Valence-Quark Picture for Proton Target in Small-x Helicity Evolution

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Outline

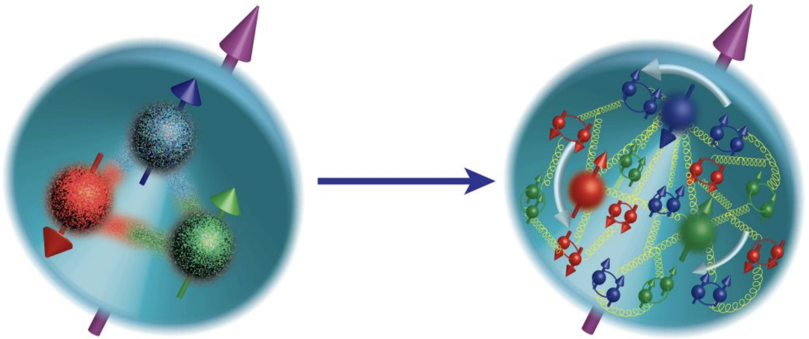
- Motivation
- Review of Small-x Helicity Evolution
- Valence Quark Model
- Sub-eikonal Gluon Vertices
- Quark Vertices
- Results and discussions

Motivation: Proton Spin Problem

- Jaffe-Manohar sum rule: $$\frac{1}{2} = S_q + S_G + L_q + L_G$$

  where the helicity of quarks (S_q) and gluons (S_G) are

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\, \Delta\Sigma(x, Q^2) \quad \text{and} \quad S_G(Q^2) = \int_0^1 dx\, \Delta G(x, Q^2)$$



$$\frac{1}{2} = S_q + S_G + L_q + L_G$$

Motivation: Proton Spin Problem

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\, \Delta\Sigma(x, Q^2)$$

$$S_G(Q^2) = \int_0^1 dx\, \Delta G(x, Q^2)$$

- More recently, the proton spin carried by quarks and gluon are estimated to be

$$S_q(Q^2 = 10\ \mathrm{GeV}^2) \approx \frac{1}{2}\int_{0.001}^{1} dx\, \Delta\Sigma(x, 10\ \mathrm{GeV}^2) \in [0.15, 0.20]$$

$$S_G(Q^2 = 10\ \mathrm{GeV}^2) \approx \int_{0.05}^{1} dx\, \Delta G(x, 10\ \mathrm{GeV}^2) \in [0.13, 0.26]$$

- They do not add to 1/2. The missing spin can come from:
  - Orbital angular momenta, L_q and L_G.
  - Small-x region of $\Delta\Sigma$ and ΔG. Scattering experiments can only access finitely small x. The limit will improve with EIC.

Small-x Helicity Evolution

- Through insertions of sub-eikonal vertices into the Wilson line, we found
  - **Two types** of vertices contribute to helicity evolution:
    - Type 1: $\sim \sigma\delta_{\sigma\sigma'} F^{12} \to V_{\underline{x}}^{\mathrm{G}[1]}$ and $\sim \sigma\delta_{\sigma\sigma'} \bar{\psi}\gamma^+\gamma_5\psi \to V_{\underline{x}}^{\mathrm{q}[1]}$
    - Type 2: $\sim \delta_{\sigma\sigma'} (\vec{D}^i - \overleftarrow{D}^i) \to V_{\underline{x}}^{i\mathrm{G}[2]}$
  - **Three polarized dipole amplitudes**:

$$V_{\underline{1}}^{\mathrm{pol}[1]} = V_{\underline{1}}^{\mathrm{G}[1]} + V_{\underline{1}}^{\mathrm{q}[1]}$$

  - $$Q(x_{10}^2, zs) = \frac{zs}{2N_c} \int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2} \right) \mathrm{Re} \left\langle \mathrm{T\,tr} \left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger} \right] + \mathrm{T\,tr} \left[V_{\underline{1}}^{\mathrm{pol}[1]} V_{\underline{0}}^{\dagger} \right] \right\rangle$$
  - $$\widetilde{G}(x_{10}^2, zs)\, S(x_{10}^2, zs) = \frac{zs}{2(N_c^2-1)} \int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2} \right) \mathrm{Re} \left\langle \mathrm{T\,Tr} \left[U_{\underline{0}} U_{\underline{1}}^{\mathrm{pol}[1]\dagger} \right] + \mathrm{T\,Tr} \left[U_{\underline{1}}^{\mathrm{pol}[1]} U_{\underline{0}}^{\dagger} \right] \right\rangle$$
  - $$G_2(x_{10}^2, zs) = \frac{\epsilon^{ij} (x_{10})_\perp^j}{x_{10}^2} \int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2} \right) \frac{zs}{2N_c} \left\langle \mathrm{tr} \left[V_{\underline{0}}^{\dagger} V_{\underline{1}}^{i\,\mathrm{G}[2]} + \left(V_{\underline{1}}^{i\,\mathrm{G}[2]} \right)^{\dagger} V_{\underline{0}} \right] \right\rangle$$

