Magnetic susceptibility of QCD matter

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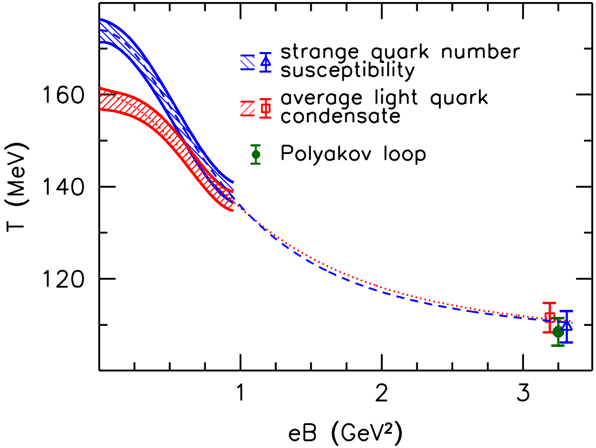
QCD phase diagram at zero density (continuum limit)



[G Endrődi, 1504.08280]

$eB = 1\,\text{GeV}^2 \approx 1.69 \cdot 10^{20}\,e\text{G} \approx (1.16 \cdot 10^{13}\text{K})^2$ $\quad (e = \sqrt{4\pi\alpha_{\text{em}}} \curvearrowright B \approx 3.3\,eB)$.

RHIC $eB < 0.04$ GeV^2, ALICE $eB < 0.3$ GeV^2. Remains a crossover at least up to $3\,\text{GeV}^2$.

Paramagnetism and diamagnetism

B: magnetic flux density (colloquially: magnetic field), H: magnetic field strength.

$$B = H + M = (1 + \chi_m)H = \mu H$$

Liquid oxygen: $\chi_m \approx 0.004$

Harvard Natural Science Demonstrations
https://sciencedemonstrations.fas.harvard.edu/presentations/paramagnetism-oxygen

Water: $\chi_m \approx -0.000009$

Nijmegen High Field Magnetic Laboratory
https://www.ru.nl/hfml/research/levitation-explained/diamagnetic-levitation

The magnetic susceptibility

free energy density

$$f_{\mathrm{tot(al)}}(B) = f_{b(\mathrm{are})}(B) + \frac{B_b^2}{2}\,, \qquad f_b(B) = -p(B) = -\frac{T}{V_3}\log\mathcal{Z}$$

magnetization

$$\mathcal{M}_b = -\frac{\partial f_b}{\partial(eB)}\left(=\frac{M}{e}\right)\,, \quad \mathcal{M}|_{B=0} = 0$$

susceptibility

$$\chi_b = \frac{\partial \mathcal{M}_b}{\partial(eB)}\bigg|_{eB=0} = -\frac{\partial^2 f_b}{\partial(eB)^2}\bigg|_{B=0} \qquad \left(\chi = \frac{1}{e^2}\frac{\chi_m}{1+\chi_m} = \frac{1}{e^2}\left(1-\frac{1}{\mu}\right)\right)$$

sign distinguishes between

- renormalized $\chi > 0$: field reduces f, paramagnet, attracts magnetic field lines.
- renormalized $\chi < 0$: field increases f, diamagnet, repels magnetic field lines.

Renormalization of f, $\mathcal{M}$ and χ

Electric charge renormalization $e_b \cdot B_b = e(\mu) \cdot B(\mu)$:

$$Z_e(\mu) = 1 + b_1 e^2 \log(\mu^2 a^2)\,, \quad e_b^2 = Z_e^{-1}(\mu) e^2(\mu)\,, \quad B_b^2 = Z_e(\mu) B^2(\mu)$$

(Free) energy density at zero temperature:

$$f_{\rm tot}(B) = f_b(B) + \underbrace{\frac{b_1 (eB)^2 \log(\mu^2 a^2)}{2} + \frac{B^2(\mu)}{2}}_{B_b^2/2} = \underbrace{f_b(B) + \frac{b_1 (eB)^2 \log(\mu^2 a^2)}{2}}_{\text{finite: } f(B;\mu)} + \frac{B^2(\mu)}{2}$$

B is a background field (no FF kinetic term in the simulated Lagrangian) $\Rightarrow b_2 = 0$.

