Strongly-interacting systems in atomic and nuclear physics

Lucas Madeira [1]
São Carlos Institute of Physics - University of São Paulo - Brazil

Critical stability of few-body quantum systems 2023 - ECT*

Grant 2023/04451-9

October 24, 2023

[1] madeira@ifsc.usp.br

Outline

- Efimov trimers: finite-range corrections
- Nuclear structure: explicit pions

- Efimov trimers: finite-range corrections
- Nuclear structure: explicit pions

Motivation

- Strongly-interacting systems appear in several areas of physics
  - Attractive interactions that can almost (or just barely) loosely bind two particles
- Universal description in the low-energy limit

- ^{4}He dimer
  - Spatial scale of Å (10^{-10} m)
  - Energy $\sim 10^{-7}$ eV
- Deuteron
  - Femtometer (10^{-15} m) scale
  - Energy is of a few MeV (10^6 eV)

- Universality allows the description of complex phenomena in terms of only a few parameters
- Although this definition is made at the two-body level, this gives rise to fascinating effects in larger systems
- Few-body effects $\rightarrow$ Many-body systems

Two-body scattering: the effective range expansion

- The s-wave scattering length and effective range are related to the low-energy phase shift $\delta_0(k)$ through[2]

$$k \cot \delta_0(k) = -\frac{1}{a} + \frac{r_0 k^2}{2} + \mathcal{O}(k^4)$$

- The two-body s-wave scattering amplitude is given by

$$\tau(k) = \frac{1}{k \cot \delta_0 - ik}$$

- From the pole of the s-wave scattering amplitude:

$$E_2 = -\frac{\hbar^2}{2m_r a^2} \quad \text{(zero-range)} \qquad \text{or} \qquad E_2 = -\frac{\hbar^2}{2m_r r_0^2}\left(1 - \sqrt{1 - \frac{2r_0}{a}}\right)^2 \quad \text{(finite-range)}$$

[2] H. A. Bethe, "Theory of the Effective Range in Nuclear Scattering", Phys. Rev. **76** (1949).
M. Macêdo-Lima et al., "Scattering length and effective range of microscopic two-body potentials", Revista Brasileira de Ensino de Física **45** (2023).

Three-body physics

JUNE 15, 1935 PHYSICAL REVIEW VOLUME 47

The Interaction Between a Neutron and a Proton and the Structure of H^3

L. H. THOMAS, *Mendenhall Laboratory, Ohio State University*

(Received April 17, 1935)

- Thomas collapse[3] (1935)
  - Deuteron and triton
  - $E_{3B} \to -\infty$ when $R \to 0$?

$$H_{2B} = T_1 + T_2 + V(r_{12}) \to E_{2B}$$

$$H_{3B} = T_1 + T_2 + T_3 + V(r_{12}) + V(r_{13}) \xrightarrow{\text{ansatz}} \boxed{E_{3B} \leqslant -\frac{[\text{constant}]}{R^2}|E_{2B}|}$$

3 L. H. Thomas, "The Interaction Between a Neutron and a Proton and the Structure of H3", Phys. Rev. **47** (1935).

Three-body physics

- Skorniakov and Ter-Martirosian (STM) equation[4] (1957)
  - Zero-range limit and momentum space
  - $a_t = a_s$ (equivalent to the problem of 3 identical bosons) $\rightarrow$ energy is not bound from below!
- Analytical solution by Minlos and Faddeev[5] (1961)
  - 3 identical bosons
  - $a \rightarrow \pm\infty$
  - Solution valid for any negative energy! $\rightarrow$ consistent with the Thomas collapse

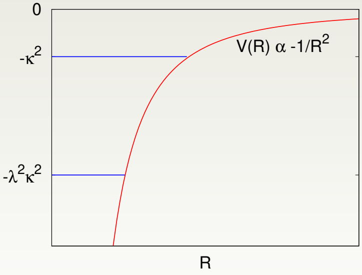
---

[4] G. V. Skorniakov et al., "Three body problem for short range forces. I. Scattering of low energy neutrons by deuterons", Sov. Phys. JETP **4** (1957).

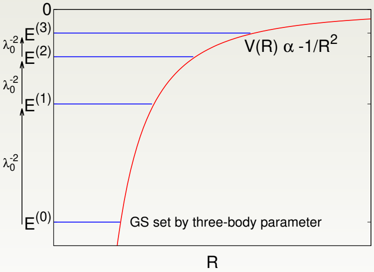
[5] L. D. Faddeev et al., "Comment on the problem of three particles with point interactions", Zh. Eksp. Teor. Fiz. **41** (1961).

Efimov physics

- Efimov (1970)
  - Real space, hyper-radius: $R = \sqrt{r^2 + \rho^2}$
  - $V(R) = -(|s_0|^2 + 1/4)/R^2$
- Scale invariance: $R \to \lambda R$
- Energy: $-\kappa^2 \to -\lambda^2\kappa^2$



- Discrete scale invariance: $R \to \lambda_0 R$
- $\lambda_0 = e^{\pi/|s_0|} \approx 22.7$
- $E^{(n)} = E^{(0)}\lambda_0^{-2n}$
- $(1/a, \kappa) \to (1/a, \kappa)/\lambda_0$



V. Efimov, "Energy levels arising from resonant two-body forces in a three-body system", Phys. Lett. B **33** (1970),
V. N. Efimov, "Weakly-bound states of 3 resonantly-interacting particles", Sov. J. Nucl. Phys. **12** (1971).

Context

- Three identical non-relativistic bosons
- Large scattering lengths
- Efimov physics

Goal

- Predict the energy of a trimer given
  - Scattering length
  - Effective range
  - A reference energy (for example at $a \to \pm\infty$)

PHYSICAL REVIEW A **104**, 033301 (2021)

Quantum Monte Carlo studies of a trimer scaling function with microscopic two- and three-body interactions

Tobias Frederico (ITA, Brazil)

Lauro Tomio (Unesp, Brazil)

Marcelo Yamashita (Unesp, Brazil)

Stefano Gandolfi (LANL, USA)

Trimer energy scaling

- The $N = 3$ boson system requires a three-body scale in the limit of a zero-range interaction to avoid the Thomas collapse
- Reference three-body energy: trimer at unitarity $E_3(1/a = 0, r_0, \nu)$
- Dimensionless quantities:

$$x = \frac{\hbar}{a\sqrt{-mE_3(0, r_0, \nu)}} \qquad y = \frac{r_0\sqrt{-mE_3(0, r_0, \nu)}}{\hbar}$$

- We define the energy ratio:

$$\boxed{F(x, y) \equiv \frac{E_3\left(1/a, r_0, \nu\right)}{E_3(0, r_0, \nu)}}$$

- Our goal is to determine $F(x, y)$ and the region where it displays universal behavior
- The zero-range limit of the scaling function has been studied extensively in the literature[6]

[6] E. Braaten et al., "Universality in few-body systems with large scattering length", Phys. Rep. **428** (2006).

