About the spin-dependent structure of the proton at very low momentum

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GENESIS 1. The beginning

- In the beginning God created the quarks (ordinary matter) and made them interact through the strong forces, the SU(3) group.
- And God said, “I do not understand a damn thing” so he said “Let there be light”, and there was light, the U(1) gauge group.

We will study the strong interactions using light at very low energies: $q^2 \to 0$.

$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p) \,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2} F_i' + \ldots$$

Observables: $ep \to ep$, $\mu p \to \mu p$, $\gamma p \to \gamma p$, atomic physics, muonic atoms, ...

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Scales (and ratios)

$m_p \sim \Lambda_\chi$
$m_\mu \sim m_\pi \sim m_r = \frac{m_\mu m_p}{m_p + m_\mu}$
$m_r \alpha \sim m_e$
$\ldots$
$Q^2 \to 0$

Tool: Effective Field Theories ≡ Factorization
Why?: There is a hierarchy of different scales
EFTs are especially useful in these situations.
1) Perturbative calculations much easier and systematic.
2) Nonperturbative information is parameterized in a model independent way.
3) Power counting.

Effective Field Theory: Non-relativistic protons, photons and (non-relativistic) electron/muons.

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Caswell-Lepage

$$\mathcal{L}_{\rm NRQED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu}$$

$$+\psi_p^\dagger\left\{ iD_0 + \frac{c_k}{2m_p}\mathbf{D}^2 + \frac{c_4}{8m_p^3}\mathbf{D}^4 + \frac{c_F^{(p)}}{2m_p}\boldsymbol{\sigma}\cdot e\mathbf{B} + \frac{c_D^{(p)}}{8m_p^2}(\mathbf{D}\cdot e\mathbf{E} - e\mathbf{E}\cdot\mathbf{D})\right.$$

$$\left. +i\frac{c_S^{(p)}}{8m_p^2}\boldsymbol{\sigma}\cdot(\mathbf{D}\times e\mathbf{E} - e\mathbf{E}\times\mathbf{D}) + c_{A_1}^{(p)}e^2\frac{\mathbf{B}^2-\mathbf{E}^2}{8m_p^3} - c_{A_2}^{(p)}e^2\frac{\mathbf{E}^2}{8m_p^3}\right\}\psi_p$$

$$+(\text{leptons})$$

$$-\frac{c_3^{(pe)}}{m_pm_e}\psi_p^\dagger\psi_p\psi_e^\dagger\psi_e + \frac{c_4^{(pe)}}{m_pm_e}\psi_p^\dagger\boldsymbol{\sigma}\psi_p\psi_e^\dagger\boldsymbol{\sigma}\psi_e + \cdots .$$

Dictionary (relation Wilson coefficients with low energy constants):

$c_F^{(p)} \to \mu_p$ anomalous magnetic moment (low energy constant)

$c_{A_i}^{(p)} \to \alpha_E, \beta_M$ Proton polarizabilities (quasi low energy constant)

$c_D \to r_p$ proton radius (quasi low energy constant)

$c_{3/4}^{(pe)} \to$ Two-photon exchange ...

$Q^2 \to 0$ INVOLVES COULOMB RESUMMATION $\to$ ATOMIC PHYSICS
$|\mathbf{q}| \sim m_\mu \alpha \sim m_e \sim 0.5$ MeV (muonic hydrogen)
$|\mathbf{q}| \sim m_e \alpha \sim 5.10^{-3}$ MeV (hydrogen)
New scales: $m_{\rm lepton}\alpha$, $m_{\rm lepton}\alpha^2$

Theoretical setup (muonic hydrogen)

We use an effective field theory, Potential Non-Relativistic QED, which describes the muonic hydrogen dynamics and profits from the hierarchy $m_\mu \gg m_\mu\alpha \gg m_\mu\alpha^2$

$$\left.\begin{array}{l}\left(i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{r}\right)\psi(\mathbf{r}) = 0 \\ +\text{corrections to the potential} \\ +\text{interaction with ultrasoft photons}\end{array}\right\} \text{potential NRQED} \qquad E \sim mv^2$$

$NRQED(m_\mu\alpha) \to pNRQED$

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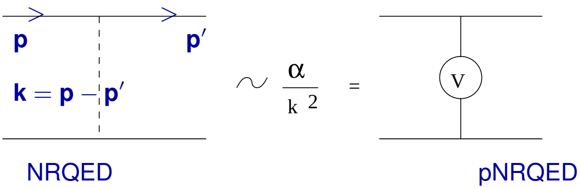
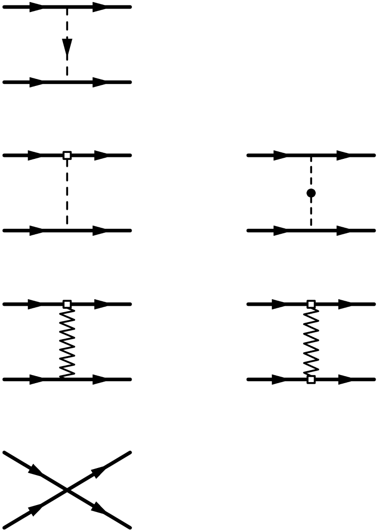
Matching NRQED to pNRQED



Positronium/muonium
Tree level



Order $1/m^2$

$$\tilde{V}^{(b)} = \frac{\pi\alpha}{2}\left[Z_p \frac{c_D^{(\mu)}}{m_\mu^2} + Z_\mu \frac{c_D^{(p)}}{m_p^2}\right],$$

$$\tilde{V}^{(c)} = -i2\pi\alpha \frac{(\mathbf{p}\times\mathbf{k})}{\mathbf{k}^2}\cdot\left\{Z_p \frac{c_S^{(\mu)}\mathbf{s}_1}{m_\mu^2} + Z_\mu \frac{c_S^{(p)}\mathbf{s}_2}{m_p^2}\right\},$$

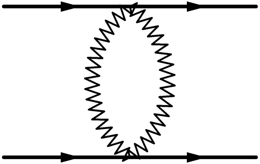
$$\tilde{V}^{(d)} = -Z_\mu Z_p 16\pi\alpha\left(\frac{d_2^{(\mu)}}{m_\mu^2} + \frac{d_2^{(\tau)}}{m_\tau^2} + \frac{d_{2,NR}}{m_p^2}\right),$$

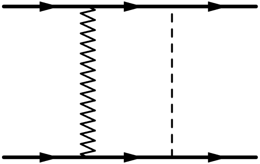
$$\tilde{V}^{(e)} = -Z_\mu Z_p \frac{4\pi\alpha}{m_\mu m_p}\left(\frac{\mathbf{p}^2}{\mathbf{k}^2} - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{\mathbf{k}^4}\right),$$

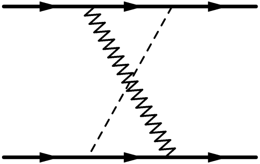
$$\tilde{V}^{(f)} = -\frac{i4\pi\alpha}{m_\mu m_p}\frac{(\mathbf{p}\times\mathbf{k})}{\mathbf{k}^2}\cdot\left(Z_p c_F^{(\mu)}\mathbf{s}_1 + Z_\mu c_F^{(p)}\mathbf{s}_2\right),$$

$$\tilde{V}^{(g)} = \frac{4\pi\alpha c_F^{(1)} c_F^{(2)}}{m_\mu m_p}\left(\mathbf{s}_1\cdot\mathbf{s}_2 - \frac{\mathbf{s}_1\cdot\mathbf{k}\mathbf{s}_2\cdot\mathbf{k}}{\mathbf{k}^2}\right),$$

$$\tilde{V}^{(h)} = -\frac{1}{m_p^2}\left\{(c_3^{pl_i} + 3c_4^{pl_i}) - 2c_4^{pl_i}\mathbf{S}^2\right\}.$$

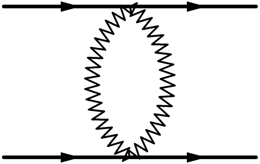


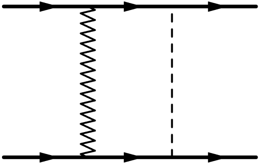


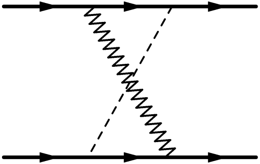


$$\tilde{V}^{(a)}_{1\,loop} = \frac{Z_\mu^2 Z_p^2 \alpha^2}{m_\mu m_p}\left(\log\frac{\mathbf{k}^2}{\mu^2} - \frac{8}{3}\log 2 + \frac{5}{3}\right),$$

$$\tilde{V}^{(b,c)}_{1\,loop} = \frac{4 Z_\mu^2 Z_p^2 \alpha^2}{3 m_\mu m_p}\left(\log\frac{\mathbf{k}^2}{\mu^2} + 2\log 2 - 1\right).$$