Small-x Helicity Evolution

- Through insertions of sub-eikonal vertices into the Wilson line, we found
  - **Three polarized dipole amplitudes**:
    - $Q(x_{10}^2, zs) = \frac{zs}{2N_c} \int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \mathrm{Re} \left\langle \mathrm{T\,tr} \left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] + \mathrm{T\,tr} \left[V_{\underline{1}}^{\mathrm{pol}[1]} V_{\underline{0}}^{\dagger}\right] \right\rangle$
    - $\widetilde{G}(x_{10}^2, zs)\, S(x_{10}^2, zs) = \frac{zs}{2(N_c^2 - 1)} \int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \mathrm{Re} \left\langle \mathrm{T\,Tr} \left[U_{\underline{0}} U_{\underline{1}}^{\mathrm{pol}[1]\dagger}\right] + \mathrm{T\,Tr} \left[U_{\underline{1}}^{\mathrm{pol}[1]} U_{\underline{0}}^{\dagger}\right] \right\rangle$
    - $G_2(x_{10}^2, zs) = \frac{\epsilon^{ij} (x_{10})_\perp^j}{x_{10}^2} \int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \frac{zs}{2N_c} \left\langle \mathrm{tr} \left[V_{\underline{0}}^{\dagger} V_{\underline{1}}^{i\,\mathrm{G}[2]} + \left(V_{\underline{1}}^{i\,\mathrm{G}[2]}\right)^{\dagger} V_{\underline{0}}\right] \right\rangle$
  - Helicity PDFs can be written as
    - $\Delta\Sigma(x, Q^2) = -\frac{N_c N_f}{2\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\right\}} \frac{dx_{10}^2}{x_{10}^2} \left[Q(x_{10}^2, zs) + 2\, G_2(x_{10}^2, zs)\right]$
    - $\Delta G(x, Q^2) = \frac{2N_c}{\alpha_s \pi^2} \left[\left(1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2}\right) G_2\left(x_{10}^2, zs = \frac{Q^2}{x}\right)\right]_{x_{10}^2 = \frac{1}{Q^2}}$

[Adamiak et al, 2308.07461]

Small-x Helicity Phenomenology

- Global analysis performed on polarized DIS and SIDIS data (226 data pts).
- Evolution begins at x_0 = 0.1, which is sensible as it resums $\alpha_s \ln^2(1/x)$.
- Initial condition inspired by the Born-level calculation:

$$F(x_{10}^2, zs) = a \ln \frac{zs}{\Lambda^2} + b \ln \frac{1}{x_{10}^2 \Lambda^2} + c$$

for amplitude $F \in \{Q_u, Q_d, Q_s, \widetilde{G}, G_2\}$ and flavor non-singlet (24 parameters).

- The evolution describes the data well, but there remains high uncertainties in the predictions, e.g. the total parton spin:

$$S_q + S_G \approx \int_{10^{-5}}^{0.1} \mathrm{d}x \left(\frac{1}{2} \Delta\Sigma + \Delta G \right)(x) = -0.64 \pm 0.60$$

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- The initial condition can benefit from **additional physical constraints.**

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[Dumitru, Miller, Venugopalan, 1808.02501]
[Dumitru, Skokov, Stebel, 2001.04516]
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- The initial condition can benefit from additional physical constraints.
- With x_0 = 0.1 at the initial condition, we are fairly close to the valence quark regime (compared to the single-log, unpolarized evolution with x_0 = 0.01).
- Model the **proton target as 3 valence quarks (uud)** and allow for a gluon emission and absorption.

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Valence Quark Model

- Polarized dipole amplitudes involve helicity-dependent averaging over (target) proton state:

$$\langle \cdots \rangle = \frac{1}{2}\,\mathcal{S}_L \sum_{\mathcal{S}_L} \frac{\langle P^+, \underline{P}, \mathcal{S}_L| \cdots |P^+, \underline{P}, \mathcal{S}_L\rangle}{\langle P^+, \underline{P}, \mathcal{S}_L|P^+, \underline{P}, \mathcal{S}_L\rangle}$$

with $\langle K^+, \underline{K}, \mathcal{S}'_L|P^+, \underline{P}, \mathcal{S}_L\rangle = \delta_{\mathcal{S}_L \mathcal{S}'_L}\, 2P^+\, 2\pi\delta(P^+ - K^+)\,(2\pi)^2\delta^2(\underline{P} - \underline{K})$

- The proton state can be written as

$$|P^+, \underline{P}, \mathcal{S}_L\rangle = \frac{1}{\sqrt{6}} \int \frac{dx_1 dx_2 dx_3}{(4\pi)^3\sqrt{x_1x_2x_3}}\, 4\pi\delta(1 - x_1 - x_2 - x_3) \int \frac{d^2q_1 d^2q_2 d^2q_3}{(2\pi)^6}\,(2\pi)^2\delta^2(\underline{q}_1 + \underline{q}_2 + \underline{q}_3)$$
$$\times \sum_{\{f_1,f_2,f_3\}=\{u,u,d\}} \sum_{\sigma_1,\sigma_2,\sigma_3} \Phi(x_1,\underline{q}_1;x_2,\underline{q}_2;x_3,\underline{q}_3)\, S(\sigma_1,f_1;\sigma_2,f_2;\sigma_3,f_3)$$
$$\times \sum_{i_1,i_2,i_3} \epsilon_{i_1i_2i_3}\, |x_1P^+, x_1\underline{P} + \underline{q}_1, i_1, \sigma_1, f_1\rangle\, |x_2P^+, x_2\underline{P} + \underline{q}_2, i_2, \sigma_2, f_2\rangle\, |x_3P^+, x_3\underline{P} + \underline{q}_3, i_3, \sigma_3, f_3\rangle$$

