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Topological Susceptibility in a Uniform Magnetic Field

Perspectives from Chiral Perturbation Theory

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Outline & References

- Introduction to the QCD θ-vacuum
  - *[Srednicki, Quantum Field Theory]*
- Topological Cumulants in a Magnetic Field & "Sum Rules" / Low-Energy Theorems
  - *[PA, Phys. Lett. B **825** (2022) 136826, Nucl. Phys. B **974** (2022) 115627, Nucl. Phys. B **982** (2022) 115823]*
- Finite Volume Corrections to QCD Observables
  - *[PA, Tiburzi, Phys. Rev. D, **107**, 094504 (2023)]*

QCD θ-Vacuum & Topological Cumulants

- QCD Lagrangian permits a θ-term, albeit small ($\theta \lesssim 10^{-11}$), that is not forbidden by any symmetries

$$Z = \int \mathscr{D}A\, \mathscr{D}q\, \mathscr{D}\bar{q} \exp\left[i \int d^4x\, \mathscr{L}_{\rm QCD}\right] \qquad \mathscr{L}_{\rm QCD} = -\frac{1}{4} G^a_{\mu\nu} G_{a\mu\nu} - \frac{g^2\theta}{32\pi^2} \widetilde{G}^{a\mu\nu} G^a_{\mu\nu} + \bar{q}\left(i\gamma \cdot D - M\right) q$$

- An axial rotation removes the θ-term at the expense of imaginary quark masses, $m_{q_f} \to m_{q_f} e^{i\theta}$

$$\mathscr{D}\bar{q}\, \mathscr{D}q \to \exp\left[-i \int d^4x \frac{g^2\Theta n}{16\pi^2} \widetilde{G}^{a\mu\nu} G_{a\mu\nu}\right] \mathscr{D}\bar{q}\, \mathscr{D}q \qquad \mathscr{L}_{\rm QCD} = -\frac{1}{4} G^a_{\mu\nu} G_{a\mu\nu} + \bar{q}\left(i\gamma \cdot D - M\, e^{i\theta\gamma_5}\right) q$$

- Topological susceptibility is intimately connected to the chiral condensate

$$\chi_t(0) = -i \int d^4x \left\langle \mathscr{T} \frac{g^2 \widetilde{G}G(x)}{32\pi^2} \frac{g^2 \widetilde{G}G(0)}{32\pi^2} \right\rangle \qquad \chi_t(0) = -\frac{1}{n^2} \langle \bar{q}(0)\, M\, q(0) \rangle + \frac{i}{n^2} \int d^4x \langle \mathscr{T}\, \bar{q}(x)\, M \gamma_5\, q(x)\, \bar{q}(0)\, M \gamma_5 q(0) \rangle$$

Key Questions & Approaches

- How do magnetic fields affect topological cumulants?
    - Possible relevance in heavy-ion collisions, magnetars & early universe
    - Additional relevance in lattice QCD simulations
        - *[Brandt, Cuteri, Endrodi, Hernandez, Marko, arXiv:2212:03385]*
- For simplicity, consider a uniform magnetic field in the low-energy regime of QCD, where effective theory approaches are valid
    - *[PA, Phys. Lett. B 825 (2021), Nucl. Phys. B 974 & 982 (2022)]*

Chiral Perturbation Theory - Ingredients

- Chiral Perturbation theory is the low energy effective theory of QCD
- The results are model-independent for $x/(4\pi F_\pi) \ll 1$ with $x = -i\partial\,, \sqrt{eH}\,, m_\pi$
- The building blocks of the effective theory are
  - *Goldstone Manifold.* $\Sigma \in SU(2)_L \times SU(2)_R / SU(2)_V$
  - *Scalar-Pseudoscalar Source.* $\chi = S + i\,P$
  - *Electromagnetic Gauge Field.* $\partial_\mu \Sigma \to D_\mu \Sigma = \partial_\mu \Sigma + iA_\mu^{\text{ext}}\,[Q, \Sigma]$
- *In massless QCD,* left-and-right-handed quarks do not mix and therefore can be rotated independently $\psi_L \to L\,\psi_L$ and $\psi_R \to R\,\psi_R$. This leads to the following symmetry in the QCD partition function
  - $U(1)_V \times SU(2)_V \times SU(2)_A \to U(1)_V \times SU(2)_V$ due to the formation of the chiral condensate, $\langle \bar{\psi}\psi \rangle \neq 0$
  - *The* $U(1)_A : \psi \to e^{i\theta_5\gamma_5}\,\psi$ *subgroup of* $U(2)_L \times U(2)_R$ *is broken by the partition function.*

Chiral Perturbation Theory - Ingredients

- Goldstone manifold, $\Sigma \in SU(2)_L \times SU(2)_R / SU(2)_V$
- The fluctuations are axial since the chiral condensate $\langle \overline{\psi}\psi \rangle$ is only invariant if $L = R \equiv V$.

$$1 \to \Sigma = 1 + \cdots$$
$$\Sigma \equiv U 1 U \in SU(2)_L \times SU(2)_R / SU(2)_V \qquad U = \exp\left(\frac{i\Phi}{2F}\right) \qquad \Phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & \pi^0 \end{pmatrix}$$

- The construction of the Lagrangian involves building a chirally invariant theory under $\Sigma \to L \Sigma R^\dagger$.

$$\mathscr{L}_2 = \frac{1}{4} F^2 \,\mathrm{Tr}\,[\, \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \,] + \frac{1}{4} F^2 \,\mathrm{Tr}\,[\, \chi \Sigma^\dagger + \chi^\dagger \Sigma \,]$$

- The mass term of QCD $\Delta\mathscr{L} = m_q \overline{\psi}\, \psi$ also explicitly breaks chiral symmetry $SU(2)_L \times SU(2)_R \to SU(2)_V$.