STM formalism

- Skorniakov and Ter-Martirosian equation

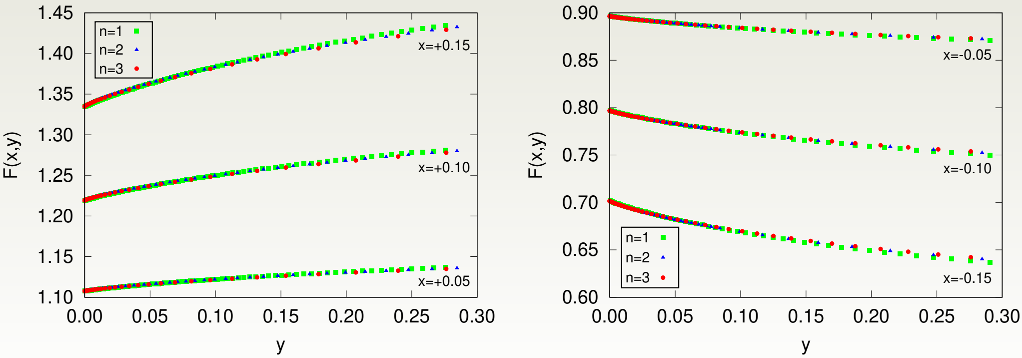
$$f(q) = -\frac{2}{\pi}\tau(k)\int_0^\infty dp\, p^2 f(p)\int_{-1}^{+1} dz\left[G_0(p,q,z;E_3) - G_0(p,q,z;-\lambda)\right]$$

- $k = i\sqrt{E_3 - 3q^2/4}$
- Three-body Green's function: $G_0(p,q,z;E) \equiv \left[E - p^2 - q^2 - pqz\right]^{-1}$
- λ: three-body short-range regularization parameter, presented as an energy subtraction point in the formalism
- The two-body s-wave scattering amplitude

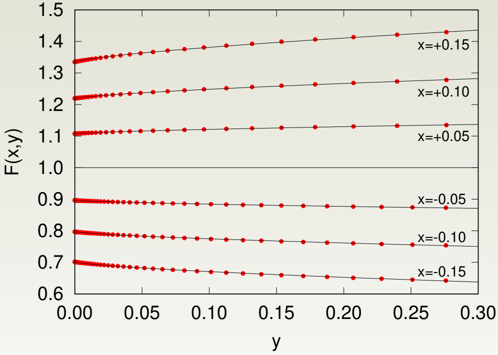
$$\tau(k) = \left(-\frac{1}{a_B} - ik\right)^{-1}\left[1 + \frac{r_0}{2}\left(\frac{1}{a_B} - ik\right)\right]$$

Scaling function

- Obtained solving the STM equation with finite range corrections
- First, second and third excited state ($n = 1, 2, 3$)
- Limit cycle



Scaling function parameterization



$$F(x,y) = 1 + c_1 x + c_2 x y^{\sigma} + c_3 x^2 + c_4 x^2 y + c_5 x^2 y^{\sigma}$$

	c_1	c_2	c_3	c_4	c_5	σ
Fit	2.106(1)	1.26(1)	0.804(4)	1.0(2)	1.2(1)	0.680(3)
Ansatz	2.107	1.35	0.804	1.03	1.05	0.641

Quantum Monte Carlo overview

- Ground-state properties of strongly-interacting many-body systems
- Main ingredients: Hamiltonian and trial wave-function

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{3}\nabla_i^2 + \sum_{i<j} V_2(r_{ij}) + V_3(R_{123})$$

- Two-body attraction:

$$V_2(r) = -\lambda_G \frac{\hbar^2 \mu_G^2}{m_r} \exp\left[-\frac{\mu_G^2 r^2}{2}\right]$$

- Three-body repulsion:

$$V_3(R_{123}) = \lambda_3 \frac{\hbar^2 \mu_3^2}{m} \exp\left[-\frac{\mu_3^2 R_{123}^2}{2}\right]$$

- $R_{123} \equiv (r_{12}^2 + r_{13}^2 + r_{23}^2)^{1/2}$

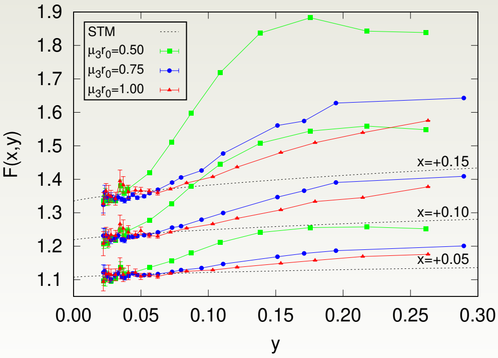
- Trial wave function:

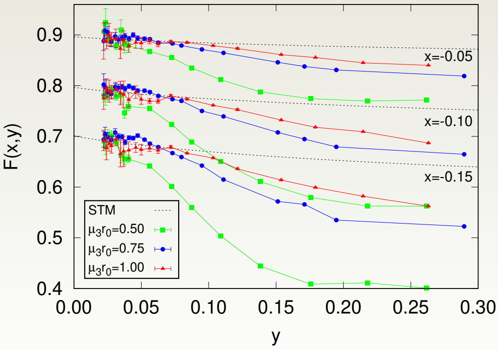
$$\psi_T(\mathbf{R}) = \left(\prod_{i=1}^{3} f_1(r_i)\right)\left(\prod_{i<j} f_2(r_{ij})\right) f_3(R_{123})$$

J. Carlson et al., "Ground-State Properties of Unitary Bosons: From Clusters to Matter", Phys. Rev. Lett. **119** (2017).

QMC results

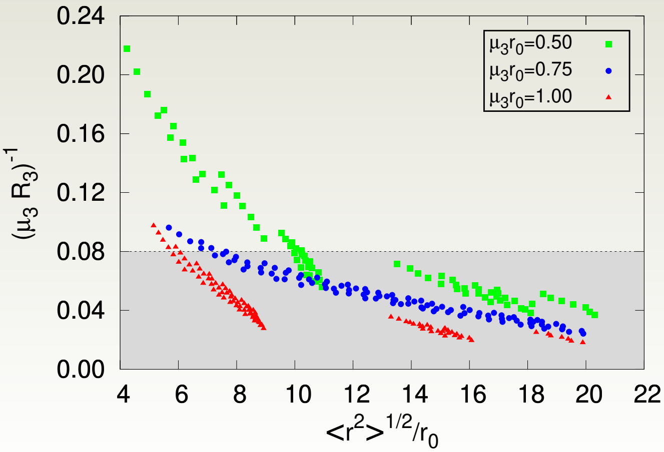
- $\mu_3 r_0$ =0.50, 0.75, 1.00
- Range of the three-body force $\propto 1/\mu_3$