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Muonic Hydrogen: electron vacuum polarization

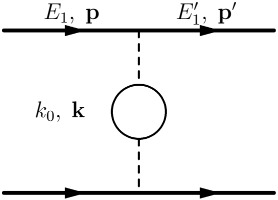

Figure : *Leading correction to the Coulomb potential due to the electron vacuum polarization.* $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ *and* $k_0 = E_1 - E_1'$.

$$\tilde{V}^{(0)} \equiv -4\pi Z_\mu Z_p \alpha_V(k) \frac{1}{\mathbf{k}^2},$$

$$\alpha_{eff}(k) = \alpha \frac{1}{1 + \Pi(-\mathbf{k}^2)},$$

where

$$\Pi(k^2) = \alpha \Pi^{(1)}(k^2) + \alpha^2 \Pi^{(2)}(k^2) + \alpha^3 \Pi^{(3)}(k^2) + \ldots$$

$$\alpha_V(k) = \alpha_{eff}(k) + \sum_{\substack{n,m=0 \\ n+m=even>0}} Z_\mu^n Z_p^m \alpha_{eff}^{(n,m)}(k) = \alpha_{eff}(k) + \delta\alpha(k), \qquad \delta\alpha(k) = O(\alpha^4)$$

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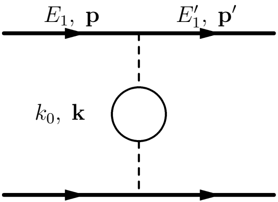

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Order $1/m^2$ from energy-dependent terms

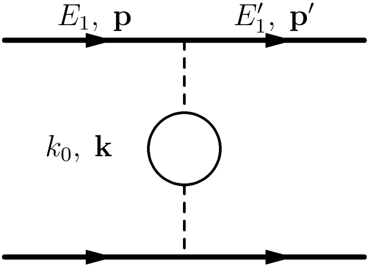


Figure : *Leading correction to the Coulomb potential due to the electron vacuum polarization.* $\mathbf{k} = \mathbf{p} - \mathbf{p}'$ *and* $k_0 = E_1 - E_1'$.

$$\delta \tilde{V}_E = -\frac{Z_\mu Z_p e^2}{4 m_\mu m_p} \frac{(\mathbf{p}^2 - \mathbf{p}'^2)^2}{\mathbf{k}^2} \frac{\alpha}{\pi} m_e^2 \int_4^\infty d(q^2) \frac{1}{(m_e^2 q^2 + \mathbf{k}^2)^2} u(q^2)\,.$$

$$u(q^2) = \frac{1}{3}\sqrt{1 - \frac{4}{q^2}}\left(1 + \frac{2}{q^2}\right)\,.$$

Theoretical setup.

Muonic hydrogen Lamb shift: $\Delta E_L \equiv E(2P_{3/2}) - E(2S_{1/2})$ and hyperfine splitting: $\Delta E_{HF} \equiv E(nS_{3/2}) - E(nS_{1/2})$

$$L_{pNRQED} = \int d^3\mathbf{r}\, d^3\mathbf{R}\, dt\, S^\dagger(\mathbf{r}, \mathbf{R}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) + e\mathbf{r}\cdot\mathbf{E}(\mathbf{R}, t) \right\} S(\mathbf{r}, \mathbf{R}, t) - \int d^3\mathbf{r}\, \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

$$V(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \ldots$$

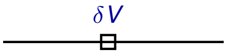
Observable: Spectrum or decays

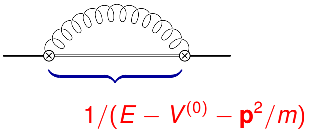
Corrections to the Green Function ($h_s^{(0)} = \mathbf{p}^2/m + V^{(0)}$)

$$G_s(E) = P_s \frac{1}{h_s^{(0)} - H_I - E} P_s = G_s^{(0)} + \delta G_s \qquad G_s^{(0)}(E) = \frac{1}{h_s^{(0)} - E}$$

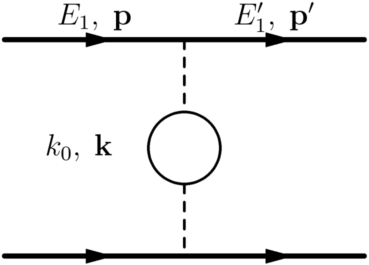
A) Ultrasoft loops (lamb shift-like): $\mathbf{x} \cdot \mathbf{E} \leftarrow$

B) Quantum mechanics perturbation theory$\leftarrow$





Vacuum polarization effects: $\mathcal{O}(m_r\alpha^3)$



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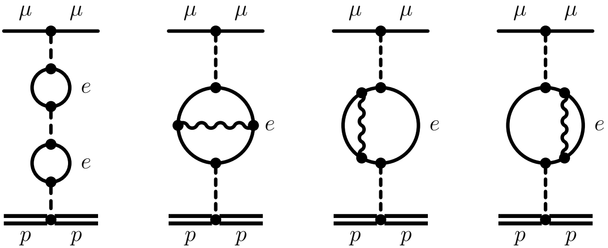
1-loop static potential

$$E_{LO} = \langle n|\delta V|n\rangle = 205.0074\ \mathrm{meV} = \mathcal{O}(m_r\alpha^3)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^4)$



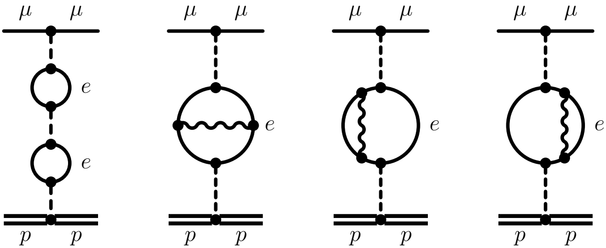
Pachuki/Borie
2-loop static potential is the same as two-loop vacuum polarization iterations
(*two loop vacuum polarization*)

$$\delta E = \langle n|\delta V|n\rangle = 1.5079 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

Quantum mechanics perturbation theory (*iteration one-loop*)

$$\delta E \sim \langle n|\delta V \frac{1}{H_C - E_n}\delta V|n\rangle = 0.151 \text{ meV} = \mathcal{O}(m_r\alpha^4)$$

Vacuum polarization effects: $\mathcal{O}(m_r\alpha^4)$



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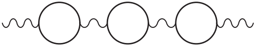
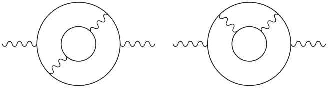
Vacuum polarization effects: $\mathcal{O}(m_r\alpha^5)$

3-loop static potential (three loop vacuum polarization, Kinoshita-Nio)

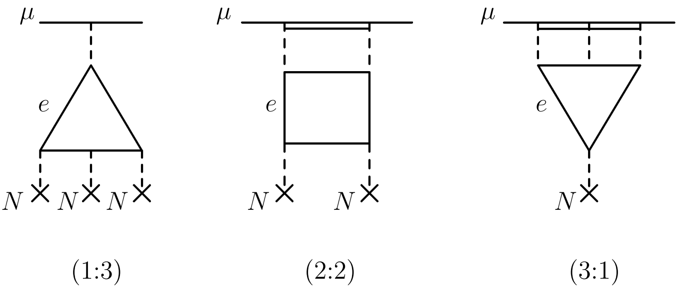
$$0.00752\ \mathrm{meV} = \mathcal{O}(m_r\alpha^5)$$

Slightly corrected by Ivanov et al.





