The short-time approximation of response functions in strongly interacting fermionic systems

ECT* Workshop on "Critical Stability of Few-Body Quantum Systems"

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Overview

The linear response of a system to an external probe gives important information about the underlying dynamics and transport properties of the system

The short-time approximation (STA) provides a means to calculate the response of strongly-correlated fermionic systems will also investigating important exclusive two-body body effects

Outline

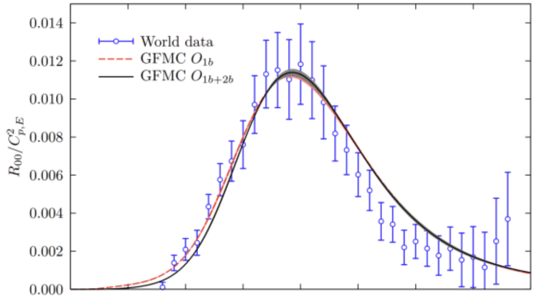
- Theory: Basics of the STA
- Application of the STA: Electron-nucleus scattering
- On-going application to nuclei: Neutrino-nucleus scattering
- Outlook: Improving the STA for nuclei
- Future many-body application: the Unitary Fermi Gas

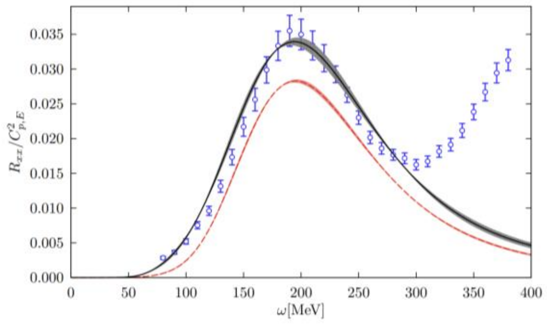
Theory: Basics of the STA

Motivation

Lovato et al PRC 91, 062501 (2015)

^{12}C electron scattering





Standard QMC approach is Euclidean response

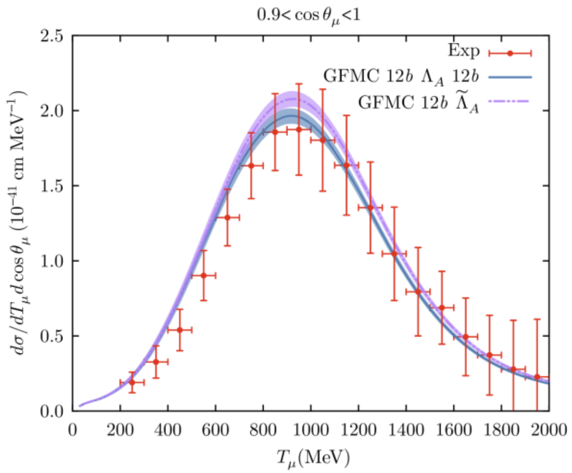
Beginning from standard response function definition:

$$R(\mathbf{q},\omega) = \sum_f \langle 0|\mathcal{O}^\dagger(\mathbf{q})|f\rangle\langle f|\mathcal{O}(\mathbf{q})|0\rangle\delta(E_f - E_0 - \omega)$$

One can evaluate the Laplace Transform and invert using Maximum Entropy techniques **[Lovato et al PRC 91, 062501 (2015)]**:

$$\widetilde{R}(\mathbf{q},\tau) = \langle 0|\mathcal{O}^\dagger(\mathbf{q})e^{-(H-E_i)\tau}\mathcal{O}(\mathbf{q})|0\rangle$$

Lovato et al PRX 10, 031068 (2020)



^{12}C CC neutrino scattering

Motivation

Pros: Retains interaction effects by propagating the full *N* particle system, fully treats correlations

Cons: Computationally intensive, sum over all final states limits one to inclusive response, sign problem in imaginary time propagation

One-body physics: Plane Wave Impulse Approximation

Factorize into a struck particle and A-1 spectator system

$$R(\mathbf{q},\omega) = \int d\mathbf{k} n(\mathbf{k}) r(\mathbf{k},\mathbf{q}) \delta\left(\omega - \frac{(\mathbf{k}+\mathbf{q})^2 - \mathbf{k}^2}{2m}\right)$$

Neglects two-body physics in the electroweak current operators

Missing Pauli blocking makes this a high-energy approximation

Valid when the momentum of the removed particle >> the typical momentum of particles in the system and final state interactions are small

Beyond PWIA: The short-time approximation

Want a method that reduces computational costs while retaining important two-body physics

Sum rules are determined by responses at τ=0, high energy physics corresponds to short imaginary time propagations

Such an approximation is obtained retaining at most two active nucleons, first developed in **Pastore et al. PRC 101, 044612 (2020)**

Propagating at most two active nucleons is computationally less expensive and thus amenable to studying heavier nuclei of experimental relevance

Sum over two-body intermediate states allows investigation of exclusive processes and could allow one to study meson production

Real time response function

Standard response definition:

$$R(\mathbf{q},\omega) = \sum_f \langle 0|\mathcal{O}^\dagger(\mathbf{q})|f\rangle\langle f|\mathcal{O}(\mathbf{q})|0\rangle\delta(E_f - E_0 - \omega)$$

Can be recast in terms of a real-time propagator:

$$R(\mathbf{q},\omega) = \int \frac{dt}{2\pi} e^{i(E_0+\omega)t}\langle 0|\mathcal{O}^\dagger e^{-iHt}\mathcal{O}|0\rangle$$

Pastore et al. PRC 101, 044612 (2020)
Andreoli et al. PRC 105, 014002 (2022)

Two-body physics: Current-current correlator

For short imaginary times, one may expand the propagator as

$$e^{-iHt} \approx 1 - iHt + \mathcal{O}(t^2) \approx 1 - i\left(\sum_i T_i + \sum_{i<j} v_{ij} + \dots\right) t + \dots$$

Making the above approximation, one only correlates two active nucleons at a time

Errors on the order $\mathcal{O}\left(\frac{\omega_{qe}^2}{B_{\text{pair}}^2}\right)$ in the region of the quasi-elastic peak

$$\omega_{qe} = \sqrt{q^2 + m^2} - m$$

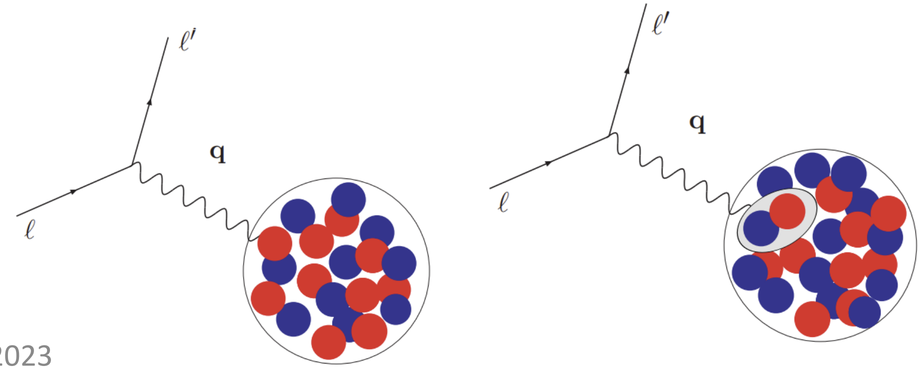
Pastore et al. PRC 101, 044612 (2020)
Andreoli et al. PRC 105, 014002 (2022)

Two-body physics: Current-current correlator

Making the approximation of two active particles *i* and *j*

$$\mathcal{O}^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}(\mathbf{q}) = \left(\sum_i \mathcal{O}_i^\dagger(\mathbf{q}) + \sum_{i<j}\mathcal{O}_{ij}^\dagger(\mathbf{q})\right) e^{-iHt}\left(\sum_{i'}\mathcal{O}_{i'}(\mathbf{q}) + \sum_{i'<j'}\mathcal{O}_{i'j'}(\mathbf{q})\right)$$
$$= \sum_i \mathcal{O}_i^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}_i(\mathbf{q}) + \sum_{i\neq j}\mathcal{O}_i^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}_j(\mathbf{q})$$
$$+ \sum_{i\neq j}\left(\mathcal{O}_i^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}_{ij}(\mathbf{q}) + \mathcal{O}_{ij}^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}_i(\mathbf{q}) + \mathcal{O}_{ij}^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}_{ij}(\mathbf{q})\right)$$

Retains important contributions coming from 1b*2b interference terms



Pastore et al. PRC 101, 044612 (2020)
Andreoli et al. PRC 105, 014002 (2022)

