Anomalous transport coefficients from lattice QCD

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- Expectations
- Lattice QCD formulation
- Results
- Summary

Introduction

- Discuss two anomalous transport coefficients:

- Chiral Magnetic Effect $\equiv$ **CME**,
  - an **electric current** in the presence of **chiral imbalance** and a **magnetic field**,
  - parallel to the magnetic field:

$$\langle J_z \rangle = \left\langle \bar{\psi}\gamma_z\psi \right\rangle = c_{\text{CME}}\, \mu_5\, qB_z\,.$$

- Chiral Separation Effect $\equiv$ **CSE**,
  - a **chirality current** in the presence of **charge imbalance** and a **magnetic field**,
  - parallel to the magnetic field:

$$\left\langle J_z^5 \right\rangle = \left\langle \bar{\psi}\gamma_5\gamma_z\psi \right\rangle = c_{\text{CSE}}\, \mu\, qB_z\,.$$

- Derivatives of the currents yield the **coefficients**:

$$c_{\text{CME}} = \left.\frac{d^2 \langle J_z \rangle}{d\mu_5 dB_z}\right|_{\mu_5=B=0}\,, \quad c_{\text{CSE}} = \left.\frac{d^2 \left\langle J_z^5 \right\rangle}{d\mu dB_z}\right|_{\mu=B=0}\,.$$

Introduction

- Using **lattice QCD**:
- The **partition function** is

$$Z = \int \mathcal{D}\mathcal{U}\mathcal{D}\bar{\psi}\mathcal{D}\psi \, \mathrm{e}^{\, iS[\mathcal{U},\bar{\psi},\psi]} = \int \mathcal{D}\mathcal{U}\mathcal{D}\bar{\psi}\mathcal{D}\psi \, \mathrm{e}^{\, -S_E[\mathcal{U},\bar{\psi},\psi]} \, ,$$

- where S_E is the **Wick-rotated**, **finite temperature** action of QCD

$$S_E = \int_0^{1/T} d\tau \int d^3x \frac{\mathrm{Tr}\, F^2(x,\tau)}{2g^2} + \sum_f \bar{\psi}^{(f)}(x,\tau) \left(\not{D} + m^{(f)} \right) \psi^{(f)}(x,\tau) \, .$$

- This is inherently **equilibrium**!
- **Observables** are

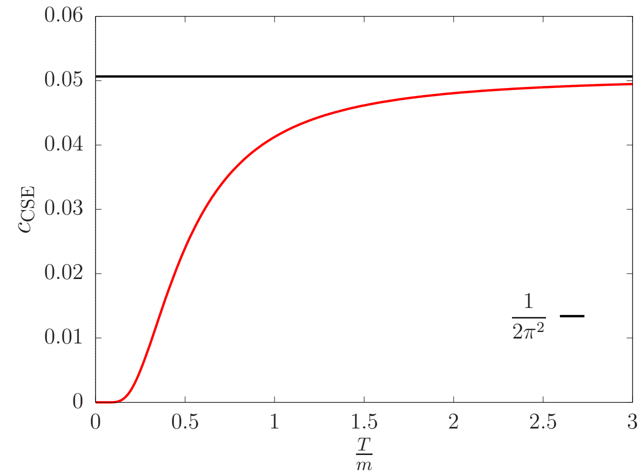
$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\mathcal{U}\mathcal{D}\bar{\psi}\mathcal{D}\psi \, \mathrm{e}^{\, -S_E[\mathcal{U},\bar{\psi},\psi]} O[\mathcal{U},\bar{\psi},\psi] \, .$$

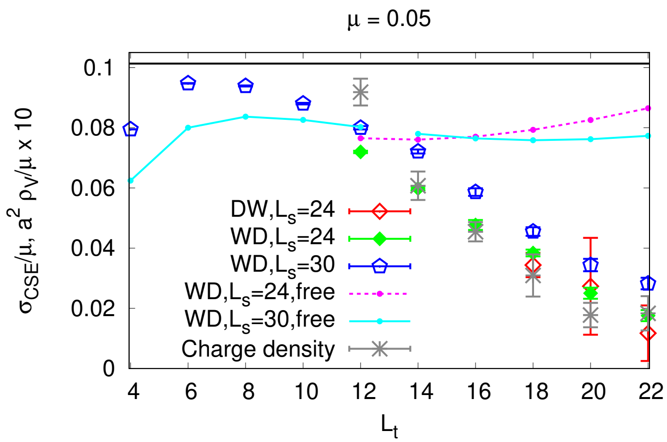
- Discretize the action, with **lattice spacing** a, and use **importance sampling Monte Carlo** integration to evaluate the path integral.