Static potential, not vacuum polarization: $\mathcal{O}(m_r\alpha^5)$



Light-by-light (Wichmann-Kroll and Delbrück) contribution very small (Karshenboim et al.)

$$\Delta E \simeq -0.0009 \text{ mev} = \mathcal{O}(m_r\alpha^5)$$

Earlier work by Borie

$1/m$ potential

$$V(\mathbf{x}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \ldots$$

$$\frac{V^{(1)}(r)}{m_\mu} \to \mathcal{O}(m_r \alpha^6)$$

relativistic corrections+vacuum polarization

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r\alpha^4, \alpha^5)$$

Lamb shift

$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)

$\mathcal{O}(m\alpha^5)$ 0.0059 (Pachucki)

relativistic corrections+vacuum polarization

$$V(\mathbf{x}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r\alpha^4, \alpha^5)$$

Lamb shift
$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)
$\mathcal{O}(m\alpha^5)$ 0.0059 (Pachucki)

relativistic corrections+vacuum polarization

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$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r\alpha^4, \alpha^5)$$

Lamb shift
$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)
$\mathcal{O}(m\alpha^5)$ 0.0169 (Pachucki and Veitia; Borie)

relativistic corrections+vacuum polarization

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

$$\frac{V^{(2)}(r)}{m_\mu^2} \to \mathcal{O}(m_r\alpha^4, \alpha^5)$$

Lamb shift
$\mathcal{O}(m\alpha^4 \times \frac{m^2}{m_p^2})$ 0.0575 (purely relativistic)
$\mathcal{O}(m\alpha^5)$ 0.018759 (Jentschura; Karshenboim&Ivanov&Korzinin; Peset&AP)

relativistic corrections+vacuum polarization

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

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Hyperfine

$\frac{m}{m_p} \times \mathcal{O}(m\alpha^4)$

$\frac{m}{m_p} \times \mathcal{O}(m\alpha^5)$ Martynenko; Indelicato; Pachucki; Borie; Peset-Pineda; ...

relativistic corrections+vacuum polarization

$$V(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = V^{(0)}(r) + \frac{V^{(1)}(r)}{m_\mu} + \frac{V^{(2)}(r)}{m_\mu^2} + \dots$$

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$\frac{m}{m_p} \times \mathcal{O}(m\alpha^4)$

$\frac{m}{m_p} \times \mathcal{O}(m\alpha^5)$ Martynenko; Indelicato; Pachucki; Borie; Peset-Pineda; ...

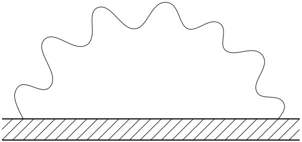
i)	$\mathcal{O}(m_r\alpha^4)$	δE_{Fermi}	182.65604
ii)	$\mathcal{O}(m_r\alpha^5)$	$V^{(2,1)} \times V_{\text{VP}}^{(0,2)}$	0.73449
iii)	$\mathcal{O}(m_r\alpha^5)$	$V_{\text{VP}}^{(2,2)}$	0.37465

Table : *The different contributions to the hyperfine splitting for the 1S in muonic hydrogen in* meV *units.*

i)	$\mathcal{O}(m_r\alpha^4)$	$\delta E_{\mathrm{Fermi}}$	22.832005
ii)	$\mathcal{O}(m_r\alpha^5)$	$V^{(2,1)} \times V_{\mathrm{VP}}^{(0,2)}$	0.074474
iii)	$\mathcal{O}(m_r\alpha^5)$	$V_{\mathrm{VP}}^{(2,2)}$	0.048275

Table : *The different contributions to the hyperfine splitting for the 2S in muonic hydrogen in* meV *units.*

Ultrasoft effects: $\mathcal{O}(m\alpha^5)$



$$\Delta E = -0.6677 \; meV$$

$$\mathcal{O}(m\alpha^5 \frac{m_\mu}{m_p}) : \qquad \Delta E = -0.045 \; meV$$

All (soft+ultrasoft):

$$\Delta E = -0.71896 \text{ meV}.$$

Start the overlap with hadronic effects.

Hadronic corrections

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.} \delta^3(\mathbf{r}) \to \Delta E \sim \frac{1}{m_p^2} D_d^{had.} (m_r \alpha)^3$$

$$D_d^{(p\mu)} = -c_3 - 16\pi\alpha d_2 + \frac{\pi\alpha}{2} c_D^{(p)}$$

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \to \frac{1}{m_p^2} D_d^{had.} (\mathbf{S}_1 + \mathbf{S}_2)^2 \delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

c_3, c_4, d_2, $c_D^{(p)}$, ... matching coefficients of NRQED.

$$\delta\mathcal{L} = \cdots + \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} - e\frac{c_D}{m_p^2} N_p^\dagger \boldsymbol{\nabla} \cdot \mathbf{E} N_p - \frac{c_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu + \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma} \mu$$

Hyperfine muonic hydrogen 1S

i)	$\mathcal{O}(m_r\alpha^4)$	$\delta E_{\rm Fermi}$	182.65604
ii)	$\mathcal{O}(m_r\alpha^5)$	$V^{(2,1)} \times V^{(0,2)}_{\rm VP}$	0.73449
iii)	$\mathcal{O}(m_r\alpha^5)$	$V^{(2,2)}_{\rm VP}$	0.37465
iv)	$\mathcal{O}(m_r\alpha^5 \frac{m_\mu}{m_p})$	$\delta \bar{E}^{\rm TPE}$	−1.161(20)
v)	$\mathcal{O}(m_r\alpha^6)$	$V^{(2,1)} \times V^{(2,1)}$	0.01457
vi)	$\mathcal{O}(m_r\alpha^6)$	$V^{(2,3)}; \delta c_4^{(-1)}$	−0.01755
vii)	$\mathcal{O}(m_r\alpha^6)$	$V^{(2,1)} \times V^{(0,3)}_{\rm VP}$	0.00556
viii)	$\mathcal{O}(m_r\alpha^6)$	$V^{(2,3)}_{\rm VP}$	0.00292
ix)	$\mathcal{O}(m_r\alpha^6 \times \frac{m_\mu}{m_p})$	$V^{(2,1)} \times V^{(2,1)}$	0.01752
x)	$\mathcal{O}(m_r\alpha^6 \times \frac{m_\mu}{m_p})$	$V^{(2,2)} \times V^{(0,2)}_{\rm VP}; c_{4,\rm TPE}$	−0.00466
	Total sum		182.623(27)

Table : *The different contributions to the hyperfine splitting for the 1S in muonic hydrogen in* meV *units. The value for the iv) and x) entries is obtained from the hydrogen hyperfine measurement.*

Hyperfine muonic hydrogen 2S

i)	$\mathcal{O}(m_r\alpha^4)$	$\delta E_{\rm Fermi}$	22.832005
ii)	$\mathcal{O}(m_r\alpha^5)$	$V^{(2,1)} \times V^{(0,2)}_{\rm VP}$	0.074474
iii)	$\mathcal{O}(m_r\alpha^5)$	$V^{(2,2)}_{\rm VP}$	0.048275
iv)	$\mathcal{O}(m_r\alpha^5 \frac{m_\mu}{m_p})$	$\delta E^{\rm TPE}$	$-0.1451(25)$
v)	$\mathcal{O}(m_r\alpha^6)$	$V^{(2,1)} \times V^{(2,1)}$	0.002581
vi)	$\mathcal{O}(m_r\alpha^6)$	$V^{(2,3)}; \delta c_4^{(-1)}$	-0.002194
vii)	$\mathcal{O}(m_r\alpha^6)$	$V^{(2,1)} \times V^{(0,3)}_{\rm VP}$	0.000375
viii)	$\mathcal{O}(m_r\alpha^6)$	$V^{(2,3)}_{\rm VP}$	0.000563
ix)	$\mathcal{O}(m_r\alpha^6 \times \frac{m_\mu}{m_p})$	$V^{(2,1)} \times V^{(2,1)}$	0.001846
x)	$\mathcal{O}(m_r\alpha^6 \times \frac{m_\mu}{m_p})$	$V^{(2,2)} \times V^{(0,2)}_{\rm VP}; c_{4,\rm TPE}$	-0.000473
	Total sum		22.8123(33)

Table : *The different contributions to the hyperfine splitting for the 2S in muonic hydrogen in* meV *units. The value for the iv) and x) entries is obtained from the hydrogen hyperfine measurement.*

HOW to determine the Two-Photon Exchange correction?

- Dispersion relations
- lattice (not yet)
- Chiral perturbation theory ($\rightarrow$ (non-analytic) m_q dependence, N_c dependence)
- ....