Chiral Perturbation Theory - Ingredients

- The leading order mass term and higher order terms are generated by introducing sources in the QCD Lagrangian that preserves chiral symmetry $\Delta\mathscr{L} = \bar{\psi}_L \chi \psi_R + \bar{\psi}_R \chi^\dagger \psi_L$ with $\chi \to L\chi R^\dagger$.

$$\mathscr{L}_4 = \frac{\ell_3 + \ell_4}{16}\mathrm{Tr}\left[\chi\Sigma^\dagger + \chi^\dagger\Sigma\right]^2 + \frac{h_1 + h_3 - \ell_4}{4}\mathrm{Tr}\left(\chi\chi^\dagger\right) - \frac{\ell_7}{16}\left[\mathrm{Tr}\left(\chi\Sigma^\dagger - \chi^\dagger\Sigma\right)\right]^2$$
$$+\frac{h_1 - h_3 - \ell_4}{16}\left\{\left[\mathrm{Tr}\left(\chi\Sigma^\dagger + \chi^\dagger\Sigma\right)\right]^2 + \left[\mathrm{Tr}\left(\chi\Sigma^\dagger - \chi^\dagger\Sigma\right)\right]^2 - 2\mathrm{Tr}\left(\Sigma\chi^\dagger\Sigma\chi^\dagger + \chi\Sigma^\dagger\chi\Sigma^\dagger\right)\right\}$$

  - ℓ_i and h_i are low and high energy constants contain divergences and in terms of the MS-bar scale Λ are

$$\ell_i = \ell_i^r(\Lambda) - \frac{\gamma_i \Lambda^{-2\varepsilon}}{2\,(4\pi)^2}\left[\frac{1}{\epsilon} + 1\right] \qquad \Lambda\frac{d}{d\Lambda}\ell_i^r = -\frac{\gamma_i}{(4\pi)^2} \qquad \ell_i^r(\Lambda) = \frac{\gamma_i}{2(4\pi)^2}\left[\bar{\ell}_i + \log\frac{m^2}{\Lambda^2}\right]$$

- In a magnetic field, further counter-terms are required

$$\Delta\mathscr{L}_4 = -\frac{4h_2 + \ell_5}{2}\mathrm{Tr}\left[F^R_{\mu\nu}F^{R\mu\nu} + F^L_{\mu\nu}F^{L\mu\nu}\right] + \ell_5\mathrm{Tr}\left[\Sigma F^L_{\mu\nu}\Sigma^\dagger F^{R\mu\nu}\right] \qquad F^R_{\mu\nu} = F^L_{\mu\nu} = -e\frac{\tau_3}{2}F_{\mu\nu}$$

