Nonrelativistic Conformal Field Theory and its applications

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ECT* workshop *The physics of strongly interacting matter: neutron stars, cold atomic gases and related systems*
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Plan

- Nonrelativistic conformal symmetry
- Nonrelativistic CFTs
- Fermions at unitarity
- Neutrons and “UnNuclear Physics”

Refs.: Y. Nishida, DTS 0706.3746
H.-W. Hammer, DTS 2103.12610
S.D. Chowdhury, R. Mishra, DTS 2309.15177

Schrödinger group

- Symmetries of the free Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi$$

- Phase rotation $M = mN$ $\quad \psi \to e^{i\alpha}\psi$
- space and time translations $\mathbf{P}, H$; rotations J_{ij}
- Galilean boosts $\mathbf{K}$ $\psi(t, \mathbf{x}) \to e^{im\mathbf{v}\cdot\mathbf{x} - \frac{i}{2}mv^2t}\,\psi(t, \mathbf{x} - \mathbf{v}t)$
- Dilatation D $\psi(t, \mathbf{x}) \to \lambda^{3/2}\psi(\lambda^2 t, \lambda\mathbf{x})$

“Proper conformal transformation”

$$C: \psi(t, \mathbf{x}) \to \frac{1}{(1+\alpha t)^{3/2}} \exp\left(\frac{i}{2}\frac{m\alpha x^2}{1+\alpha t}\right)\psi\left(\frac{t}{1+\alpha t}, \frac{\mathbf{x}}{1+\alpha t}\right)$$

Schrödinger algebra

$$[X, Y]$$

$X \setminus Y$	P_j	K_j	D	C	H
P_i	0	$-i\delta_{ij}M$	$-iP_i$	$-iK_i$	0
K_i	$i\delta_{ij}M$	0	iK_i	0	iP_i
D	iP_j	$-iK_j$	0	$-2iC$	$2iH$
C	iK_j	0	$2iC$	0	iD
H	0	$-iP_j$	$-2iH$	$-iD$	0

$$[N, \text{anything}] = 0$$

Nonrelativistic CFTs

Y. Nishida, DTS, 2007

- NRCFTs are QFTs with Schrödinger symmetry
- local operators $O(\vec{x})$ characterized by charge (mass) and dimension $[D, O(0)] = i\Delta_0 O(0), \quad [M, O(0)] = iN_O O(0)$

  example: $\psi \quad N_\psi = 1, \ \Delta_\psi = \frac{3}{2}$

  - primary operators: $[K_i, O(\vec{0})] = [C, O(\vec{0})] = 0$
- Constraints from conformal invariance:

$$\langle TO(t,\vec{x})O^\dagger(0,0)\rangle = \frac{c}{t^{\Delta_O}} \exp\left(\frac{im_O x^2}{2t}\right)$$