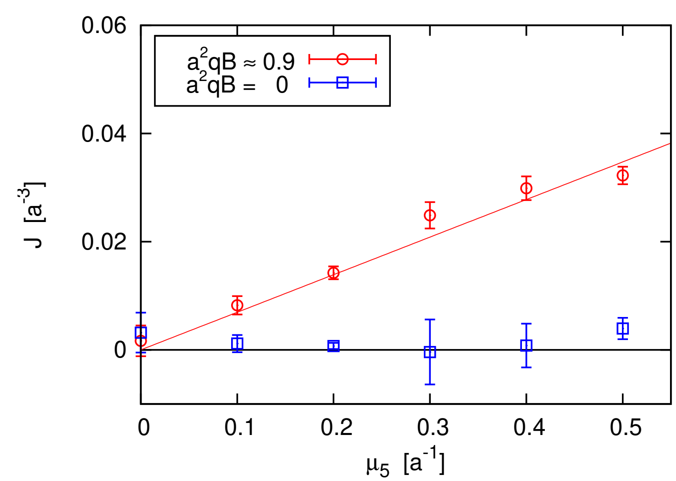
Expectations

- Perturbative calculations
  - Gluonic **interactions neglected**, except as source of chiral imbalance.
  - c_{CSE} interpolates between 0 and $1/(2\pi^2)$ as T/m goes from $0 \to \infty$. [1]
  - c_{CME} is calculated in- and out-of-equilibrium, different results 0 or $1/(2\pi^2)$. [2, 3, 4]
  - c_{CME} is sensitive to proper **regularization**! [5]

- Very few lattice results
  - c_{CSE} in $\mathrm{QC_2D}$, **compatible** with perturbative results. [6]
  - c_{CME} QCD with 2, identical flavor, Wilson fermions, **neither compatible** with 0 NOR $1/(2\pi^2)$. [7]



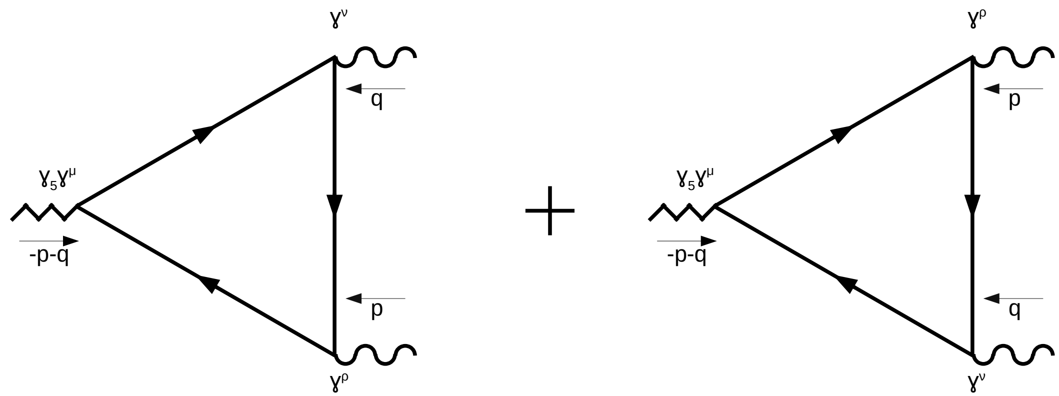




Expectations: regulator sensitivity

Text book example: the triangle anomaly

$$\Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) =$$



- Massive fermions

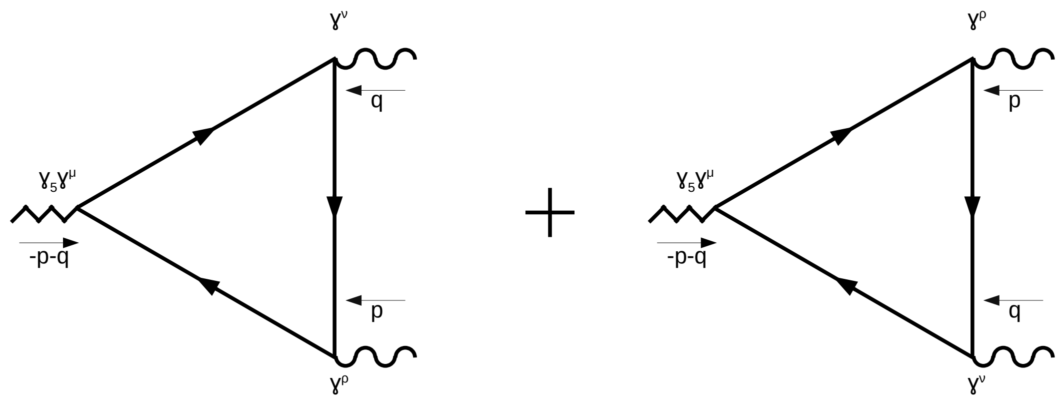
- No regularization

$$(p+q)_\mu \Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) = mP_5^{\nu\rho}(p,q)$$

Expectations: regulator sensitivity

Text book example: the triangle anomaly

$$\Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) =$$



- Massive fermions

- Pauli-Villars regularization

- New particles, with coeffs c_s and masses $m_s \to \infty$, $s = 0, 1, 2, 3$, $s = 0$ the physical fermion.