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$$\times \sum_{i_1,i_2,i_3} \epsilon_{i_1 i_2 i_3} \, |x_1 P^+, x_1 \underline{P} + \underline{q}_1, i_1, \sigma_1, f_1\rangle \, |x_2 P^+, x_2 \underline{P} + \underline{q}_2, i_2, \sigma_2, f_2\rangle \, |x_3 P^+, x_3 \underline{P} + \underline{q}_3, i_3, \sigma_3, f_3\rangle$$

- We assume that the valence quark wave function is separable: $\psi = \Phi \cdot S$, with normalization

$$\int \frac{dx_1 dx_2 dx_3}{(4\pi)^3} \, 4\pi\delta(1 - x_1 - x_2 - x_3) \int \frac{d^2q_1 d^2q_2 d^2q_3}{(2\pi)^6} \, (2\pi)^2 \delta^2(\underline{q}_1 + \underline{q}_2 + \underline{q}_3)$$
$$\times \sum_{\{f_1,f_2,f_3\}=\{u,u,d\}} \sum_{\sigma_1,\sigma_2,\sigma_3} \left| \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3) \, S(\sigma_1, f_1; \sigma_2, f_2; \sigma_3, f_3) \right|^2 = 1$$

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$$|P^+, \underline{P}, \mathcal{S}_L\rangle = \frac{1}{\sqrt{6}} \int \frac{dx_1 dx_2 dx_3}{(4\pi)^3 \sqrt{x_1 x_2 x_3}} \, 4\pi\delta(1 - x_1 - x_2 - x_3) \int \frac{d^2 q_1 d^2 q_2 d^2 q_3}{(2\pi)^6} \, (2\pi)^2 \delta^2(\underline{q}_1 + \underline{q}_2 + \underline{q}_3)$$
$$\times \sum_{\{f_1, f_2, f_3\} = \{u,u,d\}} \sum_{\sigma_1, \sigma_2, \sigma_3} \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3) \, S(\sigma_1, f_1; \sigma_2, f_2; \sigma_3, f_3)$$
$$\times \sum_{i_1, i_2, i_3} \epsilon_{i_1 i_2 i_3} \, |x_1 P^+, x_1 \underline{P} + \underline{q}_1, i_1, \sigma_1, f_1\rangle \, |x_2 P^+, x_2 \underline{P} + \underline{q}_2, i_2, \sigma_2, f_2\rangle \, |x_3 P^+, x_3 \underline{P} + \underline{q}_3, i_3, \sigma_3, f_3\rangle$$

- The flavor-spin wave function, S, is taken to be totally symmetric:

$$S(\sigma_1, f_1; \sigma_2, f_2; \sigma_3, f_3) = \frac{1}{\sqrt{18}} \big\{ [2\delta_{\sigma_1, \mathcal{S}_L} \delta_{\sigma_2, \mathcal{S}_L} \delta_{\sigma_3, -\mathcal{S}_L} - \delta_{\sigma_1, \mathcal{S}_L} \delta_{\sigma_2, -\mathcal{S}_L} \delta_{\sigma_3, \mathcal{S}_L} - \delta_{\sigma_1, -\mathcal{S}_L} \delta_{\sigma_2, \mathcal{S}_L} \delta_{\sigma_3, \mathcal{S}_L}] \, \delta^{f_1, u} \delta^{f_2, u} \delta^{f_3, d}$$
$$+ [2\delta_{\sigma_1, \mathcal{S}_L} \delta_{\sigma_2, -\mathcal{S}_L} \delta_{\sigma_3, \mathcal{S}_L} - \delta_{\sigma_1, \mathcal{S}_L} \delta_{\sigma_2, \mathcal{S}_L} \delta_{\sigma_3, -\mathcal{S}_L} - \delta_{\sigma_1, -\mathcal{S}_L} \delta_{\sigma_2, \mathcal{S}_L} \delta_{\sigma_3, \mathcal{S}_L}] \, \delta^{f_1, u} \delta^{f_2, d} \delta^{f_3, u}$$
$$+ [2\delta_{\sigma_1, -\mathcal{S}_L} \delta_{\sigma_2, \mathcal{S}_L} \delta_{\sigma_3, \mathcal{S}_L} - \delta_{\sigma_1, \mathcal{S}_L} \delta_{\sigma_2, -\mathcal{S}_L} \delta_{\sigma_3, \mathcal{S}_L} - \delta_{\sigma_1, \mathcal{S}_L} \delta_{\sigma_2, \mathcal{S}_L} \delta_{\sigma_3, -\mathcal{S}_L}] \, \delta^{f_1, d} \delta^{f_2, u} \delta^{f_3, u} \big\}$$

