Gluon-target scattering at next-to-eikonal accuracy

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Outline

- Eikonal approximation and sources of NEik corrections
- Quark propagator at NEik accuracy
- Gluon propagator at NEik accuracy
- Quark-target and gluon-target scattering at NEik accuary
- Summary and outlook

Dilute-Dense Scattering within CGC

High energy scattering within the CGC relies on two pillars:

- Semi-classical approximation :
  - dense target is represented by strong semiclassical gluon field $\mathcal{A}_a^\mu(x) = O(1/g)$ at weak coupling g with finite support.
- Eikonal approximation:
  - keeping only the leading power terms in the high energy limit.

High energy limit can be achieved by boosting the target along x^-:

$$\mathcal{A}_a^\mu(x) \mapsto \begin{cases} \gamma_t\, \mathcal{A}_a^- \left(\gamma_t\, x^+,\ \frac{x^-}{\gamma_t},\ \mathbf{x}\right) \\ \frac{1}{\gamma_t} \mathcal{A}_a^+ \left(\gamma_t\, x^+,\ \frac{x^-}{\gamma_t},\ \mathbf{x}\right) \\ \mathcal{A}_a^i \left(\gamma_t\, x^+,\ \frac{x^-}{\gamma_t},\ \mathbf{x}\right) \end{cases}$$

Eikonal approximation

The Eikonal approximation can be understood as the limit of infinite boost of $\mathcal{A}^\mu(x)$:

- Static limit; background field $\mathcal{A}^\mu(x)$ is taken to be independent of x^- due to Lorentz time dilation
  (no longitudinal momentum exchange between the target and the projectile during the interaction)

- Shockwave limit: background field $\mathcal{A}^\mu(x)$ is Lorentz contracted.
  $\Rightarrow$ interaction between the projectile partons and target occurs instantly in x^+ (No transverse motion within the target)

- There is a strong hierarchy between the components of the background field $\mathcal{A}^\mu$ under a boost of parameter γ_t along the "$-$" direction:

  $$\mathcal{A}^- = O(\gamma_t) \;\;\gg\;\; \mathcal{A}_\perp = O(1) \;\;\gg\;\; \mathcal{A}^+ = O(1/\gamma_t)$$

  and only $\mathcal{A}^-$ is taken into account.

In the Eikonal limit the background field takes the form:

$$\mathcal{A}^\mu(x^+, x^-, \mathbf{x}) \simeq \delta^{\mu -}\mathcal{A}_a^-(x^+\mathbf{x}) \propto \delta(x^+)$$

- $\big(g\mathcal{A}^-(x^+, \mathbf{x})\big)^n$ is resummed to all orders which leads to Wilson lines along x^+

Next-to-Eikonal corrections to the CGC

Next-to-Eikonal (NEik) corrections are of order $1/\gamma_t$ at the level of the boosted background field.

⋆ NEik corrections arise from relaxing either of the three approximations:

1. Interactions with the suppressed components of background field (transverse comp. $\mathcal{A}_\perp$).
2. Finite longitudinal width of the target is included by going beyond the shockwave limit $\Rightarrow$ transverse motion of the parton in the medium.
3. x^- dependence of $\mathcal{A}^\mu(x)$ is accounted for by going beyond the static limit and it is treated as gradient expansion around a common x^- value:

$$\boxed{\frac{\partial_- \mathcal{A}^-(x)}{\mathcal{A}^-(x)} = O(1/\gamma_t)}$$

⋆ An extra source for NEik corrections:

- interaction via t-channel quark exchange (interaction with quark background).

see also Kovchegov *et al.* (2016-2024), Chirilli (2019)

Power counting for the quark background field $\Psi(z)$

- Under a boost of the target of parameter γ_t along the "$-$" direction, a current associated with the target should behave as

$$J^-(z) \propto \gamma_t\,, \quad J^j(z) \propto (\gamma_t)^0\,, \quad J^+(z) \propto (\gamma_t)^{-1}\,,$$

- The quark background field of the target can be split as $\Psi(z) = \Psi^{(-)}(z) + \Psi^{(+)}(z)$, with

$$\Psi^{(-)}(z) \equiv \frac{\gamma^+\gamma^-}{2}\,\Psi(z)\,, \quad \Psi^{(+)}(z) \equiv \frac{\gamma^-\gamma^+}{2}\,\Psi(z)\,.$$

Then, the components of the background quark current write

$$\begin{aligned}
\overline{\Psi}(z)\,\gamma^-\,\Psi(z) &= \overline{\Psi^{(-)}}(z)\,\gamma^-\,\Psi^{(-)}(z),\\
\overline{\Psi}(z)\,\gamma^j\,\Psi(z) &= \overline{\Psi^{(-)}}(z)\,\gamma^j\,\Psi^{(+)}(z) + \overline{\Psi^{(+)}}(z)\,\gamma^j\,\Psi^{(-)}(z),\\
\overline{\Psi}(z)\,\gamma^+\,\Psi(z) &= \overline{\Psi^{(+)}}(z)\,\gamma^-\,\Psi^{(+)}(z)\,.
\end{aligned}$$

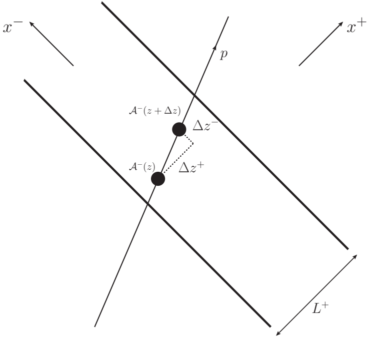
Under a boost of the target, the projections $\Psi^{(-)}(z)$ and $\Psi^{(+)}(z)$ should thus scale as

$$\Psi^{(-)}(z) \propto (\gamma_t)^{\frac{1}{2}}\,, \quad \Psi^{(+)}(z) \propto (\gamma_t)^{-\frac{1}{2}},$$

$\Rightarrow$ we keep only the leading components $\Psi^{(-)}(z)$ of $\Psi(z)$

Effect of relative z^- dependence of $\mathcal{A}^-$ insertions along one propagator:

$$\mathcal{A}^-(z^- + \Delta z^-) - \mathcal{A}^-(z^-) \simeq \Delta z^- \; \partial_- \mathcal{A}^-(z^-)$$



- Slow z^- dependence from time dilation:

$$\partial_- \mathcal{A}^- \propto \frac{1}{\gamma_t} \mathcal{A}^-$$

- Small Δz^- displacement of the trajectory within the target width L^+:

$$\Delta z^- \sim \frac{p^-}{p^+} \Delta z^+ \leq \frac{p^-}{p^+} L^+ = O\left(\frac{1}{\gamma_t}\right)$$

Double power suppression, beyond static approx and beyond shockwave approx:

$\Rightarrow$ NNEik effect within a single propagator!

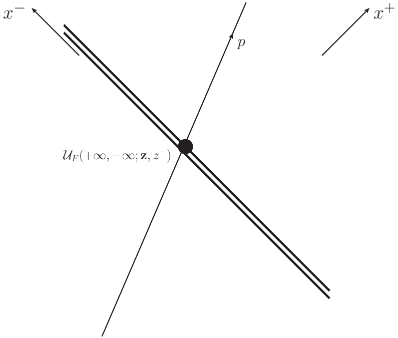
Effect of relative z^- dependence of $\mathcal{A}^-$ insertions along one propagator is NNEik.

However, dependence on average z^- is suppressed only once.

$\Rightarrow$ Use Wilson lines with overall z^- dependence

$$\partial_- \mathcal{U}_F(+\infty, -\infty; \mathbf{z}, z^-) \propto \frac{1}{\gamma_t} \mathcal{U}_F(+\infty, -\infty; \mathbf{z}, z^-)$$

$\rightarrow$ Accounts for NEik effects beyond static approx



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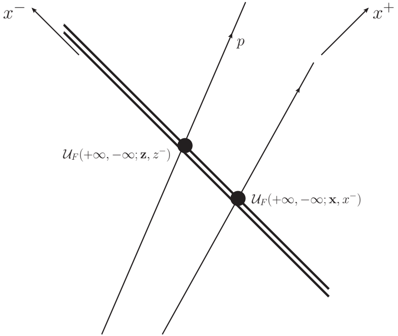
$\rightarrow$ Accounts for NEik effects beyond static approx

In particular: NEik corrections induced by the difference in z^- between different Wilson lines.