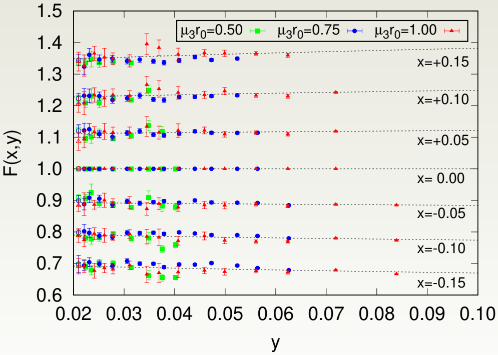
Universal window

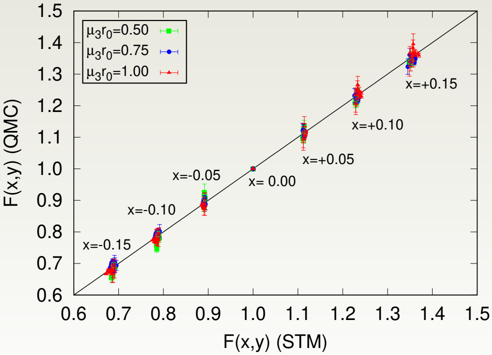
- Typical length scale associated with the trimer: $R_3 = \hbar/(-mE_3)^{1/2}$
- Range of the three-body force $\propto 1/\mu_3$
- $\langle r^2 \rangle^{1/2}/r_0$



Universal window

- Criterion: $1/(\mu_3 R_3) \leqslant 0.08$





Summary

Scaling function

- STM with finite range corrections
- Limit cycle
- Simple parametrization consistent with the zero-range behavior

$$F(x,y) \equiv \frac{E_3\left(1/a, r_0, \nu\right)}{E_3(0, r_0, \nu)}$$

Quantum Monte Carlo

- Microscopic Hamiltonian: two-body attraction and three-body repulsion
- Criterion $\rightarrow$ universal window

Outlook: atomic and nuclear physics

- Bosons
  - Larger clusters
  - Matter
- Fermions
  - Three (or more components)
  - Nuclear physics

- Efimov trimers: finite-range corrections
- Nuclear structure: explicit pions

Nucleon-nucleon interaction

- Major open problem in nuclear physics: how to construct a nucleon-nucleon (NN) interaction potential from first principles?
- Pion dynamics is constrained by chiral symmetry
- Effective Field Theory (EFT) → identify soft and hard scales, degrees of freedom and relevant symmetries
- Heavy baryon leading order chiral Lagrangian density

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \pi_i \partial^\mu \pi_i - \frac{1}{2}m_\pi^2 \pi_i \pi_i + N^\dagger \left[i\partial_0 + \frac{\nabla^2}{2M_0} - \frac{1}{4f_\pi^2}\epsilon_{ijk}\tau_i \pi_j \partial_0 \pi_k - \frac{g_A}{2f_\pi}\tau_i \sigma^j \partial_j \pi_i - M_0 \right] N$$
$$- \frac{1}{2}C_S (N^\dagger N)(N^\dagger N) - \frac{1}{2}C_T (N^\dagger \sigma_i N)(N^\dagger \sigma_i N)$$

- Standard quantum Monte Carlo simulations: pion degrees of freedom are replaced with potentials
- Our goal: to include explicit pion degrees of freedom in QMC simulations

PHYSICAL REVIEW C **98**, 034005 (2018)

Quantum Monte Carlo formalism for dynamical pions and nucleons

Lucas Madeira,[1,*] Alessandro Lovato,[2,3] Francesco Pederiva,[2,4] and Kevin E. Schmidt[1]

[1] *Department of Physics, Arizona State University, Tempe, Arizona 85287, USA*

[2] *INFN-TIFPA, Trento Institute for Fundamental Physics and Applications, 38123 Trento, Italy*

[3] *Physics Division, Argonne National Laboratory, Argonne, Illinois 60439, USA*

[4] *Dipartimento di Fisica, University of Trento, via Sommarive 14, I-38123 Povo, Trento, Italy*

Alessandro Lovato
(Argonne, USA)

Francesco Pederiva
(Trento, Italy)

Kevin Schmidt
(ASU, USA)

Pion fields in the Schrödinger picture

- Schrödinger picture: pion fields and their conjugate momenta are time independent
- Plane-wave expansion in a box of size L with periodic boundary conditions. The allowed momenta are discretized

$$\boldsymbol{k} = \frac{2\pi}{L}(n_x, n_y, n_z) \text{ with } n_i = 0, \pm 1, \pm 2, \ldots$$

- EFTs have cutoffs
- To avoid infinities, the theory is regularized introducing an ultraviolet cutoff for the three-momentum of the pions

$$\pi_i(\boldsymbol{x}) = \sqrt{\frac{2}{L^3}} \sum_{\boldsymbol{k}}{}' [\pi^c_{i\boldsymbol{k}} \cos(\boldsymbol{k}\cdot\boldsymbol{x}) + \pi^s_{i\boldsymbol{k}} \sin(\boldsymbol{k}\cdot\boldsymbol{x})]$$

$$\Pi_i(\boldsymbol{x}) = \sqrt{\frac{2}{L^3}} \sum_{\boldsymbol{k}}{}' [\Pi^c_{i\boldsymbol{k}} \cos(\boldsymbol{k}\cdot\boldsymbol{x}) + \Pi^s_{i\boldsymbol{k}} \sin(\boldsymbol{k}\cdot\boldsymbol{x})]$$

Quantum Monte Carlo Hamiltonian

- Since the number of nucleons is conserved, the Hamiltonian for the sector with A nucleons and the pion field can be written down as

$$
\begin{aligned}
H &= H_N + H_{\pi\pi} + H_{AV} + H_{WT} \\
H_N &= \sum_{i=1}^{A} \left[\frac{P_i^2}{2M_P} + M_P + \beta_K P_i^2 + \delta M \right] + \sum_{i<j}^{A} \delta_{R_0}(\boldsymbol{r}_i - \boldsymbol{r}_j)[C_S + C_T \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j] \\
H_{\pi\pi} &= \frac{1}{2} {\sum_{\boldsymbol{k}}}' \left[|\boldsymbol{\Pi}_{\boldsymbol{k}}^c|^2 + \omega_{\boldsymbol{k}}^2 |\boldsymbol{\pi}_{\boldsymbol{k}}^c|^2 + |\boldsymbol{\Pi}_{\boldsymbol{k}}^s|^2 + \omega_{\boldsymbol{k}}^2 |\boldsymbol{\pi}_{\boldsymbol{k}}^s|^2 \right]
\end{aligned}
$$