$[E] = 2$

$[p] = 1$

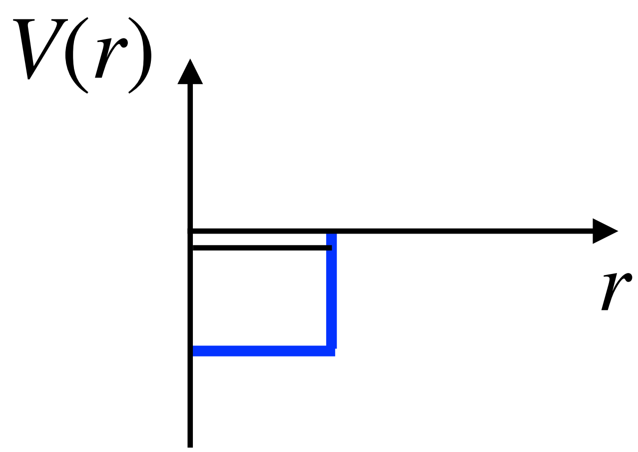
Example of NRCFTs

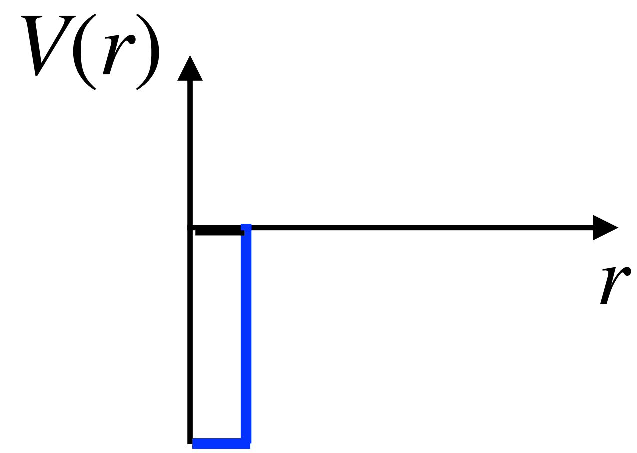
- Free particles
- nonrelativistic anyons (two spatial dimensions)
- Spin-1/2 fermions at unitarity
  - realized in cold atom experiments, but also approximately by neutrons

Example of NRCFT

- $S = \int dt\, d^d\mathbf{x} \left(i\psi^\dagger \partial_t \psi - \frac{1}{2} |\nabla \psi|^2 - \lambda \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \right)$
- NR power counting: $[E] = 2$, $[p] = 1$, $[\psi] = \frac{d}{2}$
- $d = 2 + \epsilon$: $\beta(\lambda) = \epsilon\lambda + \frac{1}{2\pi}\lambda^2$
- interacting CFT at $\lambda_* = -2\pi\epsilon$
- Does the fixed point survive at $d = 3$?
    - yes, this is the so-called “unitarity fermion”

"Unitarity regime"





- Take a potential of a certain shape, e.g.,
  $V(r) = -V_0$ for $r < r_0$, 0 for $r > r_0$

- fine-tune the depth so that there is one "bound state" at zero energy

$$V_0 = \frac{\pi^2 \hbar^2}{8m} \frac{1}{r_0^2}$$

- Then let $r_0 \to 0$: "unitarity regime" s-wave scattering saturates unitarity

Scattering length and unitarity regime

- This situation corresponds to low-energy resonant scattering in quantum mechanics
- s-wave scattering amplitude given by scattering length a and effective range r_0:

$$f(k) = \frac{1}{-ik + \frac{1}{a} + \frac{1}{2}kr_0^2}$$

- Unitarity regime: $a \to \infty$

$$r_0 \to 0$$

  - no dimensionful length scale

- The unitarity regime can be understood directly, without taking the limit $r_0 \to 0$

Quantum-mechanical approach

- Wave function of m spin-up and n spin-down fermions $\psi(\mathbf{x}_1, \ldots, \mathbf{x}_m; \mathbf{y}_1, \ldots, \mathbf{y}_n)$
- ψ antisymmetric under exchanging two $\mathbf{x}$'s or $\mathbf{y}$'s
- When one spin-up and one spin-down fermions approach each other:

$$\psi(\mathbf{x}, \mathbf{y}) = \frac{C}{|\mathbf{x}-\mathbf{y}|} + O(|\mathbf{x}-\mathbf{y}|) + \cdots$$

- $$H = -\frac{1}{2}\sum_a \frac{\partial^2}{\partial \mathbf{x}_a^2} - \frac{1}{2}\sum_a \frac{\partial^2}{\partial \mathbf{y}_a^2}$$

What are the local operators?

- First example: annihilation operator in second quantized formulation of QM
- $\langle 0 | \hat{\psi}(\mathbf{x}) | \Psi_{1-\text{body}} \rangle = \Psi(\mathbf{x})$
- This is a charge-1 operator, dimension=3/2

Charge-2 local operator

- Second-quantized formulation of QM:

$$\langle 0 | \psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y}) | \Psi_{2-\mathrm{body}} \rangle = \Psi(\mathbf{x}, \mathbf{y})$$

- Limit $\mathbf{y} \to \mathbf{x}$ does not exist:

$$\langle 0 | \psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{x}) | \Psi \rangle = \Psi(\mathbf{x}, \mathbf{x}) = \infty$$

- but one can define

$$O_2(\mathbf{x}) = \lim_{\mathbf{y}\to\mathbf{x}} |\mathbf{x} - \mathbf{y}| \psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y})$$

- then

$$\langle 0 | O_2(\mathbf{x}) | \Psi \rangle = \lim_{\mathbf{y}\to\mathbf{x}} |\mathbf{x} - \mathbf{y}| \Psi(\mathbf{x}, \mathbf{y}) = \text{finite}$$