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu\alpha) \rightarrow pNRQED$$

HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots$$

$$\mathcal{L}_\pi = \frac{F_\pi^2}{4}\mathrm{Tr}\left[D_\mu U D^\mu U\right] + \cdots \qquad U = u^2 = e^{i\frac{\Pi}{F_\pi}}$$

$$\mathcal{L}_N = N^\dagger(iv^\mu\nabla_\mu + g_A u_\mu S^\mu)N + \cdots + (\Delta) + \cdots - e\frac{c_D}{m_p^2}N_p^\dagger \boldsymbol{\nabla}\cdot\mathbf{E}N_p$$

$$D_\mu = \partial_\mu + ieQA_\mu \qquad \nabla_\mu = \partial_\mu + \Gamma_\mu \qquad u_\mu = iu^\dagger(\nabla_\mu U)u$$

$$\Gamma_\mu = \frac{1}{2}\left\{u^\dagger(\partial_\mu + ieQA_\mu)u + u(\partial_\mu + ieQA_\mu)u^\dagger\right\}$$

$$\mathcal{L}_{N,l} = \frac{1}{m_p^2}\sum_i c_{3,R}^{pl_i}\bar{N}_p\gamma^0 N_p \bar{l}_i\gamma^0 l_i + \frac{1}{m_p^2}\sum_i c_{4,R}^{pl_i}\bar{N}_p\gamma^j N_p \bar{l}_i\gamma_j l_i$$

$$\delta\mathcal{L} = \cdots + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} - e\frac{c_D}{m_p^2}N_p^\dagger\boldsymbol{\nabla}\cdot\mathbf{E}N_p - \frac{c_3}{m_p^2}N_p^\dagger N_p\mu^\dagger\mu + \frac{c_4}{m_p^2}N_p^\dagger\sigma N_p\mu^\dagger\sigma\mu$$

HBET (m_π)

$$\mathcal{L}_{HBET} = \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\pi + \mathcal{L}_{l\pi} + \mathcal{L}_{(N,\Delta)} + \mathcal{L}_{(N,\Delta)l} + \mathcal{L}_{(N,\Delta)\pi} + \mathcal{L}_{(N,\Delta)l\pi},$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F^2 + \frac{d_2}{m_p^2}F_{\mu\nu}D^2F^{\mu\nu} + \cdots$$

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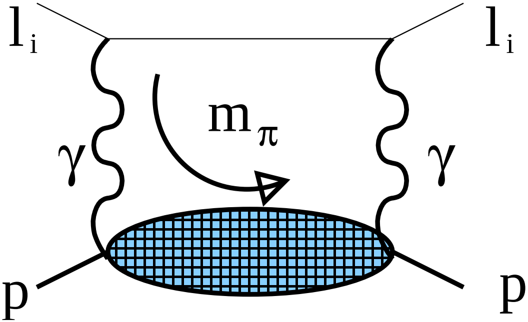
$c_3^{\rm had} =$

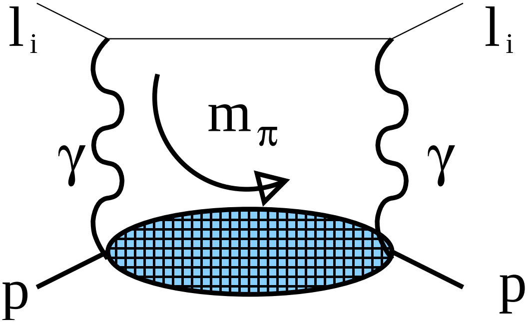


$$T^{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\langle p,s|TJ^\mu(x)J^\nu(0)|p,s\rangle\,,$$

$$T^{\mu\nu} = \left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)S_1(\rho,q^2)+\frac{1}{m_p^2}\left(p^\mu-\frac{m_p\rho}{q^2}q^\mu\right)\left(p^\nu-\frac{m_p\rho}{q^2}q^\nu\right)S_2(\rho,q^2)$$

$S_1 = ??$ $\qquad$ $S_2 = ??$

$c_3^{\rm had} =$

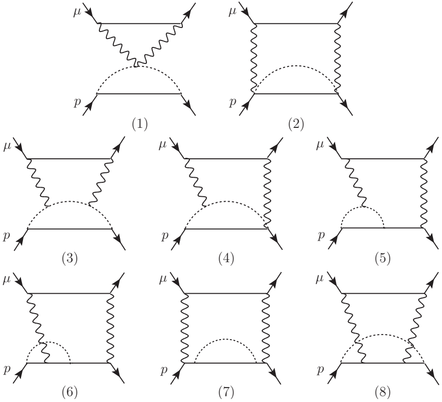


$$T^{\mu\nu} = i \int d^4x \, e^{iq\cdot x} \langle p, s | T J^\mu(x) J^\nu(0) | p, s \rangle \, ,$$

$$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) S_1(\rho, q^2) + \frac{1}{m_p^2} \left(p^\mu - \frac{m_p \rho}{q^2} q^\mu \right) \left(p^\nu - \frac{m_p \rho}{q^2} q^\nu \right) S_2(\rho, q^2)$$

$S_1 = ??$ $\qquad$ $S_2 = ??$

TWO-PHOTON EXCHANGE correction



m_μ extra suppression+χPT (Model independent)

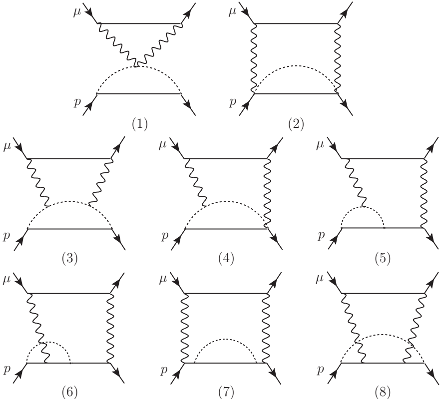
Power-like chiral enhanced ($\rightarrow \chi$PT can predict the leading order!)

$$c_3^{\rm had} \sim \alpha^2 \frac{m_\mu}{m_\pi} + \mathcal{O}\left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}}\right) \qquad \delta E \sim \mathcal{O}\left(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \frac{m_\mu}{m_\pi}\right)$$

Error ($\Delta = M_\Delta - M_p \sim 300$ MeV): LO $\times \frac{m_\pi}{\Delta} \simeq$ LO $\times \frac{1}{2}$

$\rightarrow c_3^{\rm had} = \alpha^2 \frac{m_\mu}{m_\pi} 47.2(23.6)$

TWO-PHOTON EXCHANGE correction



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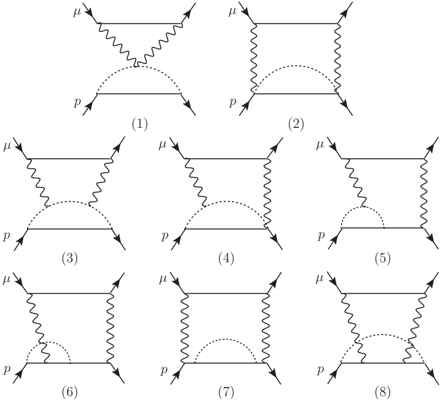

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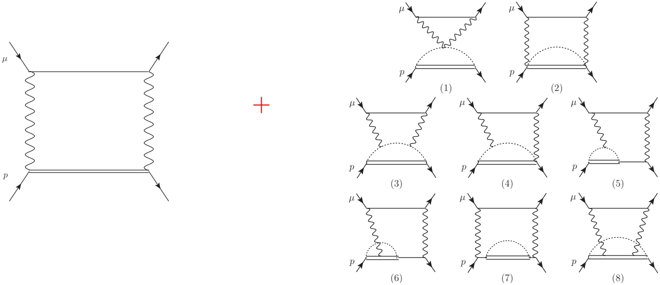
Large N_c. Including the Δ particle

Error:

$$\frac{m_\mu}{\Delta} \sim N_c \frac{m_\mu}{\Lambda_{QCD}} \to N_c \frac{m_\mu}{\Lambda_{QCD}} \sim \frac{1}{3}$$



$$c_3^{\rm had} \sim \alpha^2 \frac{m_\mu}{m_\pi}\left[1 + \# \frac{m_\pi}{\Delta} + \cdots\right] + \mathcal{O}\left(\alpha^2 \frac{m_\mu}{\Lambda_{QCD}}\right) = \alpha^2 \frac{m_\mu}{m_\pi}\begin{cases} 47.2(23.6) & (\pi), \\ 56.7(20.6) & (\pi + \Delta), \end{cases}$$

$$\Delta E_{\rm TPE} = 28.59(\pi) + 5.86(\pi\&\Delta) = 34.4(12.5)\mu{\rm eV} \ \ ({\rm Peset\&AP})\,.$$

(Model dependent: $\Delta E_{\rm TPE} = 33(2)\mu{\rm eV}$ (Birse-McGovern))

Large N_c. Including the Δ particle

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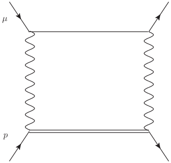