Generic expectation value

Using a complete set of two body-final states:

$$
\begin{aligned}
\langle \mathcal{O}_L^\dagger \mathcal{O}_R \rangle \quad = & \\
& \frac{N(N-1)}{2} \sum_{\alpha_1''\alpha_2''\alpha_1'\alpha_2'} \sum_{\alpha_{N-2}} \int d\mathbf{R}'' d\mathbf{r}'' d\mathbf{R}' d\mathbf{r}' d\mathbf{R}_{N-2} \\
& \times \langle 0 | \mathcal{O}_L^\dagger | \mathbf{R}'' \mathbf{r}'' \alpha_1'' \alpha_2'' \mathbf{R}_{N-2} \alpha_{N-2} \rangle \langle \mathbf{R}'' | e^{-iH_{12}^{\mathrm{CM}} t} | \mathbf{R}' \rangle \\
& \times \langle \mathbf{r}'' \alpha_1'' \alpha_2'' | e^{-iH_{12}^{\mathrm{rel}} t} | \mathbf{r}' \alpha_1' \alpha_2' \rangle \times \langle \mathbf{R}' \mathbf{r}' \alpha_1' \alpha_2' \mathbf{R}_{N-2} \alpha_{N-2} | \mathcal{O}_R | 0 \rangle
\end{aligned}
$$

Integrations over coordinates may be performed with some numerical integration scheme (Gauss-Legendre, Monte Carlo, …)

Pastore et al. PRC 101, 044612 (2020)
Andreoli et al. PRC 105, 014002 (2022)

Variational Monte Carlo

Stochastic method to solve the Schrödinger Equation $H|\Psi\rangle = E|\Psi\rangle$ for some many-body Hamiltonian (Argonne v_{18}, χEFT, Pöschl-Teller potential,…)

Generic fermion trial wave function may be written in terms of an anti-symmetric long-range term and a correlation operator

$$|\Psi_T\rangle = \hat{F}|\Phi\rangle$$

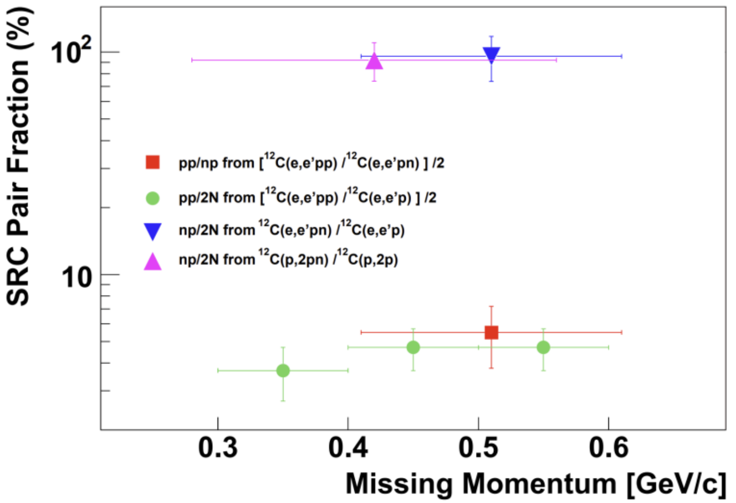
Embedded in the correlation operator are variational parameters that are optimized by minimizing the energy expectation value obtained by Monte Carlo integration:

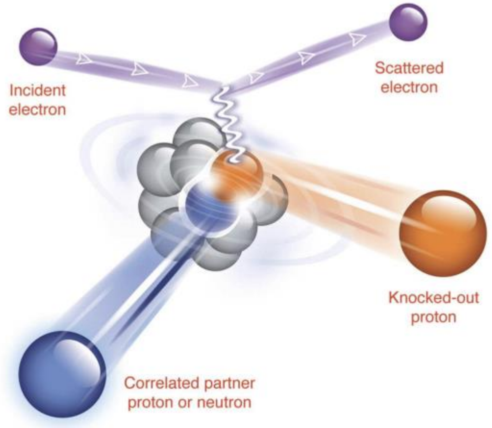
$$E_V = \frac{\langle\Psi_V|H|\Psi_V\rangle}{\langle\Psi_V|\Psi_V\rangle} \geq E_0$$

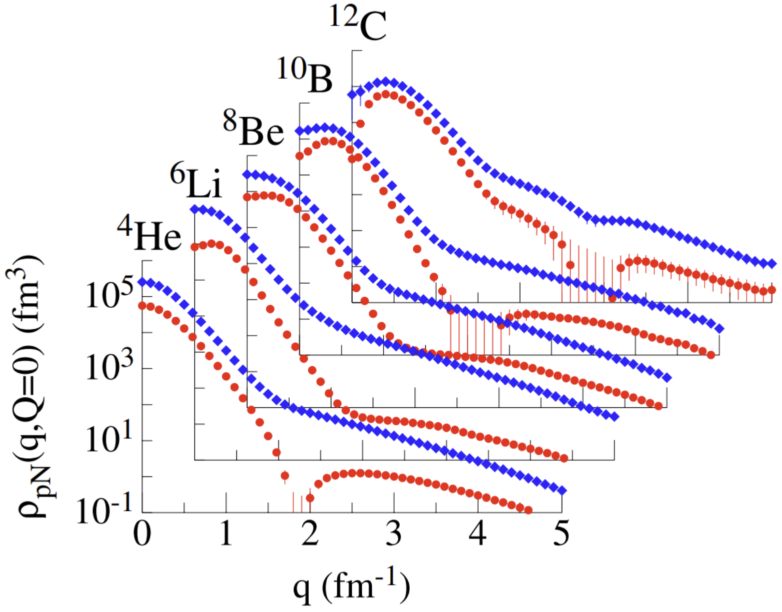
STA applied to nuclei: electron-nucleus scattering

Motivation

Subedi et al. Science 320, 1426 (2008)







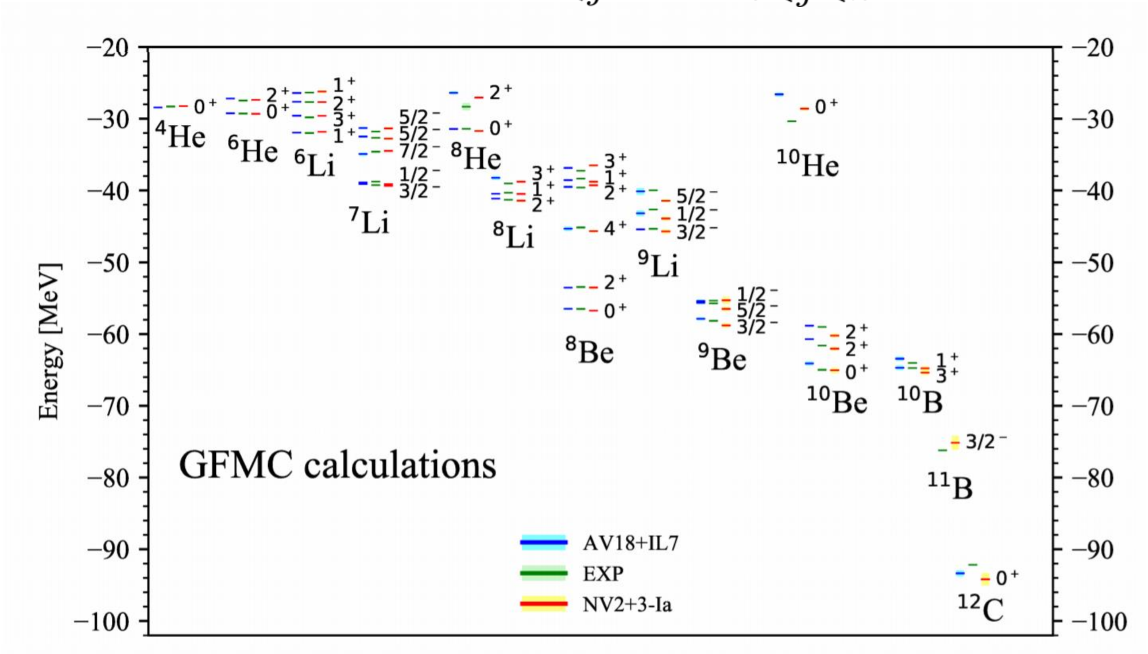
Wiringa et al. PRC 89, 024305 (2014)

Electron scattering illuminates our understand of short-range correlations (SRCs) in nuclear physics – for instance, triple coincidence experiments at JLAB show np pairs dominant (~90%) over pp/np at high relative momentum (400-500 MeV)

Studying the exclusive electron-induced processes can further inform our understanding of the nature of SRCs as well as improve models of nuclear physics

Nuclear Hamiltonian and currents

$$H = \sum_i K_i + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk}$$



Piarulli et al. PRL 120, 052503 (2018)
Wiringa et al. PRC 51, 38 (1995)

Potential correlating nucleons in pairs or triples with either phenomenological (AV18) or χEFT approach (NV2+3)

Electroweak currents in either approach are schematically given by:

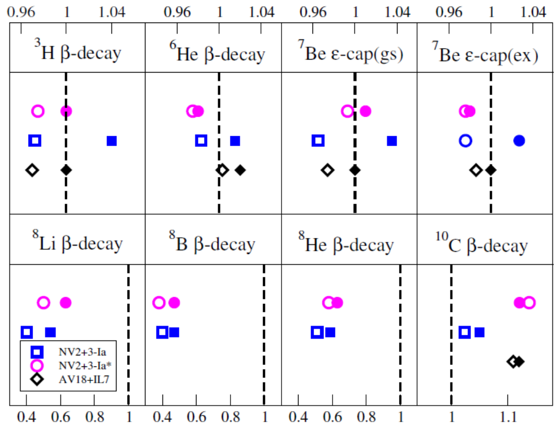
$$\rho = \sum_{i=1}^{A} \rho_i + \sum_{i<j} \rho_{ij} + \ldots$$

$$\mathbf{j} = \sum_{i=1}^{A} \mathbf{j}_i + \sum_{i<j} \mathbf{j}_{ij} + \ldots$$