Sub-Eikonal Gluon Exchange

- Start with type-1 polarized Wilson line:

$$V_{\underline{x}}^{\mathrm{G}[1]} = \frac{i\,g\,P^+}{s} \int\limits_{-\infty}^{\infty} dx^- V_{\underline{x}}[\infty, x^-]\, F^{12}(x^-, \underline{x})\; V_{\underline{x}}[x^-, -\infty]$$

Split dipole amplitude, Q_f, for each flavor f into

$$Q_f = Q_f^{\mathrm{G}} + Q_f^{\mathrm{q}}$$

according to the exchanged parton. Then,

$$Q_f^{\mathrm{G}}(x_{10}^2, zs) = \frac{zs}{2N_c} \int d^2\left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \mathrm{Re}\left\langle \mathrm{T\,tr}\left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{G}[1]\dagger}\right] + \mathrm{T\,tr}\left[V_{\underline{1}}^{\mathrm{G}[1]} V_{\underline{0}}^{\dagger}\right]\right\rangle$$

$$V_{\underline{x}}^{\mathrm{G}[1]} = \frac{i\,g\,P^+}{s} \int\limits_{-\infty}^{\infty} dx^- V_{\underline{x}}[\infty, x^-]\, F^{12}(x^-, \underline{x})\; V_{\underline{x}}[x^-, -\infty]$$

Sub-Eikonal Gluon Exchange

- The gluon field in the operator relates to color charge/current such that

$$-\vec{\nabla}^2 A^{+a}(x) = \rho^a(x) = \sum_f \bar{\psi}^f(x)\,\gamma^+ t^a \psi^f(x) \qquad \Leftrightarrow \qquad p_\perp^2\, \tilde{A}^{+a}(x^-, \underline{p}) = \tilde{\rho}^a(x^-, \underline{p})$$

$$-\vec{\nabla}^2 A^{ja}(x) = J^{ja}(x) = \sum_f \bar{\psi}^f(x)\,\gamma^j t^a \psi^f(x) \qquad \Leftrightarrow \qquad p_\perp^2\, \tilde{A}^{ja}(x^-, \underline{p}) = \tilde{J}^{ja}(x^-, \underline{p})$$

- The quark field can be written in terms of creation & annihilation operators:

$$\psi^f_{i\alpha}(x^-, \underline{x}) = \int \frac{dp^+ d^2p}{(2\pi)^3 2p^+} \sum_S \left[\hat{b}^f_{p,i,S} u^\alpha_S(p)\, e^{-ip^+x^- + i\underline{p}\cdot\underline{x}} + \hat{d}^{f\dagger}_{p,i,S} v^\alpha_S(p)\, e^{ip^+x^- - i\underline{p}\cdot\underline{x}} \right]$$

which act on the quark states within the proton.

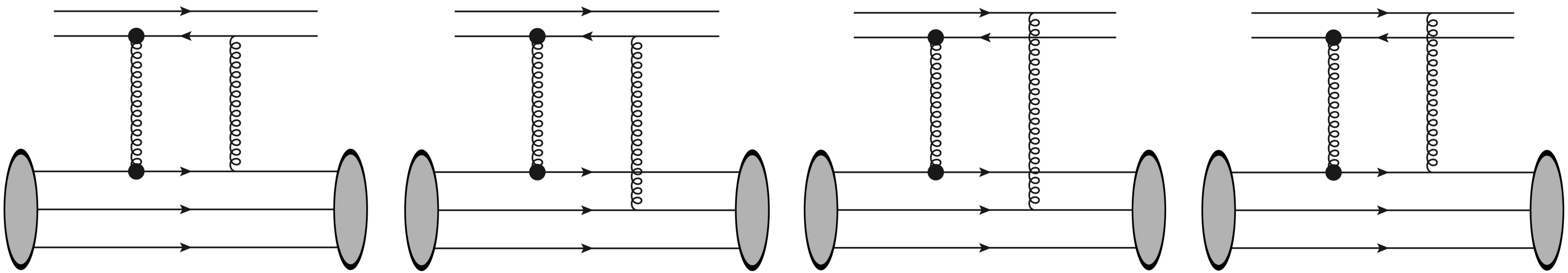
$$V_{\underline{x}}^{\mathrm{G}[1]} = \frac{i\,g\,P^+}{s} \int\limits_{-\infty}^{\infty} dx^- V_{\underline{x}}[\infty, x^-]\, F^{12}(x^-, \underline{x})\; V_{\underline{x}}[x^-, -\infty]$$

Sub-Eikonal Gluon Exchange

$$Q_f^{\mathrm{G}}(x_{10}^2, zs) = \frac{zs}{2N_c} \int d^2\left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \mathrm{Re}\left\langle \mathrm{T\,tr}\left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{G}[1]\dagger}\right] + \mathrm{T\,tr}\left[V_{\underline{1}}^{\mathrm{G}[1]} V_{\underline{0}}^{\dagger}\right]\right\rangle$$

