Sub-eikonal corrections to scattering amplitudes at high energy

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Beyond Eikonal Scattering in High Energy Physics

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Motivation

DIS hadronic tensor (with strong and electromagnetic interactions only)

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x,Q^2) + \left(P_\mu - q_\mu \frac{q\cdot P}{q^2}\right)\left(P_\nu - q_\nu \frac{q\cdot P}{q^2}\right)\frac{F_2(x,Q^2)}{P\cdot q}$$
$$+ i\,\varepsilon_{\mu\nu\lambda\sigma}\, q^\lambda S^\sigma \frac{M}{P\cdot q}\, g_1(x,Q^2) + i\,\varepsilon_{\mu\nu\lambda\sigma} q^\lambda \left(S^\sigma - P^\sigma \frac{q\cdot S}{q\cdot P}\right)\frac{M}{P\cdot q}\, g_2(x,Q^2)$$

- To extract the polarized structure functions g_1 and g_2, we need the antisymmetric part of the leptonic tensor.
- Sub-eikonal corrections provide anti-symmetric terms of the hadronic tensor using the high-energy OPE.
- Study TMD evolution from low to large x_B.

Propagation in the shock wave: Wilson line (Spectator frame)

Boost of the fields $\qquad x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}}$

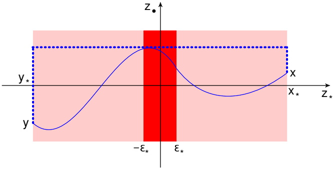
$$
\begin{aligned}
A^{-}(x^{-},x^{+},x_{\perp}) &\rightarrow \lambda A^{-}(\lambda^{-1}x^{-},\lambda x^{+},x_{\perp})\\
A^{+}(x^{-},x^{+},x_{\perp}) &\rightarrow \lambda^{-1}A^{+}(\lambda^{-1}x^{-},\lambda x^{+},x_{\perp})\\
A_{\perp}(x^{-},x^{+},x_{\perp}) &\rightarrow A_{\perp}(\lambda^{-1}x^{-},\lambda x^{+},x_{\perp})
\end{aligned}
$$

λ is the boost parameter.

$$
\langle x|\frac{i}{\hat{P}\!\!\!/ + i\epsilon}|y\rangle \;\rightarrow\; \langle x|\frac{i}{\hat{p}\!\!\!/ + g\gamma^{+}\hat{A}^{-} + i\epsilon}|y\rangle
$$

$$[\hat{p}^{+},\hat{A}^{cl}_{\mu}] = 0$$

Propagation in the shock wave: Wilson line (Spectator frame)



$$[\hat{p}^+, \hat{A}^{cl}_\mu] = 0$$

Scalar propagator

$$\langle x|\frac{i}{p^2 + 2p^+ g\hat{A}^- + i\epsilon}|y\rangle = \left[\int_0^{+\infty}\frac{đp^+}{2p^+}\theta(x^+ - y^+) - \int_{-\infty}^{0}\frac{đp^+}{2p^+}\theta(y_* - x_*)\right] e^{-ip^+(x^- - y^-)}$$

$$\times \int d^2z d^2z' \langle x_\perp|\, e^{-i\frac{\hat{p}^2_\perp}{2p^+}x^+}|z_\perp\rangle\langle z_\perp|\mathrm{Pexp}\left\{ig\int_{y^+}^{x^+} d\omega^+\, e^{i\frac{\hat{p}^2_\perp}{2p^+}\omega^+}A^-(\omega^+)e^{-i\frac{\hat{p}^2_\perp}{2p^+}\omega^+}\right\}|z'_\perp\rangle$$

$$\times\langle z'_\perp|e^{i\frac{\hat{p}^2_\perp}{2p^+}y^+}|y_\perp\rangle$$

Propagation in the shock wave: Wilson line (Spectator frame)

$$\mathrm{Pexp}\left\{ig\int_{y^+}^{x^+} d\omega^+\, e^{i\frac{\hat{p}_\perp^2}{2p^+}\omega^+} A^-(\omega^+) e^{-i\frac{\hat{p}_\perp^2}{2^+}\omega^+}\right\} \simeq A^-(\omega^+) + O(\lambda^0)$$

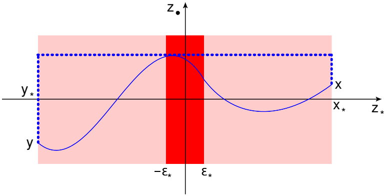
Scalar propagator

$$\langle x|\frac{i}{p^2+2p^+g\hat{A}^-+i\epsilon}|y\rangle = \left[\int_0^{+\infty}\frac{đp^+}{2p^+}\theta(x^+-y^+) - \int_{-\infty}^0\frac{đp^+}{2p^+}\theta(y^+-x^+)\right] e^{-ip^+(x^--y^-)}$$

$$\times\int d^2z d^2z' \langle x_\perp|\, e^{-i\frac{\hat{p}_\perp^2}{2p^+}x^+}|z_\perp\rangle\langle z_\perp|[x^+,y^+]|z'_\perp\rangle\langle z'_\perp|e^{i\frac{\hat{p}_\perp^2}{2p^+}y^+}|y_\perp\rangle$$

$$[x^+,y^+]_z = \mathrm{Pexp}\left\{ig\int_{y^+}^{x^+}d\omega^+ A^-(\omega^+,z_\perp)\right\}$$

Infinite boost: particle does not have time to deviate from straight line



Eikonal interactions give an infinite Wilson line

$$U_z = [\infty n_1 + z_\perp, -\infty n_1 + z_\perp] \qquad n_1 \cdot n_2 = 1$$

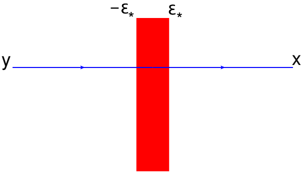
$$[x, y] = Pe^{ig\int_0^1 du(x-y)^\mu A_\mu(ux+(1-u)y)} \qquad p^\mu = p^+ n_1^\mu + p^- n_2^\mu + p_\perp^\mu$$

Quark propagator with eikonal interactions

$$\langle x|\frac{i}{\hat{\not P}+i\epsilon}|y\rangle = \left[\int_0^{+\infty}\frac{\bar{d}p^+}{2p^+}\theta(x^+-y^+) - \int_{-\infty}^{0}\frac{\bar{d}p^+}{2p^+}\theta(y^+-x^+)\right] e^{-ip^+(x^- - y^-)}$$

$$\times\frac{1}{2p^+}\,\langle x_\perp|\,e^{-i\frac{\hat{p}_\perp^2}{2p^+}x^+}\,\hat{\not p}\,\gamma^+\;U\;\hat{\not p}\,e^{i\frac{\hat{p}_\perp^2}{2p^+}y^+}\,|y_\perp\rangle$$

Shock-wave with finite width

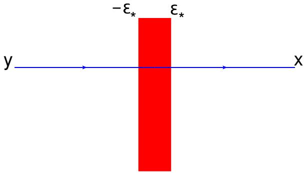


$$
\begin{aligned}
A^-(x^-,x^+,x_\perp) &\rightarrow \lambda A^-(\lambda^{-1}x^-,\lambda x^+,x_\perp) \\
A^+(x^-,x^+,x_\perp) &\rightarrow \lambda^{-1}A^+(\lambda^{-1}x^-,\lambda x^+,x_\perp \\
A_\perp(x^-,x^+,x_\perp) &\rightarrow A_\perp(\lambda^{-1}x^-,\lambda x^+,x_\perp)
\end{aligned}
$$

λ is the boost parameter

- $p^\mu = p^+ n_1^\mu + p^- n_2^\mu + p_\perp^\mu$
- small p^+ gluons are classical fields large p^+ gluons are quantum fields.
- Longitudinal size classical fields: $\epsilon^+ = \frac{2P^+}{l_\perp^2}$ with $l_\perp$ trans. mom. classical fileds
- Distance traveled by quantum fields: $z^+ = \frac{2P^+}{k_\perp^2}$ with $k_\perp$ trans. mom. classical fields
- We are in the case $|l_\perp| \sim |k_\perp|$: smal-x$_B$ regime

Shock-wave with finite width



$$A^-(x^-,x^+,x_\perp) \;\to\; \lambda A^-(\lambda^{-1}x^-,\lambda x^+,x_\perp)$$
$$A^+(x^-,x^+,x_\perp) \;\to\; \lambda^{-1}A^+(\lambda^{-1}x^-,\lambda x^+,x_\perp)$$
$$A_\perp(x^-,x^+,x_\perp) \;\to\; A_\perp(\lambda^{-1}x^-,\lambda x^+,x_\perp)$$

λ is the boost parameter

$$\hat{O} \equiv \{\hat{p}^\mu_\perp, \hat{A}^\perp_\mu\} + \{\hat{P}^-, \hat{A}^+\} - g\hat{A}^2_\perp$$

$$\langle x|\frac{i}{\hat{P}^2+i\epsilon}|y\rangle = \langle x|\frac{i}{\hat{p}^2+2p^+g\hat{A}^- + g\hat{O}+i\epsilon}|y\rangle$$
$$= \left[\int_0^{+\infty}\frac{đp^+}{2p^+}\theta(x^+-y^+) - \int_{-\infty}^0\frac{đp^+}{2p^+}\theta(y^+-x^+)\right]e^{-ip^+(x^--y^-)}$$
$$\times\langle x_\perp|\,e^{-i\frac{\hat{p}^2_\perp}{2p^+}x^+}\mathrm{Pexp}\left\{ig\int_{y^+}^{x^+}d\omega^+\,e^{i\frac{\hat{p}^2_\perp}{2p^+}\omega^+}\left(\hat{A}^-(\omega^+)+\frac{\hat{O}(\omega^+)}{2p^+}\right)e^{-i\frac{\hat{p}^2_\perp}{2p^+}\omega^+}\right\}e^{i\frac{\hat{p}^2_\perp}{2p^+}y^+}|y_\perp\rangle$$

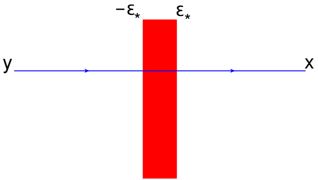
Shock-wave with finite width

$$e^{i\frac{\hat{p}_\perp^2}{2p^+}\omega^+}\left(\hat{A}^- + \frac{\hat{O}}{2p^+}\right)e^{-i\frac{\hat{p}_\perp^2}{2p^+}\omega^+} = \hat{A}^- + \frac{1}{2p^+}\Big(\{\hat{p}^i,\omega^+ F_i{}^- + (D^-\omega^+\hat{A}_i)\} + \{\hat{P}^-,\hat{A}^+\} - g\hat{A}_\perp^2\Big)$$

$$\hat{O} \equiv \{\hat{p}_\perp^\mu,\hat{A}_\mu^\perp\} + \{\hat{P}^-,\hat{A}^+\} - g\hat{A}_\perp^2$$

$$\langle x|\frac{i}{\hat{P}^2+i\epsilon}|y\rangle = \left[\int_0^{+\infty}\frac{đp^+}{2p^+}\theta(x^+-y^+) - \int_{-\infty}^0\frac{đ\alpha}{2\alpha}\theta(y^+-x^+)\right]e^{-ip^+(x^- - y^-)}$$
$$\times\langle x_\perp|\, e^{-i\frac{\hat{p}_\perp^2}{2p^+}x^+}\Big\{[x^+,y^+] + \frac{ig}{2p^+}\Big[x^+\Big(\{P_i,A^i(x^+)\} - gA_i(x^+)A^i(x^+)\Big)[x^+,y^+]$$
$$-[x^+,y^+]y^+\Big(\{P_i,A^i(y^+)\} - gA_i(y^+)A^i(y^+)\Big) + \int_{y^+}^{x^+}d\omega^+\Big(\{P^i,[x^+,\omega^+]\,\omega^+ F_i{}^-(\omega^+)\,[\omega^+,y^+]\}$$
$$+g\int_{\omega^+}^{x^+}d\,\omega'^+(\omega^+-\omega'^+)[x^+,\omega'^+]F^{i-}[\omega'^+,\omega^+]\;F_i{}^-\,[\omega^+,y^+]\Big)\Big]\Big\}e^{i\frac{\hat{p}_\perp^2}{2p^+}y^+}|y_\perp\rangle + O(\lambda^{-2})$$