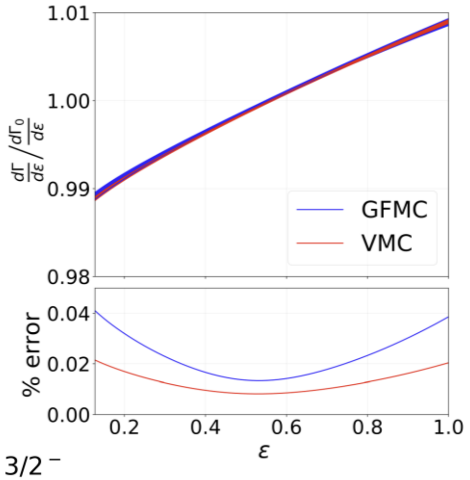
Low-energy validation of QMC framework

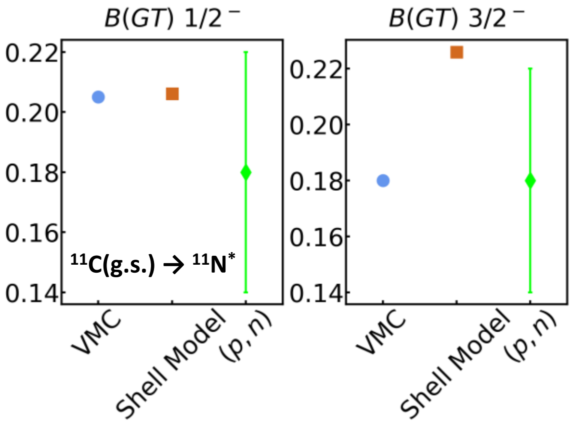
Beta decay

King et al. PRC 121, 025501 (2020)



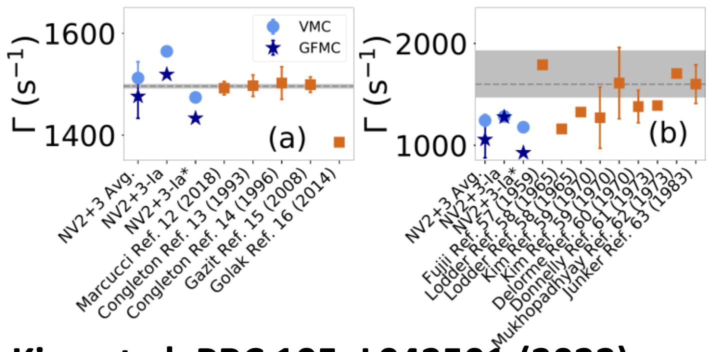
King et al. PRC 107, 015503 (2023)



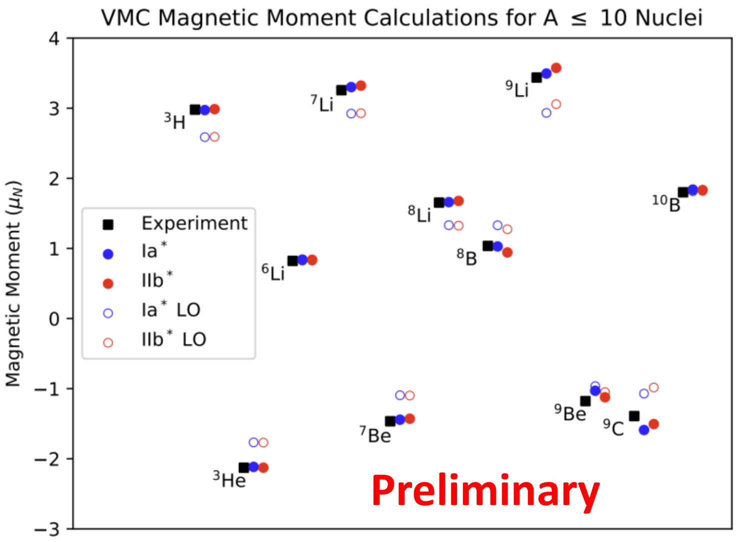


Schmitt, GBK, et al. PRC 106, 054323 (2022)

Muon capture



King et al. PRC 105, L042501 (2022)



G. Chambers-Wall, GBK, et al. (in preparation)

Two-nucleon propagator and response densities

STA turns problem of response into computing matrix elements plus two-particle propagator

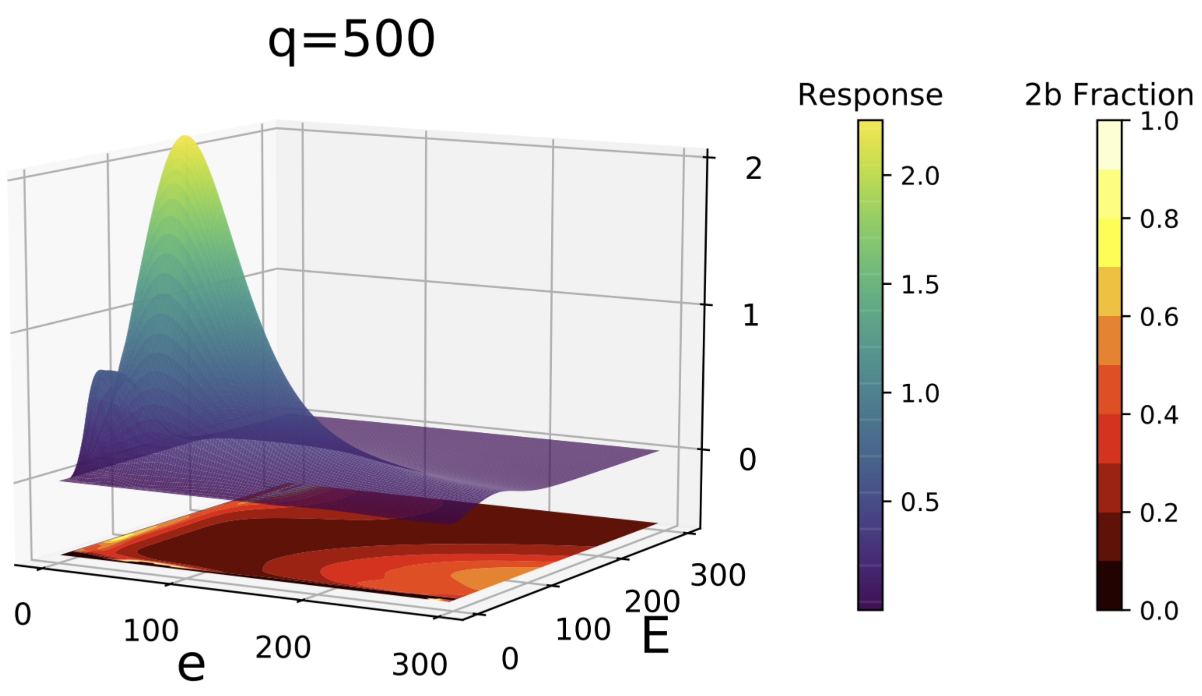
For applications to nuclei:

$$\langle \mathbf{r}'\alpha_1'\alpha_2'|e^{-iH_{12}^{\mathrm{rel}}t}|\mathbf{r}\alpha_1\alpha_2\rangle = \sum_c \int_0^\infty de e^{-iet}\phi^c_{\alpha_1'\alpha_2'}(\mathbf{r}';e)\phi^{c*}_{\alpha_1\alpha_2}(\mathbf{r};e)+e^{-ie_d t}\sum_{M_d=0,\pm1}\phi^{c_d,M_d}_{\alpha_1'\alpha_2'}(\mathbf{r}';e_d)\phi^{c_d,M_d*}_{\alpha_1\alpha_2}(\mathbf{r};e_d)$$

This allows one to conveniently express the response in terms of a density *D*:

$$R^{\mathrm{STA}}(q,\omega) = \int_0^\infty de \int_0^\infty dE_{\mathrm{CM}}\delta(\omega+E_i-e-E_{\mathrm{CM}})D(e,E_{\mathrm{CM}})$$