- Based on the diagrams shown below, a gluon-exchange contribution to Q_f reads

$$\frac{zs}{2N_c} \int d^2\left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \mathrm{Re}\left\langle \mathrm{T\,tr}\left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{G}[1]\dagger}\right]\right\rangle = -\frac{ig^2 P^+}{4N_c}\, \delta^{ab} \epsilon^{ij} \int \frac{d^2 p}{(2\pi)^2} \frac{\underline{p}^i}{p_\perp^2} \left[1 - e^{-i\underline{p}\cdot\underline{x}_{10}}\right]$$
$$\times \int\limits_{-\infty}^{\infty} dx^- \int\limits_{-\infty}^{\infty} dy^-\, \mathrm{Re}\left\langle \mathrm{T}\,\tilde{\rho}^a(x^-, \underline{p})\, \tilde{J}^{jb}(y^-, -\underline{p})\right\rangle$$



$$V_{\underline{x}}^{\mathrm{G}[1]} = \frac{i\,g\,P^+}{s} \int\limits_{-\infty}^{\infty} dx^- V_{\underline{x}}[\infty, x^-]\, F^{12}(x^-, \underline{x})\; V_{\underline{x}}[x^-, -\infty]$$

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$$Q_f^{\mathrm{G}}(x_{10}^2, zs) = \frac{zs}{2N_c} \int d^2\left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \mathrm{Re}\left\langle \mathrm{T}\,\mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{G}[1]\dagger}\right] + \mathrm{T}\,\mathrm{tr}\left[V_{\underline{1}}^{\mathrm{G}[1]} V_{\underline{0}}^{\dagger}\right]\right\rangle$$

- Finally, we evaluate the quark fields acting on the proton state to get

$$Q_f^{\mathrm{G}(0)}(x_{10}^2, zs) = \widetilde{G}^{\mathrm{G}(0)}(x_{10}^2, zs) = \frac{g^4 C_F}{6} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p_\perp^2} \left[1 - e^{-i\underline{p}\cdot\underline{x}_{10}}\right]$$
$$\times \int \frac{dx_1 dx_2 dx_3}{(4\pi)^3}\, 4\pi\delta(1 - x_1 - x_2 - x_3)\, \frac{1}{x_1} \int \frac{d^2q_1 d^2q_2 d^2q_3}{(2\pi)^6}\, (2\pi)^2 \delta^2(q_{1\perp} + q_{2\perp} + q_{3\perp})$$
$$\times \left[|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 - \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)\, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3)\right]$$

- The adjoint type-1 dipole amplitude, $\widetilde{G}$, has exactly the same gluon-exchange term as that of Q_f.

$$V_{\underline{z}}^{i\,\mathrm{G}[2]} \equiv \frac{P^+}{2s} \int_{-\infty}^{\infty} dz^- \, V_{\underline{z}}[\infty, z^-] \left[D^i(z^-, \underline{z}) - \overleftarrow{D}^i(z^-, \underline{z}) \right] V_{\underline{z}}[z^-, -\infty]$$

Sub-Eikonal Gluon Exchange

$$G_2(x_{10}^2, zs) = \frac{\epsilon^{ij}(x_{10})_\perp^j}{x_{10}^2} \int d^2\left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \frac{zs}{2N_c} \left\langle \mathrm{tr}\left[V_{\underline{0}}^\dagger V_{\underline{1}}^{i\,\mathrm{G}[2]} + \left(V_{\underline{1}}^{i\,\mathrm{G}[2]}\right)^\dagger V_{\underline{0}} \right] \right\rangle$$

- The results for G_2 contains 2 terms, one ~ A^+A^i and the other ~ A^+A^+.
  - The latter eventually leads to $\langle \rho^a \rho^b \rangle \sim \sum_{\mathcal{S}_L} \mathcal{S}_L = 0$.
  - The former is proportional to the results for Q_f.

- Eventually, we have

$$G_2^{(0)}(x_{10}^2, zs) = -\frac{ig^4 C_F}{6} \int \frac{d^2 p}{(2\pi)^2} \frac{\underline{p} \cdot \underline{x}_{10}}{p_\perp^4 x_{10}^2} \left[1 - e^{-i\underline{p}\cdot\underline{x}_{10}}\right] \int \frac{dx_1 dx_2 dx_3}{(4\pi)^3} 4\pi\delta(1 - x_1 - x_2 - x_3) \frac{1}{x_1}$$
$$\times \int \frac{d^2 q_1 d^2 q_2 d^2 q_3}{(2\pi)^6} (2\pi)^2 \delta^2(q_{1\perp} + q_{2\perp} + q_{3\perp})$$
$$\times \left[|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 - \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)\, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3) \right]$$