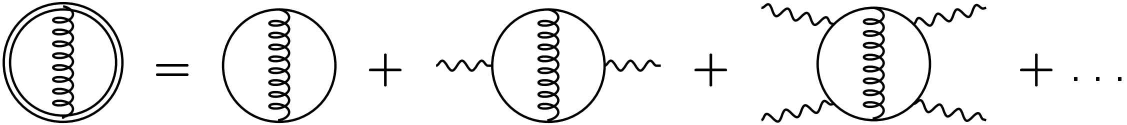
The (trivial) background energy density $B_b^2/2$ is not included in the simulated Lagrangian.

$\mu = \mu_{\rm QED} = \Lambda_{H(\text{adronic})}$.

The inverse lattice spacing a^{-1} provides the QCD UV cut-off ($\mathcal{O}[\alpha_s(a^{-1})]$ corrections to b_1).

Renormalization of f, $\mathcal{M}$ and χ II

Expand $f_b(B)$ at $T = 0$:



Similar to background field method [Abbott 81]

$\mathcal{O}(B^2)$ term: $-b_1(eB)^2 \log(\Lambda^2)/2 \Rightarrow f_{\rm tot}$ is finite [Schwinger 51]. $\mathcal{O}[(eB)^4]$ terms are finite.

Renormalization of $f_b(B)$ at $T \geq 0$:

subtract the $T = 0$ $(eB)^2$-term for $eB \ll \Lambda_H^2 \sim m_\pi^2$

($\pi^\pm$ is the lightest charged particle in the hadronic phase)

$\exists$ **no other choice** to ensure $\chi = 0 \Leftrightarrow c = 1$ in the $T = 0$ vacuum!

$f(B, T = 0) = \mathcal{O}[(eB)^4]$, $f(B, T > 0) = \mathcal{O}[(eB)^2]$.

Fit: $\Lambda_H = 115(3)(5)\,\text{MeV} \longrightarrow$ renormalized χ.

Magnetic susceptibility at very high and very low $T > 0$

Consider a **free** quark with charge q and mass m.
Using Schwinger proper time regularization:
[GB, F Bruckmann, G Endrődi, S Katz, A Schäfer 1406.0269; Elmfors et al 94]

$$\chi(T) = -N_c b_1^{\mathrm{Dirac}} \frac{q^2}{e^2} \int_0^\infty \frac{\mathrm{d}s}{s}\, e^{-m^2 s} \left\{ \Theta_3\left[\frac{\pi}{2}, e^{-1/(4sT^2)}\right] - 1 \right\} \xrightarrow{T\to\infty} +N_c b_1^{\mathrm{Dirac}} \frac{q^2}{e^2} \log\left(\frac{T^2}{m^2}\right)$$

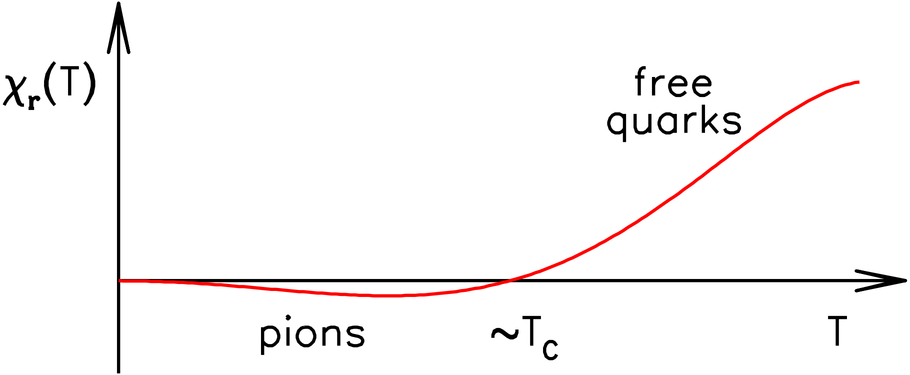
QED is not asymptotically free: $b_1^{\mathrm{Dirac}} = 1/(12\pi^2) > 0$
$\Rightarrow$ Free quarks at high T are paramagnetic.