Shock-wave with finite width: quark propagator



$$A^-(x^-,x^+,x_\perp) \rightarrow \lambda A^-(\lambda^{-1}x^-,\lambda x^+,x_\perp)$$
$$A^+(x^-,x^+,x_\perp) \rightarrow \lambda^{-1}A^+(\lambda^{-1}x^-,\lambda x^+,x_\perp)$$
$$A_\perp(x^-,x^+,x_\perp) \rightarrow A_\perp(\lambda^{-1}x^-,\lambda x^+,x_\perp)$$

λ is the boost parameter

sub-eikonal terms go like $\frac{1}{\lambda}$

$$\langle x|\frac{i}{\not{P}+i\epsilon}|y\rangle \;\rightarrow\; \langle x|\not{P}\,\frac{i}{p^2+2p^+A^-+ig\gamma^+\gamma^iF^-{}_i+\frac{1}{2}F_{ij}\sigma^{ij}+\ldots+i\epsilon}|y\rangle$$

- Note: $[\hat{p}^+,\hat{A}^{cl}_\mu]=0$

$$e^{i\frac{\hat{p}_\perp^2}{2p^+}z^+}\,\hat{A}^-(z^+)\,e^{-i\frac{\hat{p}_\perp^2}{2p^+}z^+} \simeq A^-(z^+)-\frac{z^+}{2p^+}\{p^i,F^-{}_i(z^+)\}-\frac{{z^+}^2}{8{p^+}^2}\{p^j,\{p^i,D_jF^-{}_i(z^+)\}\}+\ldots$$

Quark propagator with sub-eikonal corrections

$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}}$

$$
\langle x|\frac{i}{\hat{\not{P}}+i\epsilon}|y\rangle = \left[\int_0^{+\infty}\frac{\text{đ}p^+}{2p^+}\theta(x^+-y^+) - \int_{-\infty}^{0}\frac{\text{đ}p^+}{2p^+}\theta(y^+-x^+)\right] e^{-ip^+(x^- - y^-)}\frac{1}{2p^+}
$$

$$
\times\langle x_\perp|\, e^{-i\frac{\hat{p}_\perp^2}{2p^+}x^+}\Big\{\hat{\not{p}}\gamma^+[x^+,y^+]\hat{\not{p}} + \hat{\not{p}}\gamma^+\hat{\mathcal{O}}_1(p_\perp;x^+,y^+)\hat{\not{p}}
$$

$$
+ip^+\,\epsilon^{ij}\gamma^5\gamma_i\hat{\mathcal{O}}_j(p_\perp;x^+,y^+) - \frac{1}{2}\gamma^+[\hat{p}^j,\hat{\mathcal{O}}_j(p_\perp;x^+,y^+)] - \frac{i}{2}\epsilon^{ij}\gamma^5\gamma^+\{\hat{p}_i,\hat{\mathcal{O}}_j(p_\perp;x^+,y^+)\}
$$

$$
+ip^+\epsilon^{ij}\gamma^5\gamma_j\{p_i,\mathcal{O}^{-+}(x^+,y^+)\} - \frac{1}{2}\gamma^+[\hat{p}_\perp^2,\hat{\mathcal{O}}^{-+}(x^+,y^+)]\Big\} e^{i\frac{\hat{p}_\perp^2}{2p^+}y^+}|y_\perp\rangle + O(\lambda^{-2})
$$

- Leading-eikonal term
- Sub-eikonal terms   G.A.C JHEP 01 (2019), JHEP 06 (2021)

Operators $\hat{\mathcal{O}}_1$, $\hat{\mathcal{O}}_j$ and $\hat{\mathcal{O}}^{-+}$ *measure* the deviation from the straight line.

Quark propagator with sub-eikonal corrections

$$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}}$$

$$\hat{\mathcal{O}}_1(x^+, y^+; p_\perp) = \frac{ig}{2p^+} \int_{y^+}^{x^+} d\omega^+ \Big([x^+, \omega^+] \frac{1}{2} \sigma^{ij} F_{ij}[\omega^+, y^+] + \{\hat{p}^i, [x^+, \omega^+], \omega^+ F_i{}^-(\omega^+)[\omega^+, y^+]\}$$
$$+ g \int_{\omega^+}^{x^+} d\omega'^+ \, (\omega^+ - \omega'^+)[x^+, \omega'^+] F^{i-}[\omega'^+, \omega^+] \, F_i{}^-[\omega^+, y^+] \Big)$$

$$\hat{\mathcal{O}}_j(p_\perp; x^+, y^+) \equiv \frac{ig}{2p^+} \int_{y^+}^{x^+} d\omega^+ \Bigg[\{\hat{p}^k, [x^+, \omega^+] i F_{kj}[\omega^+, y^+]\}$$
$$+ \int_{\omega^+}^{x^+} d\omega'^+ \Big([x^+, \omega'^+] g F^{k-}[\omega'^+, \omega^+] i F_{kj}[\omega^+, {}^+] - [x^+, \omega'^+] i F_{kj}[\omega'^+, \omega^+] g F^{k-}[\omega^+, y^+]$$
$$+ [x^+, \omega'^+] \, i \, F^{-+}[\omega'^+, \omega^+] g F_j{}^-[\omega^+, y^+] - [x^+, \omega'^+] g F_j{}^-[\omega'^+, \omega^+] \, i \, F^{-+}[\omega^+, y^+] \Big) \Bigg].$$

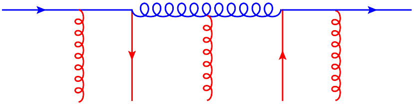
$$\hat{\mathcal{O}}^{-+}(x^+, y^+) \equiv -\frac{g}{2p^+} \int_{y^+}^{x^+} d\omega^+ [x^+, \omega^+] F^{-+}[\omega^+, y^+]$$

Quark propagator with sub-eikonal corrections

$$\hat{\mathcal{O}}_1(x^+,y^+;p_\perp) = \frac{ig}{2p^+}\int_{y^+}^{x^+} d\omega^+ \Big([x^+,\omega^+]\frac{1}{2}\sigma^{ij}F_{ij}[\omega^+,y^+] + \{\hat{p}^i, [x^+,\omega^+]\,\omega^+\,F_i{}^-(\omega^+)\,[\omega^+,y^+]\}$$
$$+g\int_{\omega^+}^{x^+} d\omega'^+\,(\omega^+-\omega'^+)[x^+,\omega'^+]F^{i-}[\omega'^+,\omega^+]F_i{}^-\,[\omega^+,y^+]\Big)$$

O_1 confirmed by T. Altinoluk, G. Beuf, A. Czajka and A. Tymowska (2021) considering forward matrix elements

Quark propagator in the background of quark fields



G.A.C JHEP 01 (2019)

$$\langle \mathrm{T}\{\psi(x)\bar{\psi}(y)\}\rangle_{\psi,\bar{\psi}}$$

$$\stackrel{x^+>0>y^+}{\ni} -\frac{g^2}{8s\pi^3(x^+y^+)^2}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\int\frac{d^2z}{(\mathcal{Z}^2+i\epsilon)^2}\left(x^+\gamma^- + \not{X}_\perp\right)$$

$$\times[\infty p_1, z^+]_z t^a\left(\gamma_\perp^\mu\psi(z^+,x_\perp)[z^+,z'^+]_z^{ab}\bar{\psi}(z'^+,z_\perp)\gamma_\mu^\perp\right)t^b[z'^+,-\infty p_1]_z\left(y^+\gamma^- + \not{Y}_\perp\right)$$

$$\mathcal{Z} \equiv \sqrt{\frac{2}{s}}\left[\frac{(x-z)_\perp^2}{x^+} - \frac{(y-z)_\perp^2}{y^+} - 2(x^- - y^-)\right]$$

Sub-eikonal corrections for gluon propagator: F^{-i} components

Light-cone gauge

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle_A = \left[-\int_0^{+\infty}\frac{\text{đ}\alpha}{2\alpha}\theta(x^+-y^+)+\int_{-\infty}^0\frac{\text{đ}p^+}{2p^+}\theta(y^+-x^+)\right]e^{-ip^+(x^--y^-)}$$
$$\times\langle x_\perp|e^{-i\frac{\hat{p}_\perp^2}{2p^+}x^+}\left(\delta^\xi_\mu-\frac{n_{2\mu}}{p^+}p^\xi\right)\mathcal{O}_\alpha(x^+,y^+)\left(g_{\xi\nu}-p_\xi\frac{n_{2\nu}}{p^+}\right)e^{i\frac{\hat{p}_\perp^2}{2p^+}y^+}|y_\perp\rangle^{ab}+i\langle x|\frac{n_{2\mu}n_{2\nu}}{{p^+}^2}|y\rangle^{ab}$$

$$\mathcal{O}_\alpha(x^+,y^+)\equiv\ [x^+,y^+]+\frac{ig}{2p^+}\int_{y^+}^{x^+}d\omega^+\Big(\{p^i,[x^+,\omega^+]\,\omega^+F_i{}^-(\omega^+)\,[\omega^+,y^+]\}$$
$$+g\int_{\omega^+}^{x^+}d\frac{2}{s}\,\omega'^+(\omega^+-\omega'^+)[x^+,\omega'^+]F^{i-}[\omega'^+,\omega^+]\ F_i{}^-[\omega^+,y^+]\Big)$$

In the background-Feynamn gauge see paper G.A.C. 2019

Sub-eikonal corrections for gluon propagator: F^{-i} components

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle_A = \left[-\int_0^{+\infty}\frac{d\!\!{}^-\alpha}{2\alpha}\theta(x^+-y^+)+\int_{-\infty}^0\frac{d\!\!{}^-p^+}{2p^+}\theta(y^+-x^+)\right]e^{-ip^+(x^--y^-)}$$

$$\times\langle x_\perp|e^{-i\frac{\hat{p}_\perp^2}{2p^+}x^+}\left(\delta^\xi_\mu-\frac{n_{2\mu}}{p^+}p^\xi\right)\mathcal{O}_\alpha(x^+,y^+)\left(g_{\xi\nu}-p_\xi\frac{n_{2\nu}}{p^+}\right)e^{i\frac{\hat{p}_\perp^2}{2p^+}y^+}|y_\perp\rangle^{ab}+i\langle x|\frac{n_{2\mu}n_{2\nu}}{{p^+}^2}|y\rangle^{ab}$$

- Applied to the single inclusive gluon production cross section at central rapidities and the light-front helicity asymmetry, in pA collisions.
  - T. Altinoluk, N. Armesto, G. Beuf, M. Martínez and C. A. Salgado (2014)
  - T. Altinoluk,a N. Armesto,a G. Beuff and A. Moscoso (2015)
- Study rapidity evolution of gluon transverse momentum dependent distribution (TMD) changes from nonlinear evolution at small $x_B \ll 1$ to linear evolution at moderate $x_B \sim 1$.
  - I. Balitsky and A. Tarasov (2015-2016)

Sub-eikonal corrections for gluon propagator

G.A.C JHEP 01 (2019)