Quantum Monte Carlo Hamiltonian

- Pion-nucleon couplings

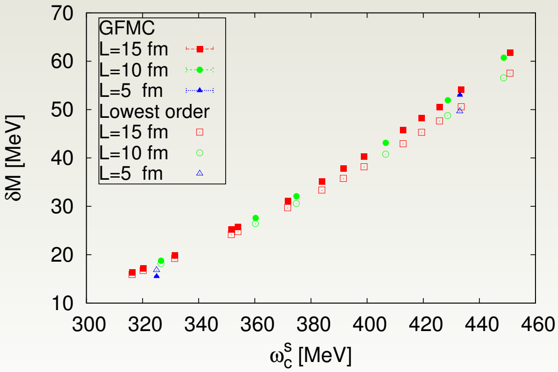
$$
\begin{aligned}
H_{AV} &= \sum_{i=1}^{A} \frac{g_A}{2f_\pi}\sqrt{\frac{2}{L^3}}\sum_{\boldsymbol{k}}{}' \left\{\boldsymbol{\sigma}_i\cdot\boldsymbol{k}\left[\boldsymbol{\tau}_i\cdot\boldsymbol{\pi}^s_{\boldsymbol{k}}\cos(\boldsymbol{k}\cdot\boldsymbol{r}_i) - \boldsymbol{\tau}_i\cdot\boldsymbol{\pi}^c_{\boldsymbol{k}}\sin(\boldsymbol{k}\cdot\boldsymbol{r}_i)\right]\right\} \\
H_{WT} &= \sum_{i=1}^{A} \frac{1}{2f_\pi^2 L^3}\boldsymbol{\tau}_i\cdot\Bigg[\sum_{\boldsymbol{k}}{}'\cos(\boldsymbol{k}\cdot\boldsymbol{r}_i)\boldsymbol{\pi}^c_{\boldsymbol{k}}\times\sum_{\boldsymbol{q}}{}'\cos(\boldsymbol{q}\cdot\boldsymbol{r}_i)\boldsymbol{\Pi}^c_{\boldsymbol{q}} \\
&+ \sum_{\boldsymbol{k}}{}'\cos(\boldsymbol{k}\cdot\boldsymbol{r}_i)\boldsymbol{\pi}^c_{\boldsymbol{k}}\times\sum_{\boldsymbol{q}}{}'\sin(\boldsymbol{q}\cdot\boldsymbol{r}_i)\boldsymbol{\Pi}^s_{\boldsymbol{q}} + \sum_{\boldsymbol{k}}{}'\sin(\boldsymbol{k}\cdot\boldsymbol{r}_i)\boldsymbol{\pi}^s_{\boldsymbol{k}}\times\sum_{\boldsymbol{q}}{}'\cos(\boldsymbol{q}\cdot\boldsymbol{r}_i)\boldsymbol{\Pi}^c_{\boldsymbol{q}} \\
&+ \sum_{\boldsymbol{k}}{}'\sin(\boldsymbol{k}\cdot\boldsymbol{r}_i)\boldsymbol{\pi}^s_{\boldsymbol{k}}\times\sum_{\boldsymbol{q}}{}'\sin(\boldsymbol{q}\cdot\boldsymbol{r}_i)\boldsymbol{\Pi}^s_{\boldsymbol{q}}\Bigg]
\end{aligned}
$$

- $\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\boldsymbol{\Pi}$ analog of $\boldsymbol{S}\cdot\boldsymbol{r}\times\boldsymbol{p}$

Mass renormalization

- Rest mass counter term as a function of the cutoff for different box sizes

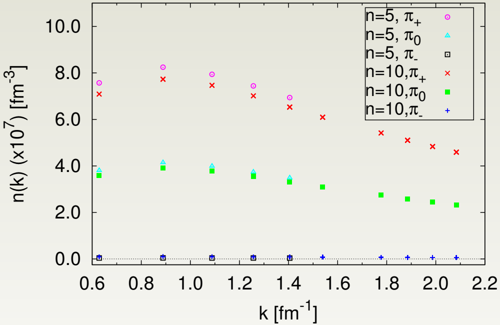
$$H_N = \left[\frac{P^2}{2M_P} + M_P + \delta M \right]$$

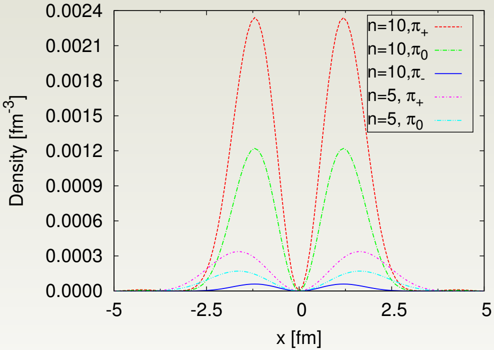


L. Madeira et al., "Quantum Monte Carlo formalism for dynamical pions and nucleons", Physical Review C **98** (2018).

The pion cloud

- Model state is a spin-up proton





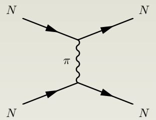
- Structure of the axial-vector coupling

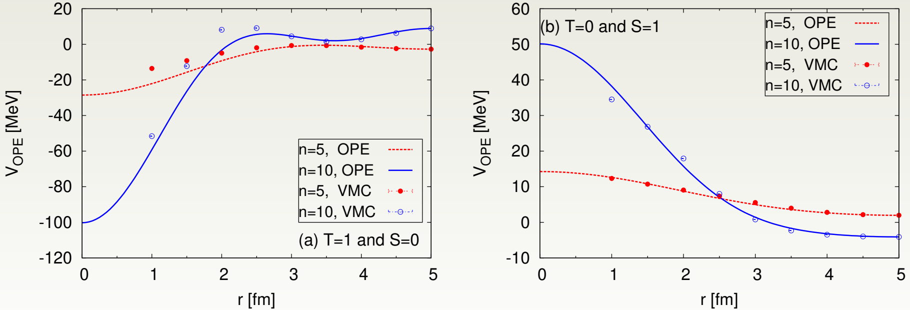
$$\tau_i \pi_i = \frac{1}{2}\tau_+(\pi_x - i\pi_y) + \frac{1}{2}\tau_-(\pi_x + i\pi_y) + \tau_z \pi_0$$

L. Madeira et al., "Quantum Monte Carlo formalism for dynamical pions and nucleons", Physical Review C **98** (2018).

One pion exchange

- Long-range behavior of the nuclear force



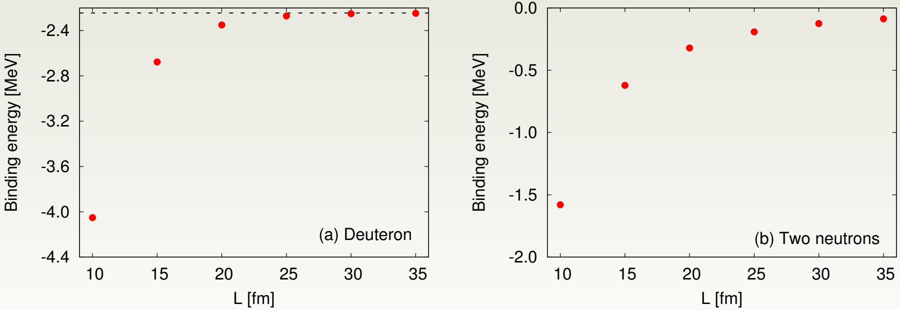


L. Madeira et al., "Quantum Monte Carlo formalism for dynamical pions and nucleons", Physical Review C **98** (2018).

Two nucleons

- We need to fit the low-energy constants in the Hamiltonian

$$H_N = \sum_{i=1}^{A}\left[\frac{P_i^2}{2M_P} + M_P + \beta_K P_i^2 + \delta M\right] + \sum_{i<j}^{A} \delta_{R_0}(\boldsymbol{r}_i - \boldsymbol{r}_j)[C_S + C_T \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j]$$