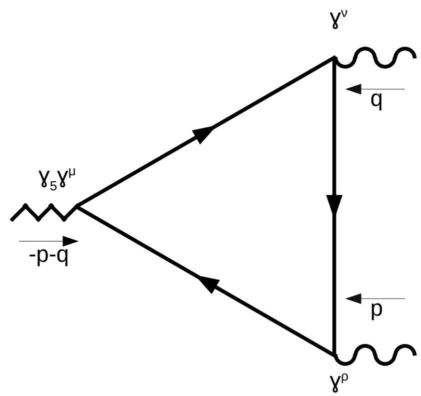
$$(p+q)_\mu \Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) = m P_5^{\nu\rho}(p,q) + \sum_{s=1} c_s m_s P_{5,s}^{\nu\rho}(p,q)$$

$$\to m P_5^{\nu\rho}(p,q) + \frac{\varepsilon^{\alpha\beta\nu\rho} q_\alpha p_\beta}{4\pi^2}$$

Expectations: regulator sensitivity

- $c_{\mathrm{CME/CSE}}$ can also be written with the triangle diagram:

$$\Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) =$$



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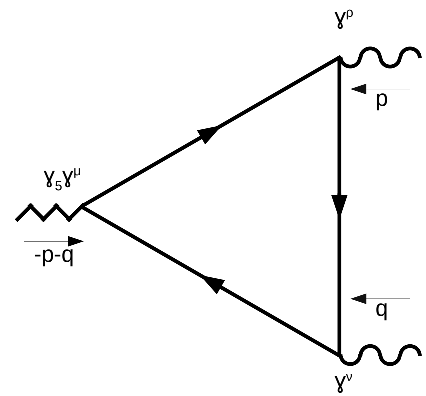


$$J_3 \sim A_3\,, \quad J_3^5 \sim A_3^5\,, \quad \boldsymbol{B}_3 = q_1 A_2\,, \quad \mu = A_0\,, \quad \mu_5 = A_0^5\,.$$

$$c_{\mathrm{CME}} = \lim_{p,q,p+q\to 0} \frac{1}{q_1}\Gamma^{023}(p+q,q,p)\,,$$

$$c_{\mathrm{CSE}} = \lim_{p,q,p+q\to 0} \frac{1}{q_1}\Gamma^{320}(p+q,q,p)\,.$$

Expectations: regulator sensitivity

$$\Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) = -\,i\sum_{s=0} c_s \int_K \frac{\mathrm{Tr}\,\left[\gamma^\mu\gamma_5(\not{K}+m_s)\gamma^\nu(\not{K}+\not{q}+m_s)\gamma^\rho(\not{K}+\not{q}+\not{p}+m_s)\right]}{(K^2-m_s^2)((K+q)^2-m_s^2)((K+q+p)^2-m_s^2)}$$

$$+\,(\{\nu,q\}\leftrightarrow\{\rho,p\})\,.$$

$$\mu_5 = A_0^5(p+q=0)\,,\quad A_2(q_0=0,\boldsymbol{q}=(q_1,0,0)) = \frac{B}{q_1} \quad\Rightarrow \Gamma^{023}_{AVV}(0,-q_1,q_1)$$

Evaluating the trace and writing out the Matsubara sum

$$\Gamma^{023}_{AVV}(0,-q,q) = -8\varepsilon^{1230}\sum_{s=0} c_s T\sum_n \int \frac{d^3k}{(2\pi)^3}\left[\frac{2q_1 m_s^2 + 2k_0(q_0k_1 - q_1k_0)}{(K^2-m_s^2)^2((K+q)^2-m_s^2)} + \frac{k_1+q_1}{(K^2-m_s^2)((K+q)^2-m_s^2)}\right]_{k_0=i\omega_n}$$

Take $q_0 \to 0$ and evaluate the Matsubara sum as well as the angle integrals

$$\Gamma^{023}_{AVV}(0,-q_1,q_1) = -\frac{1}{2\pi^2}\sum_{s=0} c_s \int_0^\infty dk\,k\left(\frac{m_s^2(\frac{1}{2}-n_F(E_k))}{E_k^3} + \frac{k^2}{E_k^2}n'_F(E_k)\right)\log\frac{(2k-q_1)^2}{(2k+q1)^2}\,.$$