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle_A = \left[-\int_0^{+\infty}\frac{d\!\!\bar{}\,\alpha}{2\alpha}\theta(x^+-y^+) + \int_{-\infty}^0\frac{d\!\!\bar{}\,p^+}{2p^+}\theta(y^+-x^+)\right]e^{-ip^+(x^- - y^-)}$$
$$\times\langle x_\perp|e^{-i\frac{\hat{p}^2_\perp}{2p^+}x^+}\left(\delta^\xi_\mu - \frac{n_{2\mu}}{p^+}p^\xi\right)\mathcal{O}_\alpha(x^+,y^+)\left(g_{\xi\nu} - p_\xi\frac{n_{2\nu}}{p^+}\right)e^{i\frac{\hat{p}^2_\perp}{2p^+}y^+}|y_\perp\rangle^{ab} + i\langle x|\frac{n_{2\mu}n_{2\nu}}{p^{+2}}|y\rangle^{ab}$$
$$+\left[-\int_0^{+\infty}\frac{d\!\!\bar{}\,p^+}{2p^+}\theta(x^+-y^+) + \int_{-\infty}^0\frac{d\!\!\bar{}\,p^+}{2p^+}\theta(y^+-x^+)\right]e^{-ip^+(x^- - y^-)}\langle x_\perp|e^{-i\frac{\hat{p}^2_\perp}{2p^+}x^+}$$
$$\times\left[\mathfrak{G}^{ab}_{1\mu\nu}(x^+,y^+;p_\perp) + \mathfrak{G}^{ab}_{2\mu\nu}(x^+,y^+;p_\perp) + \mathfrak{G}^{ab}_{3\mu\nu}(x^+,y^+;p_\perp) + \mathfrak{G}^{ab}_{4\mu\nu}(x^+,y^+;p_\perp)\right]$$
$$\times e^{i\frac{\hat{p}^2_\perp}{2p^+}y^+}|y_\perp\rangle + O(\lambda^{-2})$$

$$\mathcal{O}_\alpha(x^+,y^+) \equiv [x^+,y^+] + \frac{ig}{2p^+}\int_{y^+}^{x^+}d\omega^+\Big(\{p^i,[x^+,\omega^+]\,\omega^+\,F_i{}^-(\omega^+)\,[\omega^+,y^+]\}$$
$$+g\int_{\omega^+}^{x^+}d\,\omega'^+(\omega^+-\omega'^+)[x^+,\omega'^+]F^{i-}[\omega'^+,\omega^+]\;F_i{}^-[\omega^+,y^+]\Big)$$

Sub-eikonal corrections for gluon propagator

G.A.C JHEP 01 (2019)

$$\mathfrak{G}^{ab}_{1\mu\nu}(x^+,y^+;p_\perp) = -\frac{g\,n_{2\mu}n_{2\nu}}{4(p^+)^3}\int_{y^+}^{x^+}d\omega^+\Big[4p^i[x^+,\omega^+]F_{ij}[\omega^+,y^+]p^j$$
$$+ig\int_{\omega'^+}^{x^+}d\omega'^+\,(\omega'^+-\omega^+)[x^+,\omega'^+]iD^iF_i{}^-[\omega'^+,\omega^+]iD^jF_j{}^-[\omega^+,y^+]\Big]^{ab},$$

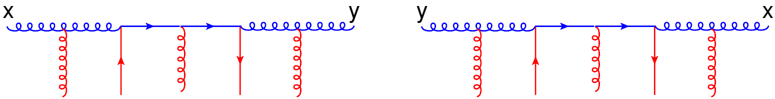
$$\mathfrak{G}^{ab}_{2\mu\nu}(x^+,y^+;p_\perp) = -\frac{g}{p^+}\delta^i_\mu\delta^j_\nu\int_{y^+}^{x^+}d\omega^+\big([x^+,\omega^+]F_{ij}[\omega^+,y^+]\big)^{ab},$$

$$\mathfrak{G}^{ab}_{3\mu\nu}(x^+,y^+;p_\perp) = \frac{g}{2(p^+)^2}\Big(\delta^j_\mu n_{2\nu}+\delta^j_\nu n_{2\mu}\Big)\int_{y^+}^{x^+}d\omega^+\,\big([x^+,\omega^+]iD^iF_{ij}[\omega^+,y^+]\big)^{ab},$$

$$\mathfrak{G}^{ab}_{4\mu\nu}(x^+,y^+;p_\perp) = -\frac{g^2}{(p^+)^2}\int_{y^+}^{x^+}d\omega^+\int_{\omega^+}^{x^+}d\omega'^+\Big(\delta^j_\nu n_{2\mu}[x^+,\omega'^+]F^{i-}[\omega'^+,\omega^+]F_{ij}[\omega^+,y^+]$$
$$+\delta^j_\mu n_{2\nu}[x^+,\omega'^+]F_{ij}[\omega'^+,\omega^+]F^{i-}[\omega^+,y^+]\Big)^{ab}$$

F_{ij} components are necessary to study, for example, spin dynamics
G.A.C JHEP 06 (2021)

Gluon propagator in the background of quark fields



$$\langle A^a_\mu(x) A^b_\nu(y)\rangle_{\psi,\bar\psi} = \Big[-\int_0^{+\infty} \frac{{\mathchar'26\mkern-12mu d}\alpha}{2\alpha}\theta(x^+ - y^+) + \int_{-\infty}^{0} \frac{{\mathchar'26\mkern-12mu d} p^+}{2p^+}\theta(y^+ - x^+)\Big] e^{-ip^+(x^- - y^-)}$$

$$\times g^2 \int_{y^+}^{x^+} dz_1^+ \int_{y^+}^{z_1^+} dz_2^+ \, \frac{1}{4p^+} \Bigg[\langle x_\perp | \, e^{-i\frac{\hat{p}_\perp^2}{2p^+}x^+} \Big(g^\xi_{\perp\mu} - \frac{n_{2\mu}}{p^+} p^\xi_\perp \Big)$$

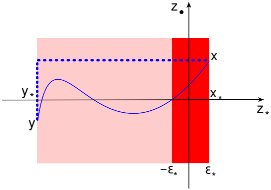
$$\times \bar\psi(z_1^+) \gamma^\perp_\xi \, \gamma^- \, [z_1^+, x^+] t^a [x^+, y^+] t^b [y^+, z_2^+] \gamma^\sigma_\perp \psi(z_2^+) \Big(g^\perp_{\sigma\nu} - p^\perp_\sigma \frac{n_{2\nu}}{p^+} \Big) e^{i\frac{\hat{p}_\perp^2}{2p^+}y^+} |y_\perp\rangle$$

$$+ \langle y_\perp | \, e^{-i\frac{\hat{p}_\perp^2}{2p^+}y^+} \Big(g^\xi_{\perp\nu} - \frac{n_{2\nu}}{p^+} p^\xi_\perp \Big) \bar\psi(z_2^+) \gamma^\perp_\xi \, \gamma^- [z_2^+, y^+] t^b [y^+, x^+] t^a [x^+, z_1^+] \gamma^\sigma_\perp \psi(z_1^+)$$

$$\times \Big(g^\perp_{\sigma\mu} - p^\perp_\sigma \frac{n_{2\mu}}{p^+} \Big) e^{i\frac{\hat{p}_\perp^2}{2p^+}x^+} |x_\perp\rangle \Bigg] + O(\lambda^{-2})$$

G.A.C JHEP 01 (2019)

Particle starts its propagation inside the shock-wave



Scalar Propagator

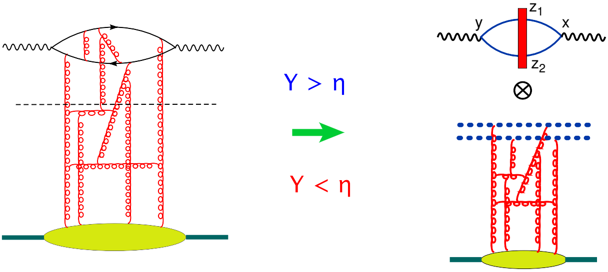
G.A.C JHEP 01 (2019)

$$
\begin{aligned}
\langle x|\frac{i}{P^2+i\epsilon}|y\rangle &= \left[\int_0^{+\infty}\frac{đp^+}{2p^+}\theta(x^+-y^+) - \int_{-\infty}^{0}\frac{đ\alpha}{2\alpha}\theta(y^+-x^+)\right]e^{-ip^+(x^- - y^-)}\\
&\times\langle x_\perp|\Bigg\{[x^+,y^+] + \frac{ig}{2p^+}\Bigg[[x^+,y^+](x^+-y^+)\Big(\{P_i,A^i(y^+)\} - gA_i(y^+)A^i(y^+)\Big)\\
&+\int_{y^+}^{x^+}d\omega^+\Big([x^+,\omega^+]\,iF^{-+}(\omega^+)[\omega^+,y^+] + [x^+,\omega^+](\omega^+-x^+)(iD^iF_i{}^-(\omega^+))[\omega^+,y^+]\\
&-2g\int_{\omega^+}^{x^+}d\omega'^+[x^+,\omega'^+](\omega'^+-x^+)F_i{}^-(\omega'^+)[\omega'^+,\omega^+]F^{i-}(\omega^+)[\omega^+,y^+]\\
&+2[x^+,\omega^+](\omega^+-x^+)F_i{}^-(\omega^+)[\omega^+,y^+]P^i\Bigg]\Bigg\}e^{-i\frac{\hat{p}_\perp^2}{2p^+}(x^+-y^+)}|y_\perp\rangle + O(\lambda^{-2})
\end{aligned}
$$

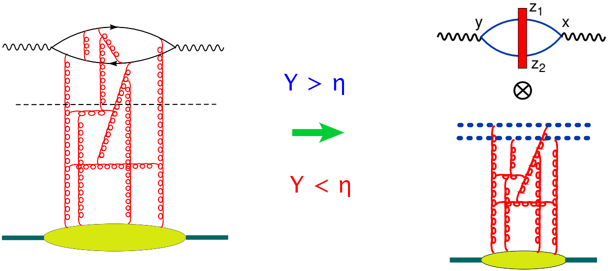
For the quark and gluon propagators see G.A.C JHEP 01 (2019)

High-Energy Operator Product Expansion

DIS amplitude is factorized in rapidity: η



$$\langle P|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|P\rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle P|\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|P\rangle + \dots$$

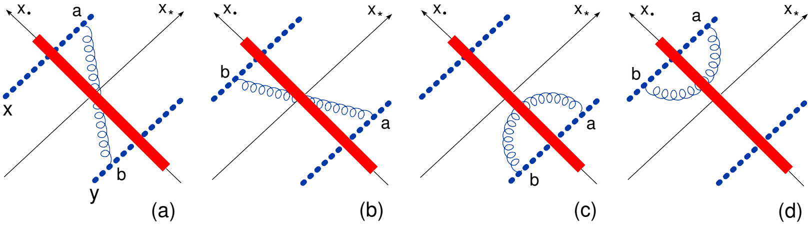


$$\langle P|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|P\rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle P|{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|P\rangle + \dots$$

- If we use a model (MV or GBW) to evaluate $\langle P|{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|P\rangle$ we can calculate the DIS cross-section.
- Energy dependence to the DIS cross section $\Rightarrow$ evolution of $\langle P|{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|P\rangle$ with respect to the rapidity parameter η.

$$\frac{d}{d\eta}\text{tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\text{LO}}\text{tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\text{tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{shockwave}} = \langle K_{\text{LO}}\text{tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{shockwave}}$$



$x_\bullet = \sqrt{\frac{s}{2}}x^-$ $\qquad$ $x_* = \sqrt{\frac{s}{2}}x^+$

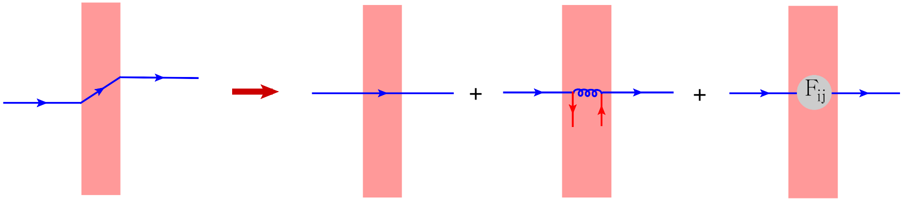
Non linear evolution equation: Balitsky-Kovchegov equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \; = \; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

- LLA for DIS in pQCD $\Rightarrow$ BFKL
  - (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$): describes proliferation of gluons.
- LLA for DIS in semi-classical-QCD $\Rightarrow$ BK eqn
  - background field method: describes recombination process.
- Note: if $x_\perp \;\to\; z_\perp$ or $y_\perp \;\to\; z_\perp$ divergences cancel out.