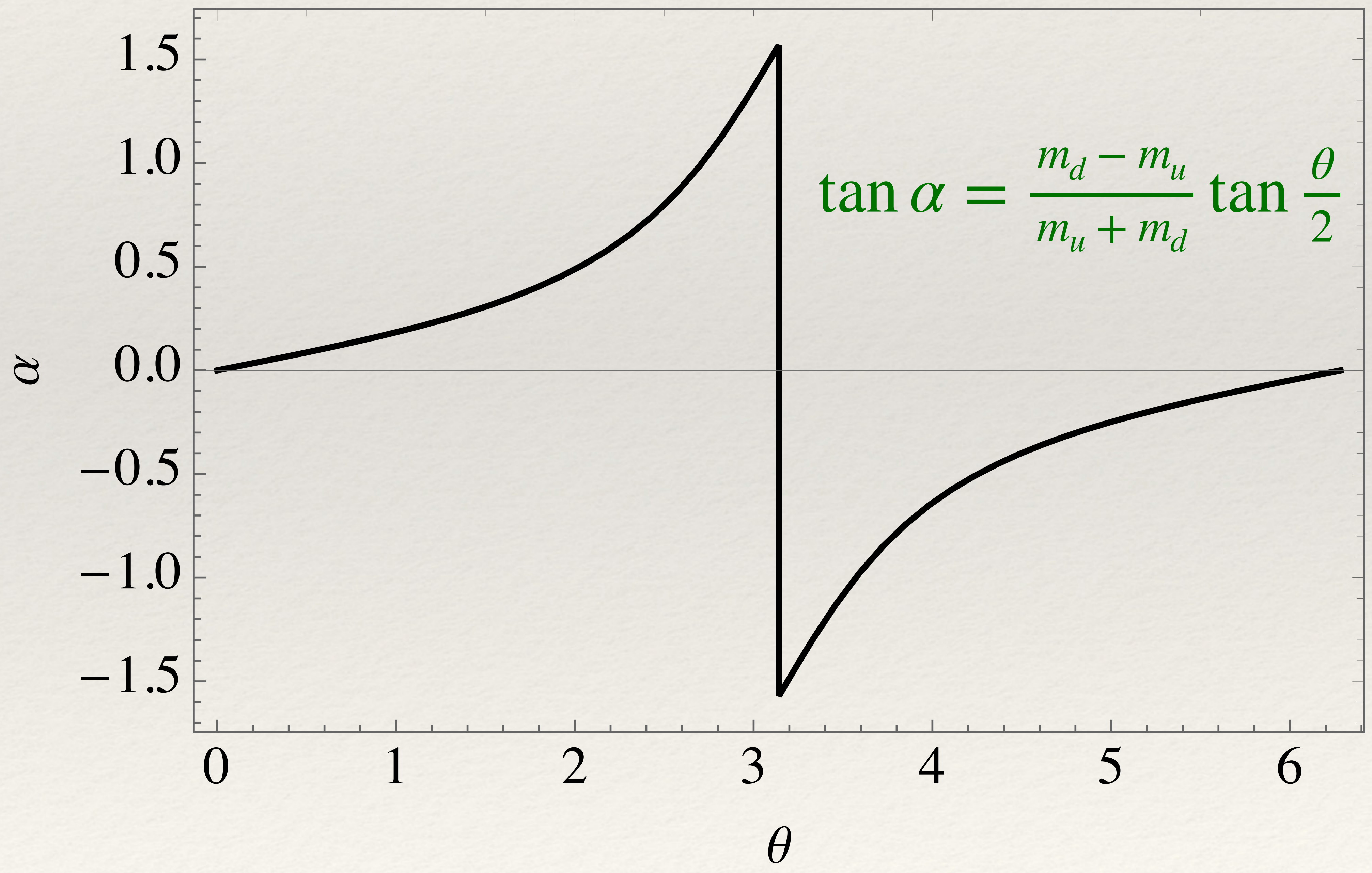
Chiral Perturbation Theory with θ-vacuum

- At finite θ, the static Lagrangian encodes the competition between identity (through average quark mass) and the third isospin direction (through the quark mass difference .

$$\mathcal{F}_{\text{tree}} = -\frac{F^2}{4}\left[\chi\,\Sigma_\alpha^\dagger + \Sigma_\alpha\,\chi^\dagger\right]$$
$$= -F^2 B_0\left[(m_u+m_d)\cos\alpha\cos\frac{\theta}{2} + (m_d-m_u)\,\hat{\phi}_3\sin\alpha\sin\frac{\theta}{2}\right]$$

$$\chi = 2\,B_0\,e^{-i\frac{\theta}{n}}\,M$$
$$M = \frac{1}{2}(m_u+m_d)\,1 + \frac{1}{2}(m_u-m_d)\,\tau_3$$

$$\Sigma_\alpha = \cos\alpha\,1 + i\sin\alpha\,\hat{\phi}_a\tau_a$$
$$\hat{\phi}_a\hat{\phi}_a = 1$$



Chiral Perturbation Theory with θ-vacuum

- The rotation induced by the theta vacuum is axial,

$$\Sigma = \mathscr{A}_\alpha \Sigma_0 \mathscr{A}_\alpha \qquad \mathscr{A}_\alpha = e^{i\frac{\alpha}{2}\tau_3} = \cos\frac{\alpha}{2} + i\,\tau_3\,\sin\frac{\alpha}{2}$$

- We require for an NLO calculation of the free energy
  - Dispersion relations in a magnetic field
  - Tree-level counter-terms from $\mathscr{L}_4$

$$U = \exp\left(\frac{i\,\Phi}{2\,F}\right) \qquad \Phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & \pi^0 \end{pmatrix}$$

- Take care to parameterize fluctuations around the rotated vacuum $\Sigma = \mathscr{A}_\alpha (U\,\Sigma_0\,U)\,\mathscr{A}_\alpha$
- In order to construct the free energy we require
  - $\mathscr{L}_2^{\text{static}}$, $\mathscr{L}_2^{\text{quadratic}}$ and $\mathscr{L}_4^{\text{static}}$
  - LO ground state value of α but not the NLO ground state (!)

Free Energy in the θ-vacuum

- The sum over Landau levels produces the standard Schwinger one-loop effective potential

$$\mathscr{F}_H(\theta) = \frac{1}{2}H_R^2 - \frac{1}{(4\pi)^2}\int_0^\infty \frac{ds}{s^3} e^{-\mathring{m}_\pi^2(\theta)s}\left[\frac{eHs}{\sinh eHs} - 1 + \frac{(eHs)^2}{6}\right]$$

- The quadratic contribution renormalizes the external field

$$H_R = Z_H H \qquad Z_H = 1 + 4e^2 h_2^r + \frac{e^2}{6(4\pi)^2}\left(\log\frac{\Lambda^2}{\mathring{m}_\pi^2(\theta)} - 1\right)$$

$$\mathring{m}_\pi^2(\theta) = \mathring{m}_\pi^2(0)\sqrt{1 - \frac{4m_u m_d}{(m_u+m_d)^2}\sin^2\frac{\theta}{2}}$$