Response density



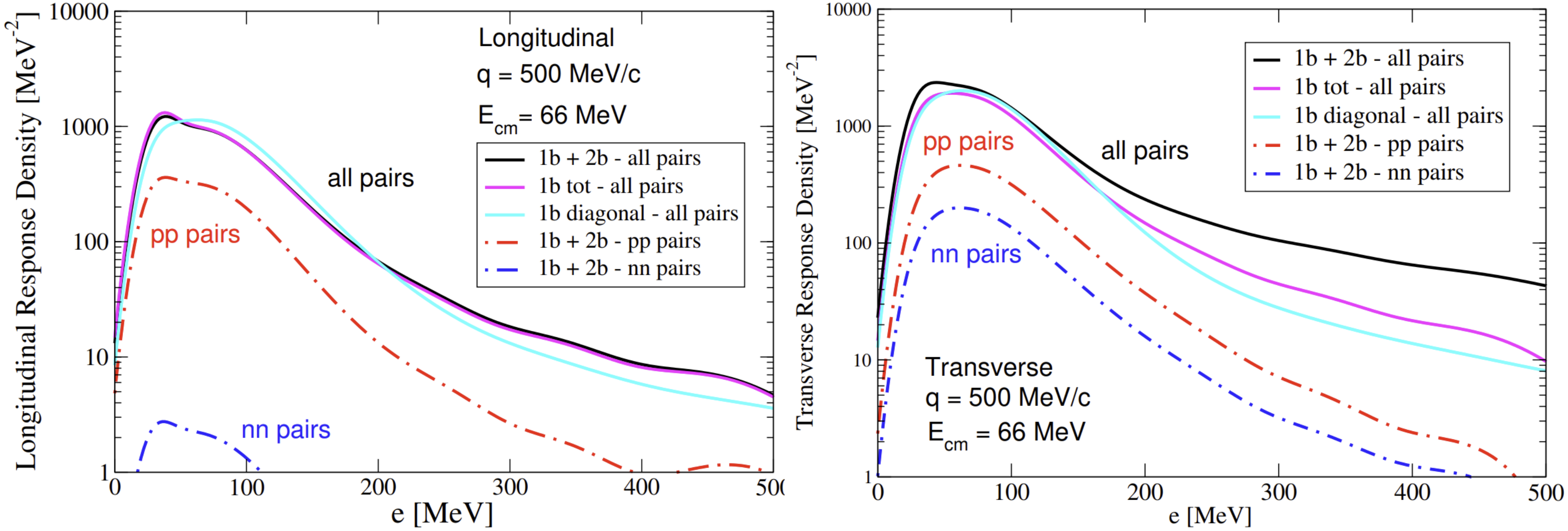
3D densities allow one to understand how the response depends on different pair kinematics

Allows for a deeper understanding of the nature of two-body contributions because energy information and two-body fraction are computable

$$R^{\mathrm{STA}}(q,\omega) = \int_0^\infty de \int_0^\infty dE_{\mathrm{CM}} \delta(\omega + E_i - e - E_{\mathrm{CM}}) D(e, E_{\mathrm{CM}})$$

Pastore et al. PRC 101, 044612 (2020)

Exclusive final state information

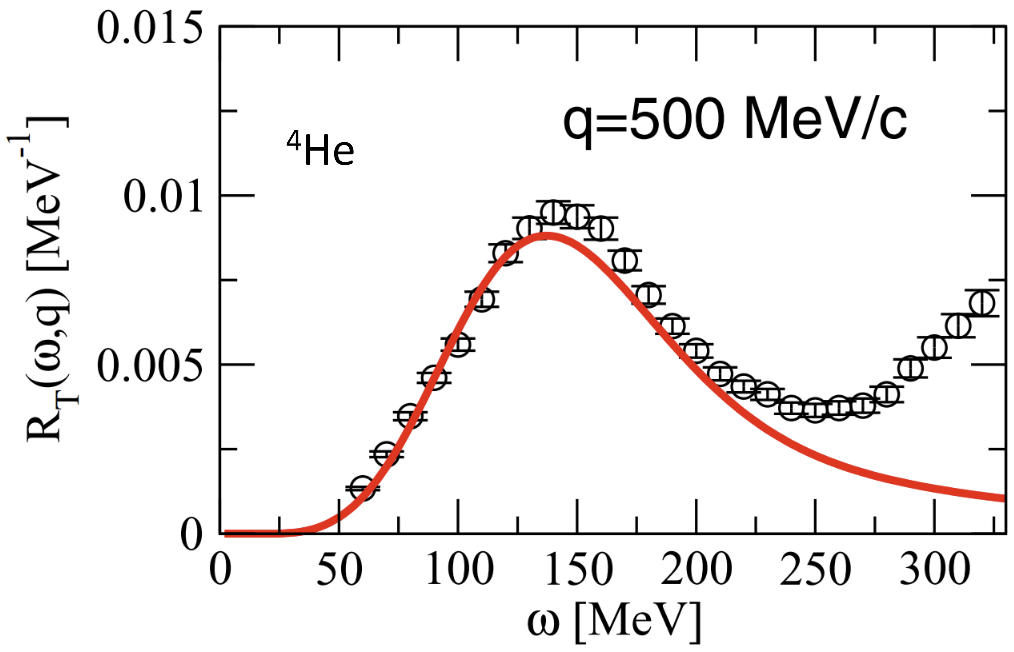


Two-body physics can be further understood by investigating exclusive responses

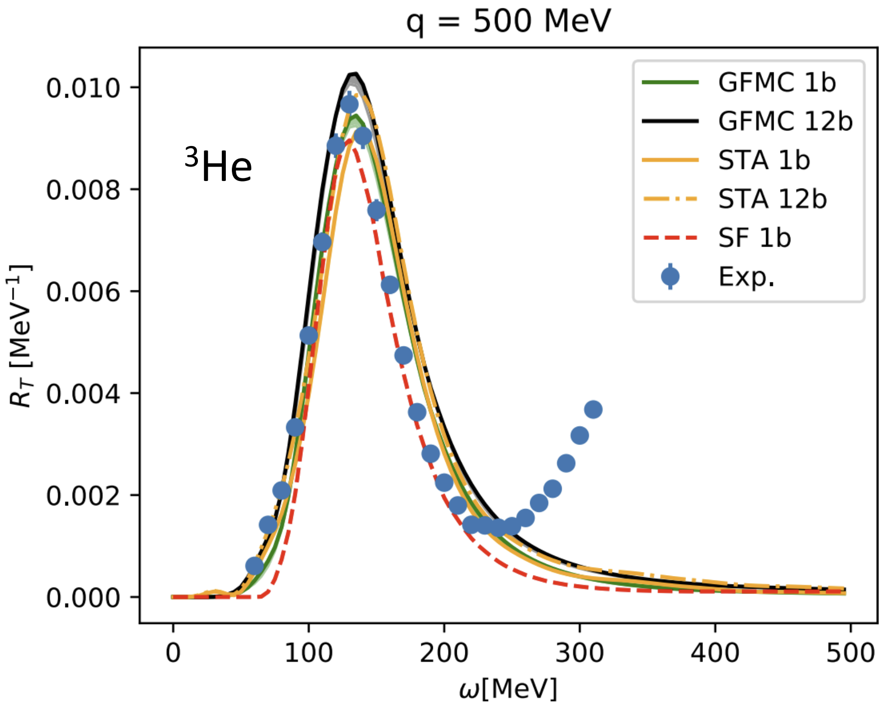
np pair dominance at high relative energy expected from momentum distributions and is borne out in the response

Pastore et al. PRC 101, 044612 (2020)

Electromagnetic responses



Pastore et al. PRC 101, 044612 (2020)



Andreoli et al. PRC 105, 014002 (2022)

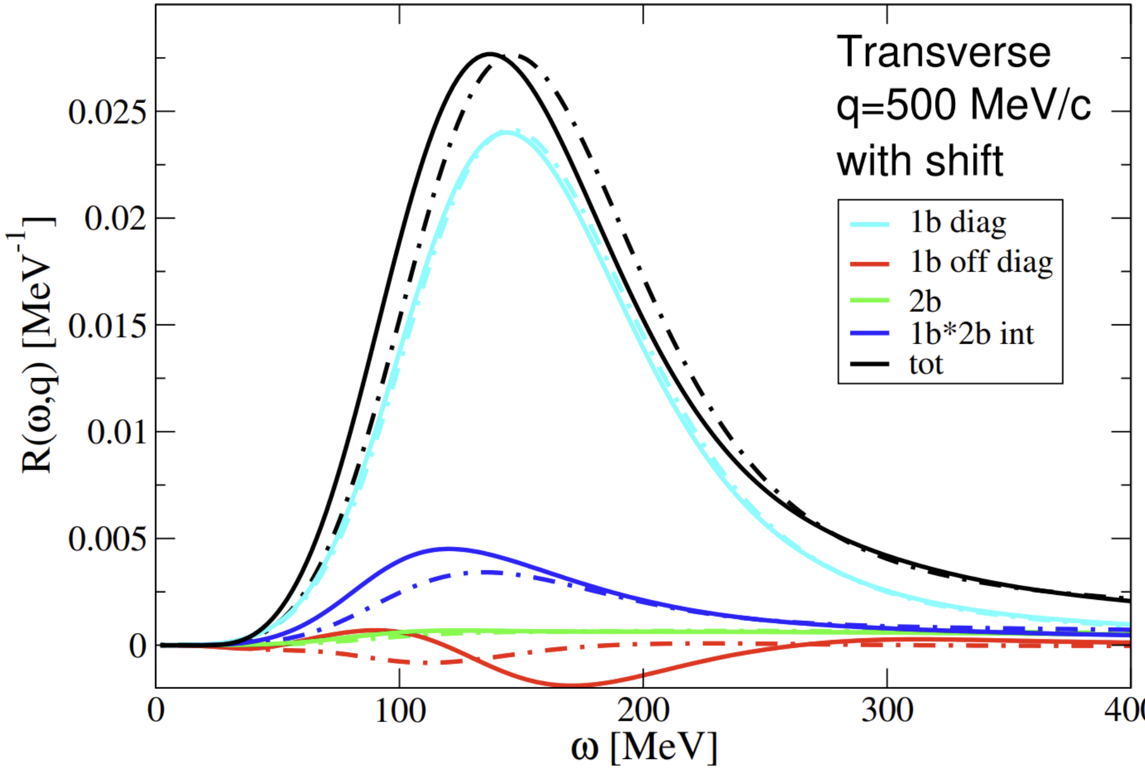
Electromagnetic responses validated for $A \leq 4$ nuclei with AV18+UIX wave functions

