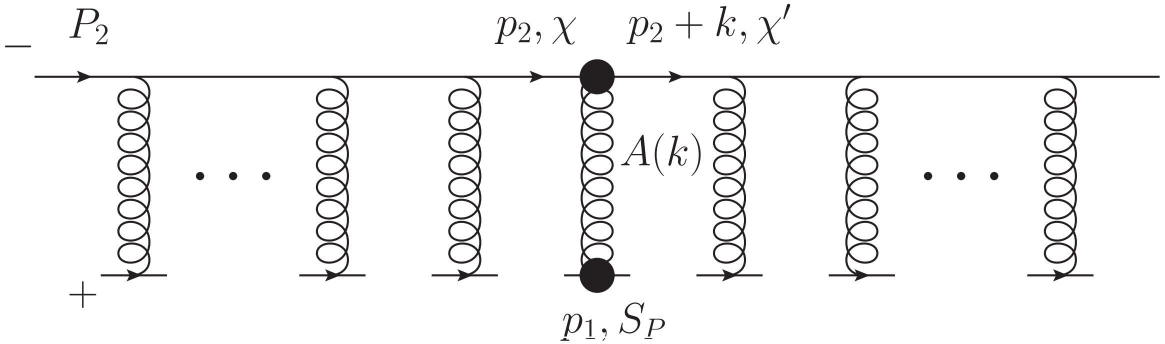
Polarized Wilson line

- It has been shown that to account for longitudinal (Kovchegov, Pitonyak and Sievert 2016) and transverse spin (Kovchegov and Sievert 2019) one needs to include corrections out to sub-sub-eikonal order
- We construct a general sub-sub-eikonal polarized Wilson line by adding all possible operator insertions



Kovchegov, MGS 2021

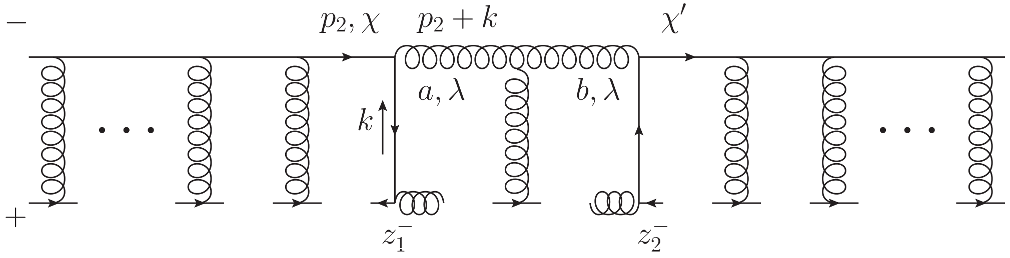
Sub-Eikonal Gluon Exchange



$$\mathcal{O}^{\text{pol G}}_{\chi',\chi}(x^-, \underline{x}) = -i\, \delta_{\chi,\chi'} \left[\overleftarrow{D}^i \frac{1}{2(P_2^- + iD^-)} \vec{D}^i + \frac{m^2}{2(P_2^- + iD^-)} \right]$$

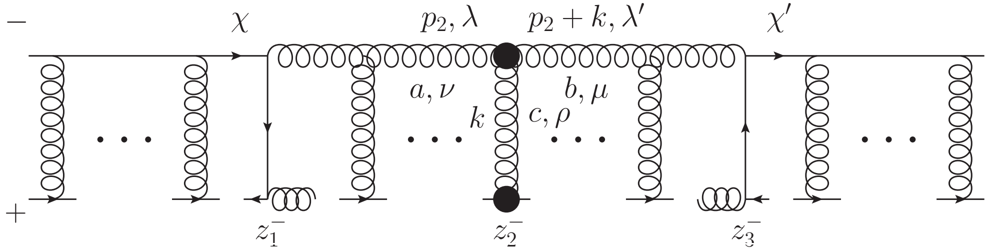
$$+ \frac{ig}{2} \left\{ \delta_{\chi,-\chi'} \left[F^{12} \frac{1}{P_2^- + iD^-} - \frac{i}{(P_2^-)^2} \epsilon^{ij}\, \overleftarrow{D}^i\, F^{-j} \right] + \chi\, \delta_{\chi,\chi'} \frac{m}{(P_2^-)^2}\, \epsilon^{ij} S^i F^{-j} + \chi\, \delta_{\chi,-\chi'} \frac{im}{(P_2^-)^2}\, S^i\, F^{-i} \right\}$$

Sub-Eikonal Quark Exchange



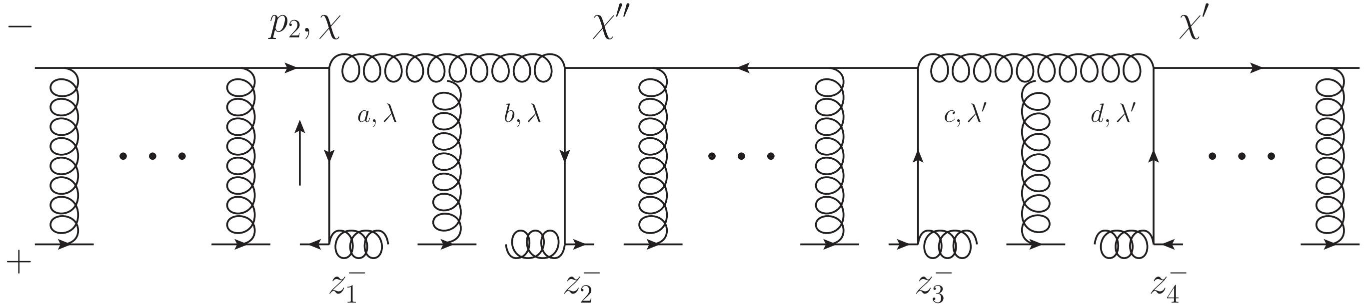
$$
\begin{aligned}
&\mathcal{O}^{\text{pol } q\bar{q}}_{\chi',\chi}(z_2^-, z_1^-; \underline{z}_2, \underline{z}_1) = -\frac{g^2 p_1^+}{8\,s}\, t^b\, \psi_\beta(z_2^-, \underline{z}_2) \Bigg[\delta^{b'b''} - \frac{i\, p_1^+\, \mathscr{D}^{b'b''-}_{\underline{z}_2}}{2s}\Bigg] U^{b''a''}_{\underline{z}_2}[z_2^-, z_1^-]\, \delta^2(\underline{z}_2 - \underline{z}_1) \\
&\times \Bigg[\delta^{a''a'} - \frac{i\, p_1^+\, \mathscr{D}^{a''a'-}_{\underline{z}_1}}{2s}\Bigg] \Bigg\{ \delta_{\chi,\chi'} \Bigg[\gamma^+\, \delta^{a'a}\, \delta^{bb'} - \frac{2mp_1^+}{s}\, \delta^{a'a}\, \delta^{bb'} \\
&\qquad - \frac{p_1^+}{s} \Bigg((\gamma^1 - i\gamma^5\gamma^2) \big[i\underline{S}\cdot \overleftarrow{\underline{\mathscr{D}}}^{bb'}_{z_2} + \gamma^5 \underline{S} \times \overleftarrow{\underline{\mathscr{D}}}^{bb'}_{z_2} \big]\, \delta^{a'a} - (\gamma^1 + i\gamma^5\gamma^2) \big[i\underline{S} \cdot \underline{\mathscr{D}}^{a'a}_{z_1} - \gamma^5 \underline{S} \times \underline{\mathscr{D}}^{a'a}_{z_1} \big]\, \delta^{bb'} \Bigg) \Bigg] \\
&- \delta_{\chi,-\chi'} \Bigg[\gamma^+\, \gamma^5\, \delta^{a'a}\, \delta^{bb'} - \frac{2mp_1^+}{s}\, i\, \gamma^1\, \gamma^2\, \delta^{a'a}\, \delta^{bb'} \\
&\qquad - \frac{p_1^+}{s} \Bigg((i\gamma^2 - \gamma^5\gamma^1) \big[i\underline{S}\cdot \overleftarrow{\underline{\mathscr{D}}}^{bb'}_{z_2} + \gamma^5 \underline{S} \times \overleftarrow{\underline{\mathscr{D}}}^{bb'}_{z_2} \big]\, \delta^{a'a} + (i\gamma^2 + \gamma^5\gamma^1) \big[i\underline{S} \cdot \underline{\mathscr{D}}^{a'a}_{z_1} - \gamma^5 \underline{S} \times \underline{\mathscr{D}}^{a'a}_{z_1} \big]\, \delta^{bb'} \Bigg) \Bigg] \\
&- \chi \delta_{\chi,\chi'} \frac{p_1^+}{s}\, \delta^{a'a}\, \delta^{bb'} \Bigg[\Big[i\gamma^5 \underline{S} \cdot \overleftarrow{\underline{D}}_{z_2} - \underline{S} \times \overleftarrow{\underline{D}}_{z_2} \Big] (1 - i\gamma^5\gamma^1\gamma^2) + \Big[i\gamma^5 \underline{S} \cdot \underline{D}_{z_1} - \underline{S} \times \underline{D}_{z_1} \Big] (1 + i\gamma^5\gamma^1\gamma^2) \Bigg] \\
&+ \chi \delta_{\chi,-\chi'} \frac{p_1^+}{s}\, \delta^{a'a} \delta^{bb'} \Bigg[\Big[i\underline{S} \cdot \overleftarrow{\underline{D}}_{z_2} - \gamma^5 \underline{S} \times \overleftarrow{\underline{D}}_{z_2} \Big] (1 - i\gamma^5\gamma^1\gamma^2) + \Big[i\underline{S} \cdot \underline{D}_{z_1} - \gamma^5 \underline{S} \times \underline{D}_{z_1} \Big] (1 + i\gamma^5\gamma^1\gamma^2) \Bigg] \Bigg\}_{\alpha\beta} \\
&\times \bar{\psi}_\alpha(z_1^-, \underline{z}_1)\, t^a + \mathcal{O}\left(\frac{1}{s^3}\right).
\end{aligned}
$$