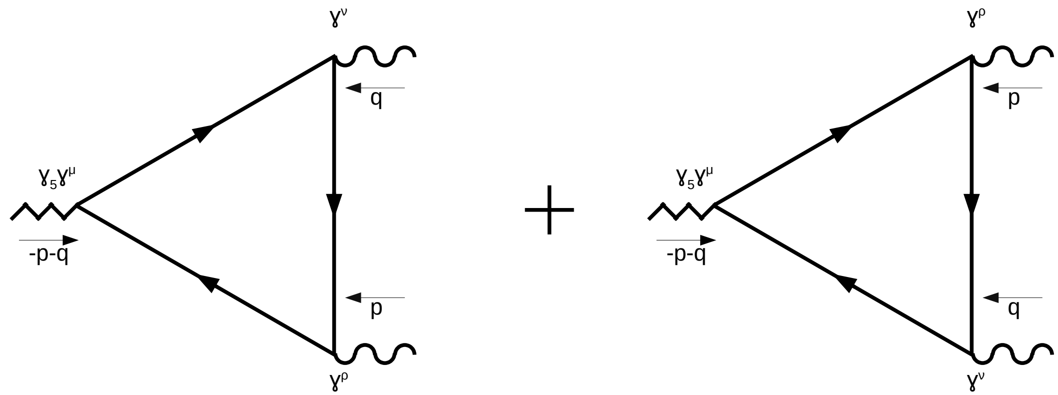
$$c_{\mathrm{CME}} = \lim_{q_1\to 0}\frac{\Gamma^{023}_{AVV}(0,-q_1,q_1)}{q_1} = \frac{1}{2\pi^2}\sum_{s=0} c_s m_s^2 \underbrace{\int_0^\infty dk \frac{1}{(k^2+m_s^2)^{3/2}}}_{1/m_s^2} + \underbrace{(T\neq 0)}_{\text{cancels!}}$$

$$= \frac{1}{2\pi^2}\Big(1+\underbrace{\sum_{s=1} c_s}_{-1}\Big) = 0\,, \qquad \text{in agreement with [4]}\,.$$

Expectations: regulator sensitivity

- $c_{\mathrm{CME/CSE}}$ can also be written with the triangle diagram:

$$\Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) =$$



$$J_3 \sim A_3\,, \quad J_3^5 \sim A_3^5\,, \quad \boldsymbol{B}_3 = q_1 A_2\,, \quad \mu = A_0\,, \quad \mu_5 = A_0^5\,.$$

$$c_{\mathrm{CME}} = \lim_{p,q,p+q\to 0} \frac{1}{q_1}\Gamma^{023}(p+q,q,p) = \frac{1}{2\pi^2} + \sum_{s=1} \frac{c_s}{2\pi^2} = 0\,,$$

$$c_{\mathrm{CSE}} = \lim_{p,q,p+q\to 0} \frac{1}{q_1}\Gamma^{320}(p+q,q,p) = -\frac{1}{\pi^2}\int_0^\infty dk\, n_F'(E_k)\,.$$

- c_{CME} is zero due to anomalous contribution!

- c_{CSE} agrees with known results [1].

Lattice QCD formulation

- We can simulate in homogeneous B background, but not at finite μ.
- Measure derivatives of the currents, at different B-s and read off linear coefficient.
- For completeness, at small B

$$c_{\mathrm{CME}} B = \left. \frac{d \left\langle J_z \right\rangle}{d\mu_5} \right|_{\mu_5=0} = \left\langle J_z J_0^5 \right\rangle_{\mu_5=0} \stackrel{?}{+} \left\langle \frac{\partial J_z}{\partial \mu_5} \right\rangle_{\mu_5=0},$$

$$c_{\mathrm{CSE}} B = \left. \frac{d \left\langle J_z^5 \right\rangle}{d\mu} \right|_{\mu=0} = \left\langle J_z^5 J_0 \right\rangle_{\mu=0} \stackrel{?}{+} \left\langle \frac{\partial J_z^5}{\partial \mu} \right\rangle_{\mu=0}.$$

- First, clarify that definitions are correct in the free case.
- Then, turn on gluonic interactions.

Lattice QCD formulation

- Regularization is important, **anomalous contributions**:

$$\text{divergence} \times (\text{cutoff} - \text{suppressed}) = \text{finite}$$

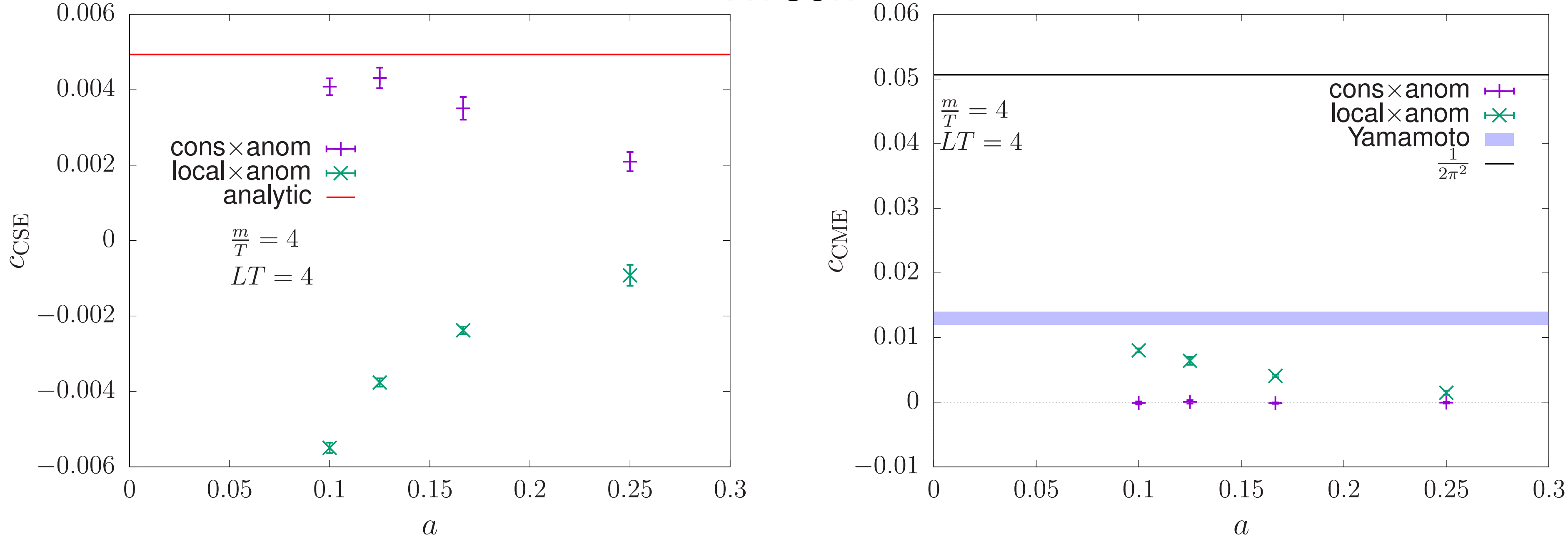
- Appears differently in different fermion discretizations.

- Wilson fermions
  - So-called doublers are given a **cutoff dependent mass** to decouple in the continuum limit ($\leftrightarrow$ PV).
  - Point-split currents are defined so **(anomalous) Ward identities are fulfilled**.
  - Local currents **violate** the (A)WI. (Used *e.g.* in spectroscopy, where it does not matter.)

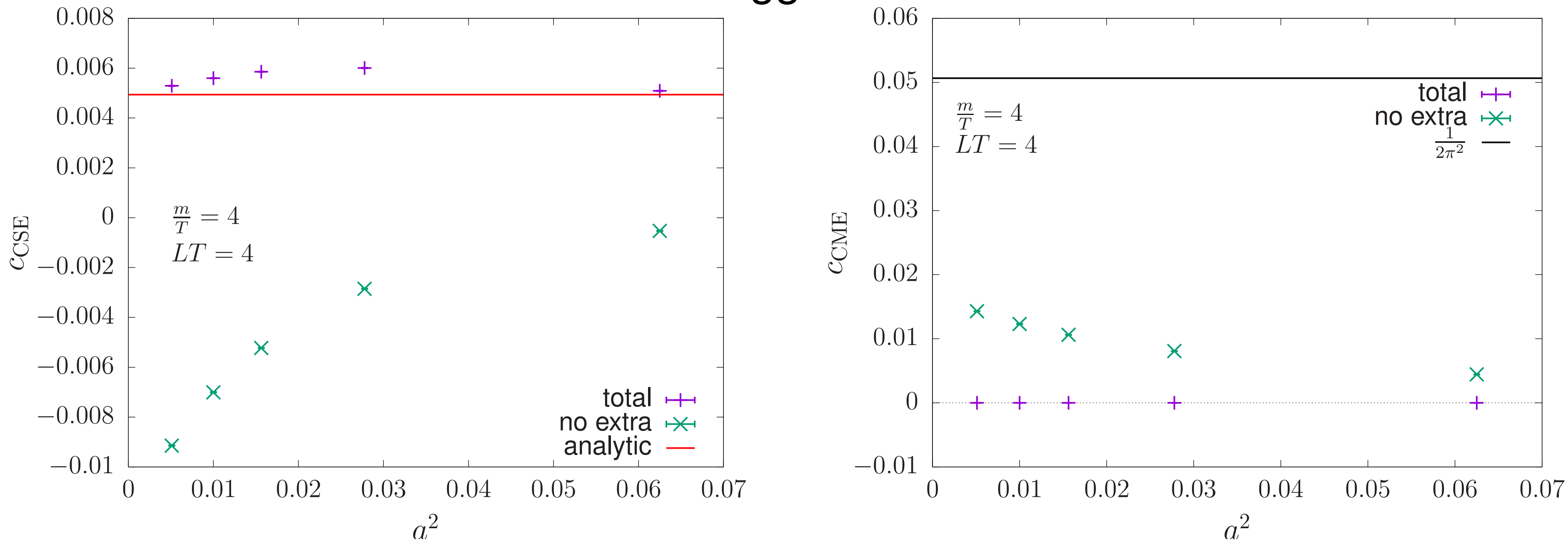
- Staggered fermions
  - Dirac and flavor structure is **mixed with coordinate dependence** to reduce doubling problem.
  - Conserved (anomalous) currents pick up **explicit (chiral) chemical potential dependence**.
  - Extra $\left\langle \frac{\partial J_z}{\partial \mu_5} \right\rangle$, $\left\langle \frac{\partial J_z^5}{\partial \mu} \right\rangle$ terms appear!