Role of interference effects

Breakdown of the current-current correlator allows one to understand the importance of different amplitudes

1b*2b interference significant enhances electromagnetic transverse response near the QE peak

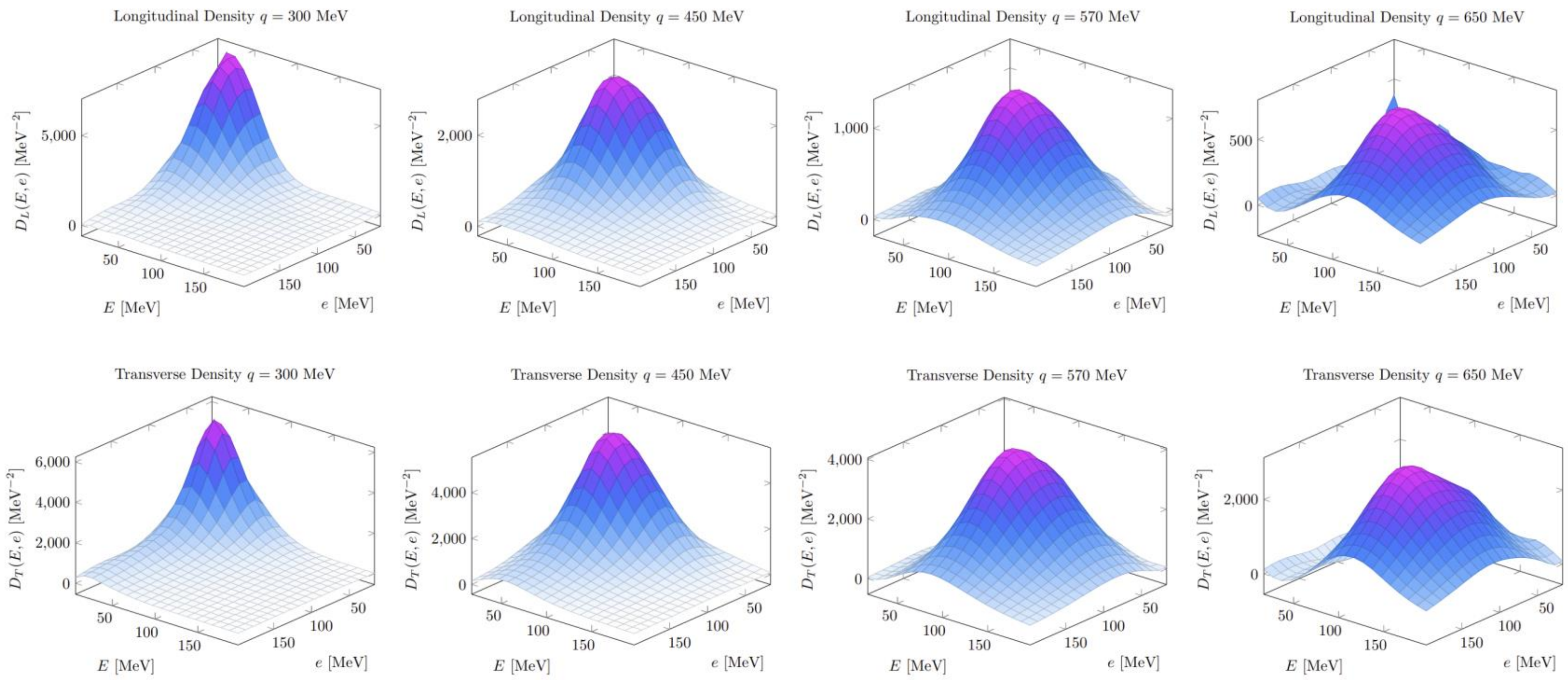
$$\mathcal{O}^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}(\mathbf{q}) = \left(\sum_i \mathcal{O}_i^\dagger(\mathbf{q}) + \sum_{i<j}\mathcal{O}_{ij}^\dagger(\mathbf{q})\right) e^{-iHt} \left(\sum_{i'} \mathcal{O}_{i'}(\mathbf{q}) + \sum_{i'<j'} \mathcal{O}_{i'j'}(\mathbf{q})\right)$$
$$= \sum_i \mathcal{O}_i^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}_i(\mathbf{q}) + \sum_{i\neq j} \mathcal{O}_i^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}_j(\mathbf{q})$$
$$+ \sum_{i\neq j}\left(\mathcal{O}_i^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}_{ij}(\mathbf{q}) + \mathcal{O}_{ij}^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}_i(\mathbf{q}) + \mathcal{O}_{ij}^\dagger(\mathbf{q})e^{-iHt}\mathcal{O}_{ij}(\mathbf{q})\right)$$



Pastore et al. PRC 101, 044612 (2020)

Recent electron scattering developments

L. Andreoli, GBK, et al. (in preparation)



Effort underway to study the electromagnetic response of ^{12}C within the STA

On-going application to nuclei: neutrino-nucleus scattering

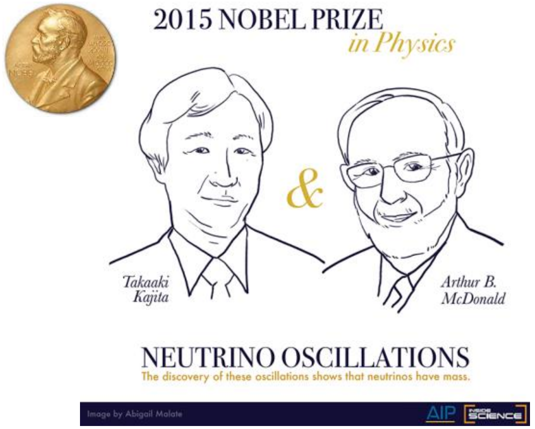
Motivation

Neutrinos produced in flavor eigenstates, travel as mass eigenstates, and due to **BSM non-zero mass** can oscillate between flavor states

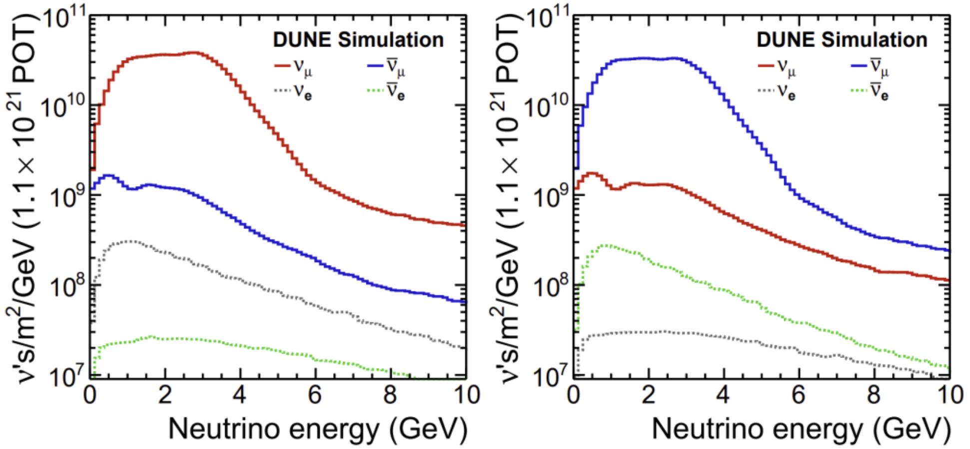
Experiments seek to measure oscillation parameters

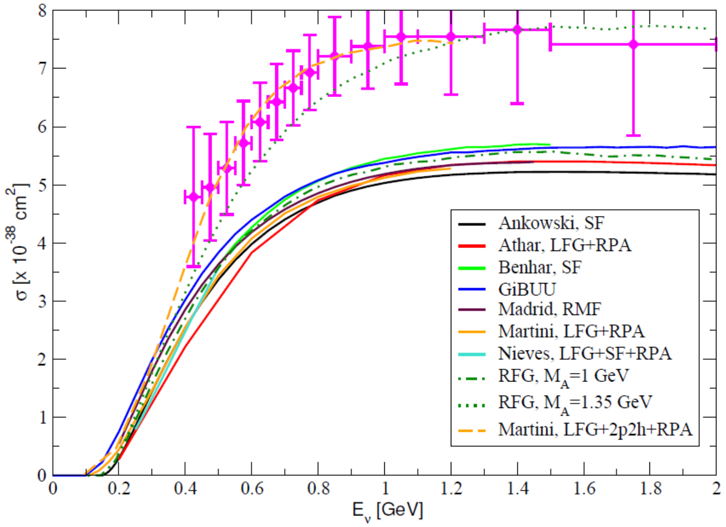
Length can be controlled in a long-baseline accelerator experiment