Dimension of O_2

- $O_2(\mathbf{x}) = \lim_{\mathbf{y}\to\mathbf{x}} |\mathbf{x}-\mathbf{y}|\,\psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y})$
- $\Delta[O_2] = 2\Delta[\psi] - 1 = 2$
- cf free theory: $\Delta[\psi\psi] = 3$

Charge-3 operator

- Need to know short distance behavior of $\Psi(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y})$
- 3-body problem solved by Efimov ~ 1970

$$\Psi(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}) \sim R^{-0.2273} f(\alpha, \hat{\rho}, \hat{r})$$

$$R^2 = |\mathbf{x}_1 - \mathbf{x}_2|^2 + |\mathbf{x}_1 - \mathbf{y}|^2 + |\mathbf{x}_2 - \mathbf{y}|^2$$

$\alpha, \hat{\rho}, \hat{r}$ = 5 hyperangles

- Charge-3 operator

$$O_3(\mathbf{x}) \sim \lim_{\mathbf{x}_2 \to \mathbf{x}} \lim_{\mathbf{y} \to \mathbf{x}} R^{0.2273} \psi_\uparrow(\mathbf{x}) \psi_\uparrow(\mathbf{x}_2) \psi_\downarrow(\mathbf{y})$$

- $\Delta[O_3] = 4.2727$

cf free theory: $[\psi_\downarrow \psi_\uparrow \nabla \psi_\uparrow] = \frac{11}{2}$

Dimension of charge-3 operators

$\Delta = \dfrac{5}{2} + s$ where s solves an equation

$l = 0$: $s\cos\left(\frac{\pi}{2}s\right) + \dfrac{4}{\sqrt{3}}\sin\left(\frac{\pi}{6}s\right) = 0$

$$\Delta = 4.666,\ 7.627,\ 9.614,\ \ldots,\ 2n + \frac{7}{2}$$

$l = 1$: $(s^2 - 1)\sin\left(\frac{\pi}{2}s\right) + \dfrac{4}{\sqrt{3}}s\cos\left(\frac{\pi}{6}s\right) - 4\sin\left(\frac{\pi}{6}s\right) = 0$

$$\Delta = 4.273,\ 6.878,\ 8.216,\ \ldots.\ \ 2n + \frac{5}{2}$$

Charge-4 operator

- Dimension of operator with particle number N>3 can only be obtained numerically
- Nishida and DTS 2007: equal to ground state energy of N unitary fermions in a harmonic trap
- $\Delta[O_4] = 5.0 \pm 0.1$ (cf. free theory: 8)

Two point functions

- One can compute two-point functions by inserting a complete set of states

$$\langle 0 | O(t, \mathbf{x}) O^\dagger(0) | 0 \rangle = \sum_n \langle 0 | O(0) | n \rangle e^{-iE_n t + i\mathbf{P}_n \cdot \mathbf{x}} \langle n | O^\dagger(0) | 0 \rangle$$

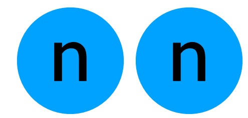
- Also constrained by Schrödinger symmetry
- 2-point function

$$\langle O(t, \mathbf{x}) O^\dagger(0, \mathbf{0}) \rangle = \frac{C}{t^{\Delta_O}} \exp\left(\frac{iM_O x^2}{2t} \right)$$