Quark propagator with sub-eikonal corrections

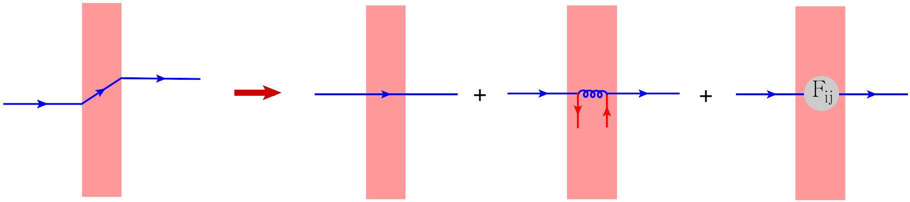


Quark propagator for g_1 structure function

$$\langle \mathrm{T}\{\psi(x)\bar{\psi}(y)\}\rangle_{A,\psi,\bar{\psi}} = \frac{-2}{s^2\pi^3(x^+y^+)^2}\int\frac{d^2z}{(\mathcal{Z}+i\epsilon)^3}\Big(x^+\gamma^- + \not{X}_\perp\Big)$$
$$\times\left\{i\sqrt{\frac{s}{2}}\gamma^+ U(z_\perp) + \frac{\mathcal{Z}}{8}\left(\gamma^+\gamma^5\frac{\mathcal{F}(z_\perp)}{\sqrt{2s}} + \gamma^\mu_\perp Q(z_\perp)\gamma^\perp_\mu\right)\right\}\Big(y^+\gamma^- + \not{Y}_\perp\Big)$$

$$\mathcal{Z} \equiv \sqrt{\frac{2}{s}}\left[\frac{(x-z)^2_\perp}{x^+} - \frac{(y-z)^2_\perp}{y^+} - 2(x^- - y^-)\right]$$

Quark propagator with sub-eikonal corrections



Quark propagator for g_1 structure function

$$\langle \mathrm{T}\{\psi(x)\bar{\psi}(y)\}\rangle_{A,\psi,\bar{\psi}} = -\frac{2}{s^2\pi^3(x^+y^+)^2}\int \frac{d^2z}{(\mathcal{Z}+i\epsilon)^3}\Big(x^+\gamma^- + \not{X}_\perp\Big)$$
$$\times\left\{ i\sqrt{\frac{s}{2}}\gamma^+ U(z_\perp) + \frac{\mathcal{Z}}{8}\left(\gamma^+\gamma^5\frac{\mathcal{F}(z_\perp)}{\sqrt{2s}} + \gamma^\mu_\perp Q(z_\perp)\gamma^\perp_\mu\right)\right\}\Big(y^+\gamma^- + \not{Y}_\perp\Big)$$

- Eikonal interaction
- Sub-eikonal interaction

Quark propagator with sub-eikonal corrections

$$\langle \mathrm{T}\{\psi(x)\bar{\psi}(y)\}\rangle_{A,\psi,\bar{\psi}} = -\frac{2}{s^2\pi^3(x^+y^+)^2}\int \frac{d^2z}{(\mathcal{Z}+i\epsilon)^3}\left(x^+\gamma^- + \not{X}_\perp\right)$$
$$\times\left\{i\sqrt{\frac{s}{2}}\gamma^+ U(z_\perp) + \frac{\mathcal{Z}}{8}\left(\gamma^+\gamma^5\frac{\mathcal{F}(z_\perp)}{\sqrt{2s}} + \gamma^\mu_\perp Q(z_\perp)\gamma^\perp_\mu\right)\right\}\left(y^+\gamma^- + \not{Y}_\perp\right)$$

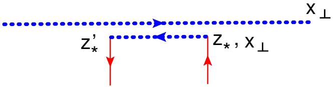
$$Q^{\alpha\beta}_{ij}(x_\perp) \equiv \frac{sg^2}{2}\int_{-\infty}^{+\infty} dz^+ \int_{-\infty}^{z^+} dz'^+ \left([\infty n_1, z^+]_x t^a \psi^\alpha(z^+, x_\perp)[z^+, z'^+]^{ab}_x \bar{\psi}^\beta(z'^+, x_\perp) t^b [z'^+, -\infty n_1]_x\right)_{ij}$$

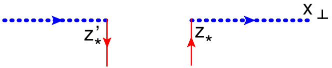
$$\mathcal{F}(z_\perp) \equiv ig\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ [\infty n_1, z^+]_z\, \epsilon^{ij} F_{ij}\left(z^+, z_{1\perp}\right)[z^+, -\infty n_1]_z\,.$$

Quark propagator with sub-eikonal corrections

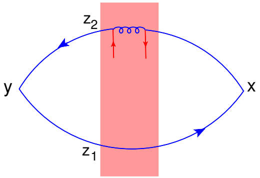
$$Q_{ij}^{\alpha\beta}(x_\perp) \equiv \frac{sg^2}{2}\int_{-\infty}^{+\infty} dz^+ \int_{-\infty}^{z^+} dz'^+ \Big([\infty p_1, z^+]_x t^a \psi^\alpha(z^+, x_\perp)[z^+, z'^+]_x^{ab} \bar{\psi}^\beta(z'_*, x_\perp) t^b [z'_*, -\infty p_1]_x \Big)_{ij}$$

$$Q_{ij}^{\alpha\beta}(x_\perp) = -g^2 \frac{s}{2}\int_{-\infty}^{+\infty} dz^+ \int_{-\infty}^{z_*} dz'^+ \Big[\frac{1}{2}\, U_x^{ij}\, \bar{\psi}^\alpha(z'^+, x_\perp)[z'^+, z^+]_x \psi^\beta(z^+, x_\perp)$$
$$+ \frac{1}{2N_c}\Big([\infty p_1, z^+]_x \psi^\beta(z^+, x_\perp) \bar{\psi}^\alpha(z'^+, x_\perp)[z'^+, -\infty] \Big)_{ij} \Big]$$





Impact Factor with sub-eikonal corrections: quark operator



$$i\,\bar\psi(z'^+,z_{2\perp})\gamma^\perp_\rho\hat Y_2\gamma^\nu\hat Y_1\not p_2\hat X_1\gamma^\mu\hat X_2\gamma^\rho_\perp\psi(z^+,z_{2\perp})$$
$$=\frac{8}{\sqrt{2s}}\bar\psi(z'^+,z_{2\perp})\,i\,\gamma^-\psi(z^+,z_{2\perp})I_1^{\mu\nu}(x^+,y^+;z_{1\perp},z_{2\perp})$$
$$-\frac{8}{\sqrt{2s}}\bar\psi(z'^+,z_{2\perp})\gamma^5\gamma^-\psi(z^+,z_{2\perp})I_5^{\mu\nu}(x^+,y^+;z_{1\perp},z_{2\perp})+O(\lambda^{-1})$$

$X_i^\mu = x^+n_1^\mu + X_{i\perp}^\mu \qquad X_{i\perp}^\mu = x_\perp^\mu - z_{i\perp}^\mu \qquad i=1,2$

$x^\mu = x^+n_1^\mu + x^-n_2^\mu + x_\perp \qquad n_1\cdot n_2 = 1$

$\gamma^+\psi = \bar{\psi}\gamma^+ = 0$ *bad* components

$$
\begin{aligned}
& i\,\bar{\psi}(z'^+, z_{2\perp})\gamma^\perp_\rho \hat{Y}_2\gamma^\nu \hat{Y}_1 \not{p}_2 \hat{X}_1\gamma^\mu \hat{X}_2\gamma^\rho_\perp \psi(z^+, z_{2\perp}) \\
&= \frac{8}{\sqrt{2s}}\,\bar{\psi}(z'^+, z_{2\perp})\, i\,\gamma^- \psi(z^+, z_{2\perp}) I_1^{\mu\nu}(x^+, y^+; z_{1\perp}, z_{2\perp}) \\
&\quad - \frac{8}{\sqrt{2s}}\,\bar{\psi}(z'^+, z_{2\perp})\gamma^5\gamma^- \psi(z^+, z_{2\perp}) I_5^{\mu\nu}(x^+, y^+; z_{1\perp}, z_{2\perp}) + O(\lambda^{-1})
\end{aligned}
$$

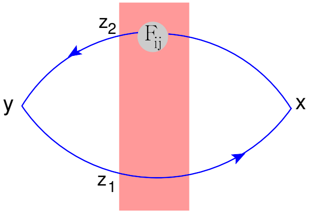
$$
I_1^{\mu\nu}(x, y; z_1, z_2) = \frac{s^2}{8}(x^+y^+)^2 \frac{\partial^2}{\partial x_\mu \partial y_\nu}\left(\mathcal{Z}_1\mathcal{Z}_2 - z^2_{12\perp}\frac{2(x-y)^2}{s\,x^+y^+}\right)
$$

$$
I_5^{\mu\nu}(x, y; z_1, z_2) = \frac{s}{2}\left(x^+\partial^\mu_x - n_2^\mu\right)\left(y^+\partial^\nu_y - n_2^\nu\right)\left[(\vec{Y}_1\times\vec{Y}_2)X_1\cdot X_2 - (\vec{X}_1\times\vec{X}_2)Y_1\cdot Y_2\right]
$$

$X_i^\mu = x^+ n_1^\mu + X^\mu_{i\perp}$ $\quad X^\mu_{i\perp} = x^\mu_\perp - z^\mu_{i\perp}$ $\quad i = 1, 2$

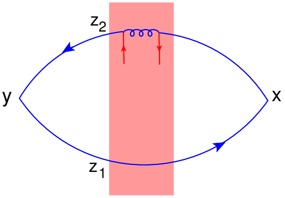
$\vec{x}\times\vec{y} = \epsilon^{ij}x_i y_j$

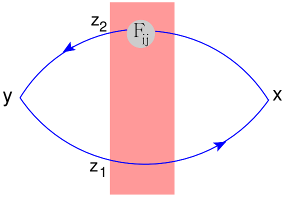
Impact Factor with F_{ij} operator



$$
\begin{aligned}
&\frac{s}{2}\mathrm{tr}\{\not{X}_1\gamma^+\not{Y}_1\gamma^\nu\not{Y}_2\gamma^+\sigma_\perp^{\alpha\beta}\not{X}_2\gamma^\mu\} \\
&= 4si\epsilon^{\alpha\beta}\left(x^+\partial_x^\mu - n_2^\mu\right)\left(y^+\partial_y^\nu - n_2^\nu\right)\left[(\vec{X}_1\times\vec{X}_2)Y_1\cdot Y_2 - (\vec{Y}_1\times\vec{Y}_2)X_1\cdot X_2\right] \\
&= 8\,i\,\epsilon^{\alpha\beta}\,I_{\mathcal{F}}^{\mu\nu}
\end{aligned}
$$

$$I_{\mathcal{F}}^{\mu\nu} = -I_5^{\mu\nu}$$





$$
\begin{aligned}
&\mathrm{T}\{\bar{\hat{\psi}}(x)\gamma^\mu\psi(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} \\
&= \int dz_1 dz_2\, \mathcal{I}^{\mu\nu}_{LO}(z_{1\perp}, z_{2\perp}; x, y)\Big[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} + \frac{\mathcal{Z}_2}{8\,s}\Big(\mathrm{Tr}\{\hat{U}^\dagger_{z_1}\hat{\mathcal{Q}}_{1z_2}\} + \mathrm{Tr}\{\hat{U}_{z_1}\hat{\mathcal{Q}}^\dagger_{1z_2}\}\Big)\Big] \\
&+\frac{1}{s}\int d^2z_1 d^2z_2\, \mathcal{I}^{\mu\nu}_5(z_{1\perp}, z_{2\perp}; x, y)\Big[\mathrm{Tr}\{\big(\hat{\mathcal{Q}}_{5z_2} + \hat{\mathcal{F}}_{z_2}\big)\hat{U}^\dagger_{z_1}\} + \mathrm{Tr}\{\big(\hat{\mathcal{Q}}^\dagger_{5z_2} + \hat{\mathcal{F}}^\dagger_{z_2}\big)\hat{U}_{z_1}\}\Big] \\
&+\mathcal{O}(\alpha_s) + \mathcal{O}(\lambda^{-2})
\end{aligned}
$$