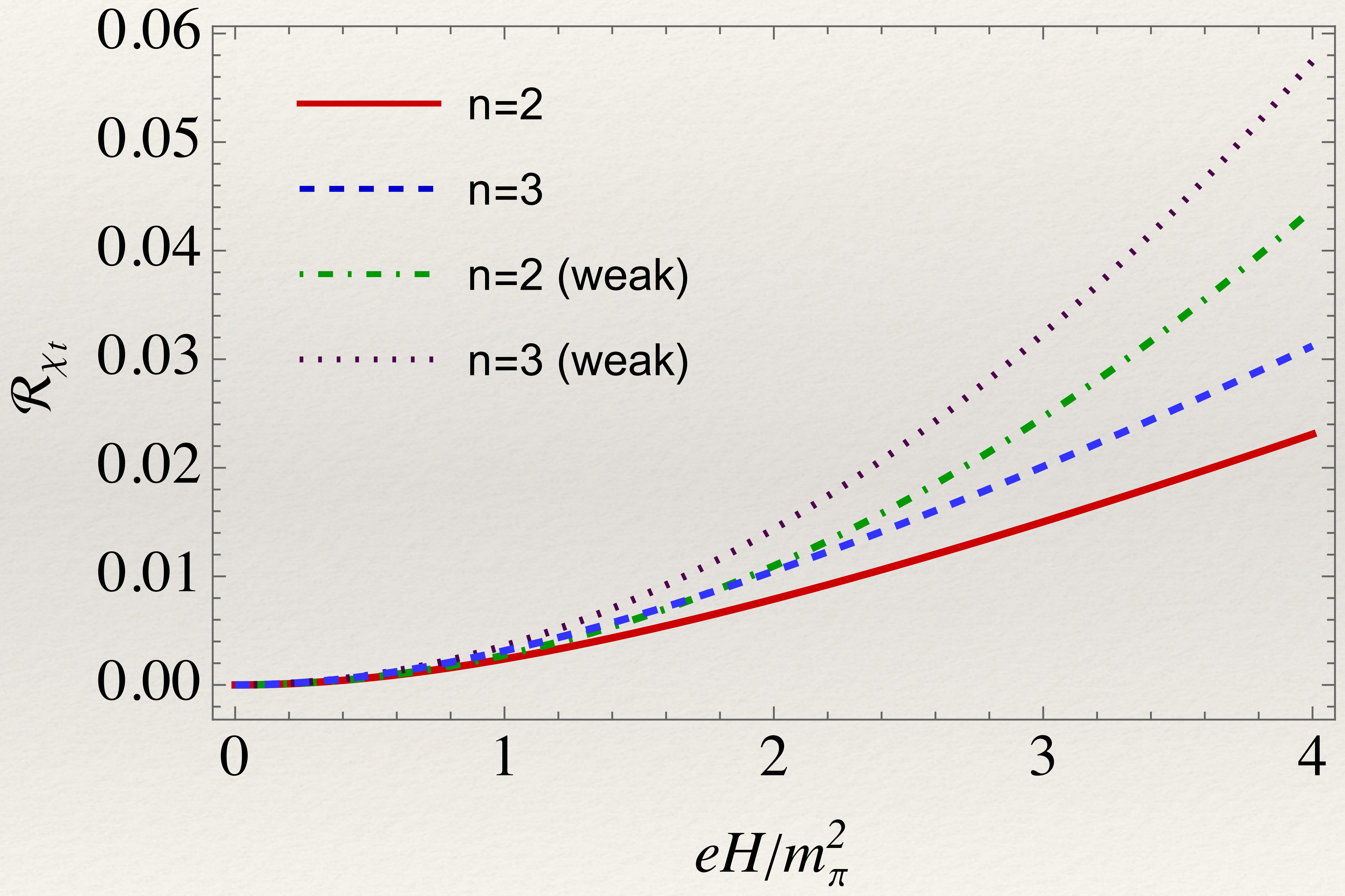
Topological Susceptibility/Sum Rule I

- The topological susceptibility is monotonous as is the chiral condensate

$$\chi_{t,H} = -B_0\,\overline{m}\int_0^\infty \frac{ds}{(4\pi s)^2} e^{-m_\pi^2 s}\left[\frac{eHs}{\sinh eHs} - 1\right]$$

$$= B_0\,\overline{m}\left[\frac{1}{6(4\pi m_\pi)^2}\right](eH)^2 + \mathcal{O}\left\{(eH)^4\right\}$$