NEik quark propagator through a gluon background field

Propagator from y before the target to x after the target:

$$S_F(x,y) = \int \frac{dq^+ d^2\mathbf{q}}{(2\pi)^3} \int \frac{dk^+ d^2\mathbf{k}}{(2\pi)^3}\, \theta(q^+)\,\theta(k^+)\, e^{-ix\cdot\check{q}}\, e^{iy\cdot\check{k}}\, \frac{(\not{\check{q}}+m)}{2q^+}\gamma^+$$

$$\times \int d^2\mathbf{z}\, e^{-i\mathbf{z}\cdot(\mathbf{q}-\mathbf{k})} \Bigg\{ \int dz^-\, e^{iz^-(q^+-k^+)}\, \mathcal{U}_F\Big(+\infty,-\infty;\mathbf{z},z^-\Big)$$

$$-2\pi\delta(q^+-k^+)\,\frac{(\mathbf{q}^j+\mathbf{k}^j)}{2(q^++k^+)} \int dz^+ \left[\mathcal{U}_F\Big(+\infty,z^+;\mathbf{z},0\Big)\overleftrightarrow{\mathcal{D}_{\mathbf{z}^j}}\,\mathcal{U}_F\Big(z^+,-\infty;\mathbf{z},0\Big)\right]$$

$$-i\,\frac{2\pi\delta(q^+-k^+)}{(q^++k^+)} \int dz^+ \left[\mathcal{U}_F\Big(+\infty,z^+;\mathbf{z},0\Big)\overleftarrow{\mathcal{D}_{\mathbf{z}^j}}\,\overrightarrow{\mathcal{D}_{\mathbf{z}^j}}\,\mathcal{U}_F\Big(z^+,-\infty;\mathbf{z},0\Big)\right]$$

$$+\frac{2\pi\delta(q^+-k^+)}{(q^++k^+)}\,\frac{[\gamma^i,\gamma^j]}{4} \int dz^+\, \mathcal{U}_F\Big(+\infty,z^+;\mathbf{z},0\Big)\, gt\cdot\mathcal{F}_{ij}(z^+,\mathbf{z},0)\,\mathcal{U}_F\Big(z^+,-\infty;\mathbf{z},0\Big)\Bigg\}\frac{(\not{\check{k}}+m)}{2k^+}$$

$$+\text{NNEik}$$

Altinoluk, Beuf, Czajka, Tymowska (2021); Altinoluk, Beuf (2022)

$$\mathcal{U}_F(x^+,y^+;\mathbf{z},z^-) \equiv \mathbf{1} + \sum_{N=1}^{+\infty} \frac{1}{N!}\,\mathcal{P}_+\left[-ig\int_{y^+}^{x^+} dz^+\, t\cdot\mathcal{A}^-(z)\right]^N$$

- Generalized Eikonal contribution: also includes the NEik non-static corrections: overall z^- dependence of the Wilson line.

NEik quark propagator through a gluon background field

Propagator from y before the target to x after the target:

$$S_F(x,y) = \int \frac{dq^+ d^2\mathbf{q}}{(2\pi)^3} \int \frac{dk^+ d^2\mathbf{k}}{(2\pi)^3}\, \theta(q^+)\,\theta(k^+)\, e^{-ix\cdot\check{q}}\, e^{iy\cdot\check{k}}\, \frac{(\check{\not{q}}+m)}{2q^+}\gamma^+$$
$$\times \int d^2\mathbf{z}\, e^{-i\mathbf{z}\cdot(\mathbf{q}-\mathbf{k})} \bigg\{ \int dz^- e^{iz^-(q^+-k^+)}\, \mathcal{U}_F\Big(+\infty,-\infty;\mathbf{z},z^-\Big)$$
$$-2\pi\delta(q^+-k^+)\,\frac{(\mathbf{q}^j+\mathbf{k}^j)}{2(q^++k^+)} \int dz^+ \left[\mathcal{U}_F\Big(+\infty,z^+;\mathbf{z},0\Big)\overleftrightarrow{\mathcal{D}_{\mathbf{z}^j}}\,\mathcal{U}_F\Big(z^+,-\infty;\mathbf{z},0\Big)\right]$$
$$-i\,\frac{2\pi\delta(q^+-k^+)}{(q^++k^+)} \int dz^+ \left[\mathcal{U}_F\Big(+\infty,z^+;\mathbf{z},0\Big)\overleftarrow{\mathcal{D}_{\mathbf{z}^j}}\,\overrightarrow{\mathcal{D}_{\mathbf{z}^j}}\,\mathcal{U}_F\Big(z^+,-\infty;\mathbf{z},0\Big)\right]$$
$$+\frac{2\pi\delta(q^+-k^+)}{(q^++k^+)}\frac{[\gamma^i,\gamma^j]}{4} \int dz^+ \mathcal{U}_F\Big(+\infty,z^+;\mathbf{z},0\Big)\, gt\cdot\mathcal{F}_{ij}(z^+,\mathbf{z},0)\,\mathcal{U}_F\Big(z^+,-\infty;\mathbf{z},0\Big)\bigg\}\frac{(\check{\not{k}}+m)}{2k^+}$$
$$+\text{NNEik}$$

Altinoluk, Beuf, Czajka, Tymowska (2021); Altinoluk, Beuf (2022)

$$\mathcal{U}_F(x^+,y^+;\mathbf{z},z^-) \equiv \mathbf{1} + \sum_{N=1}^{+\infty} \frac{1}{N!}\,\mathcal{P}_+\left[-ig\int_{y^+}^{x^+} dz^+\, t\cdot\mathcal{A}^-(z)\right]^N$$

- NEik contributions beyond the shockwave approx or due to $\mathcal{A}_\perp$.
  Last term: quark helicity coupling with longitudinal chromomagnetic field of the target $\mathcal{F}_{ij}$.

Compact notations for the decorated Wilson lines:

$$\mathcal{U}^{(1)}_{F;j}(\mathbf{z}) = \int dz^+\, \mathcal{U}_F\Big(+\infty, z^+; \mathbf{z}\Big) \overleftrightarrow{\mathcal{D}_{\mathbf{z}^j}} \mathcal{U}_F\Big(z^+, -\infty; \mathbf{z}\Big)$$

$$\mathcal{U}^{(2)}_{F}(\mathbf{z}) = \int dz^+\, \mathcal{U}_F\Big(+\infty, z^+; \mathbf{z}\Big) \overleftarrow{\mathcal{D}_{\mathbf{z}^j}}\, \overrightarrow{\mathcal{D}_{\mathbf{z}^j}} \mathcal{U}_F\Big(z^+, -\infty; \mathbf{z}\Big)$$

$$\mathcal{U}^{(3)}_{F;ij}(\mathbf{z}) = \int dz^+\, \mathcal{U}_F\Big(+\infty, z^+; \mathbf{z}\Big) gt\cdot\mathcal{F}_{ij}(z^+, \mathbf{z}) \mathcal{U}_F\Big(z^+, -\infty; \mathbf{z}\Big)$$

Propagator from y before the target to x after the target:

$$S_F(x,y) = \int \frac{dq^+ d^2\mathbf{q}}{(2\pi)^3} \int \frac{dk^+ d^2\mathbf{k}}{(2\pi)^3}\, \theta(q^+)\,\theta(k^+)\, e^{-ix\cdot\check{q}}\, e^{iy\cdot\check{k}}\, \frac{(\check{\not{q}}+m)}{2q^+}\gamma^+ \int d^2\mathbf{z}\, e^{-i\mathbf{z}\cdot(\mathbf{q}-\mathbf{k})}$$
$$\times \Bigg\{ \int dz^-\, e^{iz^-(q^+-k^+)}\, \mathcal{U}_F\Big(\mathbf{z}, z^-\Big) + 2\pi\delta(q^+-k^+) \Bigg[-\frac{(\mathbf{q}^j+\mathbf{k}^j)}{2(q^++k^+)} \mathcal{U}^{(1)}_{F;j}(\mathbf{z})$$
$$- \frac{i}{(q^++k^+)} \mathcal{U}^{(2)}_{F}(\mathbf{z}) + \frac{[\gamma^i,\gamma^j]}{4(q^++k^+)} \mathcal{U}^{(3)}_{F;ij}(\mathbf{z}) \Bigg] \Bigg\} \frac{(\check{\not{k}}+m)}{2k^+} + \text{NNEik}$$

Full NEik quark propagator through a gluon background field

Compact notations for the decorated Wilson lines:

$$\mathcal{U}^{(1)}_{F;j}(\mathbf{z}) = \int dz^+\, \mathcal{U}_F\Big(+\infty, z^+; \mathbf{z}\Big)\overleftrightarrow{\mathcal{D}_{\mathbf{z}j}}\mathcal{U}_F\Big(z^+, -\infty; \mathbf{z}\Big)$$

$$\mathcal{U}^{(2)}_{F}(\mathbf{z}) = \int dz^+\, \mathcal{U}_F\Big(+\infty, z^+; \mathbf{z}\Big)\overleftarrow{\mathcal{D}_{\mathbf{z}j}}\,\overrightarrow{\mathcal{D}_{\mathbf{z}j}}\mathcal{U}_F\Big(z^+, -\infty; \mathbf{z}\Big)$$

$$\mathcal{U}^{(3)}_{F;ij}(\mathbf{z}) = \int dz^+\, \mathcal{U}_F\Big(+\infty, z^+; \mathbf{z}\Big)gt\cdot\mathcal{F}_{ij}(z^+,\mathbf{z})\mathcal{U}_F\Big(z^+, -\infty; \mathbf{z}\Big)$$