- In momentum space

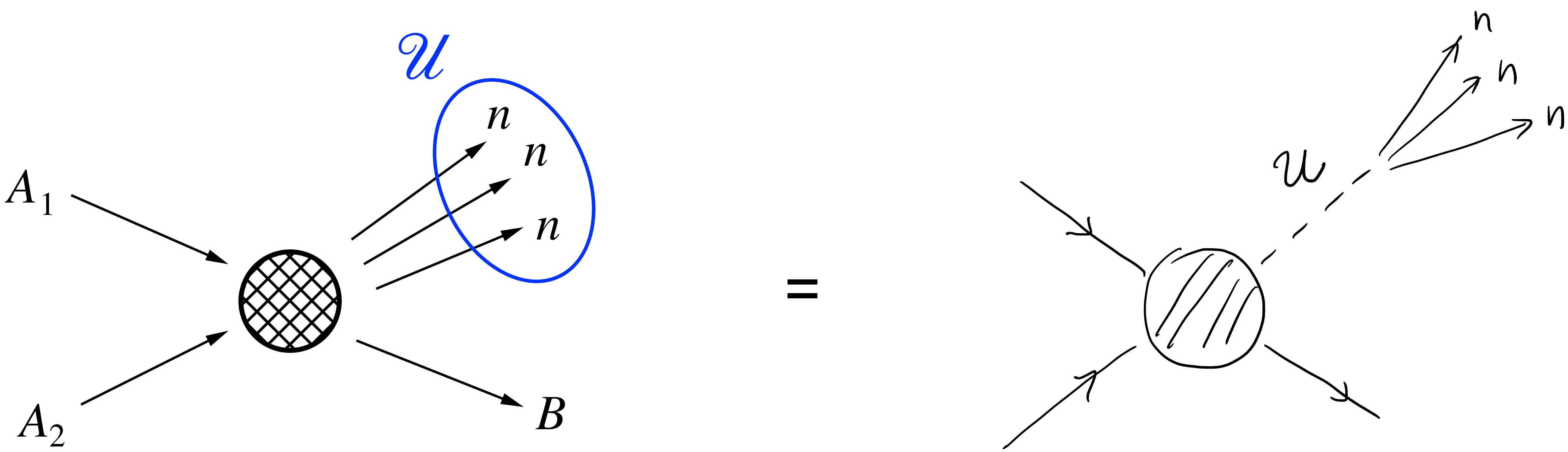
$$\langle O O^\dagger \rangle(\omega, \mathbf{p}) \sim \left(\frac{\mathbf{p}^2}{2M_O} - \omega \right)^{\Delta_O - 5/2}$$

NRCFT in real world: neutrons



- $a \approx -19$ fm, $r_0 \approx 2.8$ fm
- NRCFT in energy range between $\hbar^2/ma^2 \sim 0.1$ MeV and $\hbar^2/mr_0^2 \sim 5$ MeV
- Consequence: power-law behavior in processes with final state neutrons
- "Unnuclear Physics" Hammer, DTS 2021 nonrelativistic version of Georgi's "unparticle physics"

"UnNuclear physics"

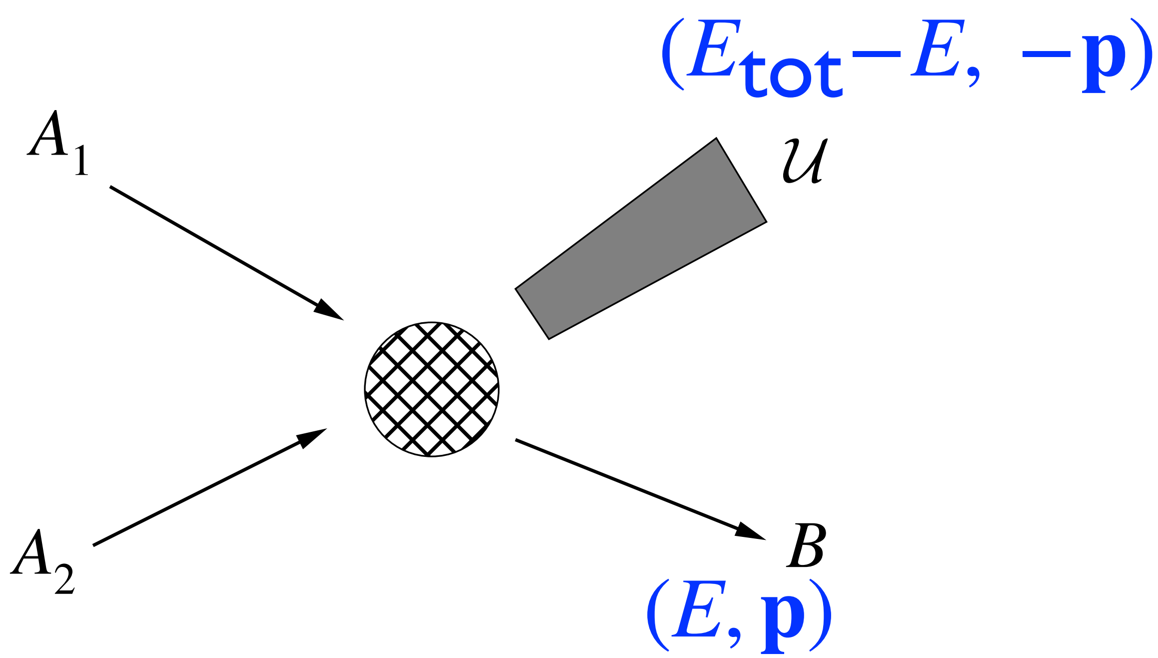


$$P(A_1 + A_2 \to B + 3n) = P(A_1 + A_2 \to B + \mathscr{U})P(\mathscr{U} \to 3n)$$