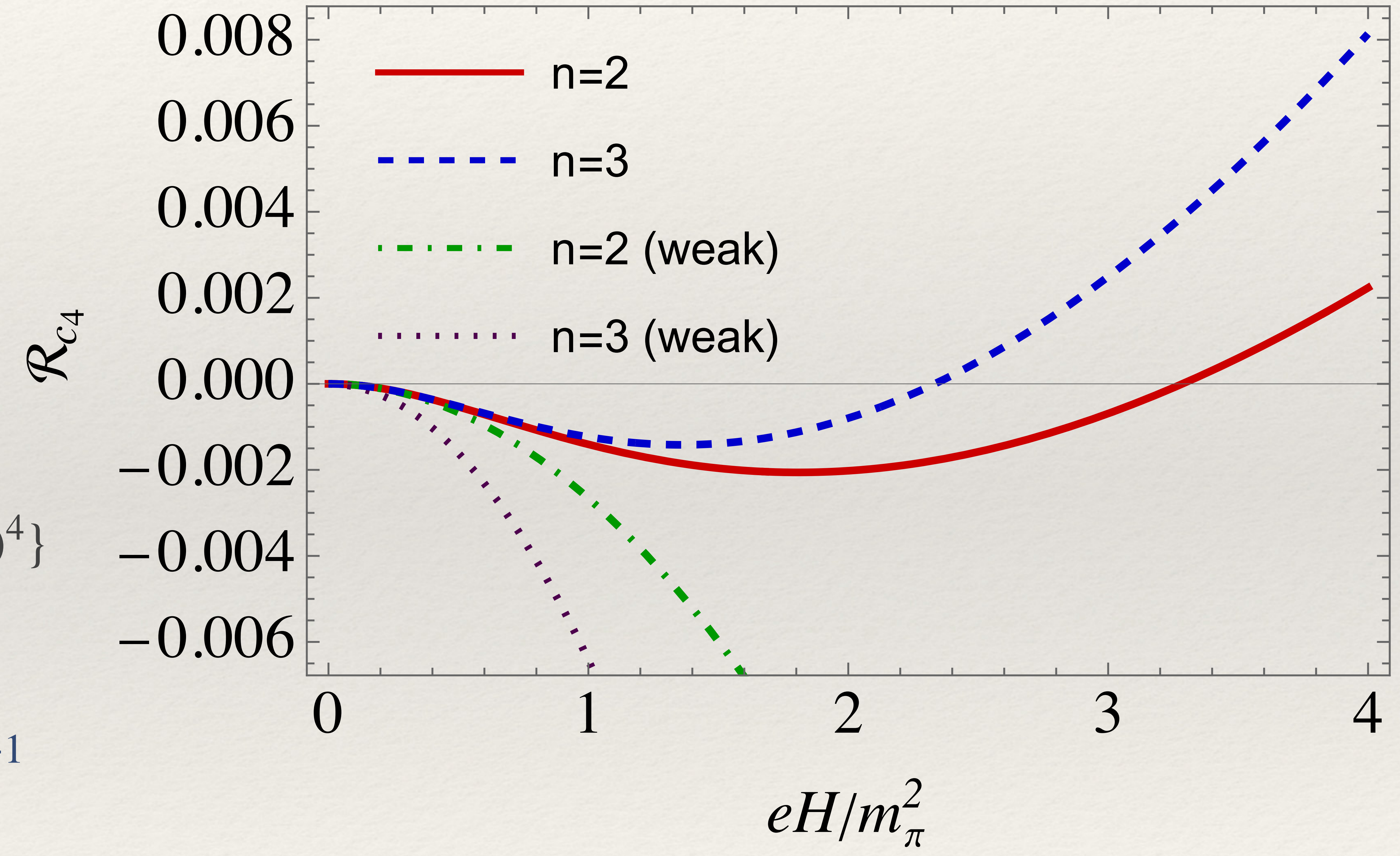
$$\chi_{t,H} = -m_{q_f}\frac{\langle \bar{q}_f q_f \rangle_H}{n}$$



Fourth Cumulant/Sum Rule II

- The fourth cumulant is negative for weak fields and positive for larger fields

$$c_{4,H} = B_0\,\overline{m}^4\left(\frac{1}{m_u^3}+\frac{1}{m_d^3}\right)\int_0^\infty \frac{ds}{(4\pi s)^2}e^{-m_\pi^2 s}\left[\frac{eHs}{\sinh eHs}-1\right]$$
$$-3B_0^2\,\overline{m}^2\int_0^\infty \frac{ds}{(4\pi)^2 s}e^{-m_\pi^2 s}\left[\frac{eHs}{\sinh eHs}-1\right]$$
$$=\left[-B_0\,\overline{m}^4\left(\frac{1}{m_u^3}+\frac{1}{m_d^3}\right)+\frac{3B_0^2\overline{m}^2}{m_\pi^2}\right]\frac{1}{6(4\pi m_\pi)^2}(eH)^2+\mathcal{O}\{(eH)^4\}$$



$$c_{4,H} = \overline{m}^4\sum_{q_f=u,d}\frac{\langle \overline{q}_f q_f\rangle_H}{m_{q_f}^3}+3\overline{m}^2\chi_{q_f,H} \qquad \overline{m}=\left(\frac{1}{m_u}+\frac{1}{m_d}\right)^{-1}$$

Finite Volume QCD in a Uniform Magnetic Field

NB. All indices are Euclidean.