Consider a pion of mass m and charge $\pm e$.

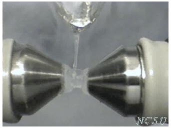
$$\chi^{\mathrm{pion}}(T) = -b_1^{\mathrm{scalar}} \underbrace{\int_0^\infty \frac{\mathrm{d}s}{s}\, e^{-m^2 s} \left\{ \Theta_3\left[0, e^{-1/(4sT^2)}\right] - 1 \right\}}_{\text{finite and positive}}$$

with $b_1^{\mathrm{scalar}} = b_1^{\mathrm{Dirac}}/4 > 0$. Sign change due to fermion $\mapsto$ boson.
$\Rightarrow$ QCD at small $T > 0$ is diamagnetic.

Expectation for the renormalized susceptibility



This is too simplistic since quarks only become "free" at $T \ggg T_c$.



Observables

- Partition function for three flavors

$$\mathcal{Z} = \int [dU]\, e^{-\beta S_g} \prod_{f=u,d,s} \left[\det M_f(q_f \cdot B, m_f)\right]^{1/4}, \qquad M_f = (\not{D}_f + m_f)\left(1 + \mathcal{O}(a^2)\right)$$

- Chiral condensates

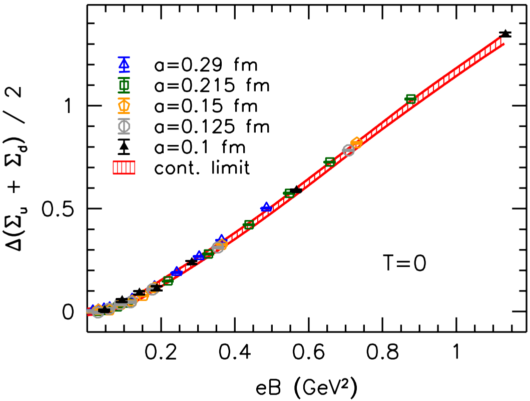
$$\bar{\psi}\psi_f = \frac{T}{V}\frac{\partial \log \mathcal{Z}}{\partial m_f} = -\frac{\partial}{\partial m_f} f_b$$

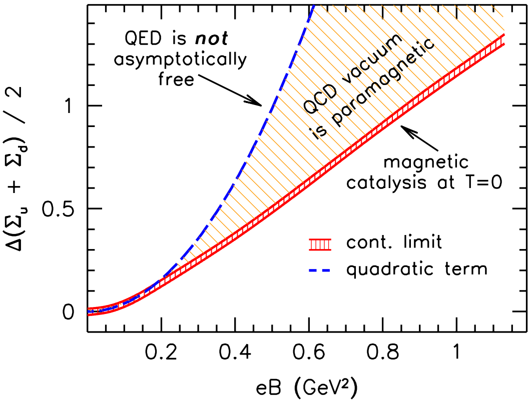
- Cancel additive QCD divergences of $\bar{\psi}\psi$ by computing

$$\Sigma_{u,d}(B,T) = \frac{2m_{ud}}{M_\pi^2 F^2}\left[\bar{\psi}\psi_{u,d}(B,T) - \bar{\psi}\psi_{u,d}(0,0)\right] + 1\,,$$
$$\Delta\Sigma_{u,d}(B,T) = \Sigma_{u,d}(B,T) - \Sigma_{u,d}(0,T)\,.$$

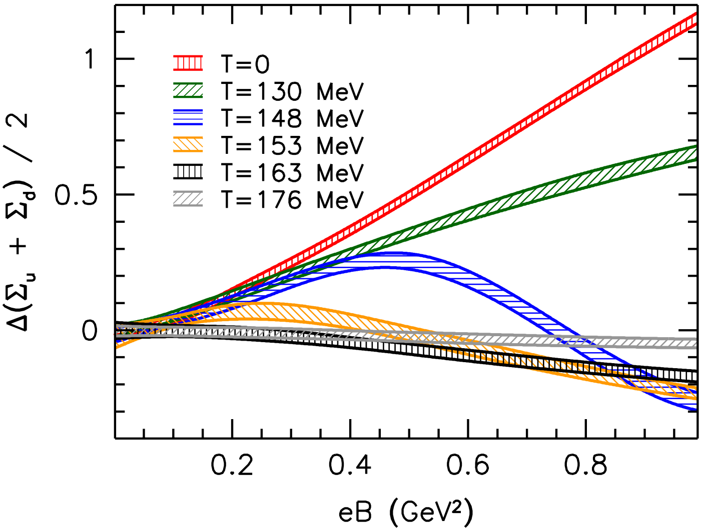
Note that $\Delta\Sigma(B,T)$ still contains a term $\propto B_b^2$ ($T=0$ uncancelled).
Also $\Delta f_b = f_b(B) - f_b(0)$ is not yet "QED-renormalized".