Sub-eikonal Impact Factors

$$\mathcal{I}_1^{\mu\nu}(x,y;z_1,z_2) = \frac{\mathcal{Z}_2}{8}\mathcal{I}^{\mu\nu}_{LO} = \frac{4}{\pi^6 s^4 (x^+y^+)^4}\frac{I_1^{\mu\nu}(x,y;z_1,z_2)}{[\mathcal{Z}_1+i\epsilon]^3[\mathcal{Z}_2+i\epsilon]^2}$$

$$\mathcal{I}_5^{\mu\nu}(x,y;z_1,z_2) = -\frac{4}{\pi^6 s^4 (x^+y^+)^4}\frac{I_5^{\mu\nu}(x,y;z_1,z_2)}{[\mathcal{Z}_1+i\epsilon]^3[\mathcal{Z}_2+i\epsilon]^2}$$

$$I_1^{\mu\nu}(x,y;z_1,z_2) = \frac{s^2}{8}(x^+y^+)^2\frac{\partial^2}{\partial x_\mu \partial y_\nu}\left(\mathcal{Z}_1\mathcal{Z}_2 - z^2_{12\perp}\frac{2(x-y)^2}{s\,x^+y^+}\right)$$

$$I_5^{\mu\nu}(x,y;z_1,z_2) = \frac{s}{2}\left(x^+\partial^\mu_x - n^\mu_2\right)\left(y^+\partial^\nu_y - n^\nu_2\right)\left[(\vec{Y}_1\times\vec{Y}_2)X_1\cdot X_2 - (\vec{X}_1\times\vec{X}_2)Y_1\cdot Y_2\right]$$

$X_i^\mu = x^+n_1^\mu + X^\mu_{i\perp} \qquad X^\mu_{i\perp} = x^\mu_\perp - z^\mu_{i\perp} \qquad i = 1,2$

$\vec{x}\times\vec{y} = \epsilon^{ij}x_iy_j$

Symmetry of the sub-eikonal Impact Factors

Sub-eikonal Impact Factors are electromagnetic gauge invariant

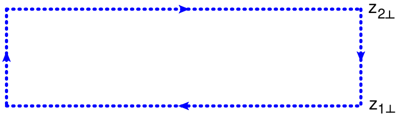
$$\partial_\mu \mathcal{I}_1^{\mu\nu}(x,y;z_1,z_2) = 0$$
$$\partial_\mu \mathcal{I}_5^{\mu\nu}(x,y;z_1,z_2) = 0$$

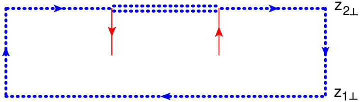
and $SL(2,C)$ Möbius invariant (inv. $x^\mu \to \frac{x^\mu}{x^2}$)

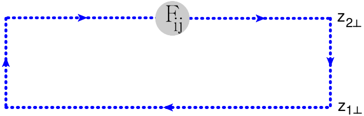
$$\int d^2z_2 d^2z_2\, \mathcal{I}_1^{\mu\nu}(x,y;z_1,z_2) \stackrel{\text{inv.}}{=} \int d^2z_2 d^2z_2\, \mathcal{I}_1^{\mu\nu}(x,y;z_1,z_2)$$
$$\int d^2z_2 d^2z_2\, \mathcal{I}_5^{\mu\nu}(x,y;z_1,z_2) \stackrel{\text{inv.}}{=} \int d^2z_2 d^2z_2\, \mathcal{I}_5^{\mu\nu}(x,y;z_1,z_2)$$

High-Energy OPE with sub-eikonal corrections

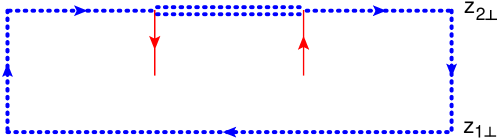
$$
\begin{aligned}
&\mathrm{T}\{\bar{\hat{\psi}}(x)\gamma^{\mu}\psi(x)\bar{\hat{\psi}}(y)\gamma^{\nu}\hat{\psi}(y)\}\\
&= \int dz_1 dz_2\, \mathcal{I}^{\mu\nu}_{LO}(z_{1\perp}, z_{2\perp}; x, y)\left[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^{\dagger}_{z_2}\} + \frac{\mathcal{Z}_2}{8\,s}\Big(\mathrm{Tr}\{\hat{U}^{\dagger}_{z_1}\hat{\mathcal{Q}}_{1z_2}\} + \mathrm{Tr}\{\hat{U}_{z_1}\hat{\mathcal{Q}}^{\dagger}_{1z_2}\}\Big)\right]\\
&+\frac{1}{s}\int d^2z_1 d^2z_2\, \mathcal{I}^{\mu\nu}_{5}(z_{1\perp}, z_{2\perp}; x, y)\left[\mathrm{Tr}\{\big(\hat{\mathcal{Q}}_{5z_2} + \hat{\mathcal{F}}_{z_2}\big)\hat{U}^{\dagger}_{z_1}\} + \mathrm{Tr}\{\big(\hat{\mathcal{Q}}^{\dagger}_{5z_2} + \hat{\mathcal{F}}^{\dagger}_{z_2}\big)\hat{U}_{z_1}\}\right]\\
&+\mathcal{O}(\alpha_s) + \mathcal{O}(\lambda^{-2})
\end{aligned}
$$



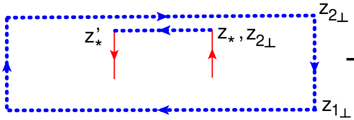




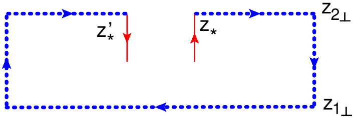
Fierz identity

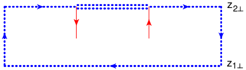


$= \frac{1}{2}$



$- \frac{1}{2N_c}$

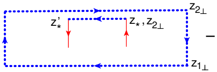




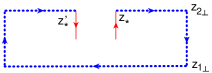
$$\mathrm{Tr}\{\hat{\mathcal{Q}}_{1x}\hat{U}_y^\dagger\} = \frac{1}{2}\,\mathrm{Tr}\{\hat{U}_y^\dagger\hat{U}_x\}\hat{\mathcal{Q}}_{1\,x} - \frac{1}{2N_c}\,\mathrm{Tr}\{\hat{U}_y^\dagger\hat{\tilde{\mathcal{Q}}}_{1x}\}$$

$$\mathrm{Tr}\{\hat{\mathcal{Q}}_{5x}\hat{U}_y^\dagger\} = \frac{1}{2}\,\mathrm{Tr}\{\hat{U}_y^\dagger\hat{U}_x\}\hat{\mathcal{Q}}_{5\,x} - \frac{1}{2N_c}\,\mathrm{Tr}\{\hat{U}_y^\dagger\hat{\tilde{\mathcal{Q}}}_{5x}\}$$

$= \frac{1}{2}$



$- \frac{1}{2N_c}$



$$\mathcal{Q}_{1x} = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\,\bar{\psi}(z'^+,x_\perp)\,i\gamma^-[z'^+,z^+]_x\psi(z^+,x_\perp)$$

$$\mathcal{Q}_{5x} \equiv -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\,\bar{\psi}(z'^+,x_\perp)\,\gamma^5\gamma^-[z'^+,z^+]_x\psi(z^+,x_\perp)$$

$$\tilde{\mathcal{Q}}_{1ij}(x_\perp) \equiv g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\Big([\infty p_1,z^+]_x\mathrm{tr}\{\psi(z^+,x_\perp)\bar{\psi}(z'^+,x_\perp)\,i\gamma^-\}[z'^+,-\infty p_1]\Big)_{ij}$$

$$\tilde{\mathcal{Q}}_{5ij}(x_\perp) \equiv g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\Big([\infty p_1,z^+]_x\mathrm{tr}\{\psi(z^+,x_\perp)\bar{\psi}(z'^+,x_\perp)\,\gamma^5\gamma^-\}[z'^+,-\infty p_1]\Big)_{ij}$$

High-Energy OPE with sub-eikonal corrections

$$
\begin{aligned}
&\mathrm{T}\{\bar{\hat{\psi}}(x)\gamma^\mu\psi(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} \\
&= \int dz_1 dz_2\, \mathcal{I}^{\mu\nu}_{LO}(z_{1\perp}, z_{2\perp}; x, y)\Big[\mathrm{Tr}\{\hat{U}^\dagger_{z_1}\hat{U}^\dagger_{z_2}\} \\
&\quad + \frac{1}{s}\frac{\mathcal{Z}_2}{16}\Big(\mathrm{Tr}\{\hat{U}^\dagger_{z_1}\hat{U}_{z_2}\}\hat{Q}_{1z_2} + \mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}\hat{Q}^\dagger_{1z_2} - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}^\dagger_{z_1}\hat{\tilde{Q}}_{1z_2}\} - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}_{z_1}\hat{\tilde{Q}}^\dagger_{1z_2}\}\Big)\Big] \\
&\quad + \frac{1}{s}\int d^2z_1 d^2z_2\, \mathcal{I}^{\mu\nu}_5(z_{1\perp}, z_{2\perp}; x, y)\Big[\mathrm{Tr}\{\hat{U}^\dagger_{z_1}\hat{U}_{z_2}\}\hat{Q}_{5z_2} + \mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}\hat{Q}^\dagger_{5z_2} \\
&\quad - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}^\dagger_{z_1}(\hat{\tilde{Q}}_{5z_2} - 2N_c\hat{\mathcal{F}}_{z_2})\} - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}_{z_1}(\hat{\tilde{Q}}^\dagger_{5z_2} - 2N_c\hat{\mathcal{F}}^\dagger_{z_2})\}\Big] \\
&\quad + \mathcal{O}(\alpha_s) + \mathcal{O}(\lambda^{-2})
\end{aligned}
$$

Eikonal term of the high-energy OPE: I. Balitsky (1996)

Sub-eikonal terms of the high-energy OPE: G.A.C. (2021)

Parametrization of the forward matrix elements

Quark distributions G.A.C. (2021)

$S^\mu \simeq \frac{\lambda}{M} P^\mu + S^\mu_\perp$ $\Delta^\mu_\perp = (x-y)^\mu_\perp$

$$\int d^2\Delta\, e^{i(\Delta,k)} \langle\langle P,S| \Big[Q_1(x_\perp)\, \mathrm{Tr}\{U_x U^\dagger_y\} + \mathrm{a.c.} \Big] |P,S\rangle\rangle = \frac{s}{2}\Big(q_1(k^2_\perp, x) + \frac{\vec{S}\times\vec{k}}{M} q_{1T}(k^2_\perp, x) \Big)$$

$$\int d^2\Delta\, e^{i(\Delta,k)} \langle\langle P,S| \Big[\mathrm{Tr}\{\tilde{Q}_1(x_\perp) U^\dagger_y\} + \mathrm{a.c.} \Big] |P,S\rangle\rangle = \frac{s}{2}\Big(\tilde{q}_1(k^2_\perp, x) + \frac{\vec{S}\times\vec{k}}{M} \tilde{q}_{1T}(k^2_\perp, x) \Big)$$

$$\int d^2\Delta e^{i(\Delta,k)} \langle\langle P,S| \Big[Q_5(x_\perp) \mathrm{Tr}\{U_x U^\dagger_y\} + \mathrm{a.c} \Big] |P,S\rangle\rangle = \frac{s}{2}\Big(\lambda q_{5L}(k^2_\perp, x) - \frac{(S,k)_\perp}{M} q_{5T}(k^2_\perp, x) \Big)$$

$$\int d^2\Delta\, e^{i(\Delta,k)} \langle\langle P,S| \Big[\mathrm{Tr}\{\tilde{Q}_5(x_\perp) U^\dagger_y\} + \mathrm{a.c} \Big] |P,S\rangle\rangle = \frac{s}{2}\Big(\lambda \tilde{q}_{5L}(k^2_\perp, x) - \frac{(S,k)_\perp}{M} \tilde{q}_{5T}(k^2_\perp, x) \Big)$$

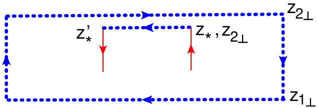
Parametrization of the forward matrix elements

Gluon distributions G.A.C. (2021)

$S^\mu \simeq \frac{\lambda}{M} P^\mu + S_\perp^\mu$ $\Delta_\perp^\mu = (x-y)_\perp^\mu$

$$\int d^2\Delta e^{i(\Delta,k)_\perp} \langle\langle P,S| \Big[\mathrm{Tr}\{\mathcal{F}(x_\perp) U_y^\dagger\} + \mathrm{a.c} \Big] |P,S\rangle\rangle = \frac{s}{2} \Big[\lambda \frac{k_\perp^2}{M^2} G_L(k_\perp^2, x) + \frac{(S,k)_\perp}{M} G_T(k_\perp^2, x) \Big]$$

$$\langle \mathrm{Tr}\{U_y^\dagger U_x\} Q_{1x} \rangle$$



Diagrams at one loop: quantum quark field

Figure: Diagrams with $\hat{Q}_{1x}$ and $\hat{Q}_{5x}$ quantum.