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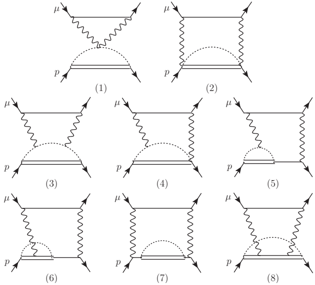

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$$\Delta E_{\rm TPE} \sim m_\mu \alpha^5 \times \frac{m_\mu^2}{(4\pi F_\pi)^2} \times \frac{m_\mu}{m_\pi} \sum_{n=0}^{\infty} c_n (N_c \sqrt{m_q})^n$$

$$\frac{\#}{\sqrt{m_q}} + ? + ?\sqrt{m_q} + \cdots$$

plus large N_c

$$\frac{\#}{\sqrt{m_q}} + \left[\#N_c + ? + \frac{?}{N_c} + \cdots\right] + \left[\#N_c^2 + ?N_c + ? + \cdots\right]\sqrt{m_q} + \cdots$$

? $\rightarrow$ Size of the counterterm in HBET

Hyperfine: Hydrogen and muonic hydrogen

Experiment:

$$E^{\rm exp}_{\rm hyd,HF}(1S) = 1420.405751768(1)\ {\rm MHz}\,,$$

$$E^{\rm exp}_{\mu p,\rm HF}(2S) = 22.8089(51)\ {\rm meV}\,.$$

Theory:

$$\frac{\delta V^{(2)}(r)}{m_\mu^2} \rightarrow \frac{1}{m_p^2} D_d^{had.} (\mathbf{S}_1 + \mathbf{S}_2)^2 \delta^3(\mathbf{r})$$

$$D_s^{had.} = 2c_4$$

c_4, matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \rightarrow NRQED(m_\mu \alpha) \rightarrow pNRQED$$

$$\delta\mathcal{L} = \cdots - \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma} \mu$$

Hyperfine: Hydrogen and muonic hydrogen

Experiment:

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c_4, matching coefficient of NRQED.

$$HBET(m_\pi/m_\mu) \to NRQED(m_\mu\alpha) \to pNRQED$$

$$\delta\mathcal{L} = \cdots - \frac{c_4}{m_p^2} N_p^\dagger \boldsymbol{\sigma} N_p \mu^\dagger \boldsymbol{\sigma}\mu$$

c_4, Spin-dependent effects

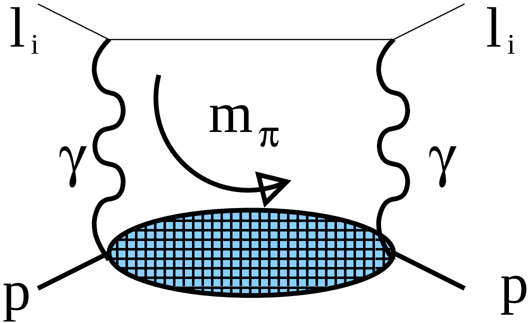


Figure : *Symbolic representation (plus permutations) of the spin-dependent correction.*

$$c_4^{pl} = -\frac{ig^4}{3}\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2}\frac{1}{k^4 - 4m_l^2 k_0^2}\left\{A_1(k_0,k^2)(k_0^2+2k^2)+3k^2\frac{k_0}{m_p}A_2(k_0,k^2)\right\}$$

Drell-Sullivan(67)

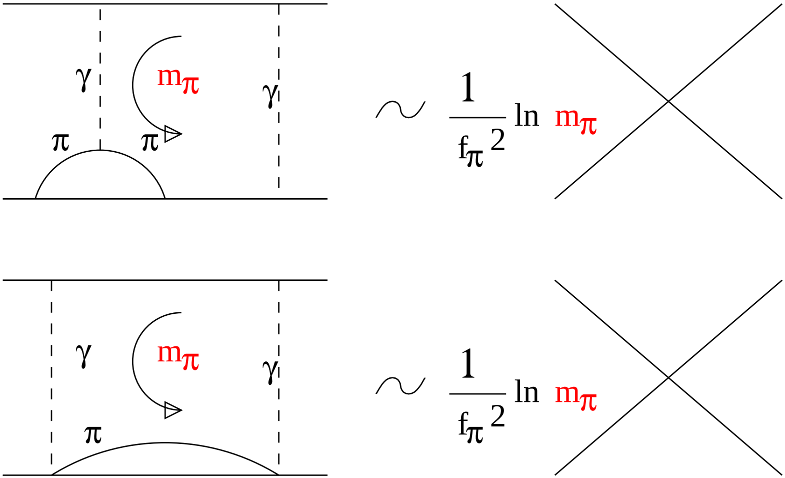
$$T^{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\langle p,s|TJ^\mu(x)J^\nu(0)|p,s\rangle\,,$$

which has the following structure ($\rho = q\cdot p/m$):

$$\begin{aligned} T^{\mu\nu} &= \left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)S_1(\rho,q^2) \\ &+\frac{1}{m_p^2}\left(p^\mu-\frac{m_p\rho}{q^2}q^\mu\right)\left(p^\nu-\frac{m_p\rho}{q^2}q^\nu\right)S_2(\rho,q^2) \\ &-\frac{i}{m_p}\,\epsilon^{\mu\nu\rho\sigma}q_\rho s_\sigma A_1(\rho,q^2) \\ &-\frac{i}{m_p^3}\,\epsilon^{\mu\nu\rho\sigma}q_\rho\left((m_p\rho)s_\sigma-(q\cdot s)p_\sigma\right)A_2(\rho,q^2) \end{aligned}$$

A_1, A_2 (χPT): Ji-Osborne; Peset-Pineda

Leading chiral logs to the hyperfine splitting



$$\delta V = 2\frac{c_4}{m_p^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r})\,.$$

$$\delta E_{HF} \sim \mathcal{O}(m_\mu \alpha^5 \times \frac{m_\mu^2}{\Lambda_\chi^2} \times \ln m_\pi)$$

The leading chiral logs can be determined for Hydrogen and muonic hydrogen hyperfine splitting (AP).

$$\begin{aligned} c_4^{pl_i} \quad &\simeq \quad \left(1 - \frac{\mu_p^2}{4}\right)\alpha^2 \ln \frac{m_{l_i}^2}{\nu^2} + \frac{b_{1,F}^2}{18}\alpha^2 \ln \frac{\Delta^2}{\nu^2} \\ &\quad + \frac{m_p^2}{(4\pi F_0)^2}\alpha^2 \frac{2}{3}\left(\frac{2}{3} + \frac{7}{2\pi^2}\right)\pi^2 g_A^2 \ln \frac{m_\pi^2}{\nu^2} \\ &\quad + \frac{m_p^2}{(4\pi F_0)^2}\alpha^2 \frac{8}{27}\left(\frac{5}{3} - \frac{7}{\pi^2}\right)\pi^2 g_{\pi N\Delta}^2 \ln \frac{\Delta^2}{\nu^2} \\ &\overset{(N_c \to \infty)}{\simeq} \quad \alpha^2 \ln \frac{m_l^2}{\nu^2} + \frac{m_p^2}{(4\pi F_0)^2}\alpha^2 \pi^2 g_A^2 \ln \frac{m_\pi^2}{\nu^2} \,. \end{aligned}$$

$$E_{\rm HF} = 4\frac{c_4^{pl_i}}{m_p^2}\frac{1}{\pi}(\mu_{l_i p}\alpha)^3 \sim m_{l_i}\alpha^5 \frac{m_{l_i}^2}{m_p^2} \times (\ln m_q, \ln \Delta, \ln m_{l_i})\,.$$

$$c_4^{pl_i} = c_{4,\rm R}^{pl_i} + c_{4,\rm point-like}^{pl_i} + c_{4,\rm Born}^{pl_i} + c_{4,\rm pol}^{pl_i} + \mathcal{O}(\alpha^3)\,.$$

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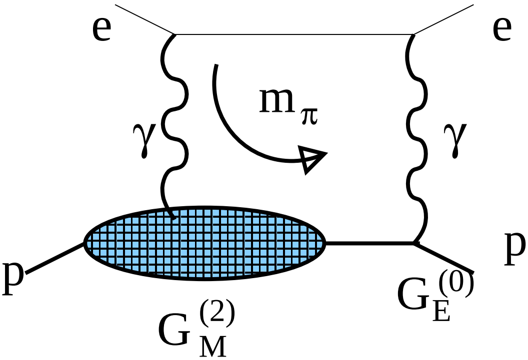
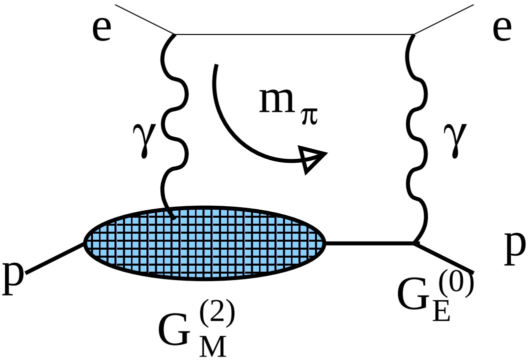