Pure Sub-Sub-Eikonal Exchanges



'Interference' of quark and gluon exchanges

$$\mathcal{O}^{\text{pol qG}\bar{\text{q}}}_{\chi',\chi} = \frac{ig^2(p_1^+)^2}{16s^2} t^b\, \psi_\beta(z_3^-, \underline{x})\, U_{\underline{x}}^{b'b}[z_3^-, z_2^-]\, \delta^2(\underline{z}_2 - \underline{x}) \Bigg\{ \delta_{\chi,\chi'} \left[\gamma^+ \underline{\overleftarrow{\mathscr{D}}}_{z_2}^{bc} \cdot \underline{\mathscr{D}}_{z_2}^{ca} + \gamma^+\gamma^5\, 2g\, \left(\mathcal{F}^{12}(z_2)\right)^{ba} \right]$$
$$- \delta_{\chi,-\chi'} \left[\gamma^+\gamma^5\, \underline{\overleftarrow{\mathscr{D}}}_{z_2}^{bc} \cdot \underline{\mathscr{D}}_{z_2}^{ca} + \gamma^+\, 2g\, \left(\mathcal{F}^{12}(z_2)\right)^{ba} \right] \Bigg\}_{\alpha\beta} U_{\underline{y}}^{aa'}[z_2^-, z_1^-]\, \delta^2(\underline{z}_2 - \underline{y})\, \bar{\psi}_\alpha(z_1^-, \underline{y})\, t^{a'}$$



Double quark-antiquark exchange

$$\mathcal{O}^{\text{pol qq}\bar{\text{q}}\bar{\text{q}}}_{\chi',\chi} = -\frac{g^4(p_1^+)^2}{64s^2} t^d \psi_\delta(z_4^-, \underline{x}) U_{\underline{x}}^{dc}[z_4^-, z_3^-] \bar{\psi}_\beta(z_2^-, \underline{x}) t^b V_{\underline{x}}^\dagger[z_3^-, z_2^-] \Big\{ \delta_{\chi,\chi'} \Big[\left(\gamma^+\right)_{\alpha\delta} \left(\gamma^+\right)_{\beta\gamma}$$
$$- \left(\gamma^+\gamma^5\right)_{\alpha\delta} \left(\gamma^+\gamma^5\right)_{\beta\gamma} \Big] + \delta_{\chi,-\chi'} \Big[\left(\gamma^+\right)_{\alpha\delta} \left(\gamma^+\gamma^5\right)_{\beta\gamma} - \left(\gamma^+\gamma^5\right)_{\alpha\delta} \left(\gamma^+\right)_{\beta\gamma} \Big] \Big\}$$
$$\times t^c \psi_\gamma(z_3^-, \underline{x}) U_{\underline{x}}^{ba}[z_2^-, z_1^-] \bar{\psi}_\alpha(z_1^-, \underline{x}) t^a$$

Full Sub-Sub-Eikonal Polarized Wilson Line

$$
\begin{aligned}
V_{\underline{x},\underline{y};\chi',\chi} = {} & V_{\underline{x}}\,\delta^2(\underline{x}-\underline{y})\,\delta_{\chi,\chi'} + \int_{-\infty}^{\infty} \mathrm{d}z^-\, d^2z\; V_{\underline{x}}[\infty,z^-]\,\delta^2(\underline{x}-\underline{z})\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi',\chi}(z^-,\underline{z})\,V_{\underline{y}}[z^-,-\infty]\,\delta^2(\underline{y}-\underline{z}) \\
& + \int_{-\infty}^{\infty} \mathrm{d}z_1^-\, d^2z_1 \int_{z_1^-}^{\infty} \mathrm{d}z_2^-\, d^2z_2 \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_2^-]\,\delta^2(\underline{x}-\underline{z}_2)\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi',\chi''}(z_2^-,\underline{z}_2)\,V_{\underline{z}_1}[z_2^-,z_1^-]\,\delta^2(\underline{z}_2-\underline{z}_1) \\
& \times \mathcal{O}^{\mathrm{pol\ G}}_{\chi'',\chi}(z_1^-,\underline{z}_1)\,V_{\underline{y}}[z_1^-,-\infty]\,\delta^2(\underline{y}-\underline{z}_1) + \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} dz_2^-\; V_{\underline{x}}[\infty,z_2^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi',\chi}(z_2^-,z_1^-;\underline{x},\underline{y})\,V_{\underline{y}}[z_1^-,-\infty] \\
& + \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} \mathrm{d}z_2^- \int_{z_2^-}^{\infty} \mathrm{d}z_3^- \int_{z_3^-}^{\infty} \mathrm{d}z_4^-\, d^2z \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_4^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi',\chi''}(z_4^-,z_3^-;\underline{x},\underline{z})\,V_{\underline{z}}[z_3^-,z_2^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi'',\chi}(z_2^-,z_1^-;\underline{z},\underline{y})\,V_{\underline{y}}[z_1^-,-\infty] \\
& + \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} \mathrm{d}z_2^- \int_{z_2^-}^{\infty} \mathrm{d}z_3^- \int_{z_3^-}^{\infty} \mathrm{d}z_4^-\; V_{\underline{x}}[\infty,z_4^-]\,\mathcal{O}^{\mathrm{pol\ qq\bar{q}\bar{q}}}_{\chi',\chi}(z_4^-,z_3^-,z_2^-,z_1^-;\underline{x})\,V_{\underline{y}}[z_1^-,-\infty]\delta^2(\underline{x}-\underline{y}) \\
& + \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} dz_2^-\, d^2z_2 \int_{z_2^-}^{\infty} dz_3^-\; V_{\underline{x}}[\infty,z_3^-]\,\delta^2(\underline{z}_2-\underline{x})\mathcal{O}^{\mathrm{pol}\quad \mathrm{qG\bar{q}}}_{\chi',\chi}(z_1^-,z_2^-,z_3^-;\underline{x},\underline{z}_2)\delta^2(\underline{z}_2-\underline{y})V_{\underline{y}}[z_1^-,-\infty] \\
& + \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} \mathrm{d}z_2^- \int_{z_2^-}^{\infty} \mathrm{d}z_3^-\, d^2z \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_3^-]\,\delta^2(\underline{x}-\underline{z})\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi',\chi''}(z_3^-;\underline{z})\,V_{\underline{z}}[z_3^-,z_2^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi'',\chi}(z_2^-,z_1^-;\underline{z},\underline{y})\,V_{\underline{y}}[z_1^-,-\infty] \\
& + \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} \mathrm{d}z_2^- \int_{z_2^-}^{\infty} \mathrm{d}z_3^-\, d^2z \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_3^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi',\chi''}(z_3^-,z_2^-;\underline{x},\underline{z})\,V_{\underline{z}}[z_2^-,z_1^-]\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi'',\chi}(z_1^-;\underline{z})\,V_{\underline{y}}[z_1^-,-\infty]\,\delta^2(\underline{y}-\underline{z})
\end{aligned}
$$