- We tuned C_S and C_T to reproduce the energies of the physical systems

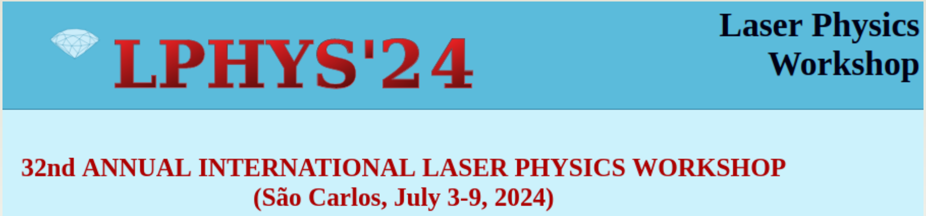
L. Madeira et al., "Quantum Monte Carlo formalism for dynamical pions and nucleons", Physical Review C **98** (2018).

Outlook

- Promising scheme to explicitly include pion contributions in QMC simulations
- One-nucleon properties
- Pion cloud: momentum and density distributions
- Two fixed nucleons $\rightarrow$ one pion exchange at large distances
- Low-energy constants
- Light-nuclei

Laser Physics 2024

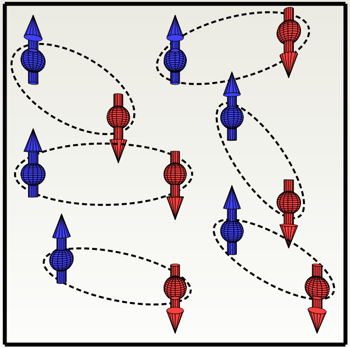
https://www.lasphys.com/workshops/lasphys24/

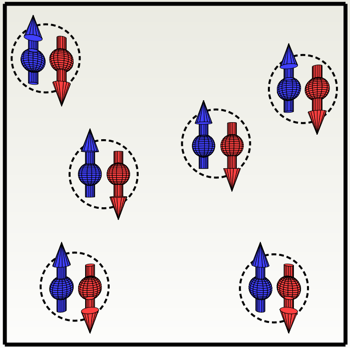


- Modern Trends in Laser Physics
- Strong Field & Attosecond Physics
- Laser Biomedical Applications
- Physics of Lasers
- Nonlinear Optics & Spectroscopy
- **Physics of Cold Trapped Atoms**
- Quantum Information Science
- Fiber Optics
- Extreme Light Technologies, Science, and Applications
- Quantum Engineering

BCS-BEC crossover

- Bardeen-Cooper-Schrieffer (BCS) theory $\rightarrow$ fermions
  - Pairing of fermions $\rightarrow$ boson-like behavior
- Bose-Einstein condensate (BEC) $\rightarrow$ bosonic fluids
  - Macroscopic occupation of a single quantum state
- BEC-BCS crossover



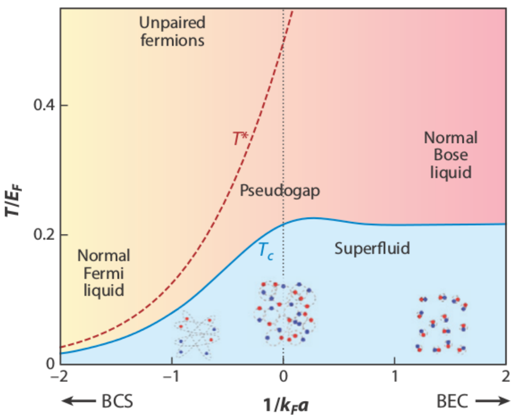


BCS-BEC crossover

- Two-body interaction between antiparallel spins

$$k \cot \delta_0(k) = -\frac{1}{a} + \underbrace{\frac{r_0 k^2}{2} + \mathcal{O}(k^4)}_{\text{neglible}}$$

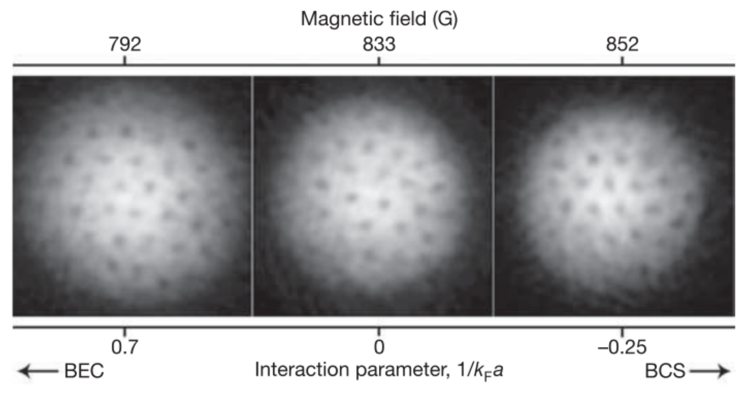
- Dimensionless quantity: $k_F a$



M. Randeria et al., "Crossover from Bardeen-Cooper-Schrieffer to Bose-Einstein Condensation and the Unitary Fermi Gas", Annual Review of Condensed Matter Physics **5** (2014).

Vortices in cold fermionic gases

- Quantized vortices → quanta of circulation: $h/(2m)$
- 3D - Experimental observation of vortices in a ^{6}Li gas



M. W. Zwierlein et al., "Vortices and superfluidity in a strongly interacting Fermi gas", Nature **435** (2005).

Vortices in cold fermionic gases

- Properties of a single vortex line using QMC methods

Silvio Vitiello
(Unicamp, Brazil)

Vanderlei Bagnato
(USP, Brazil)

Stefano Gandolfi
(LANL, USA)

Kevin Schmidt
(ASU, USA)

- LM et al., "Vortex line in the unitary Fermi gas", PRA 93 (2016)
- LM et al., "Core structure of two-dimensional Fermi gas vortices in the BEC-BCS crossover region", PRA 95 (2017)
- LM et al., "Vortices in low-density neutron matter and cold Fermi gases", PRC 100 (2019)

Cold atoms and low-density neutron matter

- Properties of neutron matter → understanding of neutron star crusts and the exterior of large neutron-rich nuclei
  - Equation of state
  - Pairing gap
- Inaccessible experimentally, unlike cold gases

- Cold atoms
  - Tunable a
  - Dilute: interparticle spacing $\gg r_{\text{eff}}$
  - Nearly zero r_{eff}
  - $a \gg r_{\text{eff}}$