Evolution equation of sub-eikonal corrections

$$\langle \mathrm{Tr}\{U_y^\dagger U_x\} Q_{1x} \rangle$$

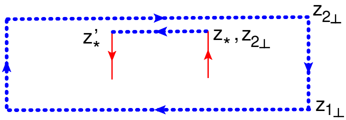
$$\langle \mathrm{Tr}\{U_y^\dagger U_x\} Q_{1x} \rangle = \frac{\alpha_s}{4\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2z \frac{\mathrm{Tr}\{U_y^\dagger U_x\}}{(x-z)_\perp^2} \left[\mathrm{Tr}\{U_x^\dagger U_z\}\, Q_{1z} - \frac{1}{N_c} \mathrm{Tr}\{U_x^\dagger \tilde{Q}_{1z}\} \right]$$

and

$$\langle \mathrm{Tr}\{U_y^\dagger U_x\} Q_{5x} \rangle = \frac{\alpha_s}{4\pi^2} \int_0^{+\infty} \frac{d\alpha}{\alpha} \int d^2z \frac{\mathrm{Tr}\{U_y^\dagger U_x\}}{(x-z)_\perp^2} \times \left[\mathrm{Tr}\{U_x^\dagger U_z\}\, Q_{5z} - \frac{1}{N_c} \mathrm{Tr}\{U_x^\dagger (\tilde{Q}_{5z} - 2N_c \mathcal{F}_z)\} \right]$$

Sanity check: operators of different parity do not mix

Evolution equation of sub-eikonal corrections

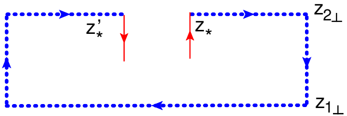


Diagrams at one loop: classical quark field (BK diagrams)

$$Q_{1x}\langle \mathrm{Tr}\{U_x U_y^\dagger\}\rangle = \frac{\alpha_s}{2\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2z \frac{(x-y)_\perp^2}{(x-z)_\perp^2 (y-z)_\perp^2} \times\Big[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c \mathrm{Tr}\{U_x U_y^\dagger\}\Big] Q_{1x}$$

It gives automatically Leading-Log resummation contribution.

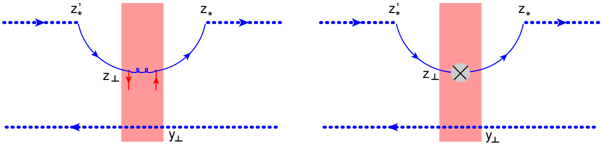
Evolution equation of sub-eikonal corrections



Diagrams at one loop: quantum quark field

Figure: Diagrams with $\tilde{Q}_{1x}$ and $\tilde{Q}_{5x}$ quantum.

Evolution equation of sub-eikonal corrections



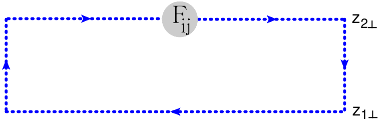
$$\langle \mathrm{Tr}\{U_y^\dagger \tilde{Q}_{1x}\}\rangle = \frac{\alpha_s}{4\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int \frac{d^2z}{(x-z)_\perp^2}\Big[\mathrm{Tr}\{U_y^\dagger U_z\}Q_{1z} - \frac{1}{N_c}\mathrm{Tr}\{U_y^\dagger \tilde{Q}_{1z}\}\Big]$$

and

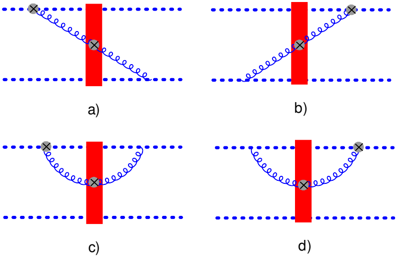
$$\langle \mathrm{Tr}\{U_y^\dagger \tilde{Q}_{5x}\}\rangle \;=\; \frac{\alpha_s}{4\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int \frac{d^2z}{(x-z)_\perp^2} \times\Big[\mathrm{Tr}\{U_y^\dagger U_z\}Q_{5z} - \frac{1}{N_c}\mathrm{Tr}\{U_y^\dagger\big(\tilde{Q}_{5z} - 2N_c\mathcal{F}_z\big)\}\Big]$$

Sanity check: operators of different parity do not mix

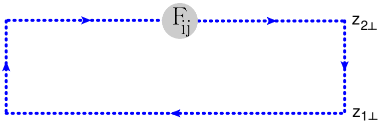
Evolution equation of sub-eikonal corrections



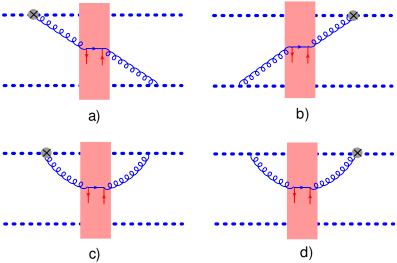
Diagrams with F_{ij} quantum



Evolution equation of sub-eikonal corrections



Diagrams with F_{ij} quantum



Evolution equation of sub-eikonal corrections

Diagrams with F_{ij} quantum

$$\langle \mathrm{Tr}\{\mathcal{F}_x\, U_y^\dagger\}\rangle = -\frac{\alpha_s}{\pi^2}\,\mathrm{Tr}\{U_x t^a U_y^\dagger t^b\}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2z$$
$$\times\Bigg[\frac{(\vec{x}-\vec{z})\times(\vec{z}-\vec{y})}{(x-z)_\perp^2\,(y-z)_\perp^2}\Big(\mathcal{Q}_{1z}^{ba}-\mathcal{Q}_{1z}^{ba\,\dagger}\Big)$$
$$-\Big(\frac{(x-z,z-y)}{(x-z)_\perp^2\,(y-z)_\perp^2}+\frac{1}{(x-z)_\perp^2}\Big)\Big(\mathcal{Q}_{5z}^{ba}+\mathcal{Q}_{5z}^{ba\,\dagger}+\mathcal{F}_z^{ba}\Big)$$
$$-4\pi^2\int\frac{d\!\!\!^{-}{}^2q}{q_\perp^2}\Big(e^{i(q,y-z)}-e^{i(q,x-z)}\Big)\delta^{(2)}(z-x)\mathcal{F}_z^{ba}\Bigg]$$

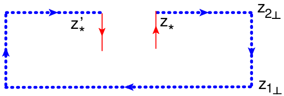
$$\mathcal{Q}_5^{ab}(z_\perp)\equiv g^2\int_{-\infty}^{+\infty}dz_{1*}\int_{-\infty}^{z_{1*}}dz_{2*}\bar{\psi}(z_{1*},z_\perp)\,\gamma^5\not{p}_1\,[z_{1*},\infty p_1]_z\,t^aU_z\,t^b\,[-\infty p_1,z_{2*}]_z\,\psi(z_{2*},z_\perp)$$

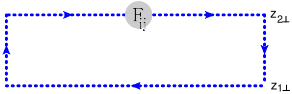
$$\mathcal{Q}_1^{ab}(z_\perp)\equiv g^2\int_{-\infty}^{+\infty}dz_{1*}\int_{-\infty}^{z_{1*}}dz_{2*}\bar{\psi}(z_{1*},z_\perp)\,i\not{p}_1\,[z_{1*},\infty p_1]_z\,t^aU_z\,t^b\,[-\infty p_1,z_{2*}]_z\,\psi(z_{2*},z_\perp)$$

Evolution equation of sub-eikonal corrections

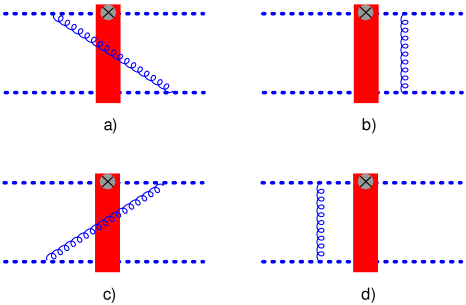
Diagrams with F_{ij} quantum

$$
\begin{aligned}
\langle\mathrm{Tr}\{\mathcal{F}_x\, U^\dagger_y\}\rangle =& \frac{\alpha_s}{2\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2z\\
&\times\Bigg\{\frac{1}{2}\frac{(\vec{x}-\vec{z})\times(\vec{z}-\vec{y})}{(x-z)^2_\perp(y-z)^2_\perp}\Bigg[\mathrm{Tr}\{U^\dagger_y\tilde{Q}_{1\,z}\}\mathrm{Tr}\{U^\dagger_z U_x\}-\mathrm{Tr}\{U_x\tilde{Q}^\dagger_{1z}\}\mathrm{Tr}\{U^\dagger_y U_z\}\\
&+\frac{1}{N_c}\Big(\mathrm{Tr}\{U_xU^\dagger_y\tilde{Q}_{1z}U^\dagger_z\}+\mathrm{Tr}\{U^\dagger_yU_xU^\dagger_z\tilde{Q}_{1z}\}-\mathrm{Tr}\{U_xU^\dagger_yU_z\tilde{Q}^\dagger_{1z}\}-\mathrm{Tr}\{U^\dagger_yU_x\tilde{Q}^\dagger_{1z}U_z\}\Big)\\
&+\frac{1}{N_c^2}\mathrm{Tr}\{U^\dagger_yU_x\}\Big(Q^\dagger_{1z}-Q_{1z}\Big)\Bigg]-\frac{1}{2}\Bigg[\frac{(x-z,z-y)}{(x-z)^2_\perp(y-z)^2_\perp}+\frac{1}{(x-z)^2_\perp}\Bigg]\\
&\times\Bigg[\mathrm{Tr}\{U^\dagger_y\big(\tilde{Q}_{5z}-2\mathcal{F}_z\big)\}\mathrm{Tr}\{U^\dagger_zU_x\}+\mathrm{Tr}\{U_x\big(\tilde{Q}^\dagger_{5z}-2\mathcal{F}^\dagger_z\big)\}\mathrm{Tr}\{U^\dagger_yU_z\}\\
&-\frac{1}{N_c}\Big(\mathrm{Tr}\{U_xU^\dagger_yU_z\tilde{Q}^\dagger_{5z}\}+\mathrm{Tr}\{U^\dagger_yU_x\tilde{Q}^\dagger_{5z}U_z\}+\mathrm{Tr}\{U_xU^\dagger_y\tilde{Q}_{5z}U^\dagger_z\}+\mathrm{Tr}\{U^\dagger_yU_xU^\dagger_z\tilde{Q}_{5z}\}\Big)\\
&+\frac{1}{N_c^2}\mathrm{Tr}\{U^\dagger_yU_x\}\Big(Q_{5z}+Q^\dagger_{5z}\Big)\Bigg]\Bigg\}
\end{aligned}
$$





Diagrams with F_{ij} or $\tilde{Q}_1$ (and $\tilde{Q}_5$) classical: BK-type diagrams