Lattice QCD formulation: free results

Wilson



Staggered



Lattice QCD formulation: chirality

Naturally

$$n_5\Big|_{\mu_5=0} = \frac{T}{V}\frac{d\log Z}{d\mu_5}\Big|_{\mu_5=0} = 0\,.$$

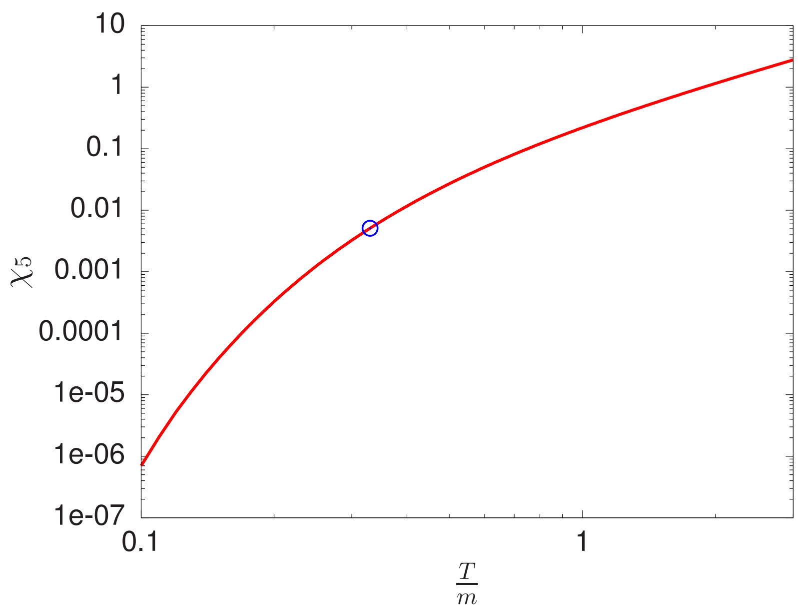
But

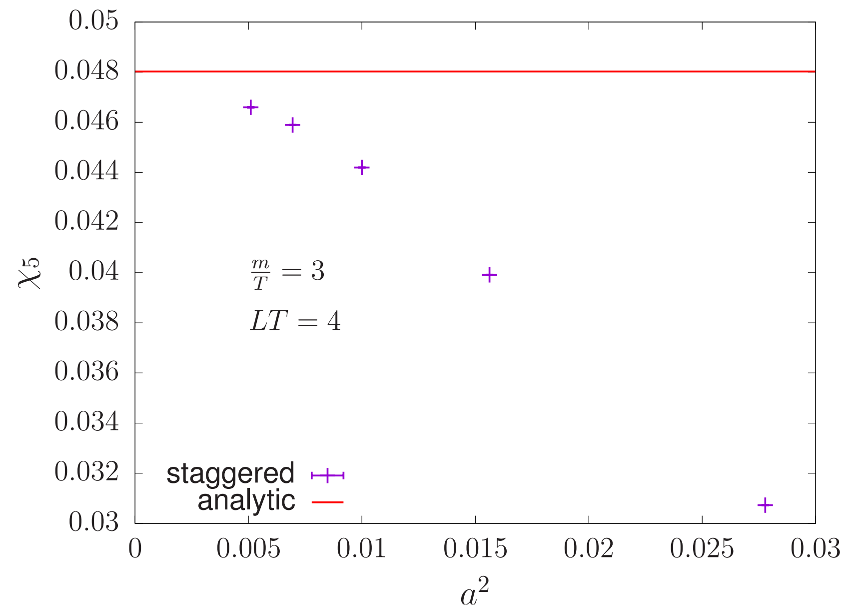
$$n_5(\mu_5) = \frac{T}{V}\frac{d^2\log Z}{d\mu_5^2}\Big|_{\mu_5=0}\mu_5 + \mathcal{O}(\mu_5^2) = \chi_5\mu_5 + \mathcal{O}(\mu_5^2)\,.$$