Accurate modeling of neutrino-nucleus cross-sections needed to infer beam energy



L. Alvarez-Ruso arXiv:1012.3871





DUNE, Eur. Phys. J. C 80, 978 (2020)

$$P_{\nu_\mu \to \nu_e} = \sin^2(2\theta)\sin^2\left(\frac{\Delta m_{21}^2 L}{2E_\nu}\right)$$

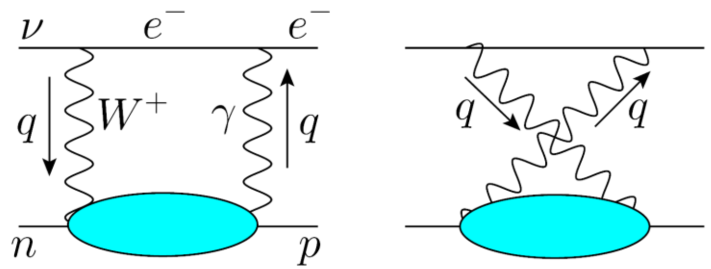
$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

Motivation

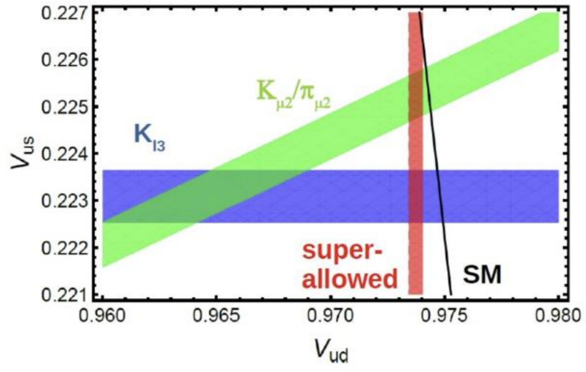
Superallowed beta decays are used to test CKM unitarity

3σ tension between experimental extractions and expectation

Connection between EM and weak nuclear response functions with beta decay radiative corrections, quasi-elastic response information required

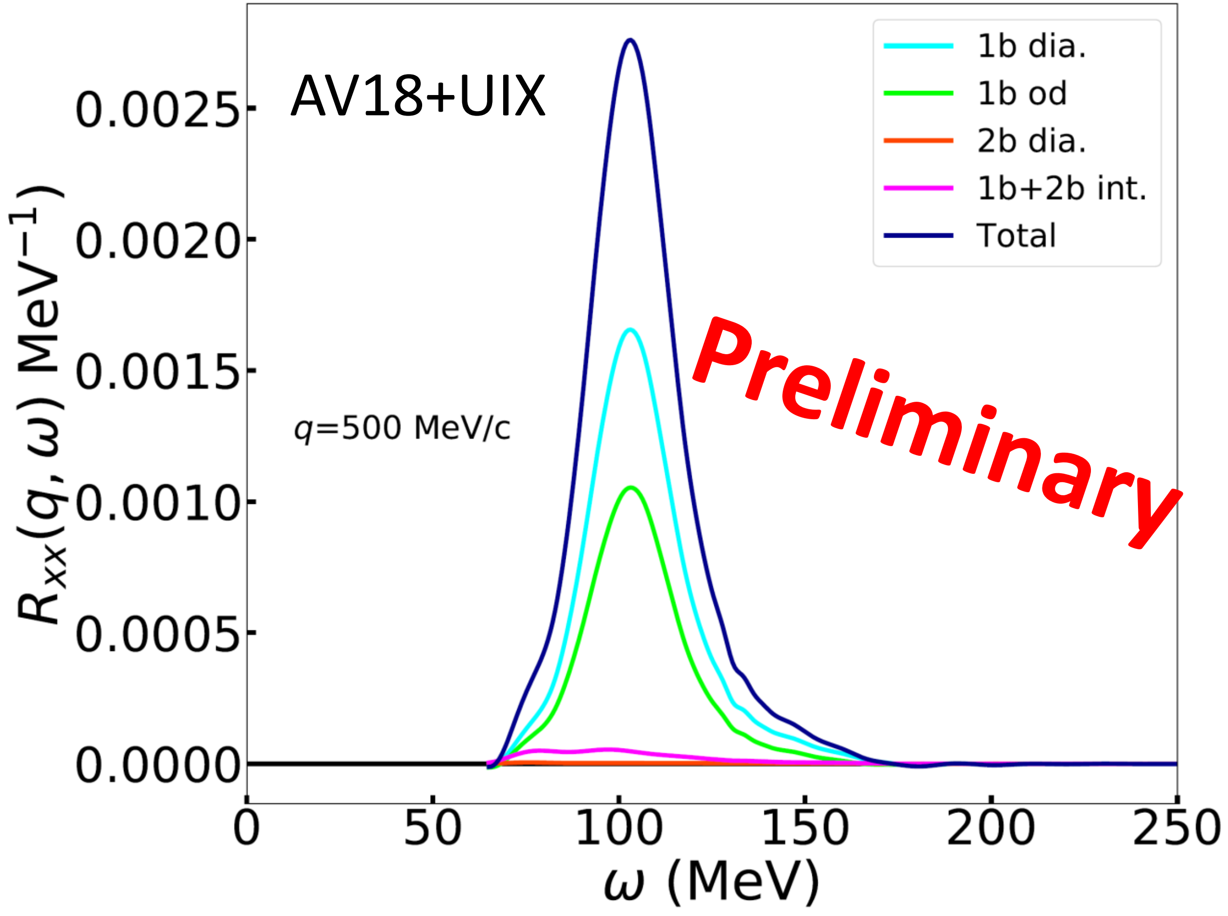


Seng et al. PRD 100, 013001 (2019)



Seng arXiv:2207.10492

Weak neutral current response of ^2H



First step toward cross section benchmark with hyperspherical harmonics calculation of the cross section in **Shen et al. PRC 86, 035503 (2012)**

Development underway to study A=4 and A=12 neutral current responses

Exporting to other many-body methods will make A=16 and A=40 accessible to the STA

Outlook: Improving the STA for nuclei

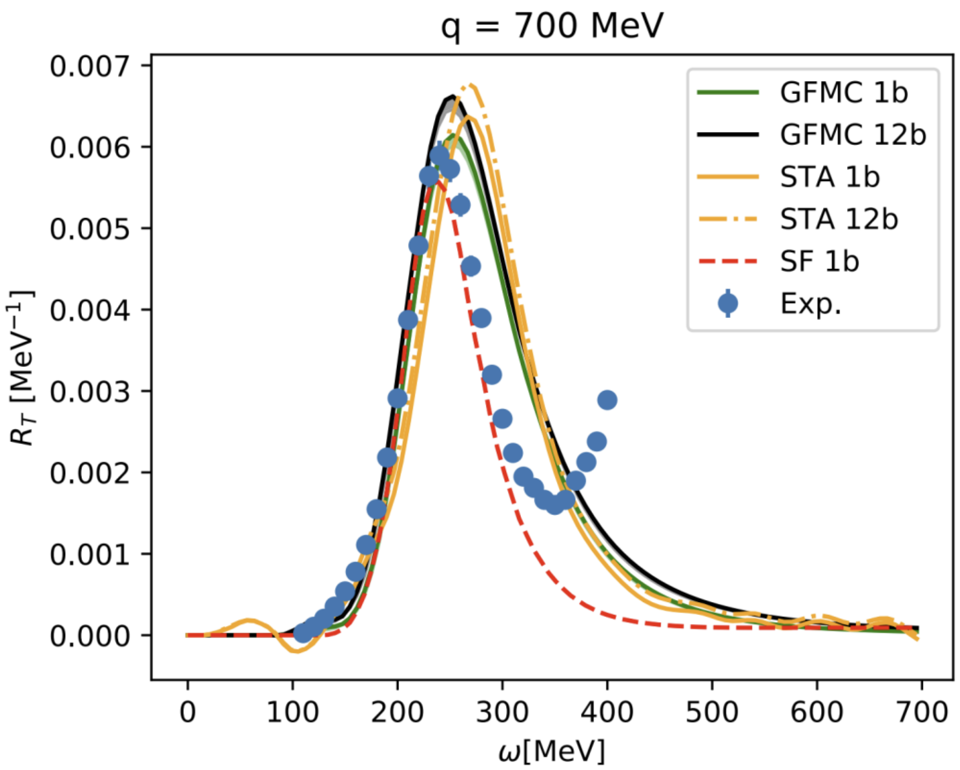
Outlook: Need for relativistic kinematics

Presently, STA and Euclidean responses break down for large *q* because of missing relativistic effects

Factorization schemes (like spectral function formalism) are amenable to relativistic final state kinematics

