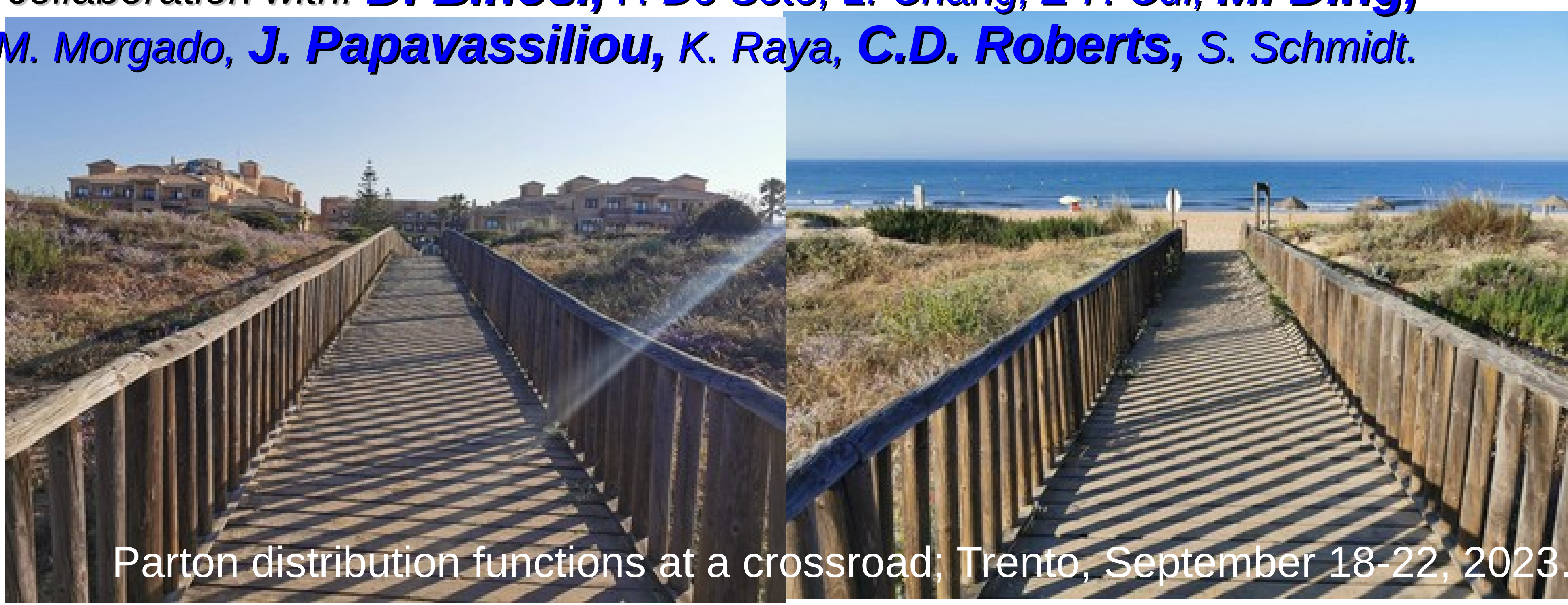
All-orders Evolution of PDFs: Principle, Practice and Predictions

J. Rodríguez-Quintero

In collaboration with: **D. Binosi,** F. De Soto, L. Chang, Z-F. Cui, **M. Ding,** J.M. Morgado, **J. Papavassiliou,** K. Raya, **C.D. Roberts,** S. Schmidt.



Parton distribution functions at a crossroad; Trento, September 18-22, 2023.