+ self-energy diagrams

Evolution equation of sub-eikonal corrections

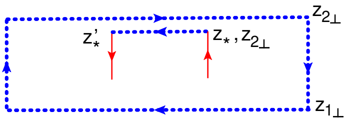
Diagrams with F_{ij} or $\tilde{Q}_1$ (and $\tilde{Q}_5$) classical: BK-type diagrams

$$\langle \mathrm{Tr}\{\tilde{Q}_{1x}\, U_y^\dagger\}\rangle \;=\; \frac{\alpha_s}{2\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2z\,\frac{(x-y)_\perp^2}{(x-z)_\perp^2\,(y-z)_\perp^2} \times\Big[\mathrm{Tr}\{U_z^\dagger\tilde{Q}_{1x}\}\,\mathrm{Tr}\{U_y^\dagger U_z\} - N_c\,\mathrm{Tr}\{U_y^\dagger\tilde{Q}_{1x}\}\Big]$$

$$\langle \mathrm{Tr}\{\tilde{Q}_{5x}\, U_y^\dagger\}\rangle \;=\; \frac{\alpha_s}{2\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2z\,\frac{(x-y)_\perp^2}{(x-z)_\perp^2\,(y-z)_\perp^2} \times\Big[\mathrm{Tr}\{U_z^\dagger\tilde{Q}_{5x}\}\,\mathrm{Tr}\{U_y^\dagger U_z\} - N_c\,\mathrm{Tr}\{U_y^\dagger\tilde{Q}_{5x}\}\Big]$$

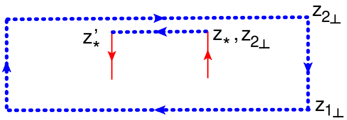
$$\langle \mathrm{Tr}\{\mathcal{F}_x\, U_y^\dagger\}\rangle \;=\; \frac{\alpha_s}{2\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2z\,\frac{(x-y)_\perp^2}{(x-z)_\perp^2\,(y-z)_\perp^2} \times\Big[\mathrm{Tr}\{U_z^\dagger\mathcal{F}_x\}\,\mathrm{Tr}\{U_y^\dagger U_z\} - N_c\,\mathrm{Tr}\{U_y^\dagger\mathcal{F}_x\}\Big]$$

Evolution equation of sub-eikonal corrections

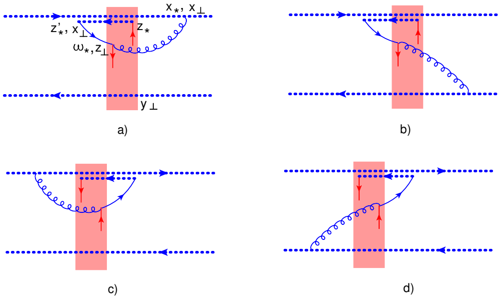


quark-to-gluon diagrams

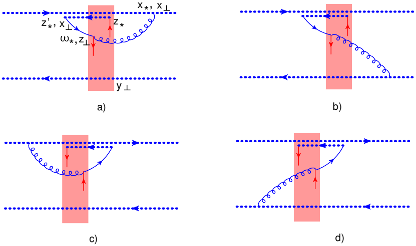
Evolution equation of sub-eikonal corrections



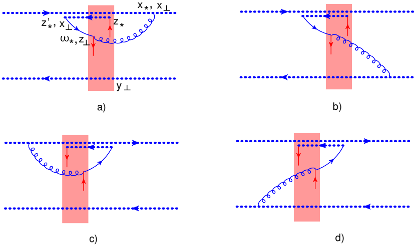
quark-to-gluon diagrams



Evolution equation of sub-eikonal corrections

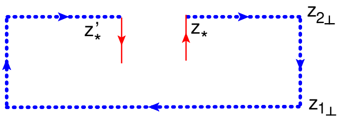


$$
\begin{aligned}
\langle \mathrm{Tr}\{U_xU_y^\dagger\}Q_{1x}\rangle &= \frac{\alpha_s}{4\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2z\left\{\frac{1}{(x-z)_\perp^2}\left[\mathrm{Tr}\left\{U_xU_y^\dagger U_z\mathcal{X}_{1zx}^\dagger\right\}\right.\right.\\
&\left.+\mathrm{Tr}\{U_zU_y^\dagger U_x\mathcal{X}_{1xz}^\dagger\}+\frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\left(\mathcal{H}_{1xz}^- + \mathcal{H}_{1zx}^+\right)\right] + \frac{(x-z,z-y)_\perp}{(y-z)_\perp^2(x-z)_\perp^2}\\
&\times\left[\mathrm{Tr}\{U_xU_y^\dagger U_z\mathcal{X}_{1zx}^\dagger\}+\mathrm{Tr}\left\{U_zU_y^\dagger U_x\mathcal{X}_{1xz}^\dagger\right\}+\frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\left(\mathcal{H}_{1xz}^-+\mathcal{H}_{1zx}^+\right)\right]\\
&\left.+\frac{(\vec{x}-\vec{z})\times(\vec{y}-\vec{z})}{(y-z)_\perp^2(x-z)_\perp^2}\left[\mathrm{Tr}\{U_xU_y^\dagger U_z\mathcal{X}_{5zx}^\dagger\}-\mathrm{Tr}\left\{U_zU_y^\dagger U_x\mathcal{X}_{5xz}^\dagger\right\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\left(\mathcal{H}_{5xz}^- - \mathcal{H}_{5zx}^+\right)\right]\right\}.
\end{aligned}
$$

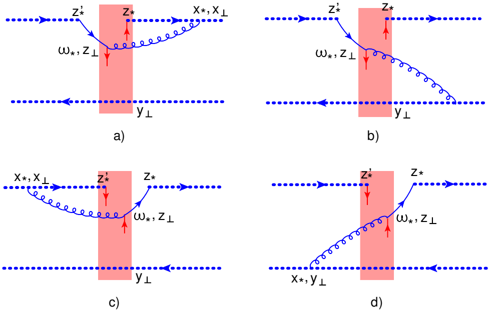


$$
\begin{aligned}
\langle \mathrm{Tr}\{U_x U_y^\dagger\} Q_{5x}\rangle = & -\frac{\alpha_s}{4\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2z \Bigg\{ \frac{1}{(x-z)_\perp^2}\Big[\mathrm{Tr}\left\{U_x U_y^\dagger U_z \mathcal{X}_{5zx}^\dagger\right\} \\
& + \mathrm{Tr}\left\{U_z U_y^\dagger U_x \mathcal{X}_{5xz}^\dagger\right\} - \frac{1}{N_c}\mathrm{Tr}\{U_y^\dagger U_x\}\Big(\mathcal{H}_{5xz}^- + \mathcal{H}_{5zx}^+\Big)\Big] + \frac{(x-z,z-y)_\perp}{(y-z)_\perp^2(x-z)_\perp^2} \\
& \times\Big[\mathrm{Tr}\{U_x U_y^\dagger U_z \mathcal{X}_{5zx}^\dagger\} + \mathrm{Tr}\left\{U_z U_y^\dagger U_x \mathcal{X}_{5xz}^\dagger\right\} - \frac{1}{N_c}\mathrm{Tr}\{U_x U_y^\dagger\}\Big(\mathcal{H}_{5xz}^- + \mathcal{H}_{5zx}^+\Big)\Big] \\
& + \frac{(\vec{x}-\vec{z})\times(\vec{y}-\vec{z})}{(y-z)_\perp^2(x-z)_\perp^2}\Big[\mathrm{Tr}\{U_z U_y^\dagger U_x \mathcal{X}_{1xz}^\dagger\} - \mathrm{Tr}\{U_x U_y^\dagger U_z \mathcal{X}_{1zx}^\dagger\} + \frac{1}{N_c}\mathrm{Tr}\{U_x U_y^\dagger\}\Big(\mathcal{H}_{1zx}^+ - \mathcal{H}_{1xz}^-\Big)\Big]\Bigg\}.
\end{aligned}
$$

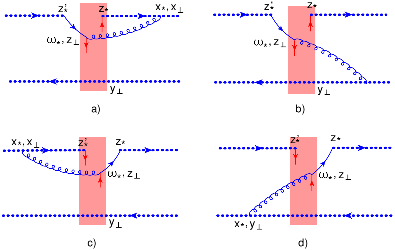
Evolution equation of sub-eikonal corrections



quark-to-gluon diagrams



Evolution equation of sub-eikonal corrections



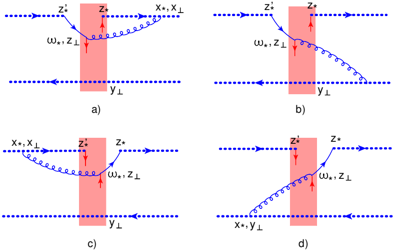
$$\langle \mathrm{Tr}\{U_y^\dagger \tilde{Q}_{1x}\}\rangle = -\frac{\alpha_s}{4\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2z$$

$$\times\Bigg\{\frac{1}{(x-z)_\perp^2}\left[\mathrm{Tr}\left\{U_zU_y^\dagger\right\}\left(\mathcal{H}^+_{1xz}+\mathcal{H}^-_{1zx}\right)-\frac{1}{N_c}\mathrm{Tr}\left\{U_y^\dagger\left(\mathcal{X}_{1xz}+\mathcal{X}_{1zx}\right)\right\}\right]$$

$$+\frac{(x-z,z-y)}{(y-z)_\perp^2(z-x)_\perp^2}\left[\mathrm{Tr}\{U_zU_y^\dagger\}\left(\mathcal{H}^+_{1xz}+\mathcal{H}^-_{1zx}\right)-\frac{1}{N_c}\mathrm{Tr}\{U_y^\dagger\left(\mathcal{X}_{1xz}+\mathcal{X}_{1zx}\right)\}\right]$$

$$+\frac{(\vec{x}-\vec{z})\times(\vec{y}-\vec{z})}{(y-z)_\perp^2(z-x)_\perp^2}\left[\mathrm{Tr}\{U_zU_y^\dagger\}\left(\mathcal{H}^-_{5zx}-\mathcal{H}^+_{5xz}\right)+\frac{1}{N_c}\mathrm{Tr}\{U_y^\dagger\left(\mathcal{X}_{5xz}-\mathcal{X}_{5zx}\right)\}\right]\Bigg\}$$

Evolution equation of sub-eikonal corrections



$$
\begin{aligned}
\langle \mathrm{Tr}\{U_y^\dagger \tilde{Q}_{5x}\}\rangle = & -\frac{\alpha_s}{4\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2z \\
& \times\Bigg\{\frac{1}{(x-z)_\perp^2}\Big[\mathrm{Tr}\left\{U_z U_y^\dagger\right\}\Big(\mathcal{H}^+_{5xz}+\mathcal{H}^-_{5zx}\Big) - \frac{1}{N_c}\mathrm{Tr}\left\{U_y^\dagger\left(\mathcal{X}_{5xz}+\mathcal{X}_{5zx}\right)\right\}\Big] \\
& + \frac{(x-z,z-y)}{(y-z)_\perp^2(z-x)_\perp^2}\Big[\mathrm{Tr}\{U_z U_y^\dagger\}\Big(\mathcal{H}^+_{5xz}+\mathcal{H}^-_{5zx}\Big) - \frac{1}{N_c}\mathrm{Tr}\Big\{U_y^\dagger\Big(\mathcal{X}_{5xz}+\mathcal{X}_{5zx}\Big)\Big\}\Big)\Big] \\
& + \frac{(\vec{x}-\vec{z})\times(\vec{y}-\vec{z})}{(y-z)_\perp^2(z-x)_\perp^2}\Big[\mathrm{Tr}\{U_z U_y^\dagger\}\Big(\mathcal{H}^+_{1xz}-\mathcal{H}^-_{1zx}\Big) + \frac{1}{N_c}\mathrm{Tr}\Big\{U_y^\dagger\left(\mathcal{X}_{1zx}-\mathcal{X}_{1xz}\right)\Big\}\Big]\Bigg\}
\end{aligned}
$$

$\mathcal{X}$-operators

$$\mathcal{X}_1(x_\perp, y_\perp) = -\sqrt{\frac{s^3}{8}} g^2 \int_{-\infty}^{+\infty} dz^+ d\omega^+ \bar{\psi}(z^+, y_\perp)[z^+, -\infty p_1]_y \, i\gamma^- \, [\infty n_1, \omega^+]_x \psi(\omega^+, x_\perp)$$

$$\mathcal{X}_1^\dagger(x_\perp, y_\perp) = g^2 \sqrt{\frac{s^3}{8}} \int_{-\infty}^{+\infty} dz^+ d\omega^+ \, \bar{\psi}(\omega^+, x_\perp)[\omega^+, \infty n_1]_x \, i\gamma^- \, [-\infty n_1, z^+]_y \psi(z^+, y_\perp)$$