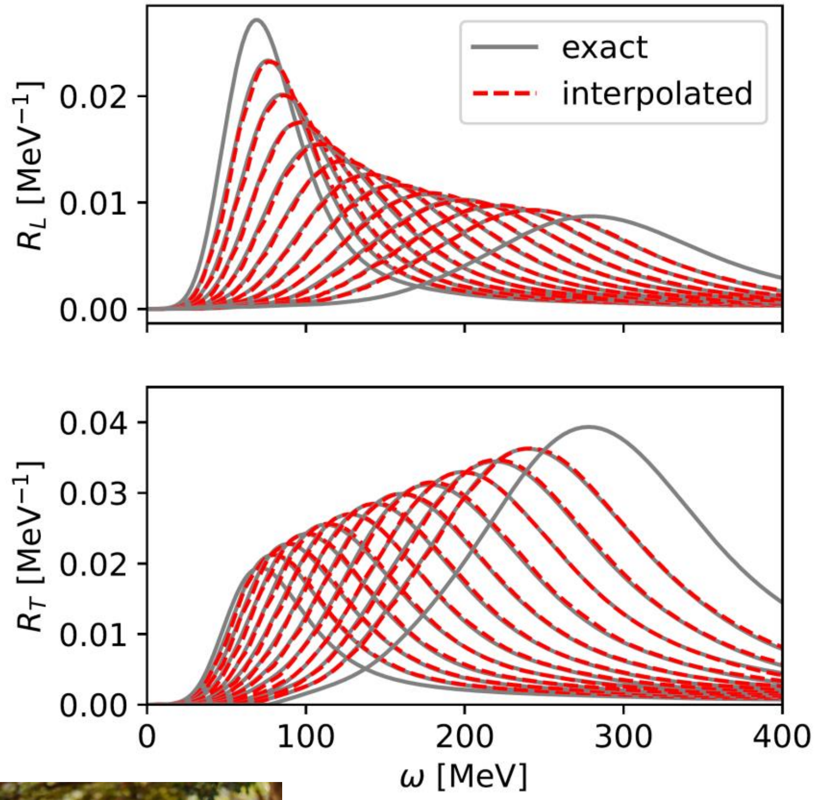
Work underway to incorporate relativistic effects in EM and weak currents

Opens the door for studying meson production, as is done in SF **[Rocco et al. PRC 100, 045503 (2019)]**



Andreoli et al. PRC 105, 014002 (2022)

Outlook: Reponses to cross sections



L. Andreoli, 2023

Interpolation scheme **developed by Lorezno Andreoli** to evaluate the response at an arbitrary *q* with only a few calculations

Sum-rule conserving scheme carried out by interpolating:

$$f(\omega, \mathbf{q}) = \frac{\int_0^{\omega} d\omega' R(\omega', \mathbf{q})}{\int_0^{\infty} d\omega' R(\omega', \mathbf{q})}$$

Response as a function of ω for arbitrary q is obtained by taking the derivative of f

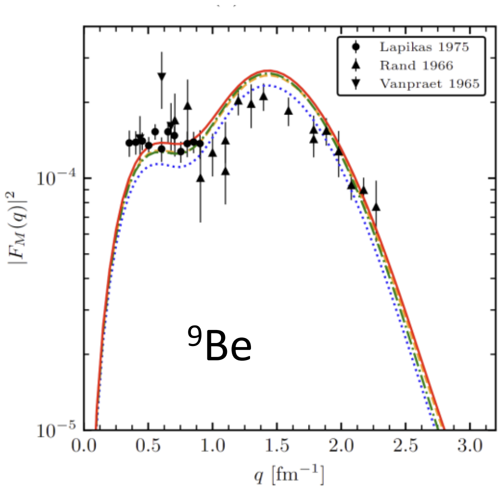
Tested in ^{4}He on a grid of $300 \leq q \leq 800$ MeV

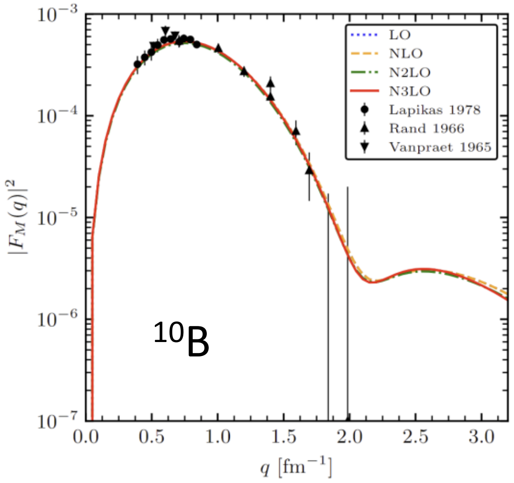
Outlook: χEFT currents

Phenomenological currents have had success describing the response, but have limitations

Chiral effective field theory interaction and current models can remedy this in the future

On-going efforts to study large *q* magnetic form factors with NV2+3 and QMC





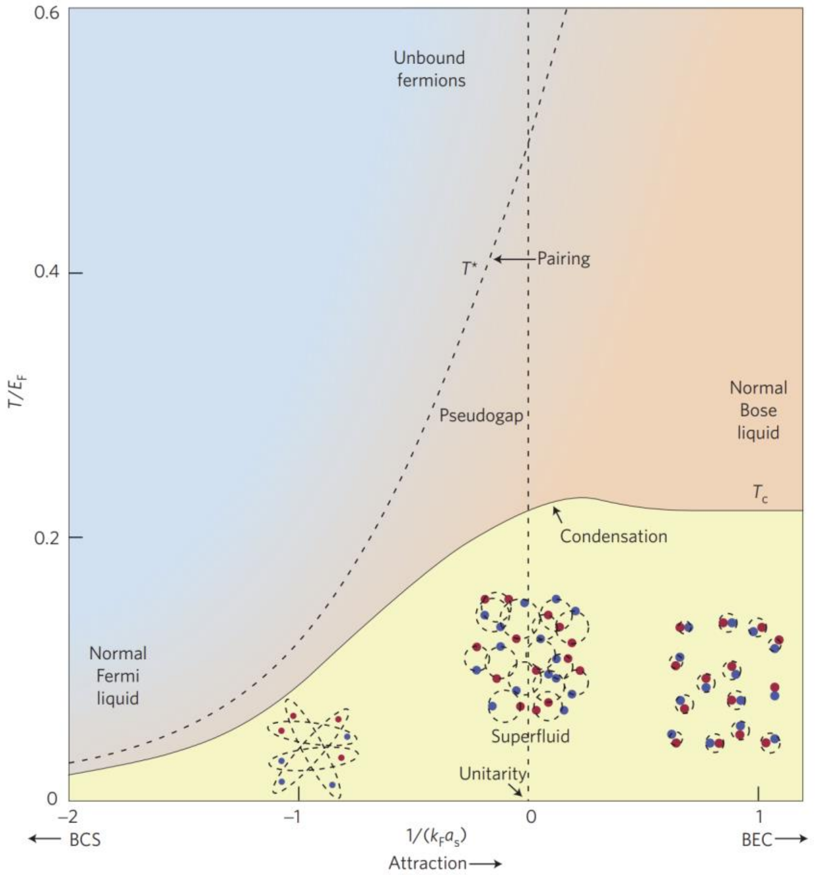
G. Chambers-Wall, GBK, et al. (in preparation)

Figures courtesy of A. Gnech

Future STA applications: The unitary Fermi Gas

Motivation

Randeria *Nature Phys 6* (2010)



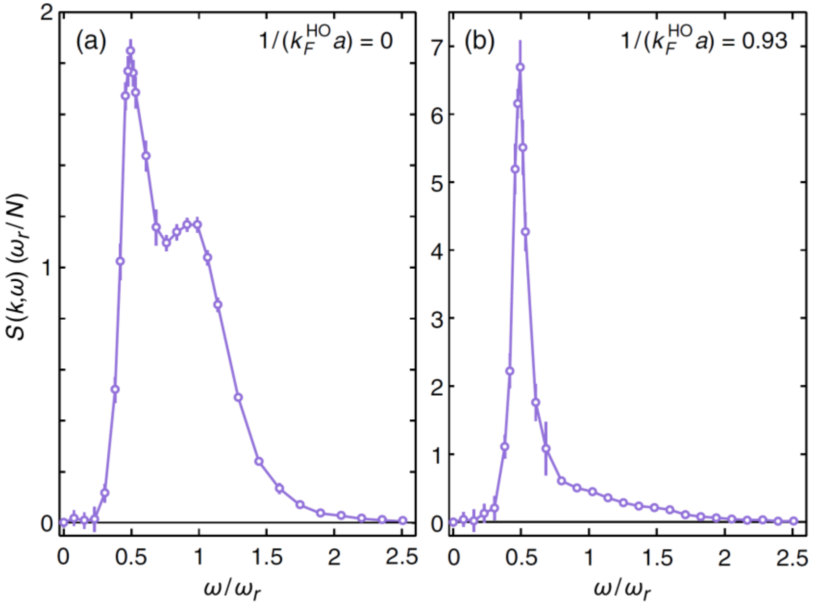
For ultracold systems, cross over from weakly interacting Fermions (pairs) to weakly repulsive Bosons (molecules)

Crossover:

$$a \to \infty$$

$$r_{\mathrm{eff}} \to 0$$

Scattering length of ultracold atomic systems is tunable with magnetic Feshbach resonances



Hoinka et al. PRL 110, 055305 (2013)