cf. Altinoluk et al (2020), Chirilli (2019) full sub-eikonal propagator

Polarized Wilson Lines in TMDs

- The Dirac matrix which projects out a linear combination of TMDs also determines which sub-eikonal operator insertions contribute to the TMDs and generally whether the leading contribution is sub-eikonal or sub-sub-eikonal

$$\bar{v}_{\chi_2}(k_2)\Gamma v_{\chi_1}(k_1)\langle \mathrm{T}\; V_{\underline{x}}^{ij}\bar{v}_{\chi_1}(k_1)V_{\underline{y}}^{\dagger\,\mathrm{pol}\,ji}v_{\chi_2}(k_2)\rangle$$

$$= \Big(a(k_1,k_2)\delta_{\chi_1,\chi_2} + b(k_1,k_2)\delta_{\chi_1,-\chi_2} + c(k_1,k_2)\chi_1\delta_{\chi_1,\chi_2} + d(k_1,k_2)\chi_1\delta_{\chi_1,-\chi_2}\Big)$$

$$\times\,\langle \delta_{\chi_1,\chi_2}\mathrm{T}\;\mathrm{tr}\left[V_{\underline{x}}V_{\underline{y}}^{\dagger}\right] + \delta_{\chi_1,\chi_2}\mathrm{T}\;\mathrm{tr}\left[V_{\underline{x}}V_{\underline{y}}^{\dagger\,\text{sub-eik.}}\right] + \delta_{\chi_1,-\chi_2} + \mathrm{T}\;\mathrm{tr}\left[V_{\underline{x}}V_{\underline{y}}^{\dagger\,\text{sub-eik.}}\right] + \ldots\rangle$$

Eikonal dipole amplitude

Spin projected pieces of polarized Wilson line in polarized dipole amplitudes

Polarized Wilson Lines in TMDs

- For γ^+ and $\gamma^+\gamma_5$ we get leading contributions from the following sub-eikonal quark and gluon exchange operators

$$V_{\underline{x}}^{\mathrm{i}} = -\frac{p_1^+}{8\,s} \int\limits_{-\infty}^{\infty} \mathrm{d}z^- \;\; V_{\underline{x}}[\infty, z^-]\,(\overleftarrow{D}_z^i - \vec{D}_z^i)\,V_{\underline{x}}[z^-, -\infty]$$

$$V_{\underline{x}}^{[2]} = \frac{i\,p_1^+}{8\,s} \int\limits_{-\infty}^{\infty} \mathrm{d}z^- \;\; V_{\underline{x}}[\infty, z^-]\,[(\overleftarrow{D}_z^i - \vec{D}_z^i)^2 - (\underline{k}_1 - \underline{k})^2]\,V_{\underline{x}}[z^-, -\infty]$$

$$-\frac{g^2\,p_1^+}{4\,s} \int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} dz_2^- \;\; V_{\underline{x}}[\infty, z_2^-]\,t^b\,\psi_\beta(z_2^-, \underline{x})\,U_{\underline{x}}^{ba}[z_2^-, z_1^-]\,\left[\frac{\gamma^+}{2}\right]_{\alpha\beta}\,\bar{\psi}_\alpha(z_1^-, \underline{x})\,t^a\,V_{\underline{x}}[z_1^-, -\infty]$$

$$V_{\underline{x}}^{\mathrm{mag}} = \frac{igp_1^+}{2s} \int\limits_{-\infty}^{\infty} \mathrm{d}z^- \;\; V_{\underline{x}}[\infty, z^-]F^{12}(z^-, \underline{x})V_{\underline{x}}[z^-, -\infty]$$

$$-\frac{g^2\,p_1^+}{4\,s} \int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} dz_2^- \;\; V_{\underline{x}}[\infty, z_2^-]\,t^b\,\psi_\beta(z_2^-, \underline{x})\,U_{\underline{x}}^{ba}[z_2^-, z_1^-]\,\left[\frac{\gamma^+\gamma^5}{2}\right]_{\alpha\beta}\,\bar{\psi}_\alpha(z_1^-, \underline{x})\,t^a\,V_{\underline{x}}[z_1^-, -\infty]$$

Polarized Wilson Lines in TMDs

- For γ^+ and $\gamma^+\gamma_5$ we get leading contributions from the following sub-eikonal quark and gluon exchange operators

$$V_{\underline{x}}^{\mathrm{i}} = -\frac{p_1^+}{8\,s}\int_{-\infty}^{\infty} \mathrm{d}z^- \;\; V_{\underline{x}}[\infty, z^-]\,(\overleftarrow{D}_z^i - \vec{D}_z^i)\,V_{\underline{x}}[z^-,-\infty]$$

covariant phase correction from free propagator

$$V_{\underline{x}}^{[2]} = \frac{i\,p_1^+}{8\,s}\int_{-\infty}^{\infty} \mathrm{d}z^- \;\; V_{\underline{x}}[\infty, z^-]\,[(\overleftarrow{D}_z^i - \vec{D}_z^i)^2 - (\underline{k}_1 - \underline{k})^2]\,V_{\underline{x}}[z^-,-\infty]$$

$$- \frac{g^2\,p_1^+}{4\,s}\int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} dz_2^- \;\; V_{\underline{x}}[\infty, z_2^-]\,t^b\,\psi_\beta(z_2^-,\underline{x})\,U_{\underline{x}}^{ba}[z_2^-,z_1^-]\left[\frac{\gamma^+}{2}\right]_{\alpha\beta}\bar{\psi}_\alpha(z_1^-,\underline{x})\,t^a\,V_{\underline{x}}[z_1^-,-\infty]$$

$$V_{\underline{x}}^{\mathrm{mag}} = \frac{igp_1^+}{2s}\int_{-\infty}^{\infty} \mathrm{d}z^- \;\; V_{\underline{x}}[\infty, z^-]F^{12}(z^-,\underline{x})V_{\underline{x}}[z^-,-\infty]$$

$$- \frac{g^2\,p_1^+}{4\,s}\int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} dz_2^- \;\; V_{\underline{x}}[\infty, z_2^-]\,t^b\,\psi_\beta(z_2^-,\underline{x})\,U_{\underline{x}}^{ba}[z_2^-,z_1^-]\left[\frac{\gamma^+\gamma^5}{2}\right]_{\alpha\beta}\bar{\psi}_\alpha(z_1^-,\underline{x})\,t^a\,V_{\underline{x}}[z_1^-,-\infty]$$