Quark Exchange

- No contribution for G_2.
- For Q_f, the quark-exchange Wilson line is

$$V_{\underline{x}}^{\mathrm{q}[1]} = \frac{g^2 P^+}{2\,s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, V_{\underline{x}}[\infty, x_2^-] \, t^b \, \psi_\beta(x_2^-, \underline{x}) \, U_{\underline{x}}^{ba}[x_2^-, x_1^-] \; \left[\gamma^+ \gamma^5\right]_{\alpha\beta} \, \bar{\psi}_\alpha(x_1^-, \underline{x}) \, t^a \, V_{\underline{x}}[x_1^-, -\infty]$$

such that

$$Q_f^{\mathrm{q}}(x_{10}^2, zs) = \frac{zs}{2N_c} \int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2} \right) \mathrm{Re} \left\langle \mathrm{T\,tr} \left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{q}[1]\dagger} \right] + \mathrm{T\,tr} \left[V_{\underline{1}}^{\mathrm{q}[1]} V_{\underline{0}}^{\dagger} \right] \right\rangle$$

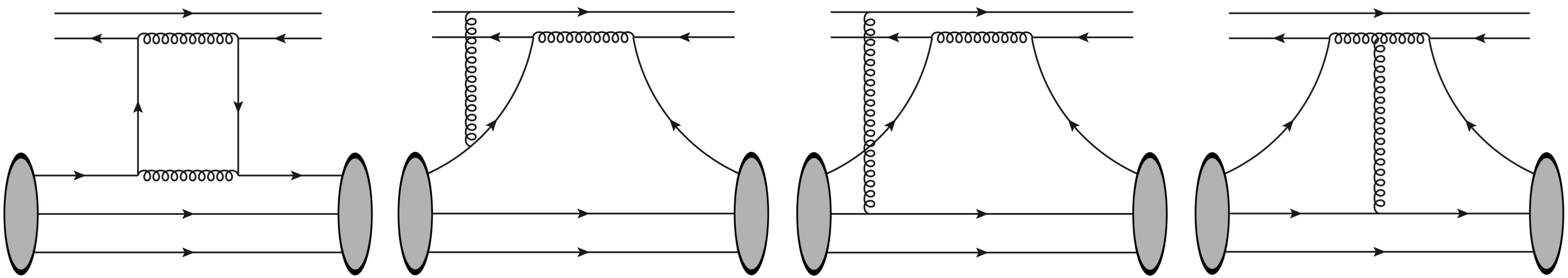
- Here, the quark fields in the sub-eikonal Wilson line yield creation and annihilation operators without the need to write out the charge or current.

$$Q_f^{\mathrm{q}}(x_{10}^2, zs) = \frac{zs}{2N_c} \int d^2\left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \mathrm{Re}\left\langle \mathrm{T\,tr}\left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{q}[1]\dagger}\right] + \mathrm{T\,tr}\left[V_{\underline{1}}^{\mathrm{q}[1]} V_{\underline{0}}^{\dagger}\right]\right\rangle$$

Quark Exchange

$$V_{\underline{x}}^{\mathrm{q}[1]} = \frac{g^2 P^+}{2\,s} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- \, V_{\underline{x}}[\infty, x_2^-]\, t^b \, \psi_\beta(x_2^-, \underline{x})\, U_{\underline{x}}^{ba}[x_2^-, x_1^-] \, \left[\gamma^+ \gamma^5\right]_{\alpha\beta} \, \bar{\psi}_\alpha(x_1^-, \underline{x})\, t^a \, V_{\underline{x}}[x_1^-, -\infty]$$

- The non-zero contributions come from the diagrams:



- Overall, we have that

$$Q_f^{\mathrm{q}(0)}(x_{10}^2, zs) = -\frac{g^4 C_F}{36 N_c}\left[4\delta^{f,u} - \delta^{f,d}\right] \int \frac{dx_1 dx_2 dx_3}{(4\pi)^3}\, 4\pi\delta(1 - x_1 - x_2 - x_3)\, \frac{1}{x_1} \int \frac{d^2q_1 d^2q_2 d^2q_3}{(2\pi)^6}\, (2\pi)^2 \delta^2(q_{1\perp} + q_{2\perp} + q_{3\perp})$$
$$\times \left\{ \int \frac{d^2p}{(2\pi)^2} \frac{1}{p_\perp^2} e^{-i\underline{p}\cdot\underline{x}_{10}} \left[|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 - \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)\, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3) \right] \right.$$
$$\left. + \int \frac{d^2p}{(2\pi)^2} \frac{1}{p_\perp^2} \left[2\,|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 + (N_c^2 + 1)\, \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)\, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3) \right] \right\}$$