Choice of Gauge. $A_\mu = (0, -Bx_2, 0, 0)$

Finite Volume QCD in magnetic field

- ChPT is a long-energy description of QCD
- Periodic boundary conditions are a conventional choice
- Gauge fields are periodic only upto a gauge-transformation

$$A_\mu = (0, -Bx_2, 0, 0) \qquad A_\mu(x + L_2\hat{x}_2) = A_\mu(x) + (0, -BL_2, 0, 0) \equiv A_\mu(x) + \partial_\mu \Lambda$$

- Wilson Lines & Flux quantization

$$W_1(x_2) = e^{-ieHL_1x_2}$$

$$\phi(x + L_1\hat{x}_1 + L_2\hat{x}_2) = \phi(x + L_1\hat{x}_1) = W_1(x_2)\phi(x)$$
$$\phi(x + L_2\hat{x}_2 + L_1\hat{x}_1) = W_1(x_2 + L_2)\phi(x + L_2\hat{x}_2) = W_1(x_2 + L_2)\phi(x)$$

$$eH = \frac{2\pi N_\Phi}{L_1 L_2}$$

Finite Volume Quark Condensate

- The infinite volume Green's function is *not* translationally invariant

$$(-D'_\mu D'_\mu + \mathring{m}_\pi^2) G_\infty(x', x) = \delta^{(4)}(x' - x)$$

$$G_\infty(x', x) = e^{ieH\Delta x_1 \bar{x}_2} \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s^2} \frac{eHs}{\sinh eHs} e^{-\mathring{m}_\pi^2 s} \exp\left[-\frac{eH\left(\Delta x_1^2 + \Delta x_2^2\right)}{4 \tanh eHs} - \frac{\Delta x_0^2 + \Delta x_3^2}{4s} \right]$$

- The finite volume Green's function is *also not* translationally invariant but worse, *perhaps more fun!*

$$G_+(x', x) = \sum_{\nu_\mu} [W_2^\dagger(x_1)]^{\nu_2} G_\infty(x' + \nu_\mu L_\mu, x) \qquad W_2(x_1) = e^{+ieHL_2 x_1}$$

Finite Volume Quark Condensate

- The quark condensate is therefore spatially variant

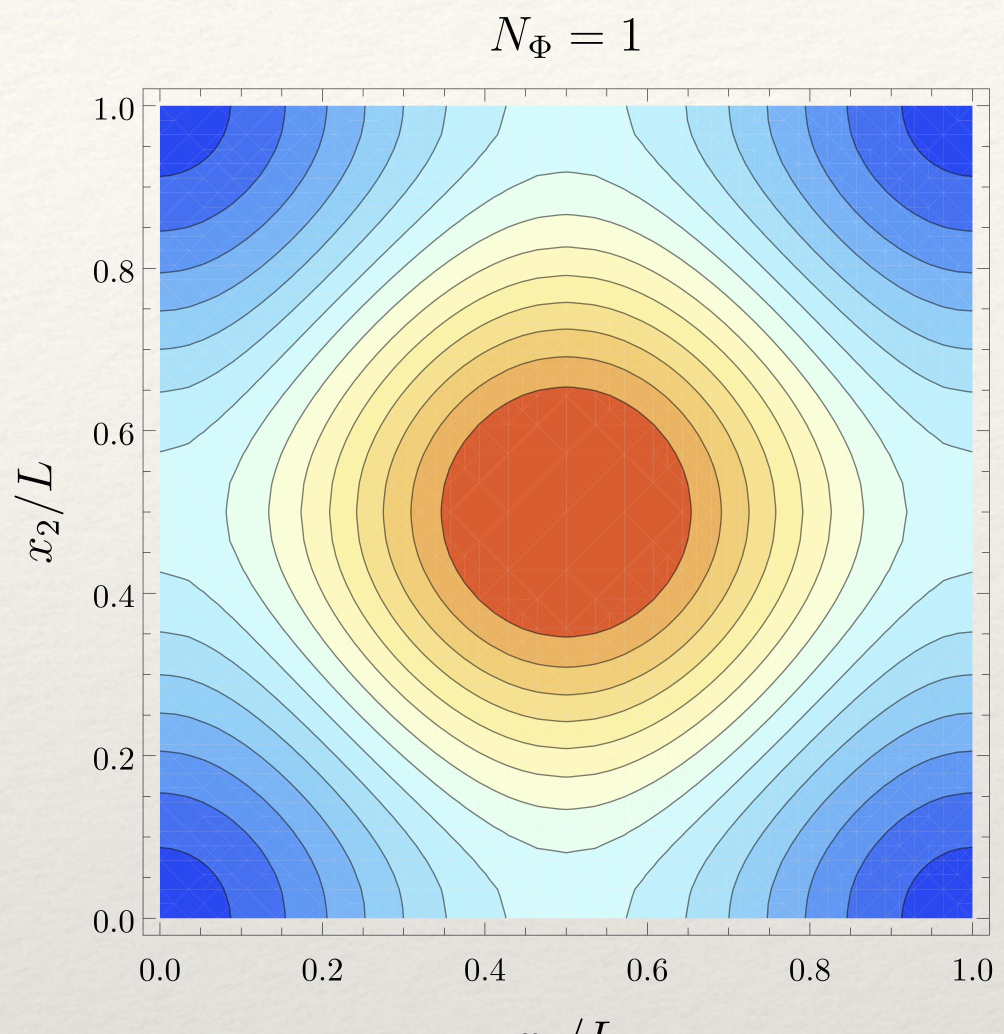
$$\mathscr{L}_{\text{mass}} = \frac{F^2}{4}\text{Tr}[\chi^\dagger \Sigma + \chi \Sigma^\dagger] = = 2B_0 m_q \left(F^2 - \frac{1}{2}\pi^0\pi^0 - \pi^-\pi^+ + \cdots \right)$$