Motivation

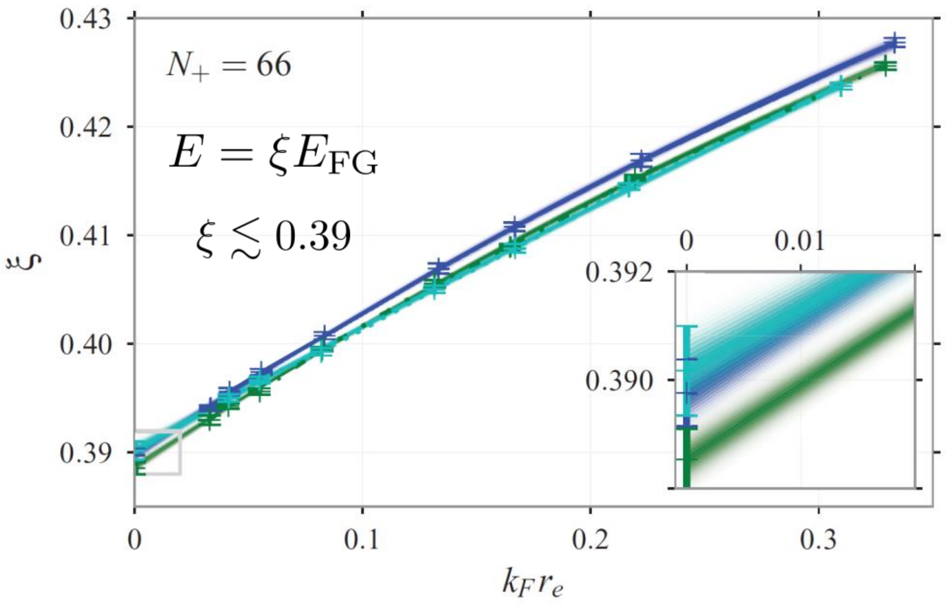
STA was developed with electron scattering in mind, but it is a general many-body approach with wide applications

Studying the unitary fermi gas has connections to several fermionic superfluid systems (neutron matter, ultracold atomic gases, superconducters, liquid He,…)

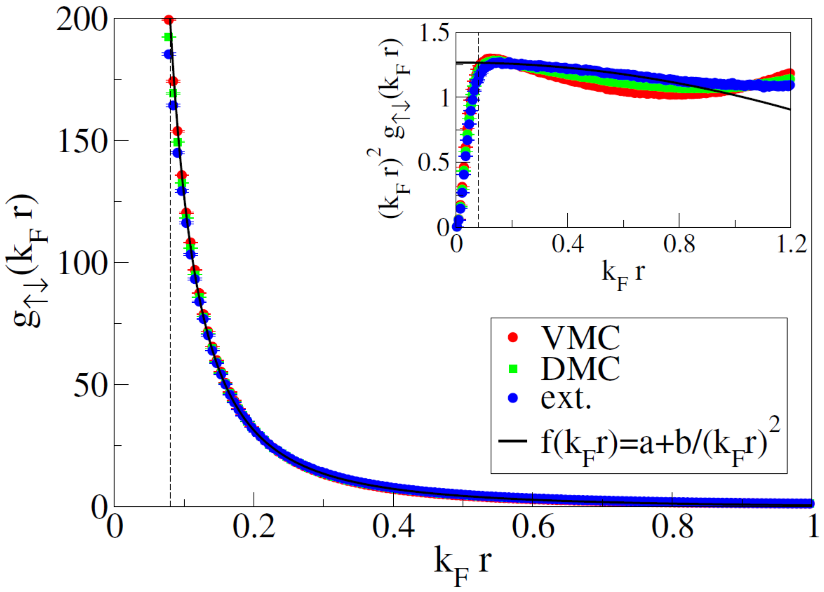
Success modelling density response in BCS, BEC, and crossover region would demonstrate the flexibility of the approach

UFG within QMC

Forbes PRA 86, 053603 (2012)



Gandolfi PRA 83, 041601 (2011)



$$g_{\uparrow\downarrow}(r) = \frac{9\pi}{20}\frac{\zeta}{k_F^2 r^2}$$

Several relations established for the UFG with connection to on Tan's contact parameter **[Ann. Phys. 323, 2952/2971/2987 (2008)]** have been investigated with Lattice and continuum QMC methods

Interaction and propagator

To study response, we adopt a Pöschl-Teller potential: $$V(r_{ij}) = -\frac{2}{m}\frac{\mu^2}{\cosh^2(\mu r_{ij})}$$

Effective range given by: $$r_{\text{eff}} = \frac{2}{\mu}$$

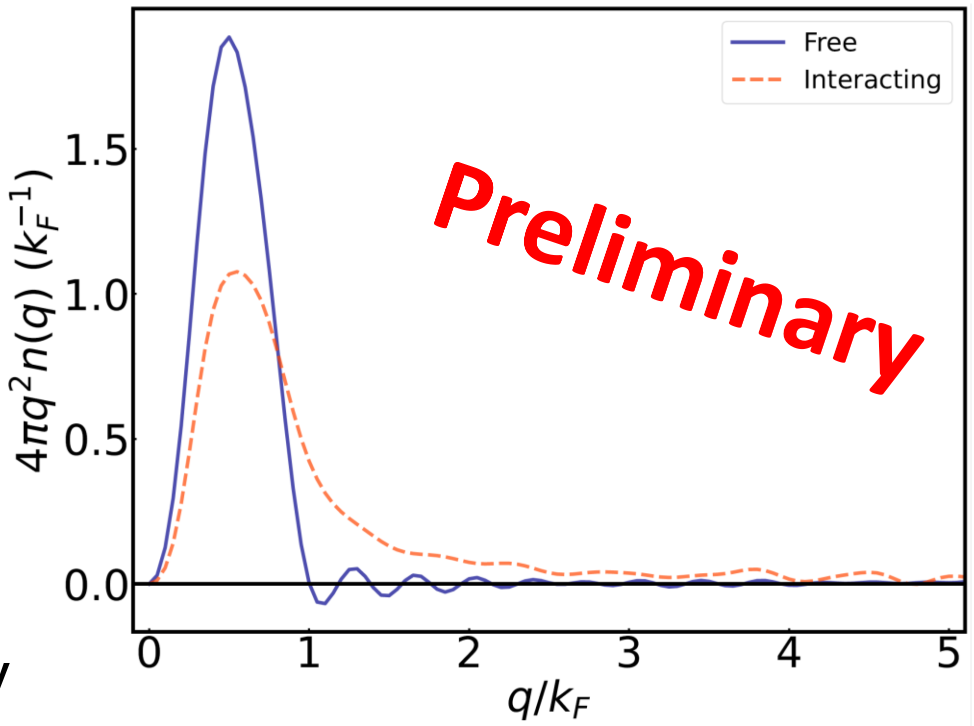
Continuum propagator is analytically solvable and easily implemented numerically

Preliminary simulations carried out with 14 particles

Progress toward structure factor

Testing computation of one-body off diagonal density matrix element through two-body momentum distributions

$$n(\mathbf{q},\mathbf{Q}) = \int d\mathbf{r}'_1 d\mathbf{r}_1 d\mathbf{r}'_2 d\mathbf{r}_2 d\mathbf{r}_3 \ldots d\mathbf{r}_N \times \Psi^*(\mathbf{r}'_1,\mathbf{r}'_2,\ldots) e^{-i\mathbf{q}\cdot(\mathbf{r}_{12}-\mathbf{r}'_{12})} e^{-i\mathbf{Q}\cdot(\mathbf{R}_{12}-\mathbf{R}'_{12})} \Psi(\mathbf{r}_1,\mathbf{r}_2,\ldots)$$



14 particles in a box

Outlook: Static and dynamic structure factors for free Fermi gas and UFG extrapolated to the unitary limit

$$a \to \infty \qquad r_{\text{eff}} \to 0$$

Investigate connection between pair/momentum distribution and opposite spin structure factor to validate STA approach

$$g_{\uparrow\downarrow}(r) = \frac{9\pi}{20}\frac{\zeta}{k_F^2 r^2}$$

$$S_{\uparrow\downarrow}(q) = \frac{3\pi}{10}\zeta\frac{k_F}{q}$$

$$n(k) = \frac{8}{10\pi}\zeta\frac{k_F^4}{k^4}$$

Outlook: Improving the QMC computation

Presently assuming periodic boundary conditions: $\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots) = \Psi(\mathbf{r}_1 + L\hat{\mathbf{x}}, \mathbf{r}_2, \ldots)$

In principle, a phase (or “twist”) can be picked up: $\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots) = e^{i\theta_x}\Psi(\mathbf{r}_1 + L\hat{\mathbf{x}}, \mathbf{r}_2, \ldots)$

$$-\pi \leq \theta_x \leq \pi$$

Averaging overall allowed twists can reduce finite size and shell effects

Applied to studies of spin susceptibility of pure neutron matter **[Riz et. al *Particles* 3, 46 (2020)]** and the electron gas **[Lin et al. PRE 64, 016702 (2001)]**

Conclusions

STA is a factorization scheme that preserves sum rules, PWIA physics, and two-particle correlations

Good for high energy responses, misses low-lying excitations or collective behavior

Applied to the electromagnetic and NC response of the deuteron

UFG density response calculations underway to test approach in other many-body system

Has far-reaching applications to general many-body systems with strong two-particle physics

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