# Instantons and chiral symmetry in hot QCD

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# Symmetries of QCD and their realization

- partition function $Z = \int \mathscr{D}U \ \prod_f \det\left(D[U] + m_f\right) \cdot e^{-S_g[U]}$
- $m_u \approx m_d \approx 0$
- Symmetries: $SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A$
  - $U(1)_A$ anomalous
  - $SU(2)_A$ spontaneously broken below $T_c$
- Order parameter of $SU(2)_A$ (Banks-Casher formula):

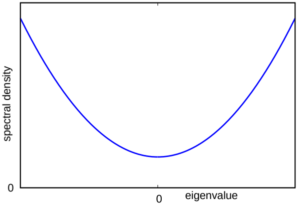
$$\langle \bar{\psi}\psi \rangle \propto \frac{1}{V}\sum_i \frac{1}{\lambda_i + m} \propto \int_{-\Lambda}^{\Lambda} d\lambda \frac{m}{\lambda^2 + m^2} \rho(\lambda) \xrightarrow[m \to 0]{} \rho(0)$$

$\lambda_i$: eigenvalues of the Dirac operator, $\rho(\lambda)$: its spectral density

# The finite temperature transition
## Standard picture

### Below $T_c$

- Chiral symmetry broken
- Order parameter: $\rho(0) \neq 0$



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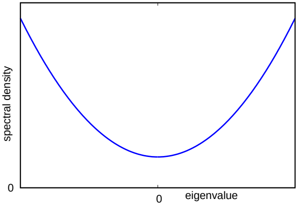
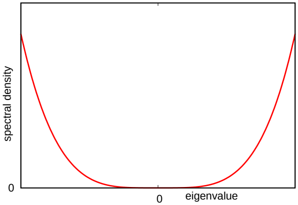



### Above $T_c$

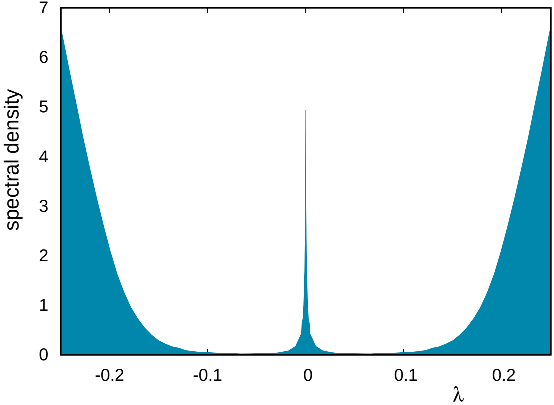
- Chiral symmetry restored
- Order parameter $\rho(0) = 0$
- (Pseudo)gap (lowest Matsubara mode)




spectral density at 0 $\Longleftrightarrow$ realization of chiral symmetry

# Spectral density at $T = 1.1\,T_c$ on the lattice

quenched: determinant (quark backreaction) omitted from Boltzmann weight




Peak at zero in the spectral density!

Edwards et al. 2000; Alexandru & Horvath 2015; Kaczmarek, Mazur, Sharma 2021

# Questions

- Why is there a peak at zero?
- How is it suppressed if the quark determinant is included?
- How does the peak influence the realization of chiral symmetry as $m \to 0$?

# Spectral peak $\leftrightarrow$ instantons?

- Instanton: special gauge field configuration with topological winding
- A "lump" of action localized in space-time
- (Anti)instanton
  $\rightarrow$ zero eigenvalue of $D(A)$ with $(-)+$ chirality eigenmode
- Topological charge $Q = n_i - n_a$
  $\rightarrow$ $|Q|$ exact zero eigenvalues
- Its fluctuations $\chi = \frac{1}{V}\langle Q^2 \rangle$ (topological susceptibility)

# Instantons above $T_c \rightarrow$ small Dirac eigenvalues

- High $T$:
  large instantons "squeezed out" in the temporal direction
- Dilute gas of instantons and antiinstantons
- Zero modes exponentially localized
- Mixing (splitting) of I-A zero modes small
- $n_i$ instantons $n_a$ antiinstantons
  $\rightarrow$ $|n_i - n_a|$ exact zero modes + mixing near zero modes