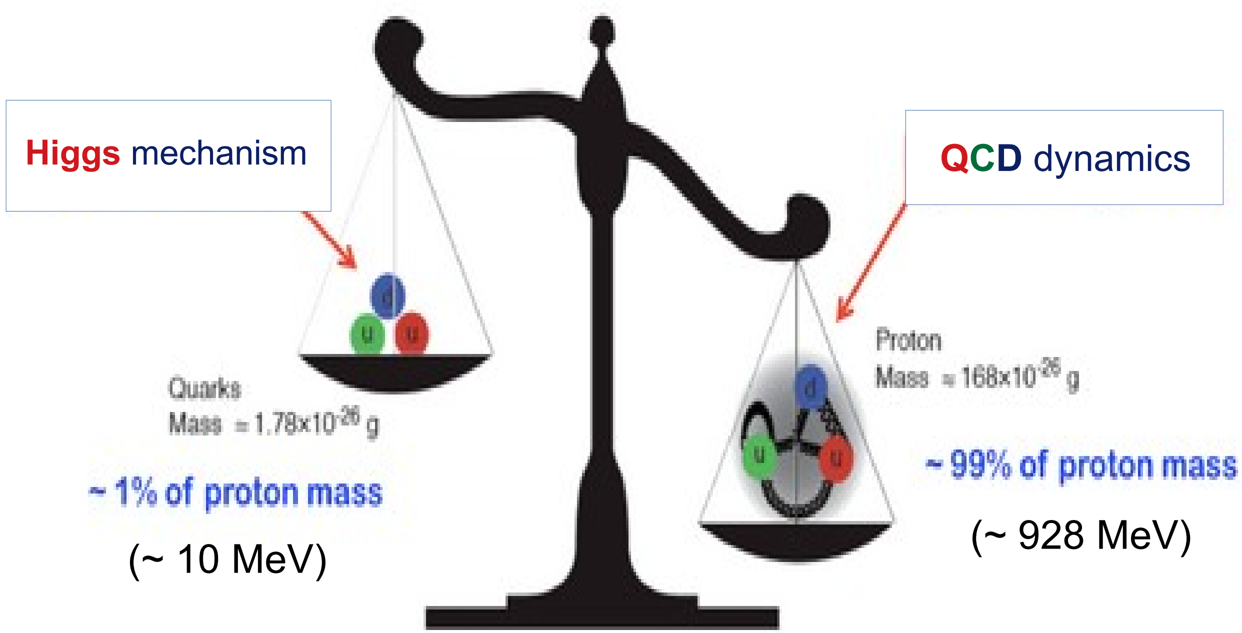
QCD: effective charges

➢ **QCD** is characterized by two **emergent** phenomena: confinement and dynamical generation of mass (DGM).

$$\mathcal{L}_{\rm QCD} = \sum_{j=u,d,s,\ldots} \bar{q}_j[\gamma_\mu D_\mu + m_j]q_j + \tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu},$$
$$D_\mu = \partial_\mu + ig\tfrac{1}{2}\lambda^a A^a_\mu,$$
$$G^a_{\mu\nu} = \partial_\mu A^a_\nu + \partial_\nu A^a_\mu - \underline{g f^{abc} A^b_\mu A^c_\nu},$$

- Quarks and gluons not *isolated* in nature.
  - Formation of colorless bound states: "**Hadrons**"
  - **1-fm scale** size of hadrons?

- Emergence of hadron masses (EHM) from QCD **dynamics**



QCD: effective charges

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Can we trace them down to fundamental d.o.f ?