Alternative expressions for the decorated Wilson lines:

$$\mathcal{U}^{(1)}_{F;j}(\mathbf{z}) = -2\int dz^+\, z^+\,\mathcal{U}_F(+\infty, z^+; \mathbf{z})[-igt\cdot\mathcal{F}_j^{\;-}(z^+,\mathbf{z})]\mathcal{U}_F(z'^+, -\infty; \mathbf{z})$$

$$\mathcal{U}^{(2)}_{F}(\mathbf{z}) = \int dz^+\int dz'^+\,(z^+ - z'^+)\,\theta(z^+ - z'^+)\mathcal{U}_F(+\infty, z^+, \mathbf{z})[-igt\cdot\mathcal{F}_j^{\;-}(z^+,\mathbf{z})]$$
$$\times\;\mathcal{U}_F(z^+, z'^+; \mathbf{z})[-igt\cdot\mathcal{F}_j^{\;-}(z'^+,\mathbf{z})]\mathcal{U}_F(z'^+, -\infty; \mathbf{z})$$

Thanks to the relation:

$$\partial_\mu\mathcal{U}_F(x^+, y^+; \mathbf{z}, z^-) + igt\cdot\mathcal{A}_\mu(x^+,\mathbf{z},z^-)\mathcal{U}_F(x^+, y^+; \mathbf{z}, z^-) - ig\mathcal{U}_F(x^+, y^+; \mathbf{z}, z^-)t\cdot\mathcal{A}_\mu(y^+,\mathbf{z},z^-)$$
$$= -ig\int_{y^+}^{x^+} dz^+\mathcal{U}_F(x^+, v^+; \mathbf{z}, z^-)t\cdot\mathcal{F}_\mu^{\;-}(z)\mathcal{U}_F(v^+, y^+; \mathbf{z}, z^-) \qquad \text{for } \mu \neq +$$

Applications with NEik quark propoagator

- quark-target scattering (unpolarized cross-section & quark-helicity asymmetry)
  Altinoluk, Beuf, Czajka, Tymowska (2021)

- DIS dijet production at NEik accuracy
  Altinoluk, Beuf, Czajka, Tymowska (2023)
  - Back-to-back limit and relation with gluon TMDs - See Guillaume's talk
  - Weak field limit and numerical analysis - See Pedro's talk

- quark-gluon dijets in DIS at NEik accuracy
  Altinoluk, Armesto, Beuf (2023)

- Inclusive DIS and SIDIS at NEik accuracy - See Swaleha's talk

Gluon propagator - basics

Gluon propagator in background field $\mathcal{A}^\mu(x)$

$$G_F^{\mu\nu}(x,y)_{\alpha\beta} = G_{0,F}^{\mu\nu}(x,y)_{\alpha\beta} + \delta G_F^{\mu\nu}(x,y)_{\alpha\beta}$$

free propagator + corrections due to interactions
with the background field

vacuum gluon propagator in momentum space:

$$\tilde{G}_{0,F}^{\mu\nu}(p) = \frac{i}{p^2 + i\epsilon}\left[-g^{\mu\nu} + \frac{p^\mu\eta^\nu + \eta^\mu p^\nu}{p\cdot\eta}\right]$$

Corrections:

- **at the (generalized) eikonal order (with z^- dependence)**

$$\delta G_F^{\mu\nu}\Big|^{\text{(g)Eik}} \equiv \delta G_F^{\mu\nu}\Big|^{(g)\text{Eik}}_{\text{pure }\mathcal{A}^-, z^-}$$

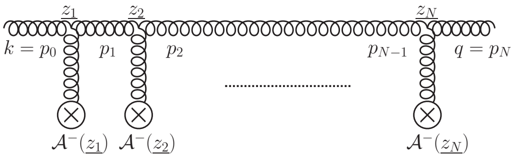
- **at the next-to-eikonal order**

$$\delta G_F^{\mu\nu}\Big|^{\text{NEik}} \equiv \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{pure }\mathcal{A}^-} + \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{single }\mathcal{A}_\perp} + \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{double }\mathcal{A}_\perp\text{ loc.}} + \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{double }\mathcal{A}_\perp,\text{non}-\text{loc}}$$

Gluon propagator in the eikonal limit

$$G_F^{\mu\nu}(x,y)\big|^{(g)\text{Eik}} = G_{0,F}^{\mu\nu}(x,y) + \delta G_F^{\mu\nu}(x,y)\big|_{\text{pure}\,\mathcal{A}^-,\text{z}^-}$$

In eikonal limit, the gluon already interacts with arbitrarily many $\mathcal{A}^-$ fields



Eikonal interactions with the medium resummed into the Wilson lines:

$$\mathcal{U}_A(x^+,y^+;\mathbf{z},z^-) \equiv \mathbf{1} + \sum_{N=1}^{+\infty} \frac{1}{N!}\,\mathcal{P}_+\left[-ig\int_{y^+}^{x^+} dz^+\, T\cdot\mathcal{A}^-(z^+,\mathbf{z},z^-)\right]^N$$

Gluon propagator in the eikonal limit

In eikonal limit, the gluon already interacts with arbitrarily many $\mathcal{A}^-$ fields

For generic x and y, with notations $\underline{k} \equiv (k^+, \mathbf{k})$, and $\check{k}$ on-shell version of k:

$$G_F^{\mu\nu}(x,y)|_{\text{Eik},z^-} = i\delta^2(x_\perp - y_\perp)\ \delta(x^+ - y^+)\eta^\mu\eta^\nu \left[\int \frac{dk^+}{2\pi}\frac{e^{-i(x^- - y^-)k^+}}{k^+k^+}\right]$$
$$+ \left\{\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\ \frac{e^{-ix\cdot\check{q}}\ e^{iy\cdot\check{k}}}{q^+ + k^+}\left[-g^{\mu\nu} + \frac{\check{k}^\mu\eta^\nu}{k^+} + \frac{\eta^\mu\check{q}^\nu}{q^+} - \frac{\eta^\mu\eta^\nu}{q^+k^+}(\check{q}\cdot\check{k})\right]\right.$$
$$\left.\times\int d^2z_\perp\ e^{-i(q_\perp - k_\perp)z_\perp}\ \int dz^-\ e^{i(q^+ - k^+)z^-}\right\}\Big[\theta(x^+ - y^+)\theta(q^+)\theta(k^+)\ U_A(x^+, y^+; z_\perp, z^-)$$
$$- \theta(y^+ - x^+)\theta(-q^+)\theta(-k^+)\ U_A^\dagger(x^+, y^+; z_\perp, z^-)\Big]$$

In the strict Eikonal limit:

$$G_F^{\mu\nu}(x,y)|_{\text{Eik}} = i\delta^2(x_\perp - y_\perp)\ \delta(x^+ - y^+)\eta^\mu\eta^\nu\left[\int\frac{dk^+}{2\pi}\frac{e^{-i(x^- - y^-)k^+}}{k^+k^+}\right]$$
$$+ \left\{\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\ \frac{e^{-ix\cdot\check{q}}\ e^{iy\cdot\check{k}}}{2k^+}\left[2\pi\ \delta(k^+ - q^+)\right]\right.$$
$$\left.\times\left[-g^{\mu\nu} + \frac{\check{k}^\mu\eta^\nu}{k^+} + \frac{\eta^\mu\check{q}^\nu}{q^+} - \frac{\eta^\mu\eta^\nu}{q^+k^+}(\check{q}\cdot\check{k})\right]\left[\int d^2z_\perp\ e^{-i(q_\perp - k_\perp)z_\perp}\right]\right\}$$
$$\times\left[\theta(x^+ - y^+)\ \theta(k^+)\ U_A(x^+, y^+; z_\perp) - \theta(y^+ - x^+)\ \theta(-k^+)\ U_A^\dagger(x^+, y^+; z_\perp)\right]$$

Full next-to-eikonal gluon propagator:

$$\delta G_F^{\mu\nu}\Big|^{\rm NEik} = \underbrace{G_F^{\mu\nu}\Big|^{\rm Eik} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{pure } \mathcal{A}^-}}_{G_F^{\mu\nu}\big|_{\text{pure } \mathcal{A}^-}} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{single } \mathcal{A}_\perp} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{double } \mathcal{A}_\perp,\,\text{loc}} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{double } \mathcal{A}_\perp,\,\text{non-loc.}}$$

From now on, always $x^+ > L^+/2$ and $y^+ < -L^+/2$: gluon propagating through whole target