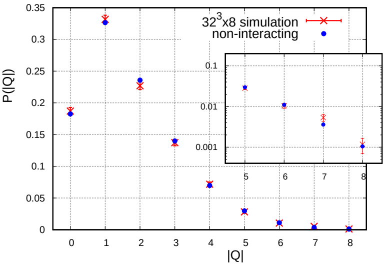
# Quenched approximation → free instanton gas

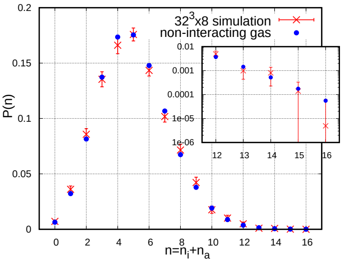
## lattice result

- Quenched approximation (quark backreaction neglected):

$$Z = \int \mathscr{D}U \prod_f \cancel{\det(D[U] + m_f)} \cdot e^{-S_g[U]}$$

- Number distribution of exact and near zero modes in the peak consistent with free instanton gas Vig R. & TGK 2021



# Including dynamical quarks

- Dynamical quarks are expected to
  - Suppress small Dirac eigenvalues ($\rightarrow$ fewer instantons)
  - Introduce I-A interaction ($\rightarrow$ pull pairs closer)
- What happens to the spectral peak and $\chi$symm if $m \rightarrow 0$?
- Direct lattice simulation with det presently not feasible

# Instanon-based random matrix model (quenched)

- Model of Dirac operator in the subspace of zero modes
- Quenched – ideal instanton gas:
  - Choose $n_\mathrm{i}$ and $n_\mathrm{a}$ from independent Poisson distributions of mean $\chi V/2$.
  - Place (anti)instantons randomly in 3d box of size $L^3$ ($V = L^3/T$).
  - Construct $(n_\mathrm{i}+n_\mathrm{a}) \times (n_\mathrm{i}+n_\mathrm{a})$ random matrix:

$$
\begin{array}{cc} n_\mathrm{i} & n_\mathrm{a} \end{array}
\\
\left(\begin{array}{c|c} 0 & iW \\ \hline iW^\dagger & 0 \end{array}\right)
$$

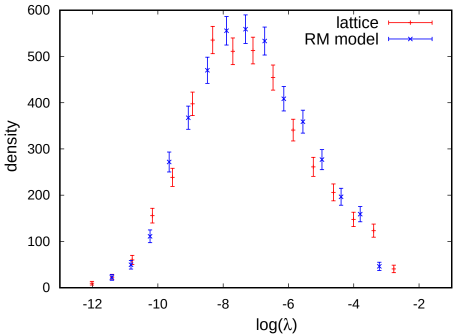
  - $w_{ij} = A \cdot \exp(-B \cdot r_{ij})$,

    $r_{ij}$ is the distance of instanton $i$ and antiinstanton $j$.

# Fit parameters to Dirac spectrum
quenched, $T = 1.1\,T_c$

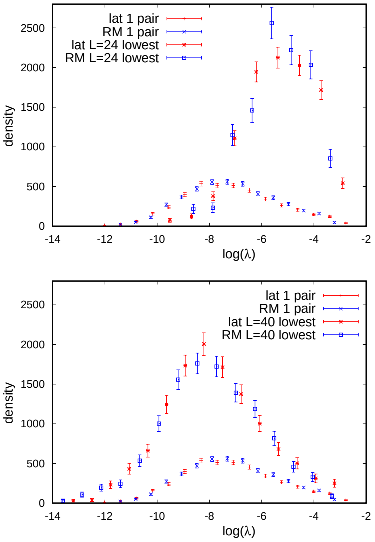
- Three parameters:
  - $\chi_0$ – instanton density: from exact zero modes $\rightarrow \chi_0 = \langle Q^2 \rangle / V$
  - $A, B$ – parameters of the exponential mixing between zero modes
- Fit $A$, $B$ to distribution of Dirac eigenvalues

  lowest eigenvalue in $V = 32^3$ configurations with only one IA pair



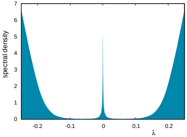
# Perferct description of quenched lattice Dirac spectrum