Magnetic catalysis at $T = 0$ (continuum limit)





Inverse magnetic catalysis ($T \geq 0$, continuum limit)



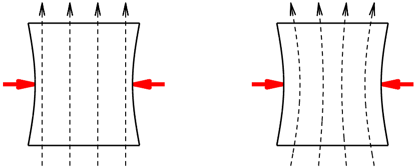
Thermodynamics in an external magnetic field

Pressure $\quad p = \frac{T}{V} \log \mathcal{Z} = -f_b \quad \Longrightarrow \quad$ equation of state

Interaction measure $\quad I = -\frac{T}{V} \frac{d \log \mathcal{Z}}{d \log a} = \epsilon - 3p = -\langle T_{\mu\mu} \rangle$

With magnetic field $\quad f_b = \epsilon - sT = \epsilon_{\text{tot}} - sT - \underbrace{\mathcal{M} \cdot eB}_{\epsilon^{\text{field}}}$

Is the pressure isotropic?



For qB fixed $\quad \langle T_{xx} \rangle = \langle T_{yy} \rangle = \langle T_{zz} \rangle = p$

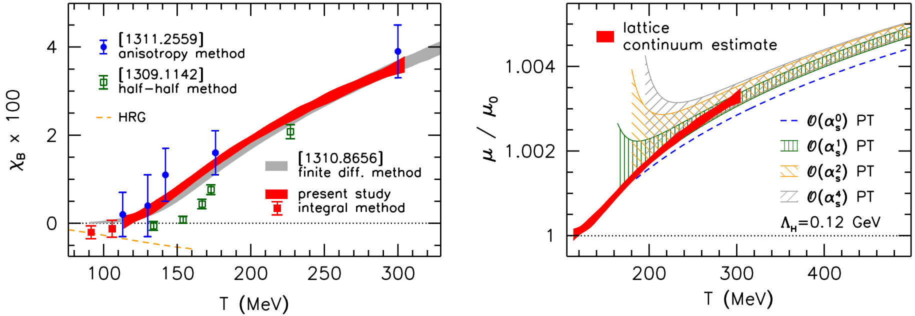
We work at fixed flux $\Phi = L_x L_y qB \quad p_z = p,\ p_x = p_y = p - \mathcal{M} \cdot eB.$

We pursued two approaches [1406.0269, 1303.1328, 1310.8145]:

generalized integral method and a microscopic determination from $T_{xx} = T_{yy}$ and T_{zz}.

Susceptibility and permeability

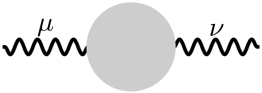
half-half: L Levkova, C DeTar, PRL 112 (14) 012002, finite diff.: C Bonati et al, PRD89 (14) 054506



Magnetic susceptibility and permeability ($f = -\frac{1}{2}(eB)^2\chi$, oxygen: $\mu \approx 1.004$):

$$\chi = \frac{\partial \mathcal{M}}{\partial (eB)}\bigg|_{B=0}, \quad B = H + e\mathcal{M}, \quad \mu = \frac{B}{H} = \frac{1}{1 - 4\pi\alpha_{\rm em}\chi}$$

eB, $\mathcal{M}$, χ are scale-independent. eH, $M = e\mathcal{M}$, $\alpha_{\rm em}$, $\mu = 1 + \chi_m$ depend on the QED scale.