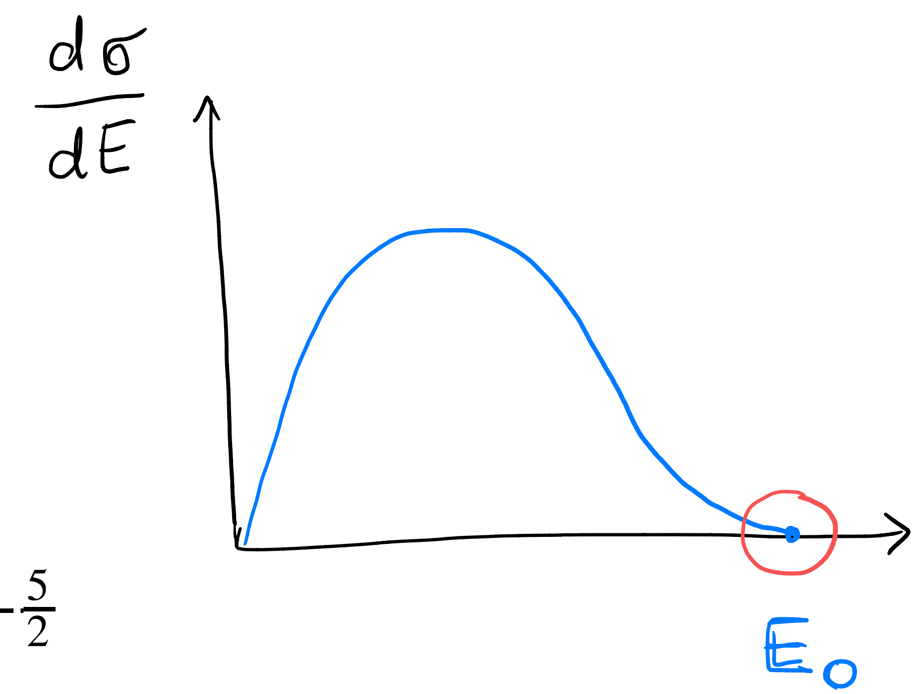
when energy scale of primary reaction is larger than $\mathscr{U} \to 3n$

$\mathscr{U}$ = "unnucleus" = field in NRCFT

Rates of unnuclear processes



$$E_{\rm tot} = E + E_{\mathcal{U}}$$



- $$\frac{d\sigma}{dE} \sim |\mathscr{M}|^2 \sqrt{E} \times \underbrace{\mathrm{Im}\, G_{\mathcal{U}}(E_{\rm tot} - E, \mathbf{p})}_{\to\ (E_0 - E)^{\Delta - \frac{5}{2}}}$$