Figure : *Symbolic representation (plus permutations) of the Zemach correction.*

$$\delta c^{pl}_{4,Zemach} = (4\pi\alpha)^2 m_p \frac{2}{3} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{\mathbf{k}^4} G_E^{(0)} G_M^{(2)} .$$



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$c_{4,\rm Born}$, Zemach magnetic radius

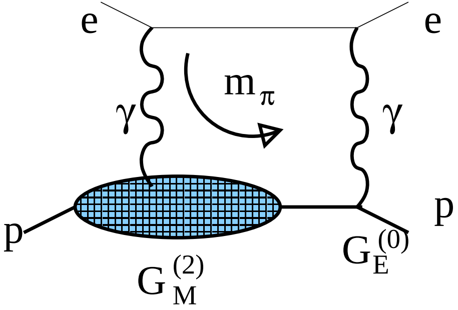

Figure : *Symbolic representation (plus permutations) of the Born correction.*

Chiral logs can be determined and constitute the leading contribution!

$$\langle r_Z \rangle = -\frac{4}{\pi}\int_0^\infty \frac{dQ}{Q^2}\left[G_E(Q^2)G_M(Q^2)-1\right] = -\frac{4}{\pi}\int_0^\infty q^{D-4}\frac{dQ}{Q^2}G_E^{(0)}(Q^2)G_M^{(2)}(Q^2)\,.$$

$$\langle r_Z \rangle = -\frac{3}{4\pi}\frac{1}{\alpha^2 M_p}c_{4,\rm Born}^{pl_i} \simeq -\frac{\pi}{2}\frac{M_p}{(4\pi F_0)^2}\left[g_A^2 \ln\frac{m_\pi^2}{\nu^2}+\frac{4}{9}g_{\pi N\Delta}^2\ln\frac{\Delta^2}{\nu^2}\right] \stackrel{(\nu=m_\rho)}{=} 1.35\ {\rm fm}\,.$$

"Experiment" $\sim 1.04 - 1.08$ fm.

$c_{4,\rm pol}$, Polarizability

Determined from dispersion relations. No subtractions needed.

Chiral computation can still give relevant information.

$$
\begin{aligned}
c^{pl_i}_{4,\rm point-like} &= \left(1-\frac{\kappa_p^2}{4}\right)\alpha^2 \ln\frac{m_{l_i}^2}{\nu^2}\,,\\
c^{pl_i}_{4,\rm pol} &= \frac{M_p^2}{(4\pi F_0)^2}\frac{\alpha^2}{\pi}\frac{8}{3}\left(\frac{7\pi}{8}-\frac{\pi^3}{12}\right)\left[g_A^2\ln\frac{m_\pi^2}{\nu^2}-\frac{8}{9}g^2_{\pi N\Delta}\ln\frac{\Delta^2}{\nu^2}\right]\\
&\quad +\frac{b^2_{1,F}}{18}\alpha^2\ln\frac{\Delta^2}{\nu^2}\,.
\end{aligned}
$$

$$
c^{pl_i}_{4,\rm point-like}+c^{pl_i}_{4,pol} \stackrel{(N_c\to\infty)}{\simeq} \alpha^2\ln\frac{m_{l_i}^2}{\nu^2}
$$

Polarizability contribution small and vanishes in the large N_c limit!!

$c_{4,\mathrm{pol}}$, Polarizability

Determined from dispersion relations. No subtractions needed.

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\begin{aligned}
c_{4,\mathrm{point-like}}^{pl_i} &= \left(1-\frac{\kappa_p^2}{4}\right)\alpha^2 \ln\frac{m_{l_i}^2}{\nu^2}\,,\\
c_{4,\mathrm{pol}}^{pl_i} &= \frac{M_p^2}{(4\pi F_0)^2}\frac{\alpha^2}{\pi}\frac{8}{3}\left(\frac{7\pi}{8}-\frac{\pi^3}{12}\right)\left[g_A^2\ln\frac{m_\pi^2}{\nu^2}-\frac{8}{9}g_{\pi N\Delta}^2\ln\frac{\Delta^2}{\nu^2}\right]\\
&\quad+\frac{b_{1,F}^2}{18}\alpha^2\ln\frac{\Delta^2}{\nu^2}\,.
\end{aligned}
$$

$$
c_{4,\mathrm{point-like}}^{pl_i}+c_{4,pol}^{pl_i} \overset{(N_c\to\infty)}{\simeq} \alpha^2\ln\frac{m_{l_i}^2}{\nu^2}
$$

Polarizability contribution small and vanishes in the large N_c limit!!

Fixing c_4^{pe}. Hydrogen

Hydrogen. By fixing the scale $\nu = m_\rho$ we obtain the following number for the total sum in the SU(2) case:

$$E_{\mathrm{HF,logarithms}}(m_\rho) = -0.031 \ \mathrm{MHz} \,,$$

which accounts for approximately 2/3 of the difference between theory (pure QED) and experiment.

$$E_{\mathrm{HF}}(QED) - E_{\mathrm{HF}}(exp) = -0.046 \ \mathrm{MHz}.$$

What is left gives the expected size of the counterterm. Experimentally what we have is $c_{4,NR}^{pe} = -48.69(3)\alpha^2$ and $c_{4,R}^{pe}(m_\rho) \simeq c_{4,R}^{p}(m_\rho) \simeq -16\alpha^2$.

$$c^{p\mu}_{4,\rm TPE} = c^{pe}_{4,\rm TPE} + [c^{p\mu}_{4,\rm TPE} - c^{pe}_{4,\rm TPE}](\chi PT) + \mathcal{O}(\alpha)\,.$$

$$\begin{aligned} c^{p\mu}_{4,\rm point-like} - c^{pe}_{4,\rm point-like} &= \left(1-\frac{\kappa_p^2}{4}\right)\ln\frac{m_\mu^2}{m_e^2} + \frac{m_\mu^2}{m_p^2}\left(1+\frac{\kappa_p}{2}(1-\frac{\kappa_p}{6})\right)\ln\frac{m_\mu^2}{\nu_{\rm pion}^2} \\ &\simeq 2.09 - 0.09 = 2.00(9)\,, \end{aligned}$$

$$c^{p\mu}_{4,\rm pol} - c^{pe}_{4,\rm pol} = \begin{cases} 0.17(9) & (\pi), \\ 0.25(10) & (\pi\&\Delta)\,, \end{cases}$$

DR (Carlson et al) $\sim -0.3(1.4)$. Relativistic χPT?

$$c^{p\mu}_{4,\rm Born} - c^{pe}_{4,\rm Born} = -\int_0^\infty dp \frac{1}{3p} G_M^{(1)}(-p^2)$$
$$\times \left[\left(\frac{p^2\kappa_p}{m_\mu^2} + \frac{32m_\mu^4 - 8m_\mu^2p^2(\kappa_p+2) - 2p^4\kappa_p}{m_\mu^2 p\left(\sqrt{4m_\mu^2+p^2}+p\right)} + 8\right) - (m_\mu \to m_e)\right]\,,$$

$G_M^{(1)}(\chi$PT): Gasser et al.; Bernard et al.

$$c^{p\mu}_{4,\rm Born} - c^{pe}_{4,\rm Born} = \begin{cases} 0 + 1.11(55) & (\pi), \\ 0 + 1.42(53) & (\pi\&\Delta)\,. \end{cases}$$

$$c^{p\mu}_{4,\mathrm{TPE}} = c^{pe}_{4,\mathrm{TPE}} + [c^{p\mu}_{4,\mathrm{TPE}} - c^{pe}_{4,\mathrm{TPE}}](\chi PT) + \mathcal{O}(\alpha)\,.$$

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Overall, combining the three contributions, we obtain

$$[c^{p\mu}_{4,\mathrm{TPE}} - c^{pe}_{4,\mathrm{TPE}}](\chi PT) = 3.68(72)$$

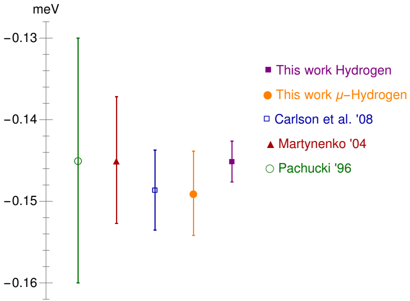

Figure : Two-photon exchange contribution to the hyperfine splitting of the 2S muonic hydrogen. Peset-Pineda

Variation of this idea has later been applied using DR (Tomalak). Error $\sim 1/2$.