$$\Pi_{\mu\nu}(q^2) = \left(q_\mu q_\nu - \delta_{\mu\nu} q^2\right) \Pi(q^2)\,,$$

$$\Pi(0) = -\partial^2 f_b / \partial(eB)^2 = \chi_b = \chi(\mu) + b_1 \log(\mu^2 a^2)$$

$$b_1 = \frac{N_c}{12\pi^2} \sum_{f=u,d,s} \frac{q_f^2}{e^2} \left[1 + \frac{\alpha_s(a^{-1})}{\pi} + \cdots\right]$$

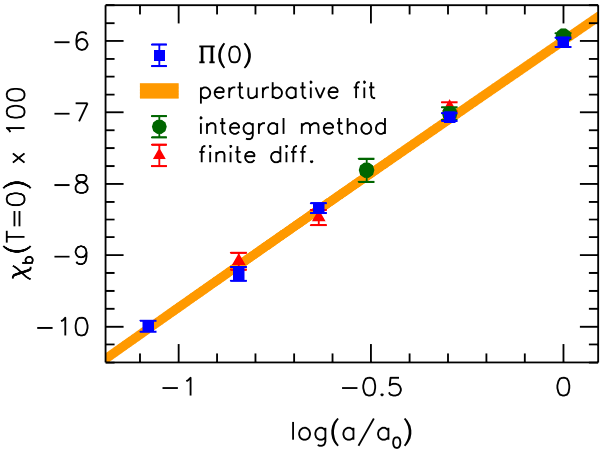
Method to determine $\chi_b = \Pi(0)$ with simulations at $B = 0$ only!

Interesting observation: Entropy density: $s = -\partial f / \partial T$

$$T \left.\frac{\partial^2 s}{\partial (eB)^2}\right|_{B=0} = -2 \frac{\partial}{\partial \log T^2} \left.\frac{\partial^2 f}{\partial (eB)^2}\right|_{B=0} = 2 \frac{\partial \chi(B=0)}{\partial \log T^2}$$

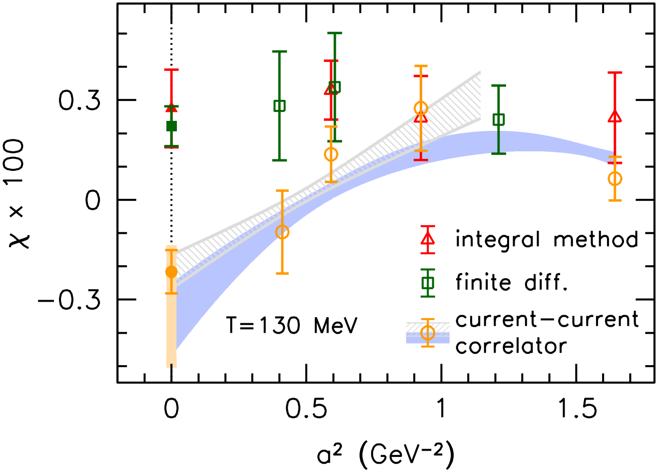
Adler function (q large, T large or μ large, $n = -\partial f / \partial \mu$):

$$D(q^2) = 12\pi^2 \frac{\partial \Pi}{\partial \log q^2} \longleftrightarrow 12\pi^2 \frac{\partial \chi_b(B=0)}{\partial \log T^2} = 6\pi^2 T \left.\frac{\partial^2 s}{\partial (eB)^2}\right|_{B=0} = 6\pi^2 \mu \left.\frac{\partial^2 n}{\partial (eB)^2}\right|_{B=0}$$