## Distribution of lowest and 2nd lowest eigenvalues – different volumes



# Random matrix model with dynamical quarks

- On the lattice $\det(D+m)^{N_\mathrm{f}}$ in Boltzmann weight
- $$\det(D+m) = \prod_{\text{zmz}}(\lambda_i+m)\times\prod_{\text{bulk}}(\lambda_i+m)$$
- Bulk weakly correlated with zero mode zone




- Approximate det with $\prod_{\text{zmz}}(\lambda_i+m)$
- Consistently included in RM model:

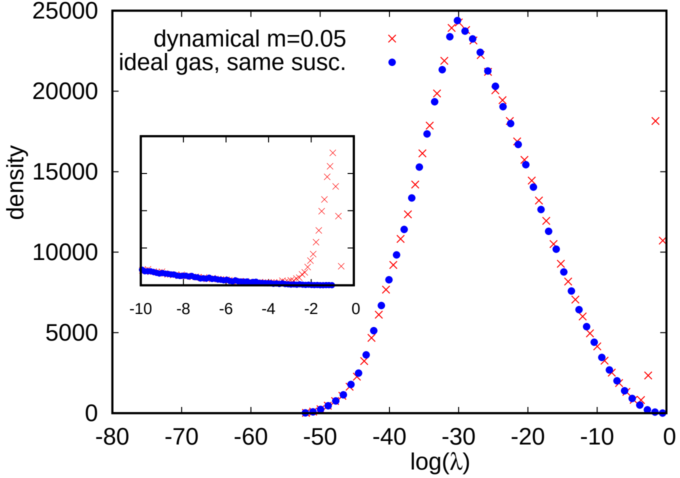
$$P(n_\mathrm{i},n_\mathrm{a}) = \underbrace{e^{-\chi_0 V}\frac{1}{n_\mathrm{i}!}\frac{1}{n_\mathrm{a}!}\left(\frac{\chi_0 V}{2}\right)^{n_\mathrm{i}+n_\mathrm{a}}}_{\text{free instanton gas}}\cdot\det(D+m)^{N_\mathrm{f}}$$

# Suppression of instantons by dynamical quarks

## results of the simulation of the model

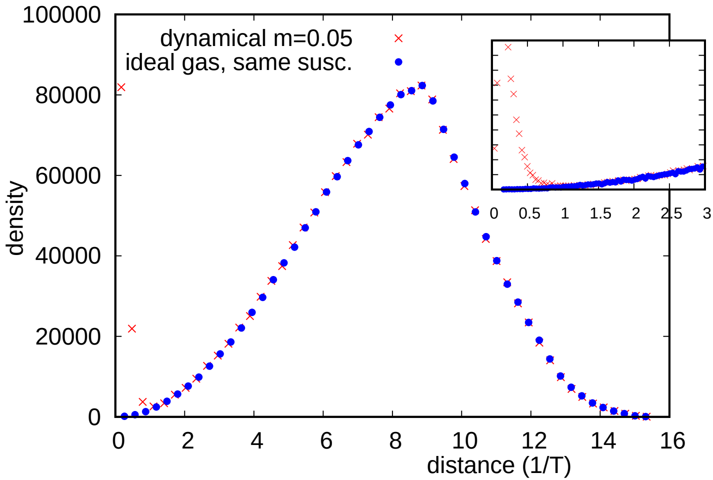
- If $|\lambda_i| \ll m \quad \Longrightarrow \quad \prod_i (\lambda_i + m) \approx m^{n_\mathrm{i}+n_\mathrm{a}}$
- $\left(\frac{\chi_0 V}{2}\right)^{n_\mathrm{i}+n_\mathrm{a}} \cdot \det(D+m)^{N_\mathrm{f}} \approx \left(\frac{m^{N_\mathrm{f}} \chi_0 V}{2}\right)^{n_\mathrm{i}+n_\mathrm{a}}$
- Distribution of number of (anti)instantons still Poisson
- Free gas, but susceptibility suppressed as $\chi_0 \to m^{N_\mathrm{f}} \chi_0$
- Instanton gas more dilute $\Rightarrow$ $|\lambda_i|$ smaller
- Even in the chiral limit $|\lambda_i| \ll m \quad \Longrightarrow$ free instanton gas