- Low-density neutron matter
  - $a \approx -18.5$ fm
  - low-density
  - $r_{\text{eff}} \approx 2.7$ fm
  - $|r_{\text{eff}}/a| \approx 0.15$

- We can compare the results to try to understand the impact of effective range

Nucleon-nucleon interactions

- One realistic phenomenological NN interaction is the Argonne AV18 potential

$$v_{ij} = \sum_{p=1}^{18} f_p(r_{ij}) O^p(\boldsymbol{r}_{ij})$$

1: 1
2: $\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j$
3: $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$
4: $(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)$
5: $\boldsymbol{S}_{ij}$
6: $(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)\boldsymbol{S}_{ij}$

7: $\boldsymbol{L} \cdot \boldsymbol{S}$
8: $(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)(\boldsymbol{L} \cdot \boldsymbol{S})$
9: $\boldsymbol{L}^2$
10: $(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)\boldsymbol{L}^2$
11: $(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)\boldsymbol{L}^2$
12: $(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)\boldsymbol{L}^2$

13: $(\boldsymbol{L} \cdot \boldsymbol{S})^2$
14: $(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)(\boldsymbol{L} \cdot \boldsymbol{S})^2$
15: $\boldsymbol{T}_{ij} = 3\boldsymbol{\tau}_{iz}\boldsymbol{\tau}_{jz} - \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j$
16: $(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)\boldsymbol{T}_{ij}$
17: $\boldsymbol{S}_{ij}\boldsymbol{T}_{ij}$
18: $\tau_{1z} + \tau_{2z}$

- Not feasible for our purposes

Low-density neutron matter and cold atoms

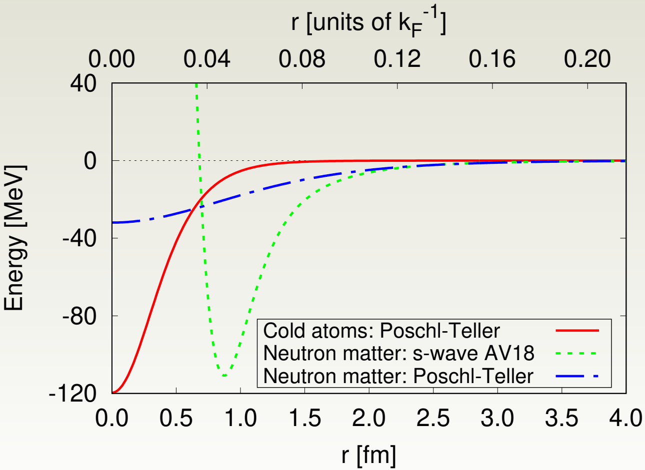
- Low-density neutron matter → the dominant interaction is *s*-wave
- *s*-wave part of AV18
- Two-component mixture of spin-up and spin-down neutrons
- Central potential for anti-parallel spins:

$$v_{S=0}(r_{ij}) = v_c(r_{ij}) - 3v_\sigma(r_{ij})$$

- This has been done for the bulk[7]

[7] A. Gezerlis et al., "Strongly paired fermions: Cold atoms and neutron matter", Phys. Rev. C **77** (2008).

Low-density neutron matter and cold atoms

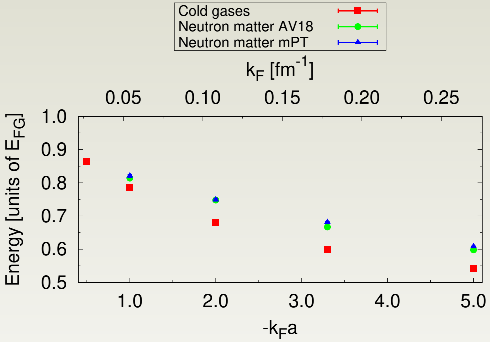


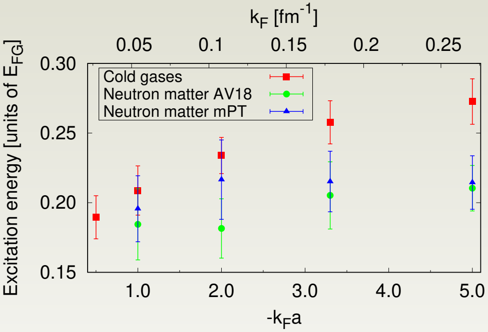
- Shape-independent approximation:

$$k \cot \delta(k) = -\frac{1}{a} + \frac{r_{\text{reff}} k^2}{2} + \mathcal{O}(k^4)$$

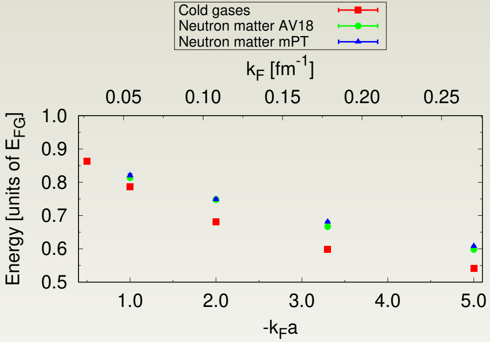
L. Madeira et al., "Vortices in low-density neutron matter and cold Fermi gases", Phys. Rev. C **100** (2019).

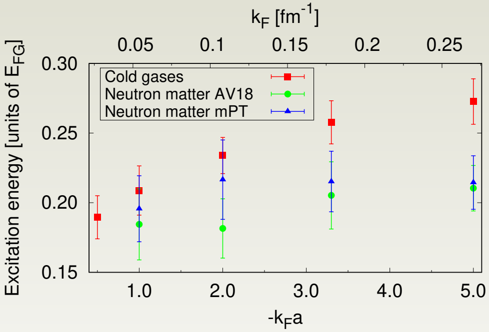
- Bulk and a single vortex





- Bulk and a single vortex





Take home message

- The effective range expansion successfully describes low-density neutron matter
- We can draw a parallel between two-component Fermi gases and neutron matter
  - However, quantitative agreement happens only for extremely dilute systems

Two-body scattering

- Two-body *s*-wave scattering with a finite range spherically symmetric potential

$$\left[-\frac{\hbar^2}{2m_r}\frac{d^2}{dr^2}+V(r)\right]u(r)=\frac{\hbar^2k^2}{2m_r}u(r)$$

- $k\to 0$ solution $u_0(r)$
- R is outside the potential range
- Scattering length a

$$\frac{1}{u_0(R)}\frac{d}{dr}u_0(r)\Big|_R=\frac{1}{R-a}$$

- Effective range r_0
- $\psi_0(r)$ is the asymptotic form of $u_0(r)$

$$r_0=2\int_0^\infty dr\left[\psi_0^2(r)-u_0^2(r)\right]$$

Zero-range trimers

- The zero-range limit of the scaling function has been studied extensively in the literature[8]

$$E_3(1/a,0,\nu) + \frac{\hbar^2}{2ma^2} = E_3(0,0,\nu)e^{\frac{\Delta(\xi)}{s_0}}, \quad \tan\xi = -\left(\frac{m|E_3(1/a,0,\nu)|}{\hbar^2}\right)^{1/2} a$$