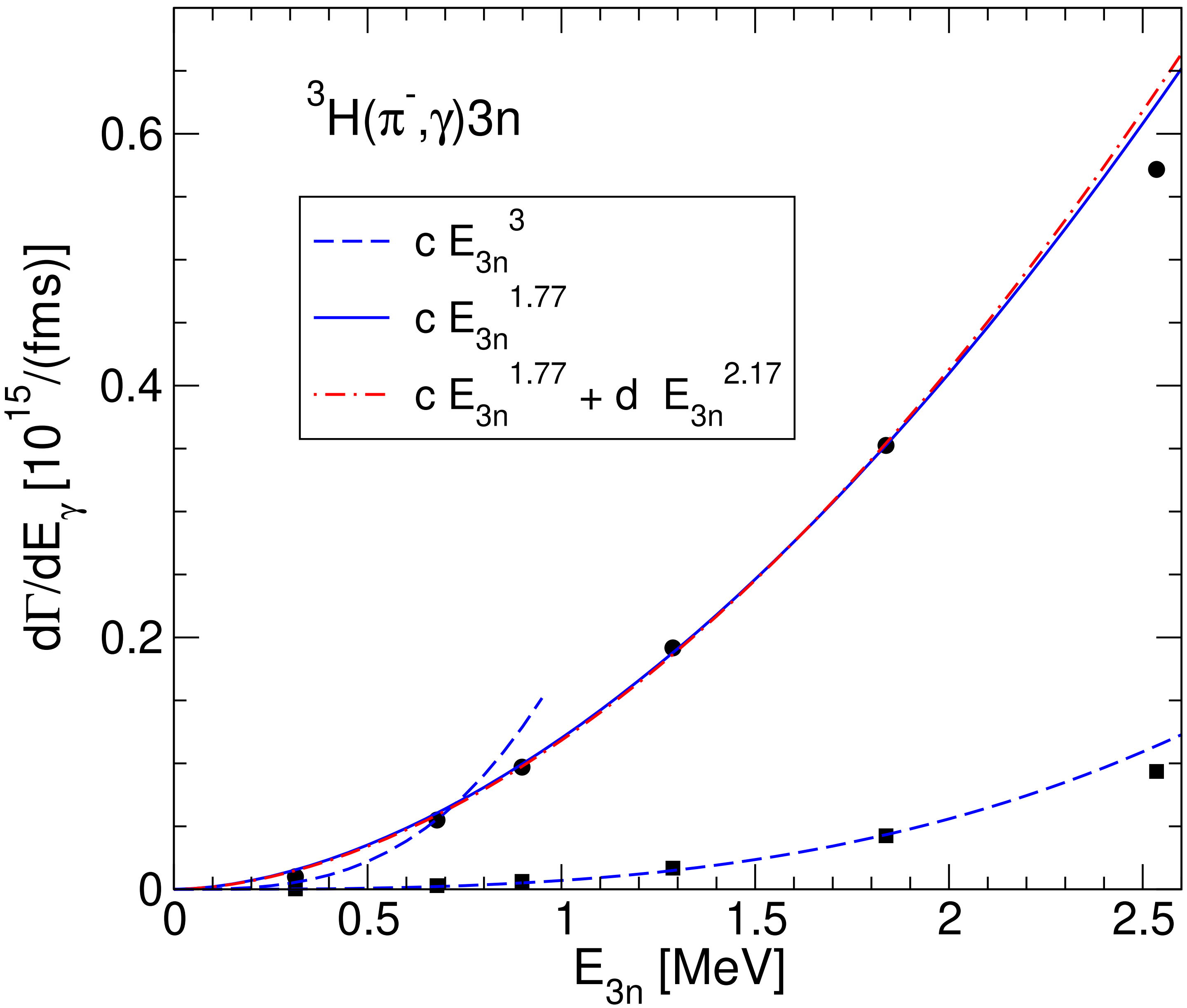
- Near end point: $\dfrac{d\sigma}{dE} \sim (E_0 - E)^{\Delta - \frac{5}{2}}$, Δ= dimension of $\mathcal{U}$