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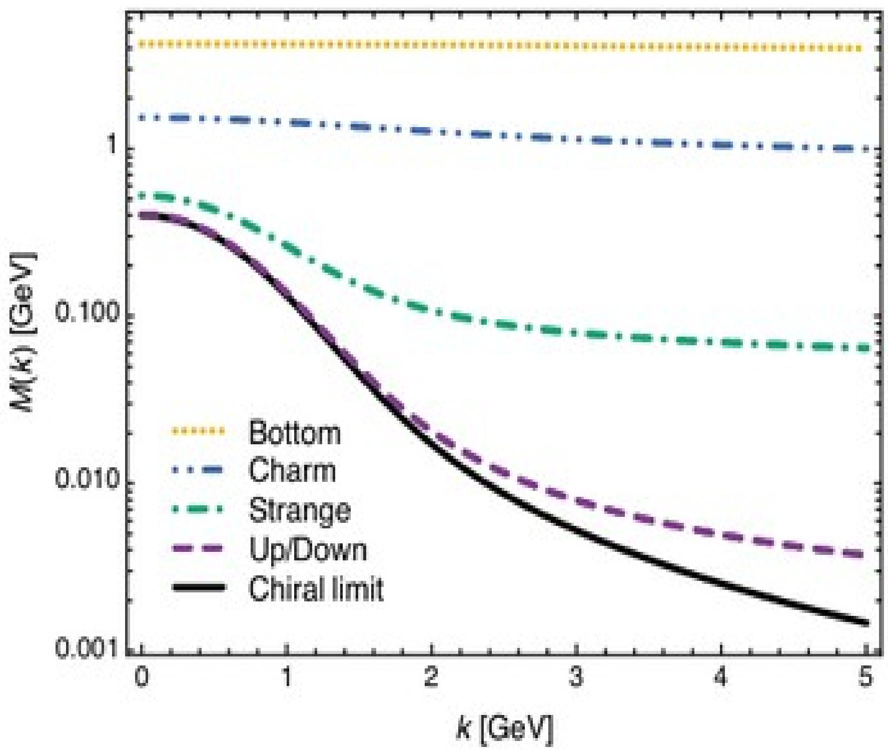
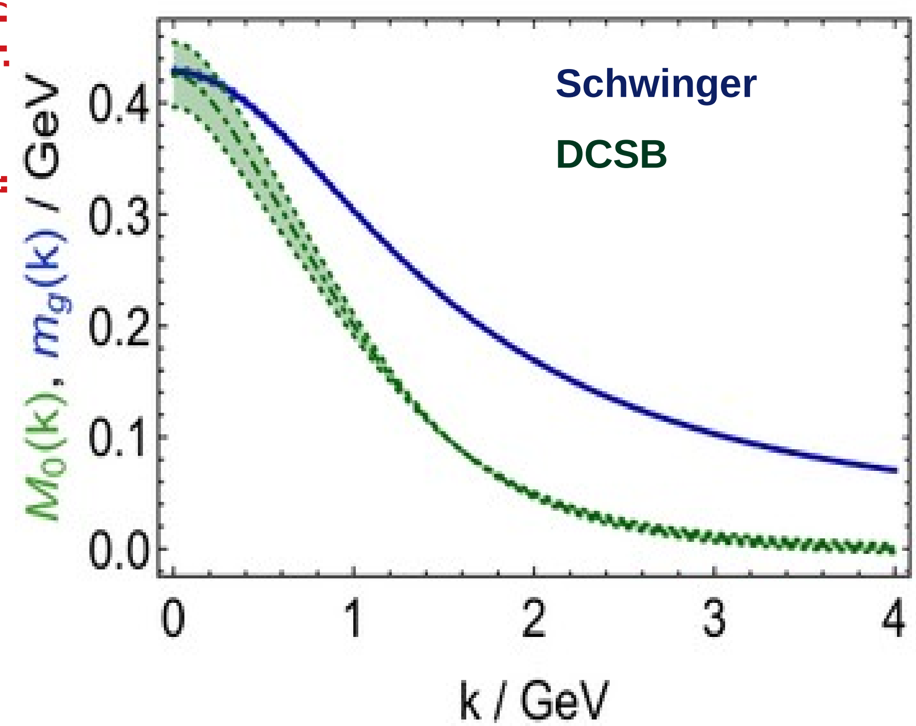
Dynamical masses

(**D**ynamical **C**hiral **S**ymmetry **B**reaking)



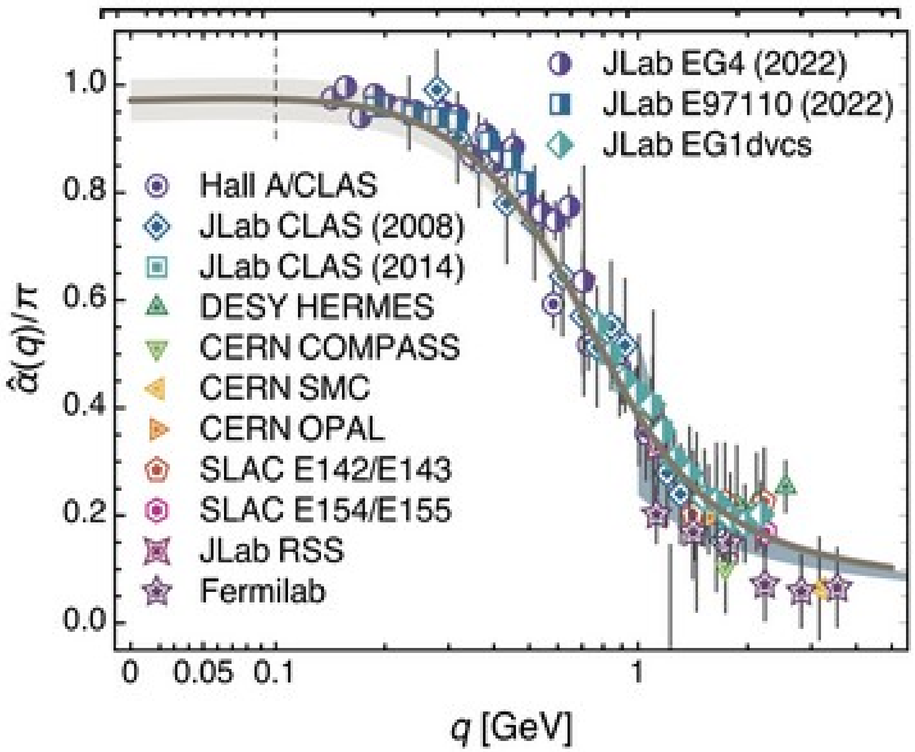
$$S_f^{-1}(p) = Z_f^{-1}(p^2)(i\gamma\cdot p + \mathrm{M_f(p^2)})$$