# Spectral density – full QCD vs. ideal instanton gas



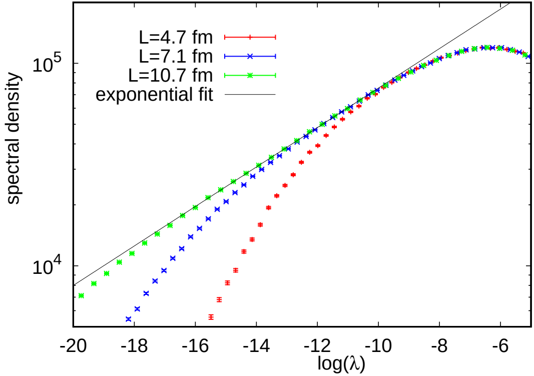
# Instanton-antiinstanton molecules

## density of closest opposite charge pairs at given distance



# Spectral density singular at the origin

(in the $V \to \infty$ limit)




If $\rho(\lambda) \propto \lambda^{\alpha}$ and $y = \log(\lambda)$ then $\tilde{\rho}(y) \propto e^{(1+\alpha)y}$ $\alpha = -0.775(5)$

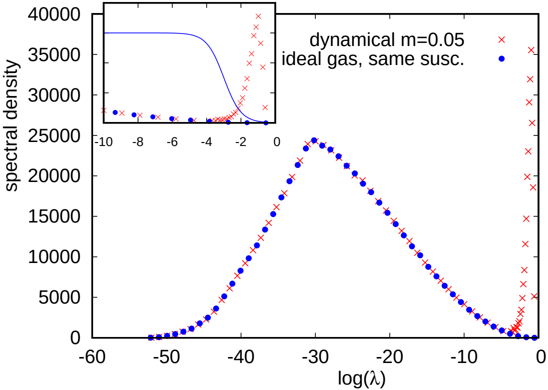
What is $\rho(0)$ for a singular spectral density?

# “Banks-Casher” for singular spectral density

## Free instanton gas contribution dominates condensate

$$\langle\bar\psi\psi\rangle \approx \langle\sum_i \frac{m}{m^2+\lambda_i^2}\rangle \approx \underbrace{\left(\text{avg. number of instantons in free gas}\right)}_{m^{N_f}\chi_0 V} \cdot \frac{1}{m} = m^{N_f-1}\chi_0 V$$

$$|\lambda_i| \ll m$$



# Fate of chiral symmetry as $m \to 0$

- Free IA gas eigenvalues $|\lambda_i| \ll m$ for any quark mass.

- $U(1)_A$ breaking susceptibility $\chi_\pi - \chi_\delta = ?$
  (Zero or nonzero? 10 year old unsettled dispute in the lattice community)

- $$\chi_\pi - \chi_\delta \approx \langle \sum_i \frac{m^2}{(m^2+\lambda_i^2)^2} \rangle \approx \underbrace{\begin{pmatrix}\text{avg. number of in-} \\ \text{stantons in free gas}\end{pmatrix}}_{m^{N_f}\chi_0 V} \cdot \frac{1}{m^2} = m^{N_f-2}\chi_0 V$$

- Contribution of IA molecules in $m \to 0$ limit: $|\lambda_i| \gg m$
  - For $N_f \leq 2$ numerically small ($\ll 1\%$ for realistic parameters).
  - Contribution to $\langle \bar{\psi}\psi \rangle \propto m$
  - Contribution to $\chi_\pi - \chi_\delta \propto m^2$

# Conclusions for QCD at $T > T_c$

- Non-interacting degrees of freedom:
  instantons (+ IA molecules)
- Dirac spectral density has singular peak at zero.
- Chiral symmetry restoration nontrivial (anomaly remains).
- Even though $SU(N_f)_\mathrm{A}$ restored,
  order of the $m \to 0$ and $V \to \infty$ limit important
- Chiral limit with $N_\mathrm{f}$ degenerate light quarks:
  - $\langle \bar{\psi}\psi \rangle \approx m^{N_\mathrm{f}-1}$ agrees with small $m$ expansion of the free energy
    Kanazawa and Yamamoto (2015)
  - $\chi_\pi - \chi_\delta \approx m^{N_\mathrm{f}-2}$