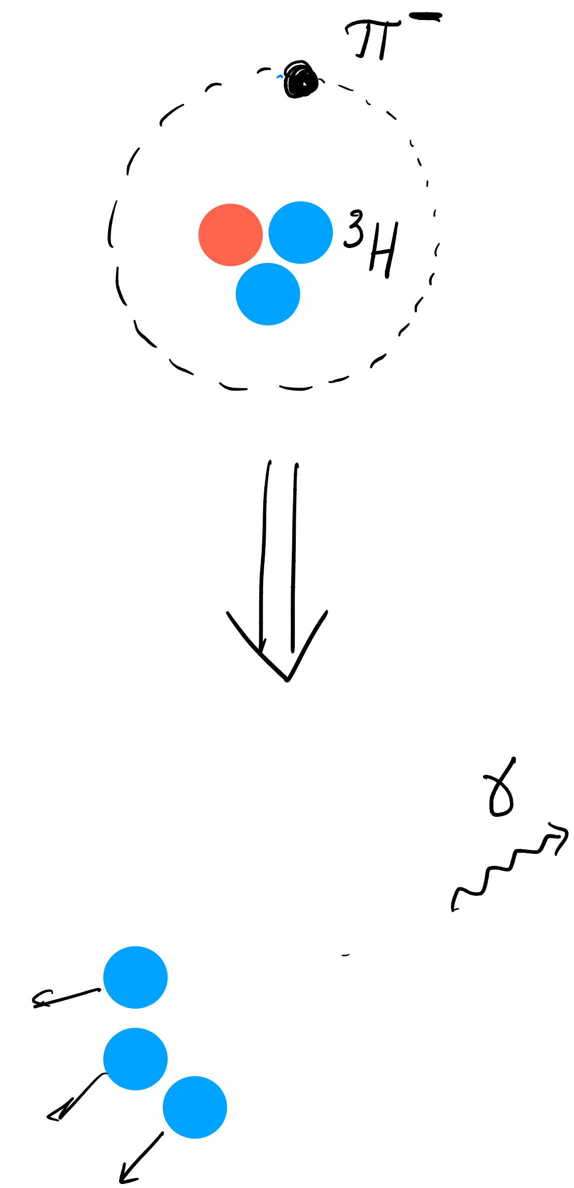
Nuclear reactions

- $\frac{d\sigma}{dE} \sim (E_0 - E)^{\alpha}$ $\qquad \alpha = \Delta - \frac{5}{2}$

	Δ	α	
• $^3H + {}^3H \to {}^4He + 2n$	2	-0.5	Watson-Migdal 1950's
• $\pi^- + {}^3H \to \gamma + 3n$	4.27	1.77	
• $^4He + {}^8He \to {}^8Be + 4n$	5.0	2.5	

Comparison with “experiment”