Δ, (ppm)	Δ_Z	Δ_R^p	$\Delta_Z + \Delta_R^p$	Δ_0^{pol}	Δ_{HFS}
this work, μH r_E, r_M^W	$-7415(84)$	$844(7)$	$-6571(87)$	$364(89)$	$-6207(127)$
this work, electron r_E, r_M^W	$-7487(95)$	$844(7)$	$-6643(98)$	$364(89)$	$-6279(135)$
this work, μH r_E, r_M^e	$-7333(48)$	$846(6)$	$-6486(49)$	$364(89)$	$-6122(105)$
this work, electron r_E, r_M^e	$-7406(56)$	$847(6)$	$-6559(57)$	$364(89)$	$-6195(109)$
Hagelstein et al. [59]				-61^{+70}_{-52}	
Peset et al. [29]					$-6247(109)$
Carlson et al. [28, 39]	-7587	835	$-6752(180)$	$351(114)$	$-6401(213)$
Martynenko et al. [38]	-7180		-6656	$410(80)$	$-6246(342)$
Pachucki [7]	-8024		-6358	$0(658)$	$-6358(658)$

Figure : From Tomalak, 2017

CONCLUSIONS

- Effective Field Theories provide with a model independent, efficient and systematic (Power counting) approach to the dynamics of NR systems and a unified framework to determine the nonperturbative effects.
- Rigorous connection between Quantum Field Theories (Wilson coefficients) and a NR Quantum-mechanical formulation of the NR systems (potentials). For instance. The proton radius is a Wilson coefficient of the effective theory. In general it is an scheme/scale dependent object.
- Unlike dispersion relations, no assumption on the high energy behavior.
- The spin-independent TPE energy shift (and the associated error) is (and can only be) computed in a model independent way with χPT. Overall number consistent with determinations from a combined use of dispersion relations and models, but individual contributions are quite different.
- χPT predicts the chiral logs of the hyperfine splitting and the difference between hydrogen and muonic hydrogen.
- Analytic understanding of the QCD dynamics: m_q and N_c dependence.

$$E_{\rm HF}(1S) = 182.623(27)\,\mathrm{meV}, \qquad E_{\rm HF}(2S) = 22.8123(33)\,\mathrm{meV}$$

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CONCLUSIONS

- Effective Field Theories provide with a model independent, efficient and systematic (Power counting) approach to the dynamics of NR systems and a unified framework to determine the nonperturbative effects.
- Rigorous connection between Quantum Field Theories (Wilson coefficients) and a NR Quantum-mechanical formulation of the NR systems (potentials). For instance. The proton radius is a Wilson coefficient of the effective theory. In general it is an scheme/scale dependent object.
- Unlike dispersion relations, no assumption on the high energy behavior.
- The spin-independent TPE energy shift (and the associated error) is (and can only be) computed in a model independent way with χPT. Overall number consistent with determinations from a combined use of dispersion relations and models, but individual contributions are quite different.
- χPT predicts the chiral logs of the hyperfine splitting and the difference between hydrogen and muonic hydrogen.
- Analytic understanding of the QCD dynamics: m_q and N_c dependence.

$$E_{\rm HF}(1S) = 182.623(27)\,\mathrm{meV}, \qquad E_{\rm HF}(2S) = 22.8123(33)\,\mathrm{meV}$$

$$8E_{\rm HF}(2S) - E_{\rm HF}(1S) = -0.124(3)\,\mathrm{meV}$$

BACK UP SLIDES

Definition of the proton radius

$$\langle p', s | J^\mu | p, s \rangle = \bar{u}(p') \left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu} q_\nu}{2m_p} \right] u(p) \,,$$

$$F_i(q^2) = F_i + \frac{q^2}{m_p^2} F_i' + \ldots$$

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2} F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2).$$

$$r_p^2 = 6 \frac{d}{dq^2} G_{E,p}(q^2)|_{q^2=0}$$

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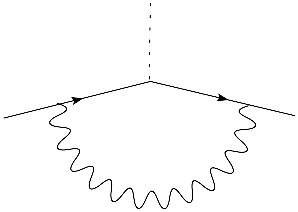
$r_p^2(\nu) = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0}$

Infrared divergent! $\rightarrow$ Wilson coefficient



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$$r_p^2(\nu) = 6\frac{d}{dq^2}G_{E,p}(q^2)|_{q^2=0} = \frac{3}{4}\frac{1}{m_p^2}\left(c_D^{(p)}(\nu) - 1\right)$$

$$c_D(\nu) = 1 + 2F_2 + 8F_1' = 1 + 8m_p^2\left.\frac{dG_{p,E}(q^2)}{d\,q^2}\right|_{q^2=0} \,,$$

Standard definition (corresponds to the experimental number):

$$r_p^2 = \frac{3}{4}\frac{1}{m_p^2}\left(c_D(\nu) - c_{D,point-like}(\nu)\right)$$

$$c_{D,point-like} = 1 + \frac{\alpha}{\pi}\left(\frac{4}{3}\ln\frac{m_p^2}{\nu^2}\right)$$

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$c_3^{\rm had}$: Two-Photon-Exchange contribution= Born+polarizability

Born:

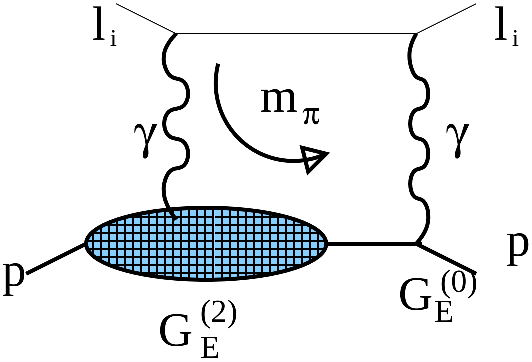

Figure : *Symbolic representation (plus permutations) of the Born $\langle r^3 \rangle$ correction.*

$$\Delta E_{\rm Born} = 0.010 \frac{\langle r^3 \rangle_{(2)}}{\rm fm^3}$$

$$\frac{\langle r^3 \rangle_{(2)}}{\rm fm^3} = \frac{48}{\pi} \int \frac{d^3k}{4\pi} \frac{1}{\mathbf{k}^6} \left(G_E^2 - 1 + \frac{1}{3} \langle r^2 \rangle \mathbf{k}^2 \right) = \frac{96}{\pi} \int \frac{d^{D-1}k}{4\pi} \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

$$\delta c_{3,\rm Born}^{pl_i} = \frac{\pi}{3} \alpha^2 m_p^2 m_\mu \langle r^3 \rangle_{(2)} = 2(\pi\alpha)^2 \left(\frac{m_p}{4\pi F_0} \right)^2 \frac{m_{l_i}}{m_\pi} \left\{ \frac{3}{4} g_A^2 + \frac{1}{8} + \frac{2}{\pi} g_{\pi N\Delta}^2 \frac{m_\pi}{\Delta} \sum_{r=0}^{\infty} C_r \left(\frac{m_\pi}{\Delta} \right)^{2r} + g_{\pi N\Delta}^2 \sum_{r=1}^{\infty} H_r \left(\frac{m_\pi}{\Delta} \right)^{2r} \right\},$$

where ($\Delta = M_\Delta - M_p \sim 300$ MeV)

$$C_r = \frac{(-1)^r \Gamma(-3/2)}{\Gamma(r+1)\Gamma(-3/2-r)} \left\{ B_{6+2r} - \frac{2(r+2)}{3+2r} B_{4+2r} \right\}, \qquad r \geq 0,$$

$$B_n \equiv \int_0^\infty dt \frac{t^{2-n}}{\sqrt{1-t^2}} \ln \left[\frac{1}{t} + \sqrt{\frac{1}{t^2} - 1} \right]$$

$$H_n \equiv \frac{n!(2n-1)!!\Gamma[-3/2]}{2(2n)!!\Gamma[1/2+n]}.$$

Including Pions and Δ particles

$$\Delta E_{\rm Born} = 0.010 \frac{\langle r^3 \rangle_{(2)}}{\rm fm^3}$$

$$\frac{\langle r^3 \rangle_{(2)}}{\rm fm^3} = \frac{48}{\pi} \int \frac{d^3k}{4\pi} \frac{1}{\mathbf{k}^6} \left(G_E^2 - 1 + \frac{1}{3} \langle r^2 \rangle \mathbf{k}^2 \right) = \frac{96}{\pi} \int \frac{d^{D-1}k}{4\pi} \frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}$$