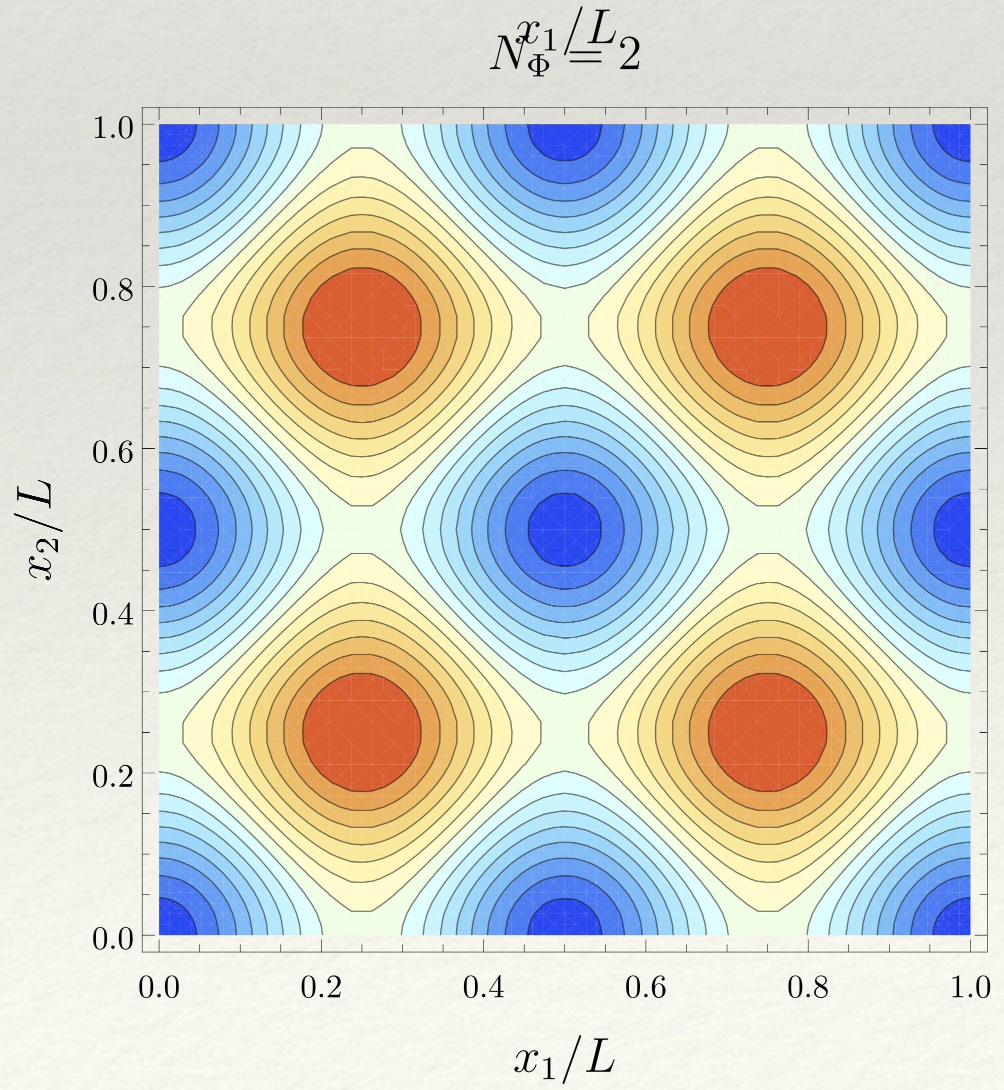
$$\langle \overline{\psi}(x)\psi(x) \rangle = -2B_0[F^2 - \tfrac{1}{2}G_0(0,0) - G_+(x,x)]$$

$$\langle \overline{\psi}\psi \rangle_0^\infty = -2B_0 F^2$$

- For asymptotically large boxes

$$R(x_\perp) = \frac{\langle \overline{\psi}(x)\psi(x)\rangle - \langle \overline{\psi}\psi\rangle^\infty}{\langle \overline{\psi}\psi\rangle_0} = -\left[\frac{5}{2} + \cos\left(\frac{2\pi N_\Phi x_1}{L}\right) + \cos\left(\frac{2\pi N_\Phi x_2}{L}\right)\right] \times \frac{m_\pi^2}{F_\pi^2}\frac{e^{-m_\pi L}}{(2\pi m_\pi L)^{3/2}}$$





Finite Volume Quark Condensate

- The quark condensate is therefore spatially variant

$$\mathscr{L}_{\text{mass}} = \frac{F^2}{4}\text{Tr}[\chi^\dagger\Sigma + \chi\Sigma^\dagger] = = 2B_0 m_q\left(F^2 - \tfrac{1}{2}\pi^0\pi^0 - \pi^-\pi^+ + \cdots\right)$$