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$$-\frac{g^2\,p_1^+}{4\,s}\int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} dz_2^- \; V_{\underline{x}}[\infty, z_2^-]\,t^b\,\psi_\beta(z_2^-, \underline{x})\,U_{\underline{x}}^{ba}[z_2^-, z_1^-]\left[\frac{\gamma^+}{2}\right]_{\alpha\beta}\bar{\psi}_\alpha(z_1^-, \underline{x})\,t^a\,V_{\underline{x}}[z_1^-, -\infty]$$

$$V_{\underline{x}}^{\mathrm{mag}} = \frac{igp_1^+}{2s}\int\limits_{-\infty}^{\infty} \mathrm{d}z^- \; V_{\underline{x}}[\infty, z^-]F^{12}(z^-, \underline{x})V_{\underline{x}}[z^-, -\infty]$$

chromomagnetic background field interaction

$$-\frac{g^2\,p_1^+}{4\,s}\int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} dz_2^- \; V_{\underline{x}}[\infty, z_2^-]\,t^b\,\psi_\beta(z_2^-, \underline{x})\,U_{\underline{x}}^{ba}[z_2^-, z_1^-]\left[\frac{\gamma^+\gamma^5}{2}\right]_{\alpha\beta}\bar{\psi}_\alpha(z_1^-, \underline{x})\,t^a\,V_{\underline{x}}[z_1^-, -\infty]$$

Polarized Wilson Lines in TMDs

- For γ^+ and $\gamma^+\gamma_5$ we get leading contributions from the following sub-eikonal quark and gluon exchange operators

$$V_{\underline{x}}^{\mathrm{i}} = -\frac{p_1^+}{8\,s}\int\limits_{-\infty}^{\infty} \mathrm{d}z^- \;\; V_{\underline{x}}[\infty, z^-]\,(\overleftarrow{D}_z^i - \vec{D}_z^i)\,V_{\underline{x}}[z^-, -\infty]$$

covariant phase correction from free propagator

$$V_{\underline{x}}^{[2]} = \frac{i\,p_1^+}{8\,s}\int\limits_{-\infty}^{\infty} \mathrm{d}z^- \;\; V_{\underline{x}}[\infty, z^-]\,[(\overleftarrow{D}_z^i - \vec{D}_z^i)^2 - (\underline{k}_1 - \underline{k})^2]\,V_{\underline{x}}[z^-, -\infty]$$

background (anti)quark field exchange

$$-\frac{g^2\,p_1^+}{4\,s}\int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} dz_2^- \;\; V_{\underline{x}}[\infty, z_2^-]\,t^b\,\psi_\beta(z_2^-, \underline{x})\,U_{\underline{x}}^{ba}[z_2^-, z_1^-]\left[\frac{\gamma^+}{2}\right]_{\alpha\beta}\bar{\psi}_\alpha(z_1^-, \underline{x})\,t^a\,V_{\underline{x}}[z_1^-, -\infty]$$

$$V_{\underline{x}}^{\mathrm{mag}} = \frac{igp_1^+}{2s}\int\limits_{-\infty}^{\infty} \mathrm{d}z^- \;\; V_{\underline{x}}[\infty, z^-]F^{12}(z^-, \underline{x})V_{\underline{x}}[z^-, -\infty]$$

chromomagnetic background field interaction

$$-\frac{g^2\,p_1^+}{4\,s}\int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} dz_2^- \;\; V_{\underline{x}}[\infty, z_2^-]\,t^b\,\psi_\beta(z_2^-, \underline{x})\,U_{\underline{x}}^{ba}[z_2^-, z_1^-]\left[\frac{\gamma^+\gamma^5}{2}\right]_{\alpha\beta}\bar{\psi}_\alpha(z_1^-, \underline{x})\,t^a\,V_{\underline{x}}[z_1^-, -\infty]$$

Polarized Wilson Lines in TMDs

- For the quark transverse spin dependent TMDs we have $\sigma^{i+}\gamma_5$ projecting out the following sub-sub-eikonal quark exchange operators (gluon exchanges are mass suppressed)

$$V_{\underline{x}}^{\mathrm{T}} \equiv \frac{g^2\,(p_1^+)^2}{16\,s^2} \int_{-\infty}^{\infty} dz_1^- \int_{z_1^-}^{\infty} dz_2^- \; V_{\underline{x}}[\infty, z_2^-]\, t^b\, \psi_\beta(z_2^-, \underline{x})\, U_{\underline{x}}^{ba}[z_2^-, z_1^-] \Big[\Big[i\gamma^5 \underline{S} \cdot \overleftarrow{\underline{D}}_x - \underline{S} \times \overleftarrow{\underline{D}}_x \Big]\, \gamma^+\gamma^- + \left[i\gamma^5 \underline{S} \cdot \underline{D}_x - \underline{S} \times \underline{D}_x \right] \gamma^-\gamma^+ \Big]_{\alpha\beta} \bar{\psi}_\alpha(z_1^-, \underline{x})\, t^a\, V_{\underline{x}}[z_1^-, -\infty],$$

$$V_{\underline{x}}^{\mathrm{T}\perp} \equiv -\frac{g^2\,(p_1^+)^2}{16\,s^2} \int_{-\infty}^{\infty} dz_1^- \int_{z_1^-}^{\infty} dz_2^- \; V_{\underline{x}}[\infty, z_2^-]\, t^b\, \psi_\beta(z_2^-, \underline{x})\, U_{\underline{x}}^{ba}[z_2^-, z_1^-] \Big[\Big[i\underline{S} \cdot \overleftarrow{\underline{D}}_x - \gamma^5 \underline{S} \times \overleftarrow{\underline{D}}_x \Big]\, \gamma^+\gamma^- + \left[i\underline{S} \cdot \underline{D}_x - \gamma^5 \underline{S} \times \underline{D}_x \right] \gamma^-\gamma^+ \Big]_{\alpha\beta} \bar{\psi}_\alpha(z_1^-, \underline{z}_1)\, t^a\, V_{\underline{x}}[z_1^-, -\infty].$$