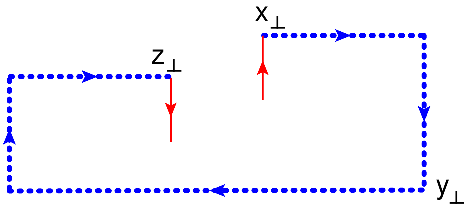
$$\mathcal{X}_5(x_\perp, y_\perp) = -g^2 \sqrt{\frac{s^3}{8}} \int_{-\infty}^{+\infty} dz^+ d\omega^+ \bar{\psi}(z^+, y_\perp)[z^+, -\infty p_1]_y \, \gamma^5 \gamma^- \, [\infty n_1, \omega^+]_x \psi(\omega^+, x_\perp)$$

$$\mathcal{X}_5^\dagger(x_\perp, y_\perp) = -g^2 \sqrt{\frac{s^3}{8}} \int_{-\infty}^{+\infty} dz^+ d\omega^+ \bar{\psi}(\omega^+, x_\perp)[\omega^+, \infty n_1]_x \, \gamma^5 \gamma^- \, [-\infty n_1, z^+]_y \psi(z^+, y_\perp)$$

$\mathcal{X}$-operators

$\mathcal{X}$-dipole operators

$\mathrm{Tr}\{U_y^\dagger \mathcal{X}_{1xz}\}$ or $\mathrm{Tr}\{U_y^\dagger \mathcal{X}_{5xz}\}$



$\mathcal{H}$-operators

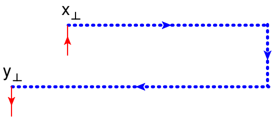
$$\mathcal{H}_1^+(x_\perp,y_\perp) = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ d\omega^+\,\bar\psi(\omega^+,y_\perp)[\omega^+,\infty n_1]_y\, i\gamma^-\ [\infty n_1,z^+]_x\psi(z^+,x_\perp)$$

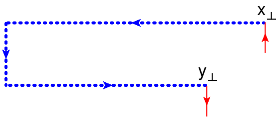
$$\mathcal{H}_5^+(x_\perp,y_\perp) = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ d\omega^+\,\bar\psi(\omega^+,y_\perp)[\omega^-,\infty n_1]_y\, \gamma^5\gamma^-\ [\infty n_1,z^+]_x\psi(z^+,x_\perp)$$

$$\mathcal{H}_1^-(x_\perp,y_\perp) = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ d\omega^+\,\bar\psi(\omega^+,y_\perp)[\omega^+,-\infty n_1]_y\, i\gamma^-\ [-\infty n_1,z^+]_x\psi(z^+,x_\perp)$$

$$\mathcal{H}_5^-(x_\perp,y_\perp) = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ d\omega^+\,\bar\psi(\omega^+,y_\perp)[\omega^+,-\infty n_1]_y\, \gamma^5\gamma^-\ [-\infty n_1,z^+]_x\psi(z^+,x_\perp)$$

TMD operators that usually appear in SIDIS and Drell-Yan processes.





Conclusions

- Quark and Gluon propagator with sub-eikonal corrections is good for
  - spin-dependent TMDs: SIDIS, Weizsäcker-Williams TMD at low-x
  - spin g_1 structure function at low-x
- New Impact factors have been derived;
- New evolution equations which describe the high-energy spin dynamics;
- New quark and gluon distributions;
- TMD type of operators appeared in DIS process.

Outlook

- Disentangle the double from the single log of energy and compare with Bartels-Ermolaev-Ryskin (and the extension obtained by Boussarie, Hatta, and Yuan (2019));
- Compare with NNLO calculation by Moch, Vermaseren and Vogt (2014);
  - Analytic continuation of local super-multiplet.
- Compare with results obtained by Kovchegov's group (20015-2021);
- Include NLO and running coupling corrections.

Outlook

Light-ray operator

Analytic continuation of light-ray operators at $j = 1$

$$F^a_{\xi+}(x)\nabla_+^{j-2}F^{a\ \xi}_{+}(x)\Big|_{x=0} = \frac{\Gamma(2-j)}{2\pi i}\int_0^{+\infty} du\ u^{1-j}\, F^a_{\xi+}(0)[0,un]^{ab}F^{b\ \xi}_{+}(un)$$

OPE in light-ray operators in QCD (Balitsky, Braun (1989))

2-point function in BFKL limit (Balitsky; Balitsky, Kazakov, Sobkov (2013-2018))

2-point function in triple Regge limit (Balitsky 2018)

A lot of activity on light-ray operators in CFT (e.g. Kravchuk, Simmons-Duffin (2018))

Appendix

The Q^2 behavior of DIS structure function is obtained from the anomalous dimension of twist-two operators

$$\mu\frac{d}{\mu}F^a_{\xi+}\nabla_+^{n-2}F^{a\,\xi}_{+} = \gamma(\alpha_s,n)F^a_{\xi+}\nabla_+^{n-2}F^{a\,\xi}_{+}$$

Dipole DIS cross-section can be written as

$$\sigma^{\gamma^* p}(x_B,Q^2) = \int d\nu\, F(\nu)\, x_B^{-\aleph(\nu)-1}\left(\frac{Q^2}{P^2}\right)^{\frac{1}{2}+i\nu}$$

$-q^2 = Q^2 \gg P^2$, and $s = (P+q)^2 \gg Q^2$

$\aleph(\gamma)$ BFKL pomeron intercept.

The n-th moment of the structure function is

$$\int_0^1 dx_B\, x_B^{n-1}\sigma^{\gamma^* p}(x_B,Q^2) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\gamma\,\frac{F(\gamma)}{n-1-\aleph(\gamma)}\left(\frac{Q^2}{P^2}\right)^{\gamma}$$

Integrating over γ-parameter we get the anomalous dimensions of the leading and higher twist operators at the *unphysical point* $n=1$.

Analytic continuation

$$\int_0^1 dx_B\, x_B^{n-1} \sigma^{\gamma^* p}(x_B, Q^2) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\gamma\, \frac{F(\gamma)}{\omega - \aleph(\gamma)} \left(\frac{Q^2}{P^2}\right)^{\gamma}$$

Analytic continuation: $n-1 \to \omega$ complex continuous variable

$\Rightarrow$ Residues $\omega = \aleph(\gamma)$; expand $\aleph(\gamma)$ for small γ and solve for γ

$$\gamma(\alpha_s, \omega) = \frac{\alpha_s N_c}{\pi\omega} + \mathcal{O}(\alpha_s^2), \quad F(\omega, Q^2) \sim \left(\frac{Q^2}{P^2}\right)^{\frac{\alpha_s N_c}{\pi\omega}}$$

Thus, we get the analytic continuation of anomalous dimension at the *unphysical point* $j \to 1$ of twist-2 gluon operator $F^a_{\xi+} \nabla^{-1} F^{\xi a}{}_{+}$

Light-ray operator

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Super-multiplet of local operators in $\mathcal{N}$=4 SYM

$$\mathcal{O}^j_\phi(x_\perp) = \int du\, \bar{\phi}^a_{AB} \nabla^j_- \phi^{ABa}(up_1 + x_\perp)$$
$$\mathcal{O}^j_\lambda(x_\perp) = \int du\, i\bar{\lambda}^a_A \nabla^{j-1}_- \lambda^a_A(up_1 + x_\perp)$$
$$\mathcal{O}^j_g(x_\perp) = \int du\, F^{a+}{}_i \nabla^{j-2}_- F^{a+i}(up_1 + x_\perp)$$

Multiplicatively renormalizable operators

$$S^j_1 = \mathcal{O}^j_g + \frac{1}{4}\mathcal{O}^j_\lambda - \frac{1}{2}\mathcal{O}^j_\phi$$
$$S^j_2 = \mathcal{O}^j_g - \frac{1}{4(j-1)}\mathcal{O}^j_\lambda + \frac{j+1}{6(j-1)}\mathcal{O}^j_\phi$$
$$S^j_3 = \mathcal{O}^j_g - \frac{j+2}{2(j-1)}\mathcal{O}^j_\lambda - \frac{(j+1)(j+2)}{2j(j-1)}\mathcal{O}^j_\phi$$

with anomalous dimensions

$$\gamma^{S_1}_j = 4[\psi(j-1) + \gamma_E] + O(\alpha_s^2), \quad \gamma^{S_2}_j = \gamma_{j+2}, \quad \gamma^{S_3}_j = \gamma^{S_1}_{j+4}$$

A. V. Belitsky, *et al* (2004)

Analytical continuation of the local super-multiplet

$$\mathcal{F}^j(x_\perp) = \int_0^\infty du\, u^{1-j}\mathcal{F}(up_1 + x_\perp)\,,$$
$$\Lambda^j(x_\perp) = \int_0^\infty du\, u^{-j}\Lambda(up_1 + x_\perp)\,,$$
$$\Phi^j(x_\perp) = \int_0^\infty du\, u^{-1-j}\Phi(up_1 + x_\perp)$$

with

$$\mathcal{F}(up_1, x_\perp) = \int dv\, F^{a-}{}_\mu(up_1 + vp_1 + x_\perp)[u+v, v]_x^{ab} F^{b-\mu}(vp_1 + x_\perp)\,,$$
$$\Lambda(up_1, x_\perp) = \frac{i}{2}\int dv\Big(- \bar{\lambda}_A^a(up_1 + vp_1 + x_\perp)[u+v, v]_x^{ab}\sigma_-\lambda_A^b(vp_1 + x_\perp)$$
$$+\bar{\lambda}_A^a(vp_1 + x_\perp)[v, u+v]_x^{ab}\sigma_-\lambda_A^b(up_1 + vp_1 + x_\perp)\Big)\,,$$
$$\Phi(u, x_\perp) = \int dv\, \phi_I^a(up_1 + vp_1 + x_\perp)[u+v, v]_x^{ab}\phi_I^b(vp_1 + x_\perp)$$

I. Balitsky, V. Kazakov, and E. Sobko (2013)

multiplicatively renorm. light-ray operators

Forward matrix elements

$$\mathcal{S}_1^j = \mathcal{F}^j + \frac{j-1}{4}\Lambda^j - j(j-1)\frac{1}{2}\Phi^j\,,$$
$$\mathcal{S}_2^j = \mathcal{F}^j - \frac{1}{4}\Lambda^j + \frac{j(j+1)}{6}\Phi^j\,,$$
$$\mathcal{S}_3^j = \mathcal{F}^j - \frac{j+2}{2}\Lambda^j - \frac{(j+1)(j+2)}{2}\Phi^j\,.$$

Notice the different coefficients between the S-operators and the $\mathcal{S}$-operators.

Correlation function in CFT at high-energy, $j \to 1$

$$\langle \mathcal{F}^j(x_\perp)\mathcal{F}^{j'}(y_\perp)\rangle = \langle \mathcal{S}_1^j(x_\perp)\mathcal{S}_1^{j'}(y_\perp)\rangle \stackrel{\text{CFT}}{=} \delta(j-j')\frac{C(\Delta,j)s^{j-1}}{[(x-y)_\perp^2]^{\Delta-1}}\mu^{-2\gamma_a}$$

Δ: canonical dimension d plus anomalous dim.
μ: normalization point.
$C(\Delta,j)$: unknown structure constant. Calculate it in the BFKL limit.

Wilson frame vs quasi-pdf frame

In the BFKL limit the two-point correlation function is UV divergent.

Regularization: point splitting $\Rightarrow$

- Wilson frame   Balitsky (2013, 2019), Balitsky, Kazhakov, Sobko (20013-2018)
  - Motivation: Give an example of actual calculation of correlation function; goal: understanding full dynamics of $\mathcal{N}=4$ SYM.
- quasi-pdf frame   G.A.C. *Quark and Gluon quasi-pdf at low-x* (in preparation)
  - Motivation: check of the calculation comparing with expected CFT general result; goal: calculate the behavior of the quasi-pdf at small-x_B.