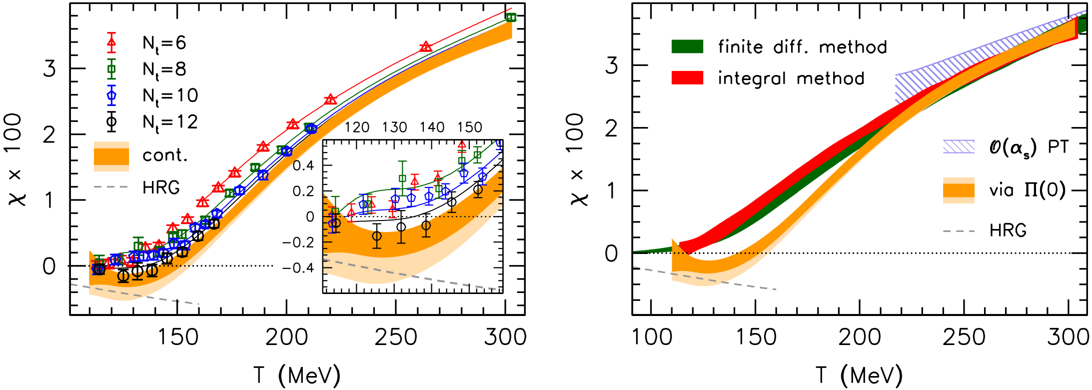
Fit: $\chi_b = b_1(a^{-1}) \cdot \left[\log(a^2/a_0^2) + \log(\mu_{\rm QED}^2 a_0^2)\right] \cdot \left[1 + z_1(a/a_0)^2\right].$

current-current method III: continuum extrapolation at $T = 130$ MeV



$\exists$ problem with previous extrapolations (including from our integral method)!

current-current method IV: results



Continuum limit for high T described by:

$$\chi(T) = b_1(\mu_{\rm therm}) \cdot \log\left(\gamma \frac{T^2}{\mu_{\rm QED}^2}\right) + \mathcal{O}(1/T^2)\,, \quad \mu_{\rm therm} \sim 2\pi T$$

Spin contribution to the magnetic susceptibility

Decomposition:

$$\chi = \sum_f \chi_f, \qquad \chi_f = \chi_f^S + \chi_f^{\rm ang},$$

$$C_f(m_f) = \frac{q_f/e}{2m_f}\,\frac{\partial}{\partial(eB)}\left\langle \bar\psi_f \sigma_{xy} \psi_f \right\rangle\Big|_{eB=0} = \frac{(q_f/e)^2}{2m_f}\tau_{fb}, \quad \sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu, \gamma_\nu].$$

$$\chi_f^S = C_f(m_f) - C_f(m_f^{\rm valence}=0)\,, \quad \left\langle \bar\psi_f \sigma_{xy} \psi_f \right\rangle = q_f B \cdot \left\langle \bar\psi_f \psi_f \right\rangle \cdot \xi_{fb} \equiv q_f B \cdot \tau_{fb}$$

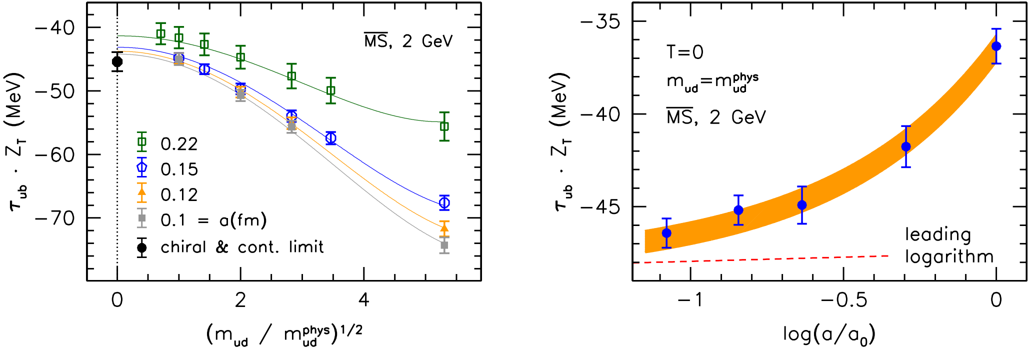
τ_{fb} undergoes additive and multiplicative renormalization:

$$\tau_f = Z_T \tau_{fb} - \tau_f^{\rm div} \equiv Z_T\left[\tau_f(T) - \tau_f(0)\right], \quad \tau_f^{\rm div} = m_f \frac{3}{4\pi^2}\log(\mu_{\rm QED}^2 a^2) + \mathcal{O}(\alpha_s).$$

The additive divergence is related to the spin contribution to the $T = 0$ term $\propto (eB)^2$ that we usually subtract.