- Relation to the two-body contact[9]

$$\frac{\partial E_3(1/a,r_0,\nu)}{\partial(1/a)} = -\frac{\hbar^2 C_2}{8\pi m}$$

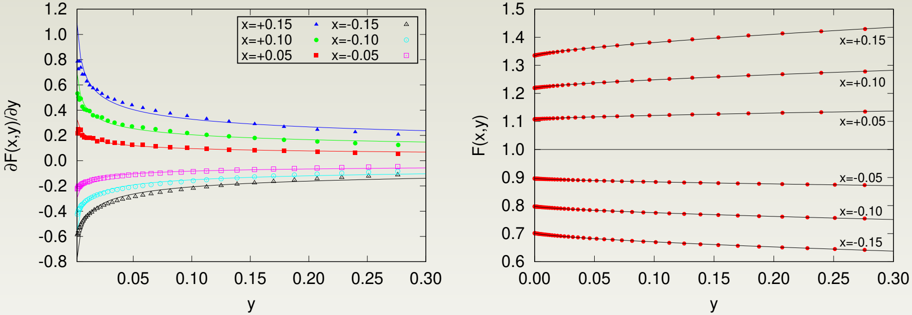
- Zero-range behavior of our scaling function

$$F(x,0) = 1 + \left(\frac{53.097}{8\pi}\right) x \approx 1 + 2.113x \qquad (|x| \ll 1)$$

[8] E. Braaten et al., "Universality in few-body systems with large scattering length", Phys. Rep. **428** (2006).

[9] Y. Castin et al., "Single-particle momentum distribution of an Efimov trimer", Phys. Rev. A **83** (2011), F. Werner et al., "General relations for quantum gases in two and three dimensions. II. Bosons and mixtures", Phys. Rev. A **86** (2012).

Scaling function parameterization



$$F(x,y) = 1 + c_1 x + c_2 x y^{\sigma} + c_3 x^2 + c_4 x^2 y + c_5 x^2 y^{\sigma}$$

	c_1	c_2	c_3	c_4	c_5	σ
Fit	2.106(1)	1.26(1)	0.804(4)	1.0(2)	1.2(1)	0.680(3)
Ansatz	2.107	1.35	0.804	1.03	1.05	0.641

Variational Monte Carlo

- VMC is based on the variational principle, and the Monte Carlo method is applied in the evaluation of the resulting multidimensional integrals
- It relies on a trial wave function Ψ_T which should mimic as many as possible properties of the true ground-state function Ψ_0
- Upper bound on the exact ground-state energy E_0

$$E_V = \frac{\int \Psi_T^*(\mathbf{R}) H \Psi_T(\mathbf{R}) d\mathbf{R}}{\int \Psi_T^*(\mathbf{R}) \Psi_T(\mathbf{R}) d\mathbf{R}} \geqslant E_0$$

- $\mathbf{R} = (\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N)$ is a $3N$-dimensional vector with the coordinates of the N particles

Variational Monte Carlo

- We rewrite this equation in the form

$$E_V = \frac{\int |\Psi_T(\mathbf{R})|^2 \left[\Psi_T(\mathbf{R})^{-1} H \Psi_T(\mathbf{R})\right] d\mathbf{R}}{\int |\Psi_T(\mathbf{R})|^2 d\mathbf{R}}$$

- Sample a set of points $\{\mathbf{R}_m : m = 1, M\}$ from the configuration-space, with the probability density

$$\mathcal{P}(\mathbf{R}) = \frac{|\Psi_T(\mathbf{R})|^2}{\int |\Psi_T(\mathbf{R})|^2 d\mathbf{R}}$$

- We define a "local energy" $E_L(\mathbf{R}) = \Psi_T(\mathbf{R})^{-1} H \Psi_T(\mathbf{R})$

$$E_V \approx \frac{1}{M} \sum_{m=1}^{M} E_L(\mathbf{R}_m)$$

- Expectation values of operators other than the Hamiltonian can be computed in analogous

Diffusion Monte Carlo

- Diffusion Monte Carlo is a method for solving imaginary-time many-body Schrödinger equation

$$-\partial_t \Phi(\mathbf{R}, t) = (H - E_T)\Phi(\mathbf{R}, t)$$

- t is real and it measures the progress in imaginary time, and E_T is an energy offset

$$\Phi(\mathbf{R}, t+\tau) = \int G(\mathbf{R} \leftarrow \mathbf{R}', \tau)\Phi(\mathbf{R}', t) d\mathbf{R}'$$

- with the Green's function

$$G(\mathbf{R} \leftarrow \mathbf{R}', \tau) = \langle \mathbf{R} | \exp\left[-\tau(H - E_T)\right] | \mathbf{R}' \rangle$$

Diffusion Monte Carlo

- The Green's function can be expressed as

$$G(\mathbf{R} \leftarrow \mathbf{R}', \tau) = \sum_i \Psi_i(\mathbf{R}) \exp\{-\tau(E_i - E_T)\} \Psi_i^*(\mathbf{R}')$$

- $\{\Psi_i\}$ and $\{E_i\}$ are the complete set of eigenstates and eigenenergies of H
- As $\tau \to \infty$, the operator $\exp\{-\tau(E_i - E_T)\}$ projects out the lowest eigenstate $|\Psi_0\rangle$ that has non-zero overlap with an initial state $|\Phi_{\text{init}}\rangle$

$$\lim_{\tau\to\infty} \langle \mathbf{R}| \exp[-\tau(H - E_T)] |\Phi_{\text{init}}\rangle = \lim_{\tau\to\infty} \int G(\mathbf{R} \leftarrow \mathbf{R}', \tau) \Phi_{\text{init}}(\mathbf{R}') d\mathbf{R}'$$
$$= \lim_{\tau\to\infty} \sum_i \Psi_i(\mathbf{R}) \exp[-\tau(E_i - E_T)] \langle \Psi_i | \Phi_{\text{init}}\rangle$$

- $E_T = E_0$ and taking the limit $\tau \to \infty$, only the $|\Psi_0\rangle$ state is projected, since the higher energy states are all exponentially damped because their energies are higher than E_0

Diffusion Monte Carlo

- Diffusion Monte Carlo is a method for solving imaginary-time many-body Schrödinger equation

$$-\partial_\tau \Phi(\mathbf{R}, \tau) = (H - E_T)\Phi(\mathbf{R}, \tau)$$

- τ is real and it measures the progress in imaginary time
- E_T is an energy offset

$$\Phi(\tau) = \exp\left[-(H - E_T)\tau\right]\psi_T$$

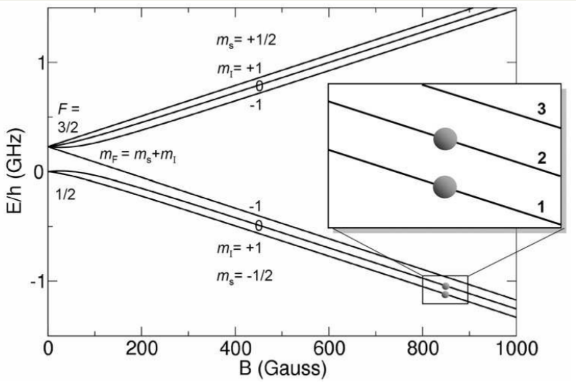
- As $\tau \to \infty$, the operator $\exp\left\{-\tau(H - E_T)\right\}$ projects out the lowest eigenstate $|\Psi_0\rangle$ that has non-zero overlap with an initial state $|\psi_\mathrm{T}\rangle$