In PV

$$\chi_5 = -\frac{1}{32\pi^2}\sum_{s=0} c_s m_s^2\left(2\log\frac{m_s^2}{m^2} - 8\right) + \frac{4}{\pi^2}\int_0^\infty dk\frac{k^2\, n_F(E_k)}{E_k}\,.$$

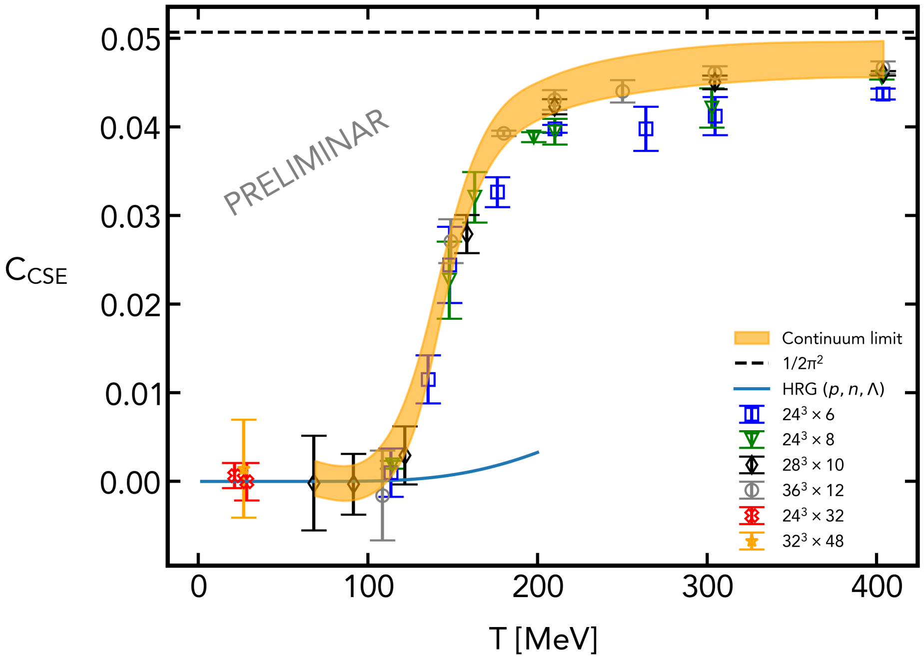
Divergent, renormalization needed, here by $T=0$ subtraction for easy comparison to lattice