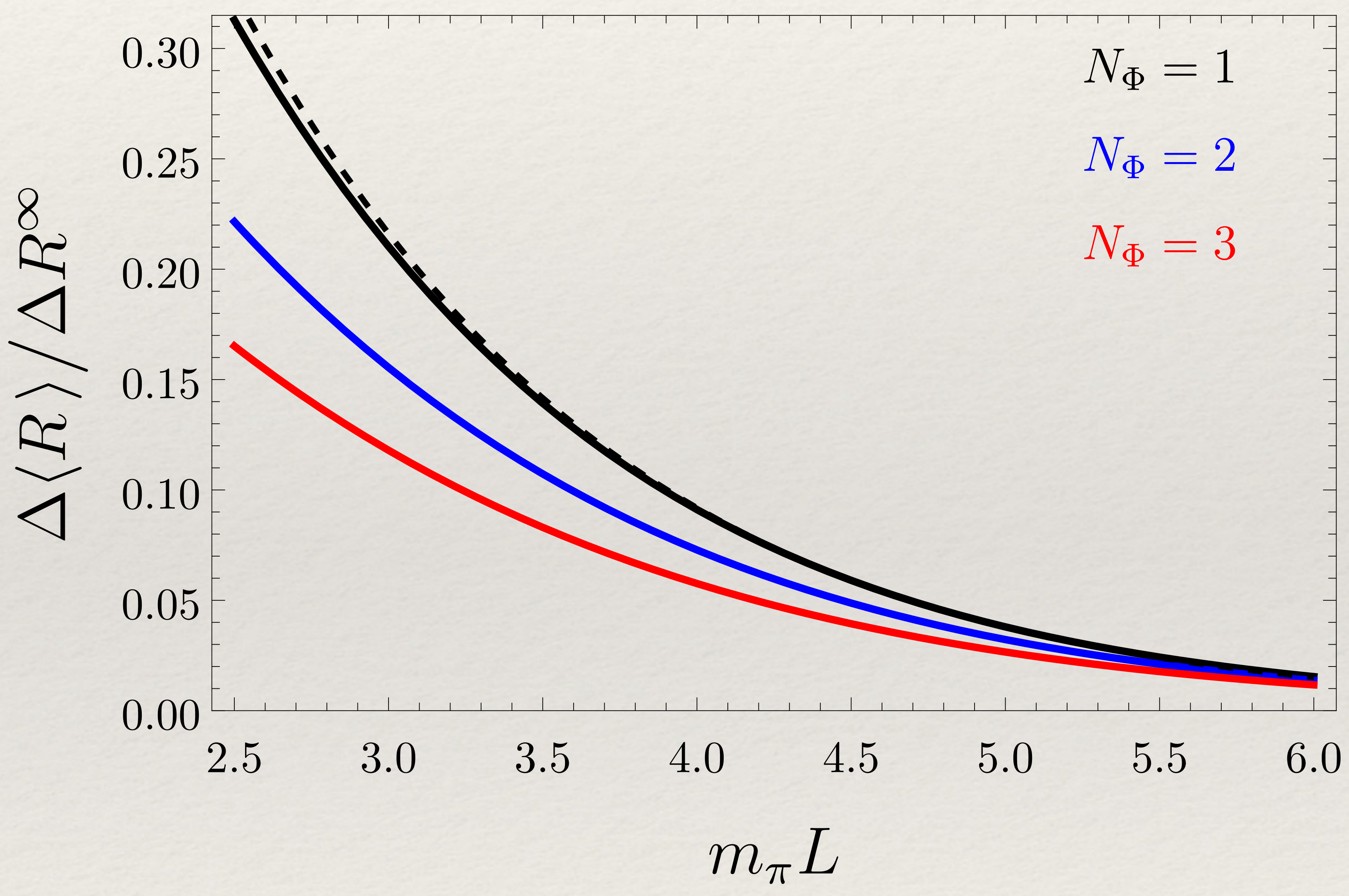
$$\langle\overline{\psi}(x)\psi(x)\rangle = -2B_0[F^2 - \tfrac{1}{2}G_0(0,0) - G_+(x,x)]$$

$$\langle\overline{\psi}\psi\rangle_0^\infty = -2B_0F^2$$

- Spatially Averaged Condensate

$$\frac{\Delta\langle R\rangle}{\Delta R^\infty} = \frac{\langle R\rangle - \langle R\rangle_{B=0}}{R^\infty - R^\infty_{B=0}}$$

$$= e^{-m_\pi L}\sqrt{2\pi m_\pi L}\left[1 - \frac{13}{8}\frac{1}{m_\pi L} + \mathcal{O}(\frac{1}{\sqrt{m_\pi L}}) + \mathcal{O}(e^{-m_\pi L}) + \mathcal{O}(\frac{N_\Phi^2}{(m_\pi L)^2}) + \cdots\right]$$



Conclusion

- Topological cumulants in a magnetic field calculated through ChPT are *model-independent* with small systematic corrections *[PA, Phys. Lett. B 825 (2021), Nucl. Phys. B 974 & 982 (2022)]*
  - For weak fields, higher order calculations are not necessary since these typically introduce large uncertainties through LECs
- Finite volume lattice corrections to the chiral condensate is approximately 5% for $m_\pi L = 5$
  - The large size of the correction is due to finite volume effects entering at the same order as finite magnetic field effects *[PA, Tiburzi, Phys. Rev. D,* ***107****, 094504 (2023)]*
  - The volume effect is worse for observables that are small at weak fields such as magnetization and magnetic susceptibility