Polarized Dipole Amplitudes

- Having obtained the relevant polarized Wilson lines, express TMDs in terms of polarized dipole amplitudes, ex. for the Sivers function

$$
-\frac{\underline{k} \times \underline{S}_P}{M_P} f_{1T}^{\perp S}(x, k_T^2)\Big|_{\text{sub-eikonal}} = \frac{16 N_c}{(2\pi)^3} \int d^2 x_{10} \frac{d^2 k_{1\perp}}{(2\pi)^3} \frac{e^{i(\underline{k}+\underline{k}_1)\cdot \underline{x}_{10}}}{\underline{k}_1^2 \, \underline{k}^2} \int\limits_{\frac{\Lambda^2}{s}}^{1} \frac{dz}{z}
$$

$$
\times \Big\{ \underline{k}_1 \cdot \underline{k} \, (k - k_1)^i \left[\epsilon^{ij} \, S_P^j \, x_{10}^2 \, F_A^S(x_{10}^2, z) + x_{10}^i \, \underline{x}_{10} \times \underline{S}_P \, F_B^S(x_{10}^2, z) + \epsilon^{ij} \, x_{10}^j \, \underline{x}_{10} \cdot \underline{S}_P \, F_C^S(x_{10}^2, z) \right]
$$

$$
+ i \, \underline{k}_1 \cdot \underline{k} \; \underline{x}_{10} \times \underline{S}_P \, F^{S\,[2]}(x_{10}^2, z) - i \, \underline{k} \times \underline{k}_1 \, \underline{x}_{10} \cdot \underline{S}_P \, F_{\text{mag}}^S(x_{10}^2, z) \Big\}
$$

$$
-\frac{\underline{k} \times \underline{S}_P}{M_P} f_{1T}^{\perp NS}(x, k_T^2)\Big|_{\text{sub-eikonal}} = \frac{16 N_c}{(2\pi)^3} \int d^2 x_{10} \frac{d^2 k_{1\perp}}{(2\pi)^3} \frac{e^{i(\underline{k}+\underline{k}_1)\cdot \underline{x}_{10}}}{\underline{k}_1^2 \, \underline{k}^2} \int\limits_{\frac{\Lambda^2}{s}}^{1} \frac{dz}{z}
$$

$$
\times \Big\{ \underline{k}_1 \cdot \underline{k} \, (k - k_1)^i \left[\epsilon^{ij} \, S_P^j \, x_{10}^2 \, F_A^{NS}(x_{10}^2, z) + x_{10}^i \, \underline{x}_{10} \times \underline{S}_P \, F_B^{NS}(x_{10}^2, z) + \epsilon^{ij} \, x_{10}^j \, \underline{x}_{10} \cdot \underline{S}_P \, F_C^{NS}(x_{10}^2, z) \right]
$$

$$
+ i \, \underline{k}_1 \cdot \underline{k} \; \underline{x}_{10} \times \underline{S}_P \, F^{NS\,[2]}(x_{10}^2, z) - i \, \underline{k} \times \underline{k}_1 \, \underline{x}_{10} \cdot \underline{S}_P \, F_{\text{mag}}^{NS}(x_{10}^2, z) \Big\}
$$

Polarized Dipole Amplitudes

- We write the TMDs in terms of impact parameter integrated dipole amplitudes

$$F^{S\,i}_{\underline{w},\underline{\zeta}}(z) = \frac{1}{2N_c} \sum_f \text{Re} \left\langle\!\left\langle \text{T tr}\left[V_{\underline{\zeta}} V^{i\,\dagger}_{\underline{w}}\right] + \text{T tr}\left[V^{i}_{\underline{w}} V^{\dagger}_{\underline{\zeta}}\right] \right\rangle\!\right\rangle,$$

$$F^{S\,[2]}_{\underline{w},\underline{\zeta}}(z) = \frac{1}{2N_c} \sum_f \text{Im} \left\langle\!\left\langle \text{T tr}\left[V_{\underline{\zeta}} V^{[2]\,\dagger}_{\underline{w};\underline{k},\underline{k}_1}\right] + \text{T tr}\left[V^{[2]}_{\underline{w};\underline{k},\underline{k}_1} V^{\dagger}_{\underline{\zeta}}\right] \right\rangle\!\right\rangle,$$

$$F^{S\,\text{mag}}_{\underline{w},\underline{\zeta}}(z) = \frac{1}{2N_c} \sum_f \text{Re} \left\langle\!\left\langle \text{T tr}\left[V_{\underline{\zeta}} V^{\text{mag}\,\dagger}_{\underline{w}}\right] + \text{T tr}\left[V^{\text{mag}}_{\underline{w}} V^{\dagger}_{\underline{\zeta}}\right] \right\rangle\!\right\rangle,$$

$$F^{NS\,i}_{\underline{w},\underline{\zeta}}(z) = \frac{1}{2N_c} \text{Re} \left\langle\!\left\langle \text{T tr}\left[V_{\underline{\zeta}} V^{i\,\dagger}_{\underline{w}}\right] - \text{T tr}\left[V^{i}_{\underline{w}} V^{\dagger}_{\underline{\zeta}}\right] \right\rangle\!\right\rangle,$$

$$F^{NS\,[2]}_{\underline{w},\underline{\zeta}}(z) = \frac{1}{2N_c} \text{Im} \left\langle\!\left\langle \text{T tr}\left[V_{\underline{\zeta}} V^{[2]\,\dagger}_{\underline{w};\underline{k},\underline{k}_1}\right] - \text{T tr}\left[V^{[2]}_{\underline{w};\underline{k},\underline{k}_1} V^{\dagger}_{\underline{\zeta}}\right] \right\rangle\!\right\rangle,$$

$$F^{NS\,\text{mag}}_{\underline{w},\underline{\zeta}}(z) = \frac{1}{2N_c} \text{Re} \left\langle\!\left\langle \text{T tr}\left[V_{\underline{\zeta}} V^{\text{mag}\,\dagger}_{\underline{w}}\right] - \text{T tr}\left[V^{\text{mag}}_{\underline{w}} V^{\dagger}_{\underline{\zeta}}\right] \right\rangle\!\right\rangle,$$

Here $\langle\!\langle \dots \rangle\!\rangle = zs \langle \dots \rangle$ with z the internal longitudinal momentum fraction and the center of mass energy squared

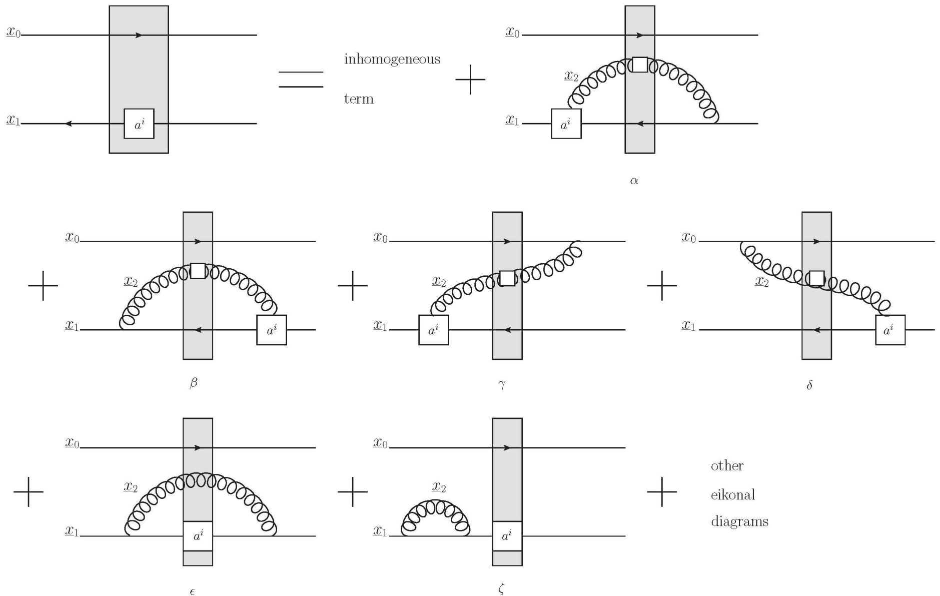
$$\int d^2 b_\perp F^{i}_{10} = \epsilon^{ij} S^j_P\, x^2_{10}\, F_A(x^2_{10}, z) + x^i_{10}\, \underline{x}_{10} \times \underline{S}_P\, F_B(x^2_{10}, z) + \epsilon^{ij}\, x^j_{10}\, \underline{x}_{10} \cdot \underline{S}_P\, F_C(x^2_{10}, z)$$

$$\int d^2 b_\perp F^{[2]}_{10} = \underline{x}_{10} \times \underline{S}_P\, F^{[2]}(x^2_{10}, z)$$

$$\int d^2 b_\perp F^{\text{mag}}_{10} = \underline{x}_{10} \cdot \underline{S}_P\, F_{\text{mag}}(x^2_{10}, z)$$