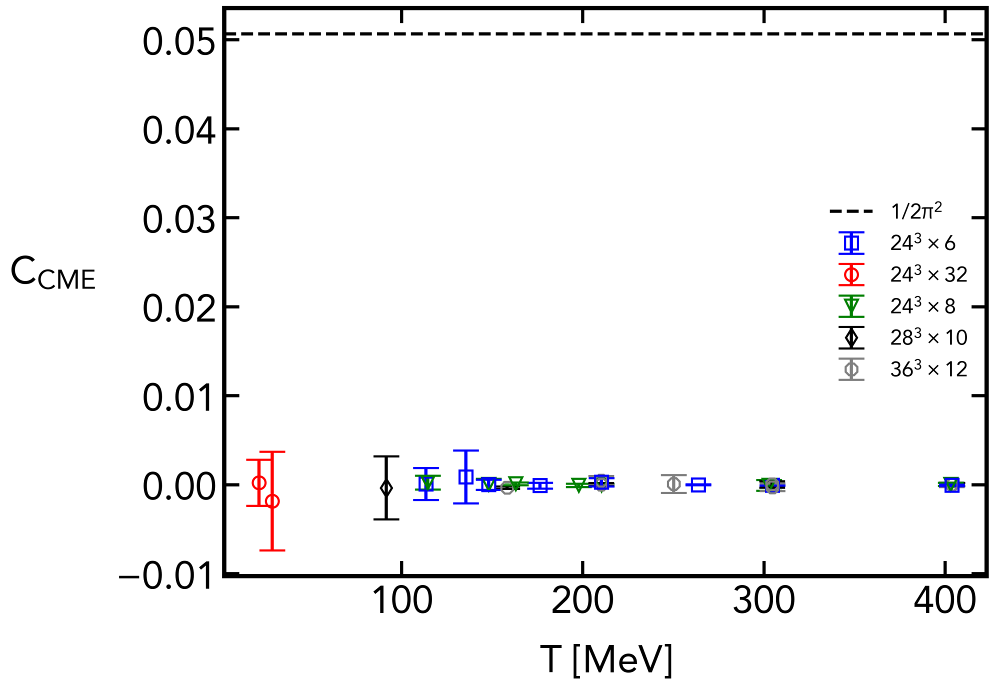




Lattice QCD formulation: interacting results

2+1 flavor, staggered, physical pion mass

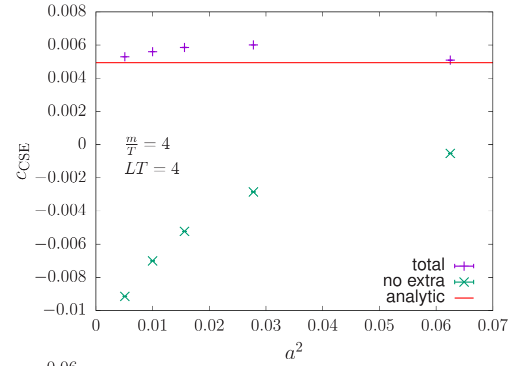


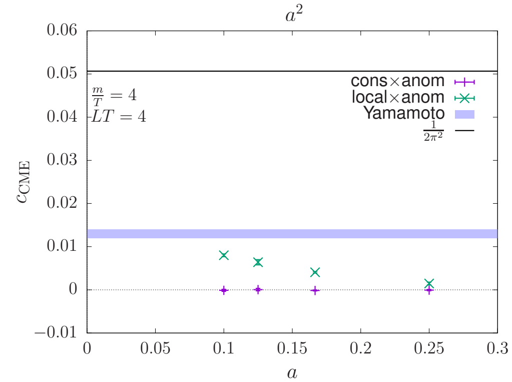


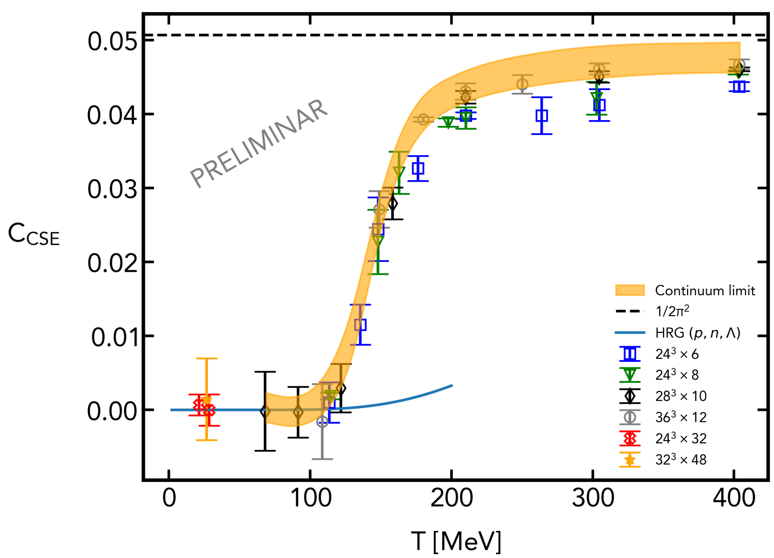
- c_{CSE} changes quickly around the transition temperature.
- Approaches the free limit at high temperature.
- c_{CME} temperature independently vanishes.

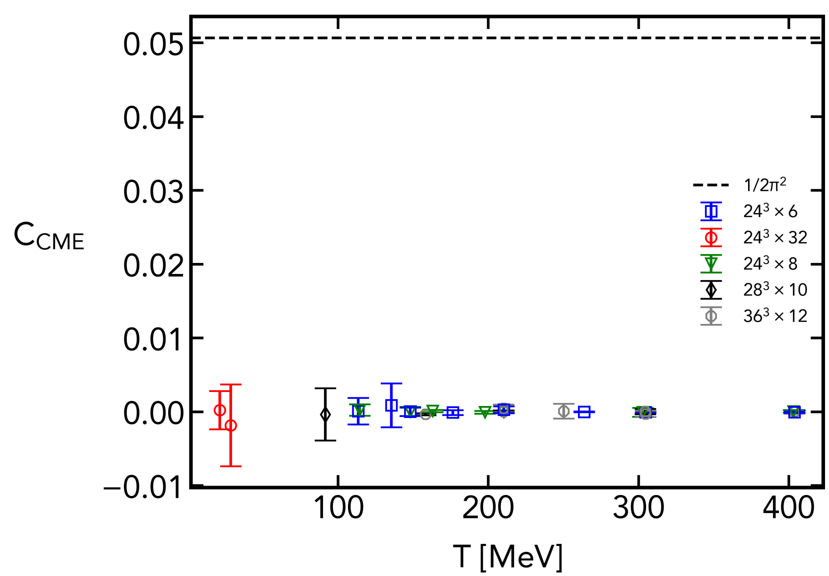
Summary

- Free results show sensitivity to proper regularization.
- It looks plausible that interacting results also suffer from this.
- We presented the first 2+1 flavor, physical point, full QCD results for c_{CSE} and c_{CME}.









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