Gluon propagator in pure $\mathcal{A}^-$ background field up to next-to-eikonal order for positive energy:

$$\begin{aligned}
\delta G_F^{\mu\nu}(x,y)\big|_{{\rm pure}A^-} = & \int \frac{d^3\underline{q}}{(2\pi)^3}\int \frac{d^3\underline{k}}{(2\pi)^3}\frac{\theta(q^+)\theta(k^+)}{q^+ + k^+}\; e^{-ix\cdot\check{q}}\; e^{iy\cdot\check{k}}\left(-g^{\mu\nu} + \frac{\check{k}^\mu\eta^\nu}{k^+} + \frac{\eta^\mu\check{q}^\nu}{q^+} - \frac{\eta^\mu\eta^\nu}{q^+k^+}(\check{q}\cdot\check{k})\right)\\
& \times \int dz^-\; e^{-iz^-(q^+-k^+)}\int d^2z_\perp\; e^{-iz_\perp(q_\perp-k_\perp)}\bigg\{\bigg[U_A\Big(\frac{L^+}{2},-\frac{L^+}{2};z_\perp,z^-\Big)-1\bigg]\\
& - \frac{q^j+k^j}{2(q^++k^+)}\int_{-\frac{L^+}{2}}^{\frac{L^+}{2}} dz^+\bigg[U_A\Big(\frac{L^+}{2},z^+;z_\perp,z^-\Big)\overrightarrow{\frac{d}{dz^j}} - \overleftarrow{\frac{d}{dz^j}}U_A\Big(z^+,-\frac{L^+}{2};z_\perp,z^-\Big)\bigg]\\
& - \frac{i}{q^++k^+}\int_{-\frac{L^+}{2}}^{\frac{L^+}{2}} dz^+\; U_A\Big(\frac{L^+}{2},z^+;z_\perp,z^-\Big)\overleftarrow{\frac{d}{dz^j}}\;\overrightarrow{\frac{d}{dz^j}}\;U_A\Big(z^+,-\frac{L^+}{2};z_\perp,z^-\Big)\bigg\} + \text{NNEik}
\end{aligned}$$

Analog to earlier results on the gluon propagator with subeikonal corrections:
Altinoluk, Armesto, Beuf, Martinez, Salgado, JHEP **1407**, 068 (2014)
Altinoluk, Armesto, Beuf, Moscoso, JHEP **1601**, 114 (2016)

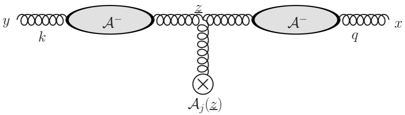
Full next-to-eikonal gluon propagator:

$$G_F^{\mu\nu}\Big|^{\text{NEik}} = G_F^{\mu\nu}\Big|^{\text{Eik}} + \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{pure }\mathcal{A}^-} + \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{single }\mathcal{A}_\perp} + \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{double }\mathcal{A}_\perp,\text{ loc.}} + \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{double }\mathcal{A}_\perp,\text{non-loc.}}$$

$$\delta G^{\mu\nu}_{F\;ab}(x,y)\big|_{\text{single}A_\perp} = \int d^4z \left[G_F^{\mu\mu'}(x,z)\big|_{\text{Eik}}\right]_{aa'} \left[\mathrm{X}^{3g}_{\mu'\nu'}(z)\right]^{a'b'} \left[G_F^{\nu'\nu}(z,y)\big|_{\text{Eik}}\right]_{b'b}$$

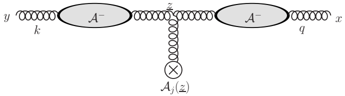
with $\mathrm{X}^{3g}_{\mu'\nu'}(z)$ is the insertion factor obtained from three gluon vertex:

$$\mathrm{X}^{3g}_{\mu'\nu'}(z) = -gf^{abc}\bigg[\Big(ig_{\mu'\nu'}\frac{\overleftarrow{d}}{dz^j}A^j_c(z)\Big) - ig_{\mu'\nu'}\,A^j_c(z)\,\frac{\overrightarrow{d}}{dz^j} - 2ig^j_{\nu'}\,A^j_c(z)\,\frac{\overrightarrow{d}}{dz^{\mu'}} - ig^j_{\nu'}\,\frac{\overleftarrow{d}}{dz^{\mu'}}\Big(A^j_c(z)\Big)$$
$$+ 2ig^j_{\mu'}\,\frac{\overleftarrow{d}}{dz^{\nu'}}\Big(A^j_c(z)\Big) + ig^j_{\mu'}A^j_c(z)\frac{\overrightarrow{d}}{dz^{\nu'}}\bigg]$$



Full next-to-eikonal gluon propagator:

$$G_F^{\mu\nu}\Big|^{\mathrm{NEik}} = G_F^{\mu\nu}\Big|^{\mathrm{Eik}} + \delta G_F^{\mu\nu}\Big|^{\mathrm{NEik}}_{\mathrm{pure}\ \mathcal{A}^-} + \delta G_F^{\mu\nu}\Big|^{\mathrm{NEik}}_{\mathrm{single}\ \mathcal{A}_\perp} + \delta G_F^{\mu\nu}\Big|^{\mathrm{NEik}}_{\mathrm{double}\ \mathcal{A}_\perp,\mathrm{loc.}} + \delta G_F^{\mu\nu}\Big|^{\mathrm{NEik}}_{\mathrm{double}\ \mathcal{A}_\perp,\mathrm{non-loc.}}$$



Subeikonal correction due to an interaction with $\mathcal{A}_\perp$ (three gluon vertex):

$$\delta G^{\mu\nu}_{F\ ab}(x,y)|_{\mathrm{single}\mathcal{A}_\perp} = g\int d^3 z \int \frac{d^3 q}{(2\pi)^3}\frac{e^{-ix\cdot\check{q}}}{2q^+}\,\theta(x^+-z^+)\theta(q^+)\int\frac{d^3k}{(2\pi)^3}\frac{e^{iy\cdot\check{k}}}{2k^+}\theta(k^+)\int dz^- e^{-iz^-(q^+-k^+)}$$
$$\times\Big[U_A(x^+,z^+;z_\perp,z^-)\Big]_{aa'}\ e^{-iq_\perp z_\perp}\Bigg\{2\Bigg[\Bigg(g^{\mu j}g^{\nu i}-\frac{\eta^\mu g^{\nu i}q^j}{q^+}-\frac{g^{\mu j}k^i\eta^\nu}{k^+}+\frac{\eta^\mu\eta^\nu k^i q^j}{q^+k^+}\Bigg)$$
$$-\Bigg(g^{\mu i}g^{j\nu}-\frac{\eta^\mu q^i g^{j\nu}}{q^+}-\frac{g^{\mu i}k^j\eta^\nu}{k^+}+\frac{\eta^\mu\eta^\nu q^i k^j}{q^+k^+}\Bigg)\Bigg]\Bigg[\frac{\overleftarrow{d}}{dz^i}\big(T\cdot A^j(z)\big)+\big(T\cdot A^j(z)\big)\frac{\overrightarrow{d}}{dz^i}\Bigg]$$
$$+\Bigg[g^{\mu\nu}-\frac{\eta^\mu\check{q}^\nu}{q^+}-\frac{\check{k}^\mu\eta^\nu}{k^+}+\frac{\eta^\mu\eta^\nu}{q^+k^+}(\check{q}\cdot\check{k})\Bigg]\Bigg[\frac{\overleftarrow{d}}{dz^j}\big(T\cdot A^j(z)\big)-\big(T\cdot A^j(z)\big)\frac{\overrightarrow{d}}{dz^j}\Bigg]\Bigg\}$$
$$\times\, e^{ik_\perp z_\perp}\Big[U_A(z^+,y^+;z_\perp,z^-)\Big]_{b'b}\,\theta(z^+-y^+)+\mathrm{NNEik}$$

Reminder: $x^+ > L^+/2$ and $y^+ < -L^+/2$: gluon propagating through the whole medium

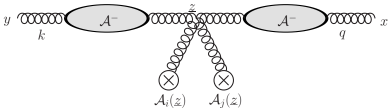
Full next-to-eikonal gluon propagator:

$$G_F^{\mu\nu}\Big|^{\text{NEik}} = G_F^{\mu\nu}\Big|^{\text{Eik}} + \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{pure }\mathcal{A}^-} + \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{single }\mathcal{A}_\perp} + \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{double }\mathcal{A}_\perp,\text{ loc.}} + \delta G_F^{\mu\nu}\Big|^{\text{NEik}}_{\text{double }\mathcal{A}_\perp,\text{non-loc.}}$$

$$\delta G_F^{\mu\nu}(x,y)\big|_{\text{double}A_\perp,\text{ loc.}} = \frac{1}{2}\int d^4z\,\Big[G_F^{\mu\mu'}(x,z)|_{\text{Eik}}\Big]_{aa'}\Big[\mathrm{X}^{4g}_{\mu'\nu'}(\underline{z})\Big]^{a'b'}\Big[G_F^{\nu'\nu}(z,y)|_{\text{Eik}}\Big]_{b'b}$$