"Higgs" masses



Gluon and quark *running masses*

QCD: effective charges



$$\alpha(k^2) = \frac{\gamma_m \pi}{\ln\left[\frac{\mathcal{M}^2(k^2)}{\Lambda^2_{\rm QCD}}\right]}\ ;\ \alpha(0) = 0.97(4)$$

where

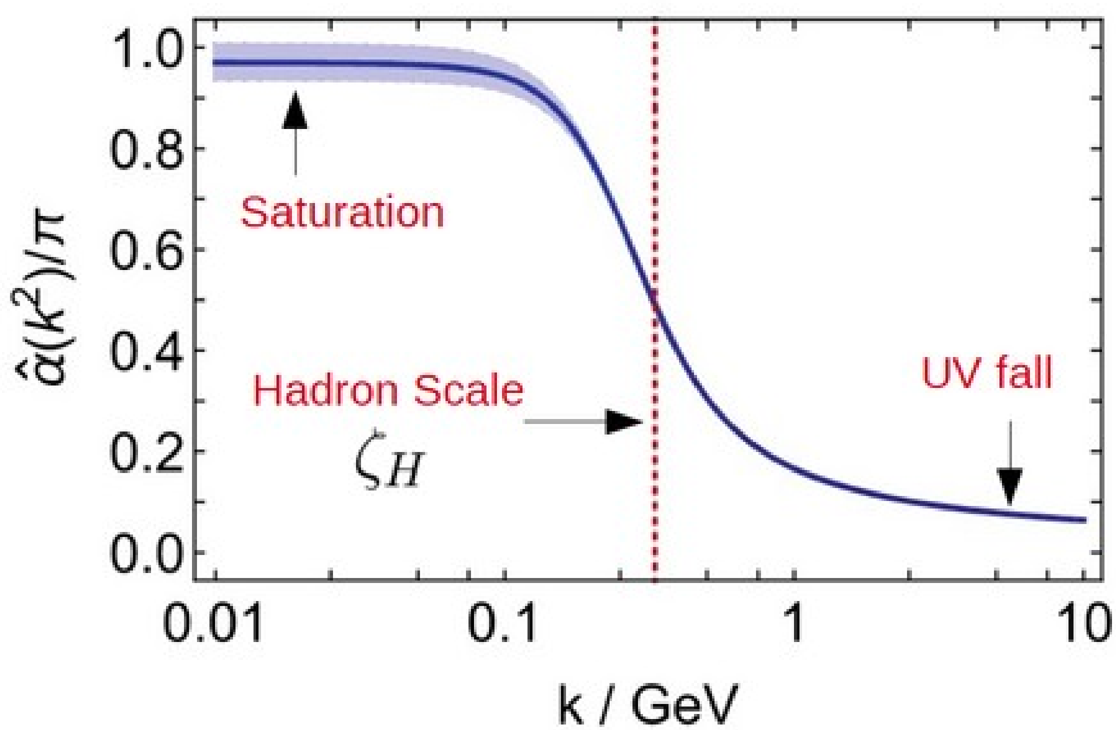
$$\mathcal{M}(k^2 = \Lambda^2_{\rm QCD}) := m_G = 0.331(2)\ \text{GeV}$$

defines the screening mass and an associated wavelength, such that larger gluon modes decouple.