Small-x Evolution

- Calculate gluon and quark emissions in the dipole amplitudes
- Sum over relevant diagrams/operators to extract evolution in large-N_c or large-N_c&N_f limit
- Obtain general operator level equations, generally similar form to Balitsky hierarchy
- BK type equations no longer close – depend on eikonal dipole amplitude



Polarized Wilson lines allow for logs as dipole size goes to zero → resum double logs $\alpha_s^n \ln^{2n}(1/x)$

Non-singlet Sivers function large-N_c linearized DLA equations

$$F_A^{NS}(x_{10}^2, z) = F_A^{NS\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z' s}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[6\, F_A^{NS}(x_{21}^2, z') - F_B^{NS}(x_{21}^2, z') + F_C^{NS}(x_{21}^2, z')\right]$$

$$F_B^{NS}(x_{10}^2, z) = F_B^{NS\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z' s}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[-2\, F_A^{NS}(x_{21}^2, z') + 5\, F_B^{NS}(x_{21}^2, z') - F_C^{NS}(x_{21}^2, z')\right],$$

$$F_C^{NS}(x_{10}^2, z) = F_C^{NS\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z' s}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[2\, F^{NS\,\text{mag}}(x_{21}^2, z') + 6\, F_C^{NS}(x_{21}^2, z')\right],$$

$$F^{NS\,\text{mag}}(x_{10}^2, z) = F^{NS\,\text{mag}\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[\Gamma^{NS\,\text{mag}}(x_{10}^2, x_{21}^2, z') + 2\,\Gamma_A^{NS}(x_{10}^2, x_{21}^2, z') - \Gamma_B^{NS}(x_{10}^2, x_{21}^2, z') + 3\,\Gamma_C^{NS}(x_{10}^2, x_{21}^2, z')\Big]$$

Dipole evolution equations

$$\Gamma_A^{NS}(x_{10}^2, x_{21}^2, z') = F_A^{NS\,(0)}(x_{10}^2, z') + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z'' s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \times \left[6\, F_A^{NS}(x_{32}^2, z'') - F_B^{NS}(x_{32}^2, z'') + F_C^{NS}(x_{32}^2, z'')\right]$$

$$\Gamma_B^{NS}(x_{10}^2, x_{21}^2, z') = F_B^{NS\,(0)}(x_{10}^2, z') + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z'' s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \times \left[-2\, F_A^{NS}(x_{23}^2, z'') + 5\, F_B^{NS}(x_{32}^2, z'') - F_C^{NS}(x_{32}^2, z'')\right],$$

$$\Gamma_C^{NS}(x_{10}^2, x_{21}^2, z') = F_C^{NS\,(0)}(x_{10}^2, z') + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z'' s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \times \left[2\, F^{NS\,\text{mag}}(x_{32}^2, z'') + 6\, F_C^{NS}(x_{32}^2, z'')\right],$$

$$\Gamma^{NS\,\text{mag}}(x_{10}^2, x_{21}^2, z') = F^{NS\,\text{mag}\,(0)}(x_{10}^2, z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\left[x_{10}^2, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{32}^2}{x_{32}^2} \times \Big[\Gamma^{NS\,\text{mag}}(x_{10}^2, x_{32}^2, z'') + 2\,\Gamma_A^{NS}(x_{10}^2, x_{32}^2, z'') - \Gamma_B^{NS}(x_{10}^2, x_{32}^2, z'') + 3\,\Gamma_C^{NS}(x_{10}^2, x_{32}^2, z'')\Big]$$

'Neighbor' dipole evolution equations

Small-x TMDs: Eikonal Sector

- The TMDs projected out by γ^+ have eikonal contributions which were known in literature
- The unpolarized quark TMD f_1 is proportional to the gluon dipole TMD in the flavor singlet sector and has asymptotics given by the QCD Reggeon (Kirschner and Lipatov 1983) in the flavor non-singlet sector
- The Sivers function $f_{1T}^{\perp}$ in the flavor singlet sector is identically zero at eikonal order, meanwhile the flavor non-singlet is given by the spin-dependent Odderon (Zhou et al 2019) and thus scales as $1/x$ with no α_s correction to its exponent in the BFKL regime

Small-x TMDs: Sub-Eikonal Sector

- The TMDs projected out by $\gamma^+\gamma_5$ have leading sub-eikonal contributions, and we also considered the sub-eikonal correction to the Sivers function
- All three TMDs $f_{1T}^{\perp}$, g_1, and g_{1T} in the flavor singlet sector are given by variations of the same polarized dipole amplitudes

Helicity studied in Cougoulic et al 2022

$$F^{S\,i}_{\underline{w},\underline{\zeta}}(z) = \frac{1}{2N_c}\sum_f \mathrm{Re/Im}\left\langle\!\!\left\langle \mathrm{T}\ \mathrm{tr}\left[V_{\underline{\zeta}}\,V^{i\,\dagger}_{\underline{w}}\right] + \mathrm{T}\ \mathrm{tr}\left[V^{i}_{\underline{w}}\,V^{\dagger}_{\underline{\zeta}}\right]\right\rangle\!\!\right\rangle_1,$$

Sub-eikonal phase term $\overleftarrow{\underline{D}}\cdot\overrightarrow{\underline{D}}$

$$F^{S\,\mathrm{mag}}_{\underline{w},\underline{\zeta}}(z) = \frac{1}{2N_c}\sum_f \mathrm{Re/Im}\left\langle\!\!\left\langle \mathrm{T}\ \mathrm{tr}\left[V_{\underline{\zeta}}\,V^{\mathrm{mag}\,\dagger}_{\underline{w}}\right] + \mathrm{T}\ \mathrm{tr}\left[V^{\mathrm{mag}}_{\underline{w}}\,V^{\dagger}_{\underline{\zeta}}\right]\right\rangle\!\!\right\rangle_1$$

'Chromomagnetic' interaction term F^{ij}

- The variations mostly change the initial conditions for evolution, resulting in different intercepts but very similar evolution equations
- Set of coupled evolution equations, solved numerically for all three TMDs and exactly for helicity

Small-x TMDs: Sub-Eikonal Sector

- In the flavor non-singlet sector these three TMDs all have different leading contributions
- The helicity TMD g_1^{NS} is essentially identical to the QCD Reggeon (t-channel quark ladder), with polarized Wilson lines containing $\bar{\psi}(z_1)\gamma^+\gamma_5\psi(z_2)$
- The Sivers function $f_{1T}^{\perp NS}$ looks very similar to the flavor singlet helicity, having the sub-eikonal phase and 'chromomagnetic' polarized dipole amplitudes (Kovchegov, MGS 2022)
- The worm-gear g_{1T} comes from a polarized quark/antiquark exchange $\bar{\psi}(z_1)\gamma^+\psi(z_2)$, with evolution driven purely by eikonal gluon emissions which ultimately does not change its naïve scaling as x^0 (MGS 2024)