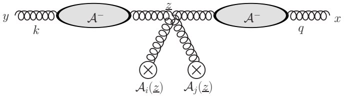
with $\mathrm{X}^{4g}_{\mu'\nu'}(z)$ is the insertion factor obtained from four gluon vertex:

$$\mathrm{X}^{4g}_{\mu'\nu'}(\underline{z}) = -ig^2\Big[f^{ea'b'}f^{edc}\left(g_{\nu'i}g_{\mu'j}-g_{\nu'j}g_{\mu'i}\right) + f^{eb'd}f^{ea'c}\left(g_{\nu'\mu'}g_{ij}-g_{\nu'j}g_{\mu'i}\right) + f^{eb'c}f^{ea'd}\left(g_{\nu'\mu'}g_{ij}-g_{\nu'i}g_{\mu'j}\right)\Big]\,A_c^i(z)\,A_d^j(z)$$



Full next-to-eikonal gluon propagator:

$$G_F^{\mu\nu}\Big|^{\rm NEik} = G_F^{\mu\nu}\Big|^{\rm Eik} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{pure } \mathcal{A}^-} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{single } \mathcal{A}_\perp} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{double } \mathcal{A}_\perp,\, \text{loc.}} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{double } \mathcal{A}_\perp, \text{non-loc.}}$$



Subeikonal correction due to an interaction with double $\mathcal{A}_\perp$-local (four gluon vertex):

$$\begin{aligned}
\delta G^{\mu\nu}_{F\,ab}(x,y)|_{\text{double}\mathcal{A}_\perp,\,\text{loc.}} &= \int \frac{d^3q}{(2\pi)^3}\, \frac{e^{-ix\cdot\check{q}}}{2q^+}\, \theta(x^+ - z^+)\theta(q^+) \int \frac{d^3k}{(2\pi)^3}\, \frac{e^{iy\cdot\check{k}}}{2k^+}\, \theta(k^+) \int dz^-\; e^{iz^-(q^+-k^+)} \\
&\times \int d^2z_\perp\; e^{-i(q_\perp - k_\perp)z_\perp} \int dz^+ \Big(ig^2\Big) \Big[U_A(x^+, z^+; z_\perp, z^-)\Big]_{aa'} \\
&\times \bigg\{ \Big[T\cdot A^i(z)\Big]\Big[T\cdot A^j(z)\Big]\bigg[-g^{\mu\nu}g_{ij} + \frac{\check{k}^\mu\eta^\nu g_{ij}}{k^+} + \frac{\eta^\mu\check{q}^\nu g_{ij}}{q^+} - \frac{\eta^\mu\eta^\nu g_{ij}}{k^+q^+}(\check{k}\cdot\check{q})\bigg] \\
&+ \bigg(-2\Big[T\cdot A_i(z), T\cdot A_j(z)\Big] + \Big[T\cdot A_i(z)\Big]\Big[T\cdot A_j(z)\Big]\bigg)\bigg(g_i^\mu g_j^\nu - \frac{k_j g_i^\mu \eta^\nu}{k^+} - \frac{\eta^\mu g_j^\nu q_i}{q^+} + \frac{\eta^\mu\eta^\nu k_j q_i}{q^+q^+}\bigg)\bigg\} \\
&\times \Big[U_A(z^+, y^+; z_\perp, z^-)\Big]_{b'b}\, \theta(z^+ - y^+) + \text{NNEik}
\end{aligned}$$

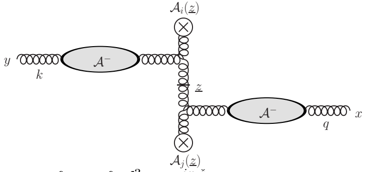
Full next-to-eikonal gluon propagator:

$$G_F^{\mu\nu}\Big|^{\rm NEik} = G_F^{\mu\nu}\Big|^{\rm Eik} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{pure }\mathcal{A}^-} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{single }\mathcal{A}_\perp} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{double }\mathcal{A}_\perp,\,\text{loc.}} + \delta G_F^{\mu\nu}\Big|^{\rm NEik}_{\text{double }\mathcal{A}_\perp,\text{non-loc.}}$$

$$\delta G_{ab}^{\mu\nu}(x,y)|_{\rm double\,A_\perp,\,non-loc.} = \int d^4z' \int d^4z'' \; \big[G_F^{\mu\mu'}(x,z')\big]_{aa'} \big[X^{3g}_{\mu'\mu''}(z')\big]^{a'a''} \times \big[G_F^{\mu''\nu''}(z',z'')\big]_{a''b''} \big[X^{3g}_{\nu''\nu'}(z'')\big]^{b''b'} \big[G_F^{\nu'\nu}(z'',y)\big]_{b'b}$$

with two $\mathrm{X}^{3g}_{\mu'\nu'}(z)$ is the insertion factors obtained from three gluon vertex.

Naively NNEik order contribution. However, the instantaneous part of the gluon propagator between z' and z'' contributes at NEik order.



$$\delta G_{ab}^{\mu\nu}(x,y)|_{\rm double\,A_\perp,\,non-loc.} = g^2 \int d^3z \int \frac{d^3q}{(2\pi)^3}\frac{e^{-ix\cdot\check{q}}}{2q^+}\;\theta(x^+-z^+)\;\theta(q^+)\int \frac{d^3k}{(2\pi)^3}\frac{e^{iy\cdot\check{k}}}{2k^+}\int dz^-\; e^{iz^-(q^+-k^+)}$$
$$\times\, U_A(x^+,z^+;z_\perp,z^-)\; e^{-iq_\perp z_\perp}\;(-i)\big[T\cdot A^i(z)\big]\;\big[T\cdot A^j(z)\big]\left[g^{\mu i}g^{j\nu} - \frac{g^{\mu i}k^j\eta^\nu}{k^+} - \frac{\eta^\mu q^i g^{j\nu}}{q^+} + \frac{\eta^\mu\eta^\nu q^i k^j}{q^+k^+}\right]$$
$$\times\, e^{ik_\perp z_\perp}\;\theta(z^+-y^+)\;\theta(k^+)\;U_A(z^+,y^+;z_\perp,z^-) + \text{NNEik}$$

Next-to-eikonal gluon propagator - full result

Full next-to-eikonal gluon propagator traversing the whole target (before to after):

$$G_F^{\mu\nu}(x,y) = \delta G_F^{\mu\nu}(x,y)|_{\text{Eik},z^-} + \delta G_F^{\mu\nu}(x,y)|_{\text{NEik}}$$

with (generalized) eikonal contribution

$$\delta G_F^{\mu\nu}(x,y)|_{Eik,z^-} = \int \frac{d^3q}{(2\pi)^3}\; e^{-ix\cdot\check{q}}\;\theta(q^+)\;\int \frac{d^3k}{(2\pi)^3}\; e^{iy\cdot\check{k}}\;\theta(k^+)\;\frac{1}{q^++k^+}\Big[-g^{\mu\nu} + \frac{\check{k}^\mu\eta^\nu}{k^+} + \frac{\eta^\mu\check{q}^\nu}{q^+} - \frac{\eta^\mu\eta^\nu}{q^+k^+}(\check{q}\cdot\check{k})\Big]$$
$$\times \int d^2z_\perp e^{-i(q_\perp-k_\perp)z_\perp}\int dz^-\; e^{i(q^+-k^+)z^-}\; U_A(x^+,y^+,z_\perp,z^-)$$

with NEik contributions $\qquad \delta G_F^{\mu\nu}(x,y)|_{\text{NEik}} = \delta G_{1,F}^{\mu\nu}(x,y) + \delta G_{2,F}^{\mu\nu}(x,y)$

$$\delta G_{1,F}^{\mu\nu}(x,y) = \int \frac{d^3\underline{q}}{(2\pi)^3}\;\frac{e^{-ix\cdot\check{q}}}{2q^+}\;\theta(q^+)\;\int \frac{d^3\underline{k}}{(2\pi)^3}\;\frac{e^{iy\cdot\check{k}}}{2k^+}\;\theta(k^+)\Big(-g^{\mu\nu} + \frac{\check{k}^\mu\eta^\nu}{k^+} + \frac{\eta^\mu\check{q}^\nu}{q^+} - \frac{\eta^\mu\eta^\nu}{q^+k^+}(\check{q}\cdot\check{k})\Big)$$
$$\times \int dz^-\; e^{iz^-(q^+-k^+)}\;\int d^2z_\perp\; e^{-iz_\perp(q_\perp-k_\perp)}\Big\{-\frac{q^j+k^j}{2}\int_{\frac{-L^+}{2}}^{\frac{L^+}{2}} dz^+$$
$$\times\Big[U_A\Big(\frac{L^+}{2},z^+;z_\perp,z^-\Big)\Big(\overrightarrow{D}^A_{z^j} - \overleftarrow{D}^A_{z^j}\Big)U_A\Big(z^+,-\frac{L^+}{2};z_\perp,z^-\Big)\Big]$$
$$- i\int_{-\frac{L^+}{2}}^{\frac{L^+}{2}} dz^+\Big[U_A\Big(\frac{L^+}{2},z^+;z_\perp,z^-\Big)\Big(\overleftarrow{D}^A_{z^j}\overrightarrow{D}^A_{z^j}\Big)\;U_A\Big(z^+,-\frac{L^+}{2};z_\perp,z^-\Big)\Big]\Big\} + \text{NNEik}$$