Then, we identify: $\zeta_H := m_G(1 \pm 0.1)$

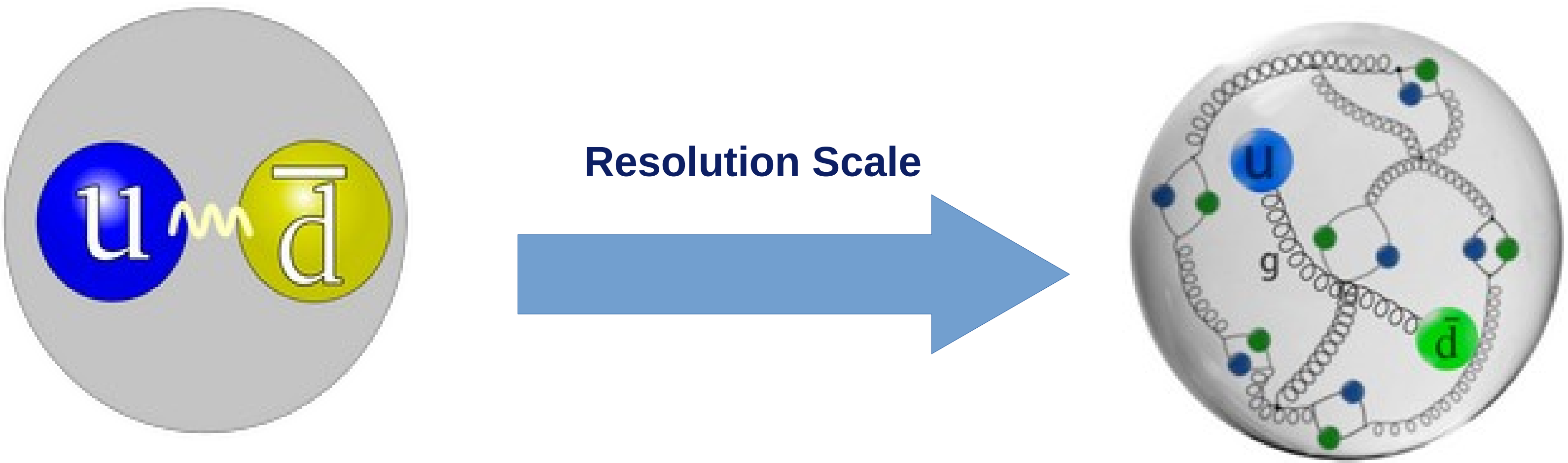
Modern continuum & lattice QCD analysis in the gauge sector delivers an analogue "Gell-Mann-Low" running charge, from which one obtains a process-independent, parameter-free prediction for the low-momentum saturation

- No landau pole
- Below a given mass scale, the interaction become scale-independent and QCD practically conformal again (as in the lagrangian).



Parton distributions: energy scales

ζ_H $\qquad\qquad\qquad$ $\zeta > \zeta_H$



- **Fully-dressed valence quarks**

  (quasiparticles)

- **Unveiling of glue and sea d.o.f.**

  (partons)

Parton distributions: energy scales

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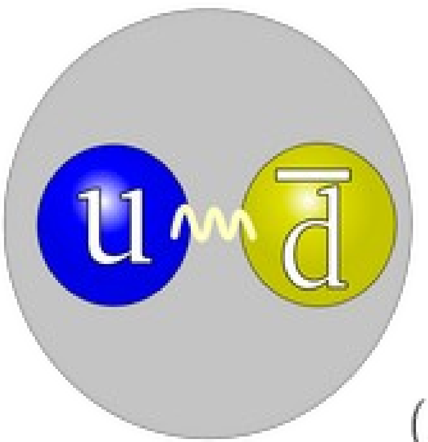

$(M_u = M_d)$

- **Fully-dressed valence quarks**
  - At this scale, **all properties** of the hadron are contained within their valence quarks.
  - **QCD constraints** are defined from here
    (e.g. large-x behavior of the PDF)

$$u^{\pi}(x;\zeta) \overset{x\simeq 1}{\approx} (1-x)^{\beta\,=\,2+\gamma(\zeta)}$$