Small-x TMDs: Sub-Sub-Eikonal Sector

- All four TMDs projected by $\sigma^{i+}\gamma_5$, namely h_{1T}, $h_{1T}^{\perp}$, $h_1^{\perp}$, and $h_{1L}^{\perp}$ come from minor variations of two polarized dipole amplitudes (Kovchegov and Sievert 2019, MGS 2024)

$$H_{10}^{1T}(z) \equiv \frac{1}{2N_c} \mathrm{Re/Im} \left\langle\!\!\left\langle \mathrm{T\ tr}\left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{T}\,\dagger}\right] \pm \mathrm{T\ tr}\left[V_{\underline{0}}^{\dagger} V_{\underline{1}}^{\mathrm{T}}\right] \right\rangle\!\!\right\rangle_2,$$

$$H_{10}^{2T}(z) \equiv \frac{1}{2N_c} \mathrm{Im/Re} \left\langle\!\!\left\langle \mathrm{T\ tr}\left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{T}\,\perp\,\dagger}\right] \pm \mathrm{T\ tr}\left[V_{\underline{0}}^{\dagger} V_{\underline{1}}^{\mathrm{T}\,\perp}\right] \right\rangle\!\!\right\rangle_2$$

$$V_{\underline{x}}^{\mathrm{T}} \equiv \frac{g^2\,(p_1^+)^2}{16\,s^2} \int_{-\infty}^{\infty} dz_1^- \int_{z_1^-}^{\infty} dz_2^-\ V_{\underline{x}}[\infty, z_2^-]\, t^b\, \psi_\beta(z_2^-, \underline{x})\, U_{\underline{x}}^{ba}[z_2^-, z_1^-] \left[\left[i\gamma^5 \underline{S}\cdot \overleftarrow{\underline{D}}_x - \underline{S} \times \overleftarrow{\underline{D}}_x \right] \gamma^+\gamma^- \right.$$
$$\left. + \left[i\gamma^5 \underline{S}\cdot \underline{D}_x - \underline{S} \times \underline{D}_x \right] \gamma^-\gamma^+ \right]_{\alpha\beta} \bar{\psi}_\alpha(z_1^-, \underline{x})\, t^a\, V_{\underline{x}}[z_1^-, -\infty],$$

$$V_{\underline{x}}^{\mathrm{T}\,\perp} \equiv -\frac{g^2\,(p_1^+)^2}{16\,s^2} \int_{-\infty}^{\infty} dz_1^- \int_{z_1^-}^{\infty} dz_2^-\ V_{\underline{x}}[\infty, z_2^-]\, t^b\, \psi_\beta(z_2^-, \underline{x})\, U_{\underline{x}}^{ba}[z_2^-, z_1^-] \left[\left[i\underline{S}\cdot \overleftarrow{\underline{D}}_x - \gamma^5 \underline{S} \times \overleftarrow{\underline{D}}_x \right] \gamma^+\gamma^- \right.$$
$$\left. + \left[i\underline{S}\cdot \underline{D}_x - \gamma^5 \underline{S} \times \underline{D}_x \right] \gamma^-\gamma^+ \right]_{\alpha\beta} \bar{\psi}_\alpha(z_1^-, \underline{z}_1)\, t^a\, V_{\underline{x}}[z_1^-, -\infty].$$

Small-x TMDs: Sub-Sub-Eikonal Sector

- For the transversity h_{1T} and pretzelosity $h_{1T}^{\perp}$ TMDs, the evolution is again that of the QCD Reggeon
  - Similar to non-singlet unpolarized and helicity TMDs, but now for both singlet and non-singlet!
- For the Boer-Mulders $h_1^{\perp}$ and worm-gear $h_{1L}^{\perp}$ the evolution is purely eikonal gluon emissions and the naïve scaling as x is unchanged

Results for flavor non-singlet outside saturation region

Leading Twist Quark TMDs				
		Quark Polarization		
		U	L	T
Nucleon Polarization	U	$f_1^{NS} \sim x^{-\sqrt{2\alpha_s C_F/\pi}}$		$h_1^{\perp NS} \sim x$
	L		$g_1^{NS} \sim x^{-\sqrt{\alpha_s N_c/\pi}}$	$h_{1L}^{\perp NS} \sim x$
	T	$f_{1T}^{\perp NS} \sim C_{\mathcal{O}} x^{-1} + C_1 x^{-3.4\sqrt{\alpha_s N_c/4\pi}}$	$g_{1T}^{NS} \sim x^0$	$h_1^{NS} \sim h_{1T}^{\perp NS} \sim x^{1-2\sqrt{\alpha_s N_c/2\pi}}$

MGS 2024

- The diagonal TMDs all receive evolution contributions equal to that of the Reggeon (Kirschner and Lipatov 1983)
- Unpolarized and helicity TMDs also studied in InfraRed Evolution Equation (IREE) framework (Ermolaev, Manaenkov and Ryskin 1996, Bartels, Ermolaev and Ryskin 1996)
- Transversity TMD also studied previously (Kirschner, Mankiewicz, Schafer, and Szymanowski 1997)
- The off diagonal terms receive no evolution power correction in their leading terms!

Results for flavor singlet outside saturation region

Leading Twist Quark TMDs				
		Quark Polarization		
		U	L	T
Nucleon Polarization	U	$f_1^{\mathrm{S}} \sim x^{-\frac{4\alpha_s N_c}{\pi}\ln(2)}$		$h_1^{\perp \mathrm{S}} \sim x$
	L		$g_1^{\mathrm{S}} \sim x^{-3.66\sqrt{\alpha_s N_c/2\pi}}$	$h_{1L}^{\perp \mathrm{S}} \sim x$
	T	$f_{1T}^{\perp \mathrm{S}} \sim x^{-2.9\sqrt{\alpha_s N_c/4\pi}}$	$g_{1T}^{\mathrm{S}} \sim x^{-2.9\sqrt{\alpha_s N_c/4\pi}}$	$h_1^{\mathrm{S}} \sim h_{1T}^{\perp \mathrm{S}} \sim x^{1-2\sqrt{\frac{\alpha_s N_c}{2\pi}}}$

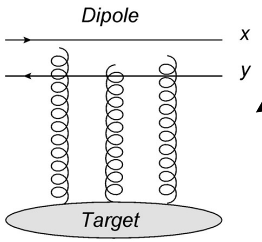
Adamiak, Tawabutr and MGS *in preparation*

- Linearized DLA asymptotics for all 8 leading twist TMDs now known!
- Surprisingly, still have some off-diagonal TMDs with no linear evolution power corrections

Protection from evolution?

- One quite interesting feature is that multiple off-diagonal TMDs receive no corrections to their naïve power law
- Based on the known spin-dependent Odderon contribution to the Sivers function (Boer et al 2016), we had previously speculated that there could be protection for T-odd TMDs

$\mathcal{O}(x,y) \sim$



$$d^{abc} = \mathrm{tr}[t^a\{t^b, t^c\}]$$

$$\left\langle \mathrm{T}\,\mathrm{tr}[V_{\underline{\zeta}} V_{\underline{w}}^\dagger] + \bar{\mathrm{T}}\,\mathrm{tr}[V_{\underline{\zeta}} V_{\underline{w}}^\dagger] \right\rangle = 2N_c \left(\mathcal{S}_{\underline{\zeta w}} + i\,\mathcal{O}_{\underline{\zeta w}}\right)$$

$$\mathcal{O}(x,y) \xrightarrow[\text{small-}x]{} \left(\frac{1}{x}\right)^{1-g(\alpha_s N_c)} \sim \left(\frac{1}{x}\right)^{1-0}$$

Bartel, Lipatov and Vacca 2000

- New results show that this protection applies to other TMDs, perhaps some other symmetry?