Quark Exchange

- For $\widetilde{G}$, in order to satisfy the quark-exchange terms of

$$\widetilde{G}(x_{10}^2, zs)\, S(x_{10}^2, zs) = \frac{zs}{2(N_c^2-1)} \int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \mathrm{Re} \left\langle \mathrm{T\,Tr}\left[U_{\underline{0}} U_{\underline{1}}^{\mathrm{pol}[1]\dagger}\right] + \mathrm{T\,Tr}\left[U_{\underline{1}}^{\mathrm{pol}[1]} U_{\underline{0}}^{\dagger}\right]\right\rangle$$

at large N_c&N_f, we need

$$\widetilde{G}^{\mathrm{q}}(x_{10}^2, zs) = \frac{zs}{2N_c} \int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \mathrm{Re} \left\langle \mathrm{T\,tr}\left[V_{\underline{0}} W_{\underline{1}}^{\mathrm{q}[1]\dagger}\right] + \mathrm{T\,tr}\left[W_{\underline{1}}^{\mathrm{q}[1]} V_{\underline{0}}^{\dagger}\right]\right\rangle$$

where

$$W_{\underline{y}}^{\mathrm{q}[1]} = \frac{g^2 P^+}{8s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^-\, V_{\underline{y}}[\infty, x_2^-]\, \psi_\beta(x_2^-, \underline{y})\, \left[\gamma^+\gamma_5\right]_{\alpha\beta} \bar{\psi}_\beta(x_1^-, \underline{y})\, V_{\underline{y}}[x_1^-, -\infty]$$

Quark Exchange

- We need

$$\widetilde{G}^{\rm q}(x_{10}^2, zs) = \frac{zs}{2N_c} \int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2} \right) \mathrm{Re} \left\langle \mathrm{T\,tr} \left[V_{\underline{0}} W_{\underline{1}}^{\mathrm{q}[1]\dagger} \right] + \mathrm{T\,tr} \left[W_{\underline{1}}^{\mathrm{q}[1]} V_{\underline{0}}^{\dagger} \right] \right\rangle$$

  where

$$W_{\underline{y}}^{\mathrm{q}[1]} = \frac{g^2 P^+}{8s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, V_{\underline{y}}[\infty, x_2^-] \, \psi_\beta(x_2^-, \underline{y}) \left[\gamma^+ \gamma_5 \right]_{\alpha\beta} \bar{\psi}_\beta(x_1^-, \underline{y}) \, V_{\underline{y}}[x_1^-, -\infty]$$

- For each contributing diagram, the result for $\widetilde{G}$ differs from that of Q_f by color factors. Overall, we have

$$\begin{aligned} \widetilde{G}^{\mathrm{q}(0)}(x_{10}^2, zs) &= \frac{g^4 C_F}{96 N_c} \int \frac{dx_1 dx_2 dx_3}{(4\pi)^3} \, 4\pi \delta(1 - x_1 - x_2 - x_3) \, \frac{1}{x_1} \int \frac{d^2 q_1 d^2 q_2 d^2 q_3}{(2\pi)^6} \, (2\pi)^2 \delta^2(q_{1\perp} + q_{2\perp} + q_{3\perp}) \\ &\times \left\{ \int \frac{d^2 p}{(2\pi)^2} \, \frac{4}{p_\perp^2} \, e^{-i \underline{p} \cdot \underline{x}_{10}} \left[|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 - \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3) \, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3) \right] \right. \\ &\left. - \int \frac{d^2 p}{(2\pi)^2} \, \frac{1}{p_\perp^2} \left[|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 - 4 \, \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3) \, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3) \right] \right\} \end{aligned}$$

Results: Q_f

$$Q_f^{\mathrm{G}(0)}(x_{10}^2, zs) = \widetilde{G}^{\mathrm{G}(0)}(x_{10}^2, zs) = \frac{g^4 C_F}{6} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p_\perp^2} \left[1 - e^{-i\underline{p}\cdot\underline{x}_{10}}\right]$$
$$\times \int \frac{dx_1 dx_2 dx_3}{(4\pi)^3} 4\pi\delta(1 - x_1 - x_2 - x_3) \frac{1}{x_1} \int \frac{d^2q_1 d^2q_2 d^2q_3}{(2\pi)^6} (2\pi)^2 \delta^2(q_{1\perp} + q_{2\perp} + q_{3\perp})$$
$$\times \left[|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 - \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)\, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3)\right]$$

$$Q_f^{\mathrm{q}(0)}(x_{10}^2, zs) = -\frac{g^4 C_F}{36 N_c} \left[4\delta^{f,u} - \delta^{f,d}\right] \int \frac{dx_1 dx_2 dx_3}{(4\pi)^3} 4\pi\delta(1 - x_1 - x_2 - x_3) \frac{1}{x_1} \int \frac{d^2q_1 d^2q_2 d^2q_3}{(2\pi)^6} (2\pi)^2 \delta^2(q_{1\perp} + q_{2\perp} + q_{3\perp})$$
$$\times \left\{ \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p_\perp^2} e^{-i\underline{p}\cdot\underline{x}_{10}} \left[|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 - \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)\, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3)\right] \right.$$
$$\left. + \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p_\perp^2} \left[2\, |\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 + (N_c^2 + 1)\, \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)\, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3)\right] \right\}$$