$$\delta G_{2,F\,ab}^{\mu\nu}(x,y) = \int \frac{d^3q}{(2\pi)^3}\;\frac{e^{-ix\cdot\check{q}}}{2q^+}\;\theta(q^+)\int \frac{d^3k}{(2\pi)^3}\;\frac{e^{iy\cdot\check{k}}}{2k^+}\theta(k^+)\;\Big(g^{\mu j}g^{\nu i} - \frac{\eta^\mu g^{\nu i}q^j}{q^+} - \frac{g^{\mu j}k^i\eta^\nu}{k^+} + \frac{\eta^\mu\eta^\nu k^i q^j}{q^+k^+}\Big)$$
$$\times\int dz^-\; e^{iz^-(q^+-k^+)}\int d^2z_\perp e^{-i(q_\perp-k_\perp)z_\perp}\int dz^+ U_A\Big(\frac{L^+}{2},z^+;z_\perp,z^-\Big)\; gT\cdot F_{ij}\; U_A\Big(z^+,-\frac{L^+}{2};z_\perp,z^-\Big)$$

Next-to-eikonal gluon propagator - full result

Full next-to-eikonal gluon propagator traversing the whole target (before to after):
(final after defining the decorated Wilson lines:)

$$U^{(1)}_{R;j}(z_\perp, z^-) = \int dz^+ \; U_R(+\infty, z^+, z_\perp, z^-)(\overrightarrow{D}_{z^j} - \overleftarrow{D}_{z^j})U_R(z^+, -\infty, z_\perp, z^-)$$
$$= -2\int dz^+ \; z^+ U_R(+\infty, z^+, z_\perp, z^-)[-igT_R \cdot \mathcal{F}^-_j(z^+, z_\perp z^-)]U_R(z'^+, -\infty, z_\perp, z^-)$$

$$U^{(2)}_{R}(z_\perp, z^-) = \int dz^+ \; U_R(+\infty, z^+, z_\perp, z^-)\overleftarrow{D}_{z^j}\overrightarrow{D}_{z^j}U_R(z^+, -\infty, z_\perp, z^-)$$
$$= \int dz^+ \int dz'^+ (z^+ - z'^+)\theta(z^+ - z'^+)U_R(+\infty, z^+, z_\perp, z^-)[-igT_R \cdot \mathcal{F}^-_j(z^+ z_\perp, z^-)]$$
$$\times U_R(z^+, z'^+, z_\perp, z^-)[-igT_R \cdot \mathcal{F}^-_j(z'^+, z_\perp, z^-)]U_R(z'^+, -\infty, z_\perp, z^-)$$

$$U^{(3)}_{R;ij}(z_\perp, z^-) = \int dz^+ \; U_R(+\infty, z^+, z_\perp, z^-)gT_R \cdot \mathcal{F}_{ij}(z)U_R(+\infty, z^+, z_\perp, z^-)$$

Special cases for quark and gluon propagators

- inside-to-after quark prop. - Swaleha's talk

$$S_F(x,y)|^{IA,q}_{Eik} = \int \frac{d^3q}{(2\pi)^3} \frac{\theta(q^+)}{2q^+} \, e^{-ix\check{q}} \, (\check{\not{q}} + m) \, U_F(x^+, y^+, y_\perp, y^-) \, \Big[1 - \frac{\gamma^+\gamma^i}{2q^+} i\overleftarrow{D}^F_{y^i}\Big] \, e^{iy^-q^+} \, e^{-iy_\perp q_\perp}$$

- before-to-inside quark prop.

$$S_F(x,y)_{\beta,\alpha}|^{\mathrm{BI,q}}_{\mathrm{Eik}} = \int \frac{d^3k}{(2\pi)^3} \frac{\theta(k^+)}{2k^+} \, e^{-ix^-k^+} \, e^{iy\cdot\check{k}} \, e^{ix_\perp k_\perp} \, \Big[1 - \frac{i\gamma^+\gamma^i}{2k^+} \overrightarrow{D}^F_{x^i}\Big] (\check{\not{k}} + m) \, U_F(x^+, y^+, x_\perp, x^-)_{\beta\alpha}$$

- inside-to-inside quark prop. - Swaleha's talk

$$S_F(x,y)_{\beta,\alpha}|^{\mathrm{II,q}}_{\mathrm{Eik}} = \int \frac{dk^+}{2\pi} \frac{\theta(k^+)}{2k^+} \, e^{-i(x^- - y^-)k^+} \bigg\{ \Big(k^+\gamma^- + m + i\gamma^i \overrightarrow{D}^F_{x^i} \Big) \frac{\gamma^+}{2k^+}$$
$$\times \bigg[\int d^2z_\perp \, \delta^2(x_\perp - z_\perp) \, \delta^2(z_\perp - y_\perp) \, U_F(x^+, y^+, z_\perp, x^-)_{\beta\alpha} \bigg] \Big(k^+\gamma^- + m - i\gamma^j \overleftarrow{D}^F_{y^j} \Big) \bigg\}$$

- inside-to-after gluon prop.

$$\delta G^{\mu\nu}_F(x,y)|^{\mathrm{IA}}_{\mathrm{Eik}} = \int \frac{d^3q}{(2\pi)^3} \, \theta(q^+) \, \frac{e^{-ix\cdot\check{q}} \, e^{iy^-q^+}}{2q^+} \, U_A(x^+, y^+, y_\perp, y^-)$$
$$\times \bigg\{ \bigg[-g^\mu_j g^{j\nu} + \frac{\eta^\mu g^\nu_j q^j}{q^+} + \Big(\frac{ig^\mu_j \eta^\nu}{q^+} + \frac{i\eta^\mu\eta^\nu}{q^+q^+} q^j \Big) \Big(\overleftarrow{D}^A_{y^j} - iq^j \Big) \bigg] \bigg\} \, e^{-iq_\perp y_\perp}$$

- before-to-inside gluon prop.

$$G^{\mu\nu}_F(x,y)_{ab}|^{BI}_{Eik} = \int \frac{d^3k}{(2\pi)^3} \, e^{iy\check{k}} \frac{\theta(k^+)}{2k^+} \, e^{-ix^-k^+} \, e^{ix_\perp k_\perp}$$
$$\times \bigg[-g^\mu_i g^{i\nu} + \frac{g^\mu_i \eta^\nu k^i}{k^+} + \Big(\frac{\eta^\mu g^{i\nu}}{k^+} - \frac{\eta^\mu\eta^\nu k^i}{k^+k^+} \Big) \big(i\overrightarrow{D}^A_{x^i} - k^i \big) \bigg] \, U_A(x^+, y^+, x_\perp, x^-)$$

- inside-to-inside gluon prop.

$$G^{\mu\nu}_F(x,y)|^{II}_{Eik} = \int \frac{dk^+}{2\pi} \frac{\theta(k^+)}{2k^+} \, e^{-i(x^- - y^-)k^+} \, \int d^2z_\perp \, \delta^2(z_\perp - y_\perp)$$
$$\times \bigg[-g^\mu_i g^{i\nu} - \frac{i\eta^\mu g^\nu_i}{k^+} \overrightarrow{D}^A_{x^i} + \frac{ig^\mu_j \eta^\nu}{k^+} \overleftarrow{D}^A_{y^j} + \frac{\eta^\mu\eta^\nu g^j_i}{k^+k^+} \overrightarrow{D}^A_{x^i} \overleftarrow{D}^A_{y^j} \bigg] \delta^2(x_\perp - z_\perp) \, U_A(x^+, y^+, z_\perp, x^-)$$

Gluon-target scattering at NEik accuracy

Total partonic level cross-section for gluon-target scattering at NEik accuracy

$$\frac{d\sigma^{gA\to g+X}}{dP.S.} = \frac{d\sigma^{gA\to g+X}}{dP.S.}\bigg|_{g\,\text{backg.}} + \frac{d\sigma^{gA\to q+X}}{dP.S.}\bigg|_{q\,\text{backg.}}$$