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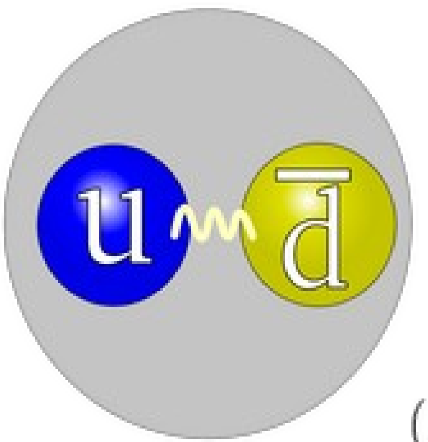

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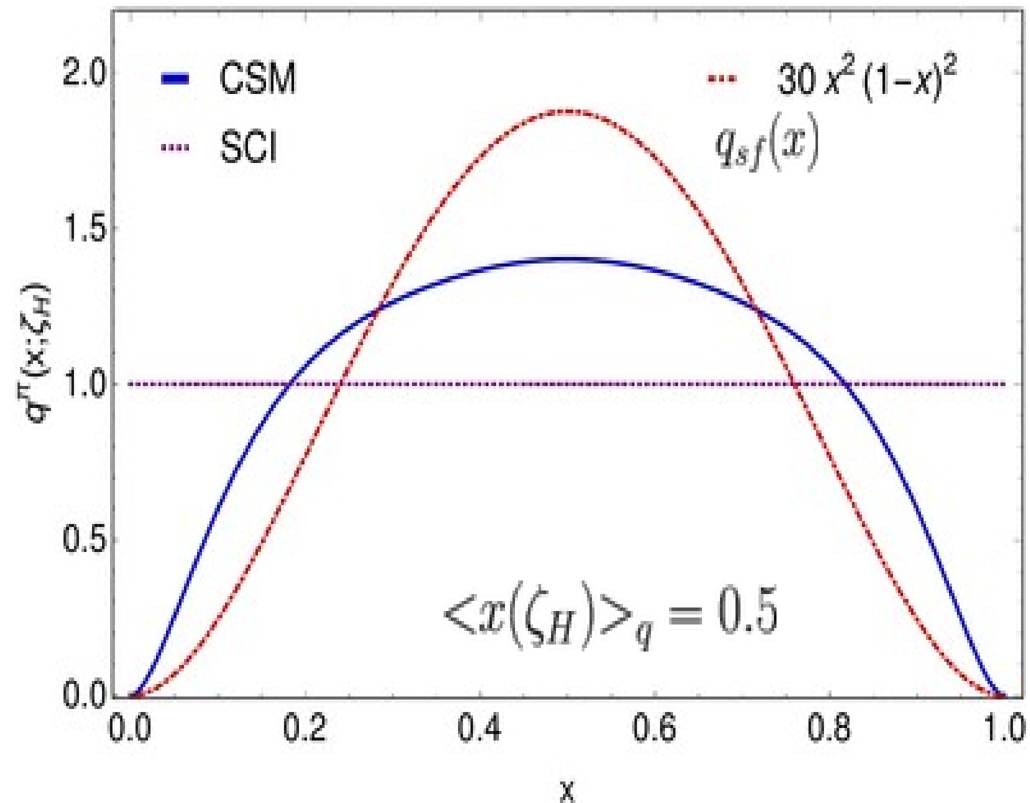

- **CSM** results produce:
  - **EHM-induced** dilated distributions
  - Soft end-point behavior

Cui:2020tdf

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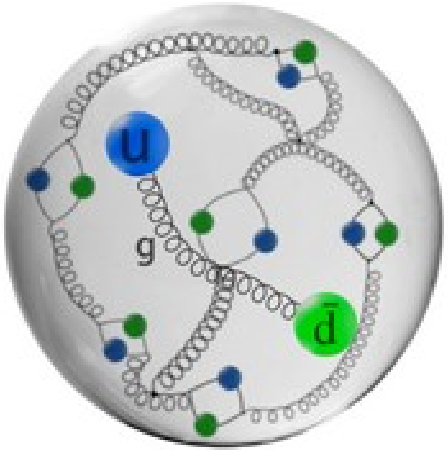