Results: $\widetilde{G}$

$$Q_f^{\mathrm{G}(0)}(x_{10}^2, zs) = \widetilde{G}^{\mathrm{G}(0)}(x_{10}^2, zs) = \frac{g^4 C_F}{6} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p_\perp^2} \left[1 - e^{-i\underline{p}\cdot\underline{x}_{10}}\right]$$
$$\times \int \frac{dx_1 dx_2 dx_3}{(4\pi)^3} 4\pi\delta(1 - x_1 - x_2 - x_3) \frac{1}{x_1} \int \frac{d^2 q_1 d^2 q_2 d^2 q_3}{(2\pi)^6} (2\pi)^2 \delta^2(q_{1\perp} + q_{2\perp} + q_{3\perp})$$
$$\times \left[|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 - \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)\, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3) \right]$$

$$\widetilde{G}^{\mathrm{q}(0)}(x_{10}^2, zs) = \frac{g^4 C_F}{96 N_c} \int \frac{dx_1 dx_2 dx_3}{(4\pi)^3} 4\pi\delta(1 - x_1 - x_2 - x_3) \frac{1}{x_1} \int \frac{d^2 q_1 d^2 q_2 d^2 q_3}{(2\pi)^6} (2\pi)^2 \delta^2(q_{1\perp} + q_{2\perp} + q_{3\perp})$$
$$\times \left\{ \int \frac{d^2 p}{(2\pi)^2} \frac{4}{p_\perp^2} e^{-i\underline{p}\cdot\underline{x}_{10}} \left[|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 - \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)\, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3) \right] \right.$$
$$\left. - \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p_\perp^2} \left[|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 - 4\, \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)\, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3) \right] \right\}$$

Results: G_2

$$G_2^{(0)}(x_{10}^2, zs) = -\frac{ig^4 C_F}{6} \int \frac{d^2p}{(2\pi)^2} \frac{\underline{p} \cdot \underline{x}_{10}}{p_\perp^4 \, x_{10}^2} \left[1 - e^{-i\underline{p}\cdot\underline{x}_{10}}\right] \int \frac{dx_1 dx_2 dx_3}{(4\pi)^3} \, 4\pi\delta(1 - x_1 - x_2 - x_3) \, \frac{1}{x_1}$$

$$\times \int \frac{d^2q_1 d^2q_2 d^2q_3}{(2\pi)^6} \, (2\pi)^2 \delta^2(q_{1\perp} + q_{2\perp} + q_{3\perp})$$

$$\times \left[|\Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3)|^2 - \Phi(x_1, \underline{q}_1; x_2, \underline{q}_2; x_3, \underline{q}_3) \, \Phi^*(x_1, \underline{q}_1 + \underline{p}; x_2, \underline{q}_2 - \underline{p}; x_3, \underline{q}_3) \right]$$

Momentum-Space Wave Function (Φ)

- We use the non-relativistic limit of the proton wave function from [Schlumpf, hep-ph/9212250] (also used in [Dumitru, Mäntysaari, Paatelainen, 2103.11682]), which
  - ➢ separates the flavor-spin part from the momentum-space part
  - ➢ models the momentum-space wave function as Gaussian in transverse momenta with the width fixed by a scale related to longitudinal momentum fraction

$$\Phi(x_i, q_{i\perp}) = N \, \exp\left[-\frac{1}{2\beta^2}\sum_{i=1}^{3}\frac{q_{i\perp}^2 + M^2}{x_i}\right]$$

where β and M were fixed in [Schlumpf, hep-ph/9212250] by form factor.

Discussions

- For each type-1 amplitude (Q_f and $\widetilde{G}$), we have 2 kinds of contribution:
  - ➢ Terms $\sim \int \frac{d^2p}{(2\pi)^2} \frac{1}{p_\perp^2} e^{-i\underline{p}\cdot\underline{x}_{10}}$ which gives $\ln \frac{1}{x_{10}^2 \Lambda^2}$ and the constant term
  - ➢ Terms $\sim \int \frac{d^2p}{(2\pi)^2} \frac{1}{p_\perp^2}$ which upon putting $zsx_{10}^2 \gg 1$ gives $\ln \frac{zs}{\Lambda^2}$ and the constant term
  - ➢ Coefficients of the log terms can be numerically determined.
  - ➢ Coefficients of the constant terms relate among all type-1 amplitudes.

Discussions

- For the type-2 amplitude, G_2, we only have the term $\sim \int \frac{d^2p}{(2\pi)^2} \frac{1}{p_\perp^2} e^{-i\underline{p}\cdot\underline{x}_{10}}$, which yields $\ln \frac{1}{x_{10}^2 \Lambda^2}$ and the constant term.
- Overall, this greatly reduces the number of parameters for flavor singlet amplitudes from 15 to at most 5, which come from the coefficients of the constant terms in the 5 amplitudes: Q_u, Q_d, Q_s, $\widetilde{G}$, G_2 specifically from the terms independent of the dipole size, x_{10}.
- The infrared scale, Λ, can be determined from fit to our numerical integral.

Conclusion

- The valence quark model provides physical inputs to put additional constraints to the initial conditions at moderate x_0 = 0.1 for flavor-singlet dipole amplitudes in helicity evolution.
- Flavor non-singlet results should be the same for quark exchange and qualitatively similar for gluon exchange.
- For complete results, stay tuned.
- Future work: apply these initial conditions to small-x helicity global analysis