Contribution from gluon background

LSZ reduction formula:

$$S_{g_2\leftarrow g_1} = \lim_{x^+\to+\infty}(-1)(2p_2^+)\int d^2x\int dx^-\, e^{+ix\cdot\check{p}_2}\,\epsilon_\mu^{\lambda_2}(p_2)^* \times \lim_{y^+\to-\infty}(-1)(2p_1^+)\int d^2y\int dy^-\, e^{-iy\cdot\check{p}_1}\,\epsilon_\nu^{\lambda_1}(p_1)\, G_F^{\mu\nu}(x,y)_{a_2\,a_1}$$

factor out $2p_1^+\,(2\pi)\delta(p_1^+ - p_2^+)$ to get the scattering amplitude $iM_{a_2a_1}$:

$$\begin{aligned} iM_{ab}^{\lambda_1\lambda_2}(\underline{k},q_\perp,z^-) = \int d^2z_\perp e^{-i(q_\perp-k_\perp)z_\perp}\Bigg\{ & {\epsilon_{\lambda_2}^i}^*\,\epsilon_{\lambda_1}^i\;\Bigg[U_A\Big(\frac{L^+}{2},\frac{-L^+}{2},z_\perp,z^-\Big) \\ & + \frac{1}{2k^+}\Bigg(-\frac{q^j+k^j}{2}\int_{\frac{-L^+}{2}}^{\frac{L^+}{2}}dz^+\Big[U_A\Big(\frac{L^+}{2},z^+;z_\perp,z^-\Big)\Big(\overrightarrow{D}_{z^j}^A-\overleftarrow{D}_{z^j}^A\Big)\,U_A\Big(z^+,-\frac{L^+}{2};z_\perp,z^-\Big)\Big] \\ & - i\int_{-\frac{L^+}{2}}^{\frac{L^+}{2}}dz^+\Big[U_A\Big(\frac{L^+}{2},z^+;z_\perp,z^-\Big)\Big(\overleftarrow{D}_{z^j}^A\overrightarrow{D}_{z^j}^A\Big)\,U_A\Big(z^+,-\frac{L^+}{2};z_\perp,z^-\Big)\Big]\Bigg)\Bigg] \\ & + \frac{{\epsilon_{\lambda_2}^j}^*\,\epsilon_{\lambda_1}^i}{2k^+}\int_{\frac{-L^+}{2}}^{\frac{L^+}{2}}dz^+\,U_A\Big(\frac{L^+}{2},z^+;z_\perp,z^-\Big)\,gT\cdot F_{ij}\,U_A\Big(z^+,-\frac{L^+}{2};z_\perp,z^-\Big)\Bigg\} \end{aligned}$$

Gluon-target scattering at NEik accuracy

Gluon background contribution to the gluon-target scattering

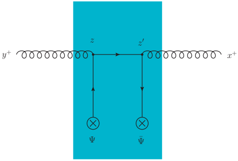
$$\left\langle \frac{d\sigma^{gA\to g+X}}{dP.S.}\Big|_{g\,\text{backg.}} \right\rangle = 2k^+ \int dr^- \; e^{ir^-(q^+-k^+)} \frac{1}{(2\pi)^2} \frac{1}{2(N_c^2-1)} \times \sum_{\lambda_1,\lambda_2} \sum_{a,b} \left\langle \mathcal{M}_{ab}^{\lambda_1\lambda_2}\left(\frac{-r^-}{2}, k^+, k_\perp\right)^\dagger \mathcal{M}_{ab}^{\lambda_1\lambda_2}\left(\frac{r^-}{2}, k^+, k_\perp\right) \right\rangle$$

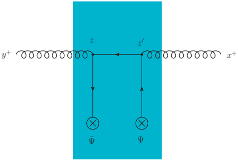
with $dP.S. = \frac{d^2q}{(2\pi)^2} \frac{dq^+}{2q^+(2\pi)}$

$$\frac{d\sigma^{gA\to g+X}}{dP.S.}\Big|_{g\,\text{backg.}} = \frac{1}{(2\pi)^2} \frac{1}{(N_c^2-1)} \int dr^- \; e^{ir^-(q^+-k^+)} \int d^2z'_\perp \int d^2z_\perp \; e^{-i(q_\perp-k_\perp)(z_\perp-z'_\perp)} \times \mathrm{Tr}\Bigg[2k^+ U_A^\dagger\left(z'_\perp, \frac{-r^-}{2}\right) U_A\left(z_\perp, \frac{r^-}{2}\right) + U_A^\dagger\left(z'_\perp, \frac{-r^-}{2}\right) \times \left(\left(-\frac{q^j+k^j}{2} \right) U_{A;j}^{(1)}\left(z_\perp, \frac{r^-}{2}\right) - iU_A^{(2)}\left(z_\perp, \frac{r^-}{2}\right) \right) + \left(-\left(\frac{q^l+k^l}{2}\right) U_{A;l}^{(1)\dagger}\left(z'_\perp, \frac{-r^-}{2}\right) + iU_A^{(2)\dagger}\left(z'_\perp, \frac{-r^-}{2}\right) \right) U_A\left(z_\perp, \frac{r^-}{2}\right) \Bigg]$$

After summation over the gluon polarizations, F_{kl} terms come with $\delta^{li}\delta^{ki}$ factor. Therefore, $U_A^{(3)}$ terms vanish at the cross section level.

Contribution from the quark background





The propagator reads

$$M_F^{\mu\nu}(x,y)|^{gqg} = \int d^4z \int d^4z' \; G_F^{\mu\mu'}(x,z')_{aa'}|^{IA}_{Eik} \; \overline{\Psi}^-_\beta(z') \; \{(-igt^{a'}\gamma_{\mu'}) \; S_F(z',z)|^{II,q}_{Eik}(-igt^{b'}\gamma_{\nu'})\}_{\beta\alpha}$$
$$\times \, \Psi^-_\alpha(z) \; (G_F^{\nu'\nu}(z,y))_{b'b}|^{BI}_{Eik}$$

Amplitude can be computed following the steps:

$$i\mathcal{M}^{\lambda_1\lambda_2}_{ab}(k,q)|^{gqg} = \frac{1}{2k^+}\int d^2z_\perp \; e^{-i(q_\perp - k_\perp)z_\perp} \int dz^+ \int dz'^+ \theta(z'^+ - z^+)\Big\{U_A(\frac{L^+}{2}, z'^+, z_\perp, z^-)_{aa'}$$
$$\times \, \overline{\Psi}_\beta(z'^+, z_\perp, z^-) \; \epsilon^{i}_{\lambda_2}{}^* \; \epsilon^j_{\lambda_1}(\frac{\gamma^i\gamma^-\gamma^j}{2})\{(-igt^{a'})U_F(z'^+, z^+, z_\perp, z^-)(-igt^{b'})\}_{\beta\alpha}$$
$$\times \, \Psi_\alpha(z) U_A(z^+, \frac{-L^+}{2}, z_\perp, z^-)_{b'b}\Big\}$$

Contribution from the quark background

Only the interference terms contribute at NEik order

$$\left\langle \frac{d\sigma^{gA\to g+X}}{dP.S.}\right\rangle\Big|^{gqg} = 2k^+ \int dr^-\ e^{ir^-(q^+-k^+)} \frac{1}{(2\pi)^2}\frac{1}{2(N_c^2-1)} \sum_{\lambda_1\lambda_2}\sum_{a,b}$$
$$\times \left[\left\langle \mathcal{M}\Big(\frac{-r^-}{2},k^+,k_\perp\Big)^\dagger\Big|_{Eik}\mathcal{M}\Big(\frac{r^-}{2},k^+,k_\perp\Big)\Big|_{NEik}\right\rangle + \left\langle \mathcal{M}\Big(\frac{-r^-}{2},k^+,k_\perp\Big)^\dagger\Big|_{NEik}\mathcal{M}\Big(\frac{r^-}{2},k^+,k_\perp\Big)\Big|_{Eik}\right\rangle\right]$$

Contribution to the cross section:

$$\left\langle \frac{d\sigma^{gA\to g+X}}{dP.S.}\bigg|_{gqg}\right\rangle = \int dr^-\ e^{ir^-(q^+-k^+)}\frac{1}{(2\pi)^2}\frac{1}{2(N_c^2-1)}\int d^2z'_\perp \int d^2z_\perp\ e^{-i(q_\perp-k_\perp)(z_\perp-z'_\perp)}\bigg\{\int dz^+\int dz_1^+$$
$$\times\,\theta(z_1^+-z^+)\Big\langle\Big\{U_A^\dagger\Big(\frac{L^+}{2},\frac{-L^+}{2},z'_\perp,\frac{-r^-}{2}\Big)_{ab}U_A\Big(\frac{L^+}{2},z_1^+,z_\perp,\frac{r^-}{2}\Big)_{aa'}\overline{\Psi}_\beta\Big(z_1^+,z_\perp,\frac{r^-}{2}\Big)$$
$$\times\,\gamma^-\Big[(-igt^{a'})U_F\Big(z_1^+,z^+,z_\perp,\frac{r^-}{2}\Big)(-igt^{b'})\Big]_{\beta\alpha}\Psi_\alpha\Big(\underline{z},\frac{r^-}{2}\Big)U_A\Big(z^+,\frac{-L^+}{2},z_\perp,\frac{r^-}{2}\Big)_{b'b}\Big\}\Big\rangle$$
$$+\int dz'^+\int dz_1^+\theta(z_1^+-z'^+)\Big\langle\Big\{U_A^\dagger\Big(z'^+,\frac{-L^+}{2},z'_\perp,\frac{-r^-}{2}\Big)_{b'b}\overline{\Psi}_\alpha\Big(\underline{z}',\frac{-r^-}{2}\Big)$$
$$\times\Big[(igt^{b'})U_F^\dagger\Big(z'^+,z_1^+,z'_\perp,\frac{-r^-}{2}\Big)(igt^{a'})\Big]_{\beta\alpha}\gamma^-\Psi_\beta\Big(z_1^+,z'_\perp,\frac{-r^-}{2}\Big)$$
$$\times\,U_A^\dagger\Big(\frac{L^+}{2},z_1^+,z'_\perp,\frac{-r^-}{2}\Big)_{aa'}U_A\Big(\frac{L^+}{2},\frac{-L^+}{2},z_\perp,\frac{r^-}{2}\Big)_{ab}\Big\}\Big\rangle\bigg\}$$

Antiquark contribution can be computed in a similar way.