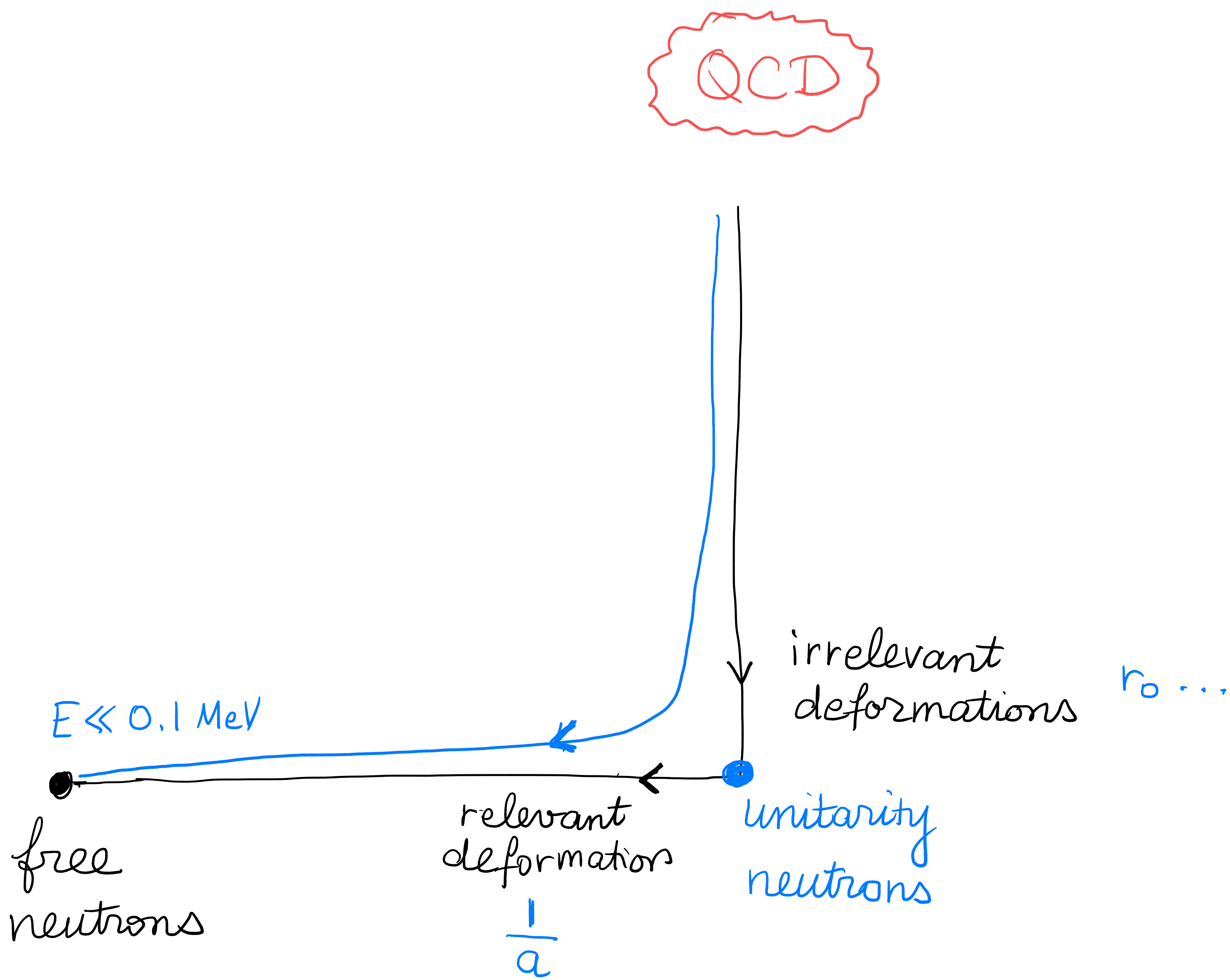




$$\pi^- + {}^3\mathrm{H} \rightarrow \gamma + 3n$$

Golak et al. PRC 98, 054001 (2018)

NRCFT as transient fixed point



Deformations of NRCFT

- Dimensional counting

$$S = \int \underbrace{dt}_{-2} \; \underbrace{d^3x}_{-3} \left(\mathscr{L}_{\rm CFT} + \text{deformations}\right)$$

- operators with dim < 5: relevant; dim > 5: irrelevant
- one relevant deformation: $[O_2^\dagger O_2] = 4$
- Leading Galilean-invariant irrelevant deformation:

$$O_2^\dagger \left(\partial_t + \frac{\nabla^2}{4}\right) O_2 \qquad \text{dim} = 6$$

Away from conformality

- Finite scattering length and effective range can be treated as perturbation away from NRCFT

$$L = L_{\text{CFT}} + \frac{1}{a} O_2^\dagger O_2 - r_0 O_2^\dagger \left(i\partial_t + \tfrac{1}{4}\nabla^2 \right) O_2$$

- Contribution to $\langle O_3 O_3^\dagger \rangle$ can be computed using conformal perturbation theory
S.D. Chowdhury, R. Mishra, DTS 2309.15177

- $$\frac{d\sigma}{dE} \sim \omega^{\Delta - 5/2} \left(1 + \frac{c_1}{a_0\sqrt{m\omega}} + c_2 r_0 \sqrt{m\omega} \right)$$

Conformal perturbation theory

- $$\langle O_3(x) O_3^\dagger(0) \rangle = \frac{1}{Z} \int \mathscr{D}\psi \, O_3(x) O_3^\dagger(0) e^{iS_{\mathrm{CFT}} + \frac{i}{a} \int_y O_2^\dagger(y) O_2(y)}$$

$$= \langle O_3(x) O_3^\dagger(0) \rangle_0 + \frac{1}{a} \int dy \, \langle O_3(x) O_2^\dagger(y) O_2(y) O_3^\dagger(0) \rangle_0$$

$|1\rangle\langle 1|$ $\quad$ $|3\rangle\langle 3|$

$$\frac{d\sigma}{dE} \sim \omega^{\Delta - 5/2} \left(1 + \frac{c_1}{a\sqrt{m\omega}} \right)$$

$$\Delta = 4.273$$
$$c_1 = 2.642$$

Effective-range correction

- Effective range correction proportional to

$$\frac{\Gamma\left(\frac{d}{2}+s-2\right)}{\Gamma\left(3-\frac{d}{2}\right)\Gamma(d-3)} = 0$$

- Vanishes at physical dimension $d = 3$, but not in fractional spatial dimension
  - we do not understand this
  - Also observed for high-charge operators Beane Orlando Reffert (2024)

Open questions

- Can one resum all $1/a$ correction and determine the correlator along the whole RG flow?
- Can be done for O_2

$$\langle O_2 O_2^\dagger \rangle = \left(\sqrt{\frac{p^2}{4} - p_0} - \frac{1}{a} \right)^{-1}$$

but needs to be down for charge-3 operators

- Charge-4 operators and higher?
- High-charge limit Beane Orlando Reffert 2403.18898

Conclusion

- NR conformal field theories have Schrödinger symmetry
- Example: fermions at unitarity
- Approximately realized by neutrons in nuclear physics
- Leads to a power-law behavior of differential cross sections of certain processes near threshold
- Nonrelativistic conformal perturbation theory
- the full power of NRCFT still to be explored