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Including Pions and Δ particles

	$\langle r^3 \rangle$	$\langle r^5 \rangle$	$\langle r^6 \rangle$	$\langle r^3 \rangle_{(2)}$
π	0.4980	1.619	5.203	0.9960
$\pi\&\Delta$	0.4071	1.522	4.978	0.8142
Dipole	0.7706	1.775	3.325	2.023
Kelly	0.9838	3.209	7.440	2.526
Distler et al.	1.16(4)	8.0(1.2)(1.0)	29.8(7.6)(12.6)	2.85(8)

Table : *The first two rows give the prediction from the effective theory (Peset&AP). The third row corresponds to the standard dipole fit with* $\langle r^2 \rangle = 0.6581$ *fm*^{3}*. The fourth and fifth rows correspond to different parameterizations of experimental data. For completeness, we also quote* $\langle r^3 \rangle_{(2)} = 2.71$ *fm*^{3} *from Friar.*

μeV	DR	*Pachucki*	*Carlson et al*	HBET	*Peset&AP*(π)	$(\pi\&\Delta)$
$\Delta E_{\rm Born}$		23.2(1.0)	24.7(1.6)		10.1(5.1)	8.3(4.3)

Table : *Predictions for the Born contribution to the* $n = 2$ *Lamb shift. The first two entries correspond to dispersion relations. The last two entries are the predictions of HBET: The 3rd entry is the prediction of HBET at leading order (only pions) and the last entry is the prediction of HBET at leading and next-to-leading order (pions and Deltas).*

The proton radius in *ep* scattering from χPT

Hessels, Horbatsch, AP

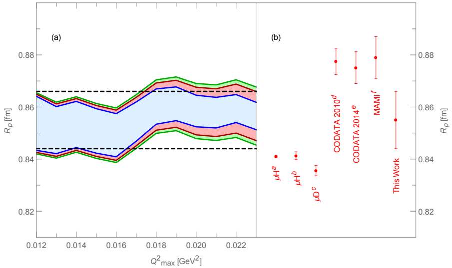
$$G_E(Q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} Q^{2n} \langle r^{2n} \rangle$$

- Extrapolation from $|\mathbf{q}| \sim 100$ MeV to $|\mathbf{q}| = 0$
- dependence on the fitting functions: normalization factors, full data set ...

Higher moments diverge in the chiral limit

$$\langle r^{2k} \rangle \sim m_\pi^{2-2k}$$

Extrapolation controlled by χPT (at low Q^2): $r_p \sim 0.855$.
Bigger values for the moments produce larger values of r_p.



c_3 Polarizability effects

(μeV)	[1]	[2]	[3]	[4]	BχPT(π)	HBET(π)	(π&Δ)
ΔE_{pol}	12(2)	11.5	7.4(2.4)	15.3(5.6)	$8.2(^{+1.2}_{-2.5})$	18.5(9.3)	26.2(10.0)

Table : *Polarizability contribution to the $n = 2$ Lamb shift. The first four entries use dispersion relations for the inelastic term and different modeling functions for the subtraction term. [1] Pachucki, [2] Martynenko, [3] Carlson&Vanderhaeghen, [4] Gorchtein et al.. The 5th entry is the prediction obtained using BχPT (Alarcon et al.). The last two entries are the predictions of HBET (Nevado&AP and Peset&AP).*

Polarizability=Inelastic+subtraction

$$c_{3,\mathrm{sub}}^{pl_i} = -e^4 M_p m_{l_i} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} (3k_{0,E}^2 + \mathbf{k}^2) S_1(0, -k_E^2)$$

$$c_{3,\mathrm{inel}}^{pl_i} = -e^4 M_p m_{l_i} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} \times \left\{ (3k_{0,E}^2 + \mathbf{k}^2)(S_1(ik_{0,E}, -k_E^2) - S_1(0, -k_E^2)) - \mathbf{k}^2 S_2(ik_{0,E}, -k_E^2) \right\}$$

$$\Delta E^{(\mathrm{sub})}(\pi\mathrm{-loop}) = -1.62\ \mu\mathrm{eV}\ ; \qquad \Delta E^{(\mathrm{sub})}(\pi\Delta\mathrm{-loop}) = -1.23\ \mu\mathrm{eV}.$$

$$\delta c_{3,\mathrm{sub}}^{pl_i} \sim -\delta c_{3,\mathrm{inel}}^{pl_i} \simeq -\frac{4}{3}\alpha^2 \frac{m_{l_i}}{\Delta} b_{1,F}^2 \ln(\nu/m_{l_i}) \to \Delta E^{(\mathrm{sub})}|_{\nu=m_\rho} \sim -11.4\ \mu\mathrm{eV}$$

Polarizability=Inelastic+subtraction

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$$c_{3,\rm inel}^{pl_i} = -e^4 M_p m_{l_i} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_{l_i}^2 k_{0,E}^2} \times \left\{ (3k_{0,E}^2 + \mathbf{k}^2)(S_1(ik_{0,E}, -k_E^2) - S_1(0, -k_E^2)) - \mathbf{k}^2 S_2(ik_{0,E}, -k_E^2) \right\}$$

$$\Delta E^{(\rm sub)}(\pi{-}\rm loop) = -1.62\ \mu eV\ ; \qquad \Delta E^{(\rm sub)}(\pi\Delta{-}\rm loop) = -1.23\ \mu eV.$$

$$\delta c_{3,\rm sub}^{pl_i} \sim -\delta c_{3,\rm inel}^{pl_i} \simeq -\frac{4}{3}\alpha^2 \frac{m_{l_i}}{\Delta} b_{1,F}^2 \ln(\nu/m_{l_i}) \to \Delta E^{(\rm sub)}|_{\nu=m_\rho} \sim -11.4\ \mu eV$$

	$\mathcal{O}(m_r\alpha^3)$	$V_{\rm VP}^{(0)}$	205. 00737
	$\mathcal{O}(m_r\alpha^4)$	$V_{\rm VP}^{(0)}$	1. 50795
	$\mathcal{O}(m_r\alpha^4)$	$V_{\rm VP}^{(0)}$	0. 15090
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm VP}^{(0)}$	0. 00752
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm LbL}^{(0)}$	−0. 00089(2)
	$\mathcal{O}(m_r\alpha^4 \times \frac{m_\mu^2}{m_p^2})$	$V^{(2,1)} + V^{(3,0)}$	0. 05747
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm VP}^{(2,2)} + V^{(2,1)} \times V_{\rm VP}^{(0,2)}$	0. 01876
	$\mathcal{O}(m_r\alpha^5)$	$V_{\rm no-VP}^{(2,2)} + \text{ultrasoft}$	−0. 71896
	$\mathcal{O}(m_r\alpha^6 \times \ln(\frac{m_\mu}{m_e}))$	$V^{(2,3)}; c_D^{(\mu)}$	−0. 00127
	$\mathcal{O}(m_r\alpha^6 \times \ln\alpha)$	$V_{\rm VP}^{(2,3)}; c_D^{(\mu)}$	−0. 00454
	$\mathcal{O}(m_r\alpha^4 \times m_r^2 r_p^2)$	$V^{(2,1)}; c_D^{(p)}$	$-5.\ 1975\frac{r_p^2}{\rm fm^2}$
	$\mathcal{O}(m_r\alpha^5 \times m_r^2 r_p^2)$	$V_{\rm VP}^{(2,2)} + V^{(2,1)} \times V_{\rm VP}^{(0,2)}; c_D^{(p)}$	$-0.\ 0282\frac{r_p^2}{\rm fm^2}$
	$\mathcal{O}(m_r\alpha^6 \ln\alpha \times m_r^2 r_p^2)$	$V^{(2,3)}; c_D^{(p)}$	$-0.\ 0014\frac{r_p^2}{\rm fm^2}$
	$\mathcal{O}(m_r\alpha^5 \times \frac{m_r^2}{m_\rho^2})$	$V_{\rm VP_{had}}^{(2)}; d_2^{\rm had}$	0. 0111(2)
	$\mathcal{O}(m_r\alpha^5 \times \frac{m_r^2}{m_\rho^2}\frac{m_\mu}{m_\pi})$	$V^{(2)}; c_3^{\rm had}$	0. 0344(125)