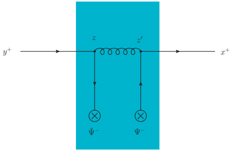
Total partonic level cross-section for quark-target scattering at NEik accuracy

$$\frac{d\sigma^{qA\to q+X}}{dP.S.} = \frac{d\sigma^{qA\to q+X}}{dP.S.}\bigg|_{g\,\text{backg.}} + \frac{d\sigma^{qA\to q+X}}{dP.S.}\bigg|_{q\,\text{backg.}}$$

Contribution from gluon background computed in static limit computed.

Altinoluk, Beuf, Czajka, Tymowska (2021)

Contribution from quark background field

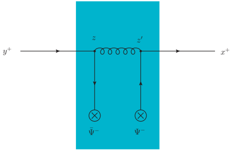


Propagator is given as:

$$M_F(x,y)|^{qgq}_{NEik} = \int d^4z \int d^4z' \left[S_F(x,z')|^{IA,q}_{Eik}\,(-igt^a\gamma_\mu)\right]_{\beta\beta'} \Psi^-_{\beta'}(z') \left[G^{\mu\nu}_F(z',z)|^{II}_{Eik}\right]_{ab}$$
$$\times\,\overline{\Psi^-(z)}_{\alpha'} \left[(-igt^b\gamma_\mu)\,S_F(z,y)|^{BI,q}_{Eik}\right]_{\alpha'\alpha}$$

Quark-target scattering at NEik accuracy

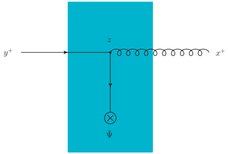
Contribution from quark background field



Propagator $\to$ S matrix $\to$ Amplitude $\to$ X section:

$$\frac{d\sigma^{qA\to q+X}}{dP.S.}\bigg|_{q-\text{backg.}} = \int dr^- e^{ir^-(q^+-k^+)} \frac{1}{(2\pi)^2}\frac{1}{2N_c}\int d^2z_\perp \int d^2z'_\perp\; e^{-i(z_\perp-z'_\perp)(q_\perp-k_\perp)}\int dz^+\int dz_1^+(-g^2)$$
$$\times\bigg\langle\bigg[U_F^\dagger\bigg(\frac{L^+}{2},\frac{-L^+}{2},z'_\perp,\frac{-r^-}{2}\bigg)\bigg]_{\beta\alpha}\bigg[U_F\bigg(\frac{L^+}{2},z_1^+,z_\perp,\frac{r^-}{2}\bigg)t^a\bigg]_{\beta\beta'}$$
$$\times\,\overline{\Psi}_{\alpha'}\bigg(\underline{z},\frac{r^-}{2}\bigg)\;\gamma^-\Psi_{\beta'}\bigg(z_1^+,z_\perp,\frac{r^-}{2}\bigg)\bigg[U_A(z_1^+,z^+,z_\perp,\frac{r^-}{2})\bigg]_{ab}\bigg[t^bU_F(z^+,\frac{-L^+}{2},z_\perp,\frac{r^-}{2})\bigg]_{\alpha'\alpha}\bigg\rangle + c.c.$$

At NEik accuracy contribution come from interference with eikonal quark propagator.

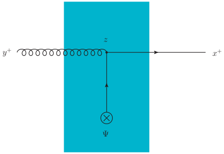


Propagator:

$$M^{\mu}_{a,\alpha}(x,y)|^{gq} = \int d^4z\; G_F^{\mu\mu'}(x,z)_{a,b}|^{IA}_{Eik}\; \overline{\Psi^-_{\beta}(z)}\; \big[(-igt^b\gamma_{\mu'})\; S_F(z,y)|^{BI}_{Eik}\big]_{\beta\alpha}$$

X-section (amplitude square):

$$\frac{d\sigma^{qA\to g+X}}{dP.S.} = \frac{1}{(2\pi)^2}\frac{g^2}{2N_c}\int dr^-\; e^{ir^-(q^+-k^+)}\int d^2z_\perp\;\int d^2z'_\perp\; e^{i(q_\perp-k_\perp)(z'_\perp-z_\perp)}\int dz'^+\int dz^+$$
$$\times\Big\langle\Big[U_F^\dagger\Big(-\frac{L^+}{2},z'_\perp,z'^+,\frac{-r^-}{2}\Big)t^{b'}\Big]_{\beta'\alpha}\overline{\Psi}_\beta\Big(\underline{z},\frac{r^-}{2}\Big)\gamma^-\Psi_{\beta'}\Big(\underline{z}',\frac{-r^-}{2}\Big)$$
$$\times U_A^\dagger\Big(z'^+,\frac{L^+}{2},z'_\perp,\frac{-r^-}{2}\Big)_{ab'}\; U_A\Big(\frac{L^+}{2},z^+,z_\perp,\frac{r^-}{2}\Big)_{ab}\Big[t^b\; U_F\Big(z^+,-\frac{L^+}{2},z_\perp,\frac{r^-}{2}\Big)\Big]_{\beta\alpha}\Big\rangle$$



Propagator:

$$M^{\nu}_{b,\beta}(x,y)|^{qg} = \int d^4z \left[S_F(x,z)|^{IA,q}_{Eik}\;(-igt^a\gamma_{\nu'})\right]_{\beta\alpha}\;\Psi^-_{\alpha}(z)\;\left[G_F^{\nu'\nu}(z,y)|^{BI}_{Eik}\right]_{ab}$$

X-section (amplitude square):

$$\frac{d\sigma^{gA\to q+X}}{dP.S.} = \frac{2q^+}{2k^+}\int dr^-\; e^{ir^-(q^+-k^+)}\frac{1}{(2\pi)^2}\frac{1}{2(N_c^2-1)}\int d^2z'_\perp \int d^2z_\perp\; e^{i(z'_\perp - z_\perp)(q_\perp - k_\perp)}\int dz'^+\int dz^+$$
$$\times\left\langle U_A^\dagger\Big(\frac{-L^+}{2}, z'^+, z'_\perp, \frac{-r^-}{2}\Big)_{a'b}\;\left[(gt^{a'})U_F^\dagger\Big(z'^+, \frac{L^+}{2}, z'_\perp, \frac{-r^-}{2}\Big)\right]_{\beta\alpha'}\overline{\Psi}_{\alpha'}\Big(z', \frac{-r^-}{2}\Big)\;\gamma^-\right.$$
$$\left.\times\;\Psi_\alpha\Big(z, \frac{r^-}{2}\Big)\;\left[U_F\Big(\frac{L^+}{2}, z^+, z_\perp, \frac{r^-}{2}\Big)(gt^a)\right]_{\beta\alpha}\;U_A\Big(z^+, \frac{-L^+}{2}, z_\perp, \frac{r^-}{2}\Big)_{ab}\right\rangle$$

Summary and final remarks

- Full next-to-eikonal (NEik) expression for the gluon propagator through the background field derived
  - Corrections due to transverse motion of the gluon while crossing the Lorentz contracted target
  - Corrections due to interaction with the $\mathcal{A}_\perp$ components of the background field
  - Corrections beyond the static limit including the effects of x^- dependence of the $\mathcal{A}_\mu$.
- Gauge covariant expression: covariant derivatives and field strength insertions in Wilson lines
- Computed $gA \to g + X$, $qA \to q + X$, $gA \to q + X$ and $qA \to g + X$ X-sections.
- The NEik quark and gluon propagator are building blocks for scattering processes at NEik

  inclusive DIS and SIDIS (in preparation) - T. Altinoluk, G. Beuf, S. Mulani

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