The tensor coefficient

τ_{fb} diverges for $m_u > 0$ (as discussed above).



$$f_\gamma^\perp = \tau^{\overline{\mathrm{MS}}}(\mu = 2\,\mathrm{GeV}) = \lim_{a\to 0} \lim_{m_u \to 0} Z_T \tau_u = -45.4(1.5)\,\mathrm{MeV}$$

= Normalization of the photon distribution amplitude.

Probability amplitude of a real photon to dissociate into a massless quark-antiquark pair of flavour u in the infinite momentum frame.

Decomposition of the susceptibility I

$$\chi = \chi^{\mathrm{ang}}(\mu_{\mathrm{QCD}}) + \sum_f \chi_f^S(\mu_{\mathrm{QCD}})$$

A free case calculation shows that $\chi^S(T=0)=0$.

This implies that $\chi^{\mathrm{ang}}(T=0) = \chi(T=0) - \chi^S(T=0) = 0$.

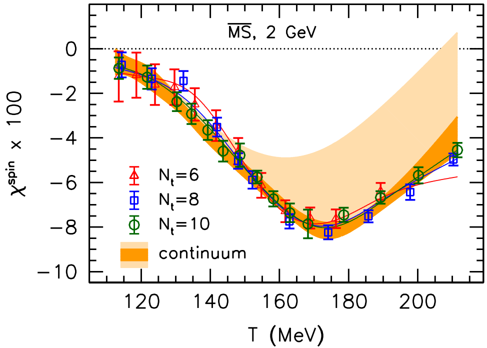
We use the $\overline{\mathrm{MS}}$ scheme with $\mu_{\mathrm{QCD}} = 2\,\mathrm{GeV}$.

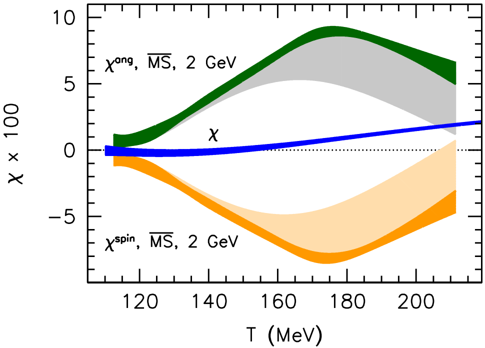
Further decompositions are possible:

$$\chi = \chi_{\mathrm{glue}} + \sum_f \chi_f = \chi_{\mathrm{glue}} + \sum_f \left(\chi_f^{\mathrm{ang}} + \chi_f^S\right) ,$$

in analogy to the decomposition of the transverse spin of a hadron in deep inelastic scattering.

Decomposition of the susceptibility II: results





At very high temperatures (free fermion gas): $\chi^S \to \frac{3}{2}\chi > 0\,, \quad \chi^{\text{ang}} \to -\frac{1}{2}\chi < 0\,.$

At $T \not\gg \Lambda_{\text{QCD}}$ in the strongly interacting medium: Pauli-diamagnetism (negative g-factor)!

This is possible since at least at $T < T_c$ the quarks are bound within hadrons and are not the relevant degrees of freedom.

Summary

- The QCD crossover temperature decreases slightly if the medium is exposed to a constant magnetic field. At very large fields it will become a first order phase transition.
- Complex, non-monotonic dependence of $\bar{\psi}\psi$ on B and T (inverse magnetic catalysis).
- Once the QED running coupling (magnetic catalysis) is accounted for, the response at $T \gtrsim 150$ MeV is paramagnetic.
- At low temperatures QCD is slightly diamagnetic.
- At the point $T = 0$, due to the B^4 term, QCD is paramagnetic. ($\chi = 0$ at $T = 0$ by definition since $c = 1$.)
- Equation of state re-computed, using the new current-current method (not shown here).
- The so-called tensor coefficient τ has been determined. (Magnetic susceptibility of the chiral condensate $\xi = \tau / \langle\bar{\psi}\psi\rangle = \langle\bar{\psi}\sigma_{xy}\psi\rangle / (qF_{xy}\langle\bar{\psi}\psi\rangle)$.)
- χ factorizes into spin contributions and a remainder (angular momentum and gluons). The latter drives paramagnetism (!) at large but not too large T.
- Do similar strongly coupled fermion systems exist in condensed matter physics???