Experiments

- Typical experimental parameters
  - Total number of atoms: 10^5 to 10^7
  - $k_F^{-1} \sim 1\ \mu$m
  - $E_F/k_B \sim 100$ nanoK
  - Temperatures $\sim 0.1 E_F/k_B$
- ^{6}Li, ^{40}K, ^{173}Yb

Two-component mixture

- Energy levels of ^{6}Li atoms in a magnetic field



$$m_s = +1/2, m_I = +1 \quad |6\rangle$$
$$m_s = +1/2, m_I = 0 \quad |5\rangle$$
$$m_s = +1/2, m_I = -1 \quad |4\rangle$$
$$m_s = -1/2, m_I = -1 \quad |3\rangle$$
$$m_s = -1/2, m_I = 0 \quad |2\rangle = |\downarrow\rangle$$
$$m_s = -1/2, m_I = +1 \quad |1\rangle = |\uparrow\rangle$$

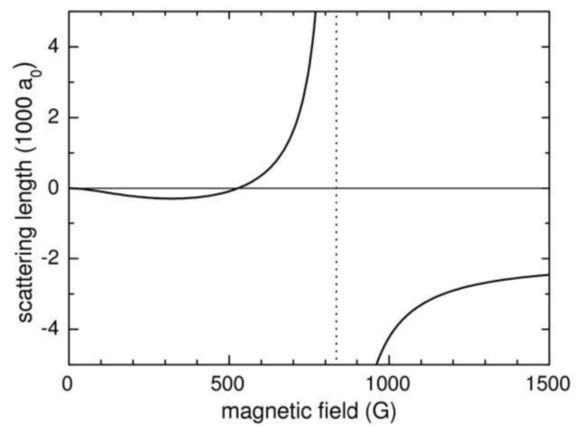
[10] Ultracold Fermi Gases, Proceedings of the International School of Physics "Enrico Fermi", Course CLXIV, Varenna, 20 - 30 June 2006, edited by M. Inguscio, W. Ketterle, and C. Salomon

Feshbach resonance

- Dramatic increase in the scattering length at the resonance

$$a(B) = a_{\text{bg}} \left(1 + \frac{\Delta B}{B - B_0}\right)$$

- $B_0 = 834$ G
- $\Delta B = 300$ G
- $a_{\text{bg}} \approx -0.75\ \mu\text{m}$



11

[11] Ultracold Fermi Gases, Proceedings of the International School of Physics "Enrico Fermi", Course CLXIV, Varenna, 20 - 30 June 2006, edited by M. Inguscio, W. Ketterle, and C. Salomon

QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \bar{q}(i\gamma^{\mu}\mathcal{D}_{\mu} - \mathcal{M})q - \frac{1}{4}\mathcal{G}_{\mu\nu,a}\mathcal{G}^{\mu\nu}_{a}$$

- Quark fields: q
- Gauge-covariant derivative: $\mathcal{D}_{\mu}$
- Gluon field strength tensor: $\mathcal{G}_{\mu\nu,a}$
- Quark mass matrix: $\mathcal{M}$
- Quark masses:

$$\begin{aligned} m_u &= 2.5 \pm 0.8\text{MeV} \\ m_d &= 5.0 \pm 0.9\text{MeV} \\ m_s &= 101 \pm 25\text{MeV} \end{aligned}$$

- Small if compared to a typical hadronic scale $m \approx 1$ GeV

Chiral symmetry

- QCD Lagrangian in the limit of vanishing quark masses

$$\mathcal{L}^0_{\text{QCD}} = \bar{q} i \gamma^\mu \mathcal{D}_\mu q - \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}^{\mu\nu}_a$$

- Right- and left-handed quark fields: $q_R = P_R q \qquad q_L = P_L q$
- Projectors:

$$P_R = \frac{1}{2}(1 + \gamma_5) \qquad P_L = \frac{1}{2}(1 - \gamma_5)$$

$$\mathcal{L}^0_{\text{QCD}} = \bar{q}_R i \gamma^\mu \mathcal{D}_\mu q_R + \bar{q}_L i \gamma^\mu \mathcal{D}_\mu q_L - \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}^{\mu\nu}_a$$

Chiral symmetry

- If we restrict ourselves to up and down quarks only, we see that $\mathcal{L}^0_{\mathrm{QCD}}$ is invariant under the global unitary transformations

$$q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} \mapsto g_R q_R = \exp\left(-i\Theta_i^R \frac{\tau_i}{2}\right) \begin{pmatrix} u_R \\ d_R \end{pmatrix}$$

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \mapsto g_L q_L = \exp\left(-i\Theta_i^L \frac{\tau_i}{2}\right) \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

- The conclusion is that right- and left-handed components of massless quarks do not mix
- This is the chiral symmetry

Explicit symmetry breaking

- Quark mass matrix for up and down quarks only

$$\mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = \frac{1}{2}(m_u + m_d)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(m_u - m_d)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \frac{1}{2}(m_u + m_d)\mathbf{1} + \frac{1}{2}(m_u - m_d)\tau_3.$$

- Isospin is an exact symmetry if $m_u = m_d$
- Both terms break chiral symmetry, but $m_u, m_d \ll$ typical hadronic scale of 1 GeV
- Explicit symmetry breaking due to non-vanishing quark masses is small

Spontaneous symmetry breaking

- A continuous symmetry is said to be spontaneously broken if a symmetry of the Lagrangian is not realized in the ground-state of the system
- Chiral symmetry $\to$ conserved charges, in particular:

$$Q_i^A = \int d^3x q^\dagger(t,\boldsymbol{x})\gamma_5 \frac{\tau_i}{2} q(t,\boldsymbol{x}), \text{ with } \frac{dQ_i^A}{dt} = 0$$

- The Q_i^A commutes with the Hamiltonian and it has negative parity
- Positive parity hadron there must be a negative one as well $\to$ "parity doublets" are not observed in nature
- A spontaneously broken global symmetry $\to$ massless Goldstone bosons with the quantum numbers of the broken generators
- The broken generators are the Q_i^A $\to$ pseudoscalar
- The Goldstone bosons $\to$ isospin triplet of the pseudoscalar pions
- The pion masses are not exactly zero because the masses of u and d quarks do not vanish either, but this explains why pions are so light