Discussion

- We now know the small-x asymptotics of all 8 leading-twist quark TMDs in the large-N_c DLA/LLA
- Interesting patterns have emerged
- Many equations ready for phenomenological implementation
  - Significant progress made applying small-x for helicity TMDs by JAM (cf. JAM Adamiak et al 2023 and Daniel's talk from Tuesday), also some work on the transversity TMD (JAM Cocuzza et al 2023)
- All N_c results require JIMWLK type evolution equation (cf. Cougoulic and Kovchegov 2019)
- Results can be derived for gluon TMDs as well
- Formalism can be extended to GPDs/GTMDs and potentially higher twist parton distributions (cf. Guillaume's talk from Monday)
- Polarized Wilson lines can be used to calculate more general spin-dependent processes at high energy (cf. Ming's talk earlier today)

Conclusions

- We have developed a general framework for including sub-eikonal effects into the small-x formalism
  - Similar formalisms for sub-eikonal corrections have been developed by various other authors, cf. Altinoluk et al 2016-2021, Chirilli 2019-2021
- We have obtained the small-x asymptotics for all eight leading-twist quark TMDs
- More improvements and applications of the formalism are in progress!

Backup Slides

Sub-eikonal power counting

- Eikonal distributions $q(x, k_T) \sim \frac{1}{x}$, no COM energy suppression
- Sub-eikonal distributions $q(x, k_T) \sim x^0, \frac{1}{s}$ energy suppression
- Sub-sub-eikonal distributions $q(x, k_T) \sim x, \frac{1}{s^2}$ energy suppression

Sub-Eikonal Phase Operator

- Start with the free particle propagator, then expand in inverse powers of the large momentum

$$\int_{-\infty}^{\infty} \frac{dp_2^+}{4\pi} e^{-\frac{i}{2}p_2^+(x^- - y^-)} \frac{i}{p_2^2 - m^2 + i\epsilon} = \frac{1}{2p_2^-} e^{-i\frac{\underline{p}_2^2 + m^2}{2p_2^-}(x^- - y^-)}$$

$$= \frac{1}{2p_2^-} \exp\left\{-i\frac{\underline{p}_2^2 + m^2}{2p_2^-} \int_{y^-}^{x^-} dz^-\right\} = \frac{1}{2p_2^-}\left[1 - i\frac{\underline{p}_2^2 + m^2}{2p_2^-}\int_{y^-}^{x^-} dz^- + \ldots\right]$$

- Promoting momentum to covariant derivatives yields the operator in the polarized Wilson line

‘Large-N_c’ Operator Equations

- Example from transversity and pretzelosity TMDs

$$H_{10}^{1T,S}(z) = H_{10}^{1T,S\,(0)}(z) + \frac{\alpha_s N_c}{2\pi^2}\int_{\frac{\Lambda^2}{s}}^{z}\frac{\mathrm{d}z'}{z'}\int \mathrm{d}^2x_2\,\frac{x_{10}^2}{x_{21}^2 x_{20}^2}\,\mathrm{Re}\left\langle\!\!\left\langle \frac{1}{N_c^2}\,\mathrm{T}\;\mathrm{tr}\left[V_{\underline{0}}t^a V_{\underline{1}}^{\mathrm{T}\dagger}t^b\right]\,(U_{\underline{2}})^{ba}\right.\right.$$
$$\left.\left. - \frac{C_F}{N_c^2}\,\mathrm{T}\;\mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{\mathrm{T}\dagger}\right] + \frac{1}{N_c^2}\,\mathrm{T}\;\mathrm{tr}\left[V_{\underline{0}}^{\dagger}t^b V_{\underline{1}}^{\mathrm{T}}t^a\right]\,(U_{\underline{2}})^{ba} - \frac{C_F}{N_c^2}\,\mathrm{T}\;\mathrm{tr}\left[V_{\underline{0}}^{\dagger}\,V_{\underline{1}}^{\mathrm{T}}\right]\right\rangle\!\!\right\rangle_2(z')$$
$$+ \frac{\alpha_s N_c}{2\pi^2}\int_{\frac{\Lambda^2}{s}}^{z}\frac{\mathrm{d}z'}{z'}\int \frac{\mathrm{d}^2x_2}{x_{21}^2}\,\mathrm{Re}\left\langle\!\!\left\langle \frac{1}{N_c^2}\mathrm{T}\;\mathrm{tr}\left[t^b V_{\underline{0}}t^a V_{\underline{2}}^{\mathrm{T}\dagger}\right]U_{\underline{1}}^{ba} + \frac{1}{N_c^2}\mathrm{T}\;\mathrm{tr}\left[t^a V_{\underline{0}}^{\dagger}t^b V_{\underline{2}}^{\mathrm{T}}\right]U_{\underline{1}}^{ab}\right\rangle\!\!\right\rangle_2(z')$$

- We see that we will obtain both higher numbers of Wilson lines in the correlators and dependence on eikonal correlators

Large-N_c DLA Evolution Equations

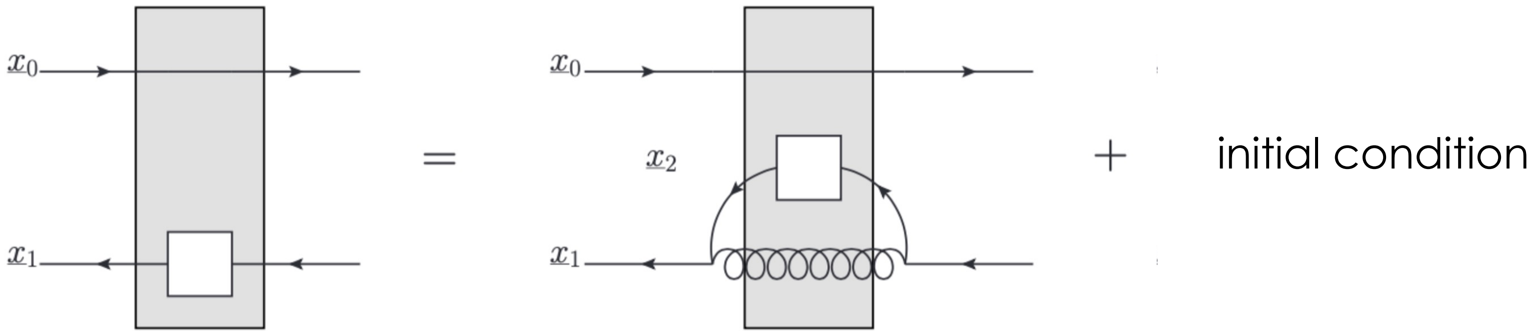
- Fully simplifying at large-N_c gives products of polarized dipole amplitudes and eikonal dipole amplitudes
- Linearizing and taking DLA we have closed equations for the polarized dipole amplitudes which we can solve numerically or in some cases exactly

$$H_{10}^{1T,S}(z) = H_{10}^{1T,S\,(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\max\{1/x_{10}^2, \Lambda^2\}/s}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[H_{12}^{1T,S}(z') - \Gamma_{10,21}^{1T,S}(z') \right] + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2} H_{21}^{1T,S}(z') \,,$$

$$\Gamma_{10,21}^{1T,S}(z') = \Gamma_{10,21}^{1T,S\,(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{\max\{1/x_{10}^2, \Lambda^2\}/s}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[H_{23}^{1T,S}(z'') - \Gamma_{10,32}^{1T,S}(z'') \right] + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2} H_{32}^{1T,S}(z'')$$

Reggeon Type Evolution

- Evolution driven by polarized quark emissions



$$H_{10}^{1T}(zs) = H_{10}^{1T\,(0)}(zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2} H_{21}^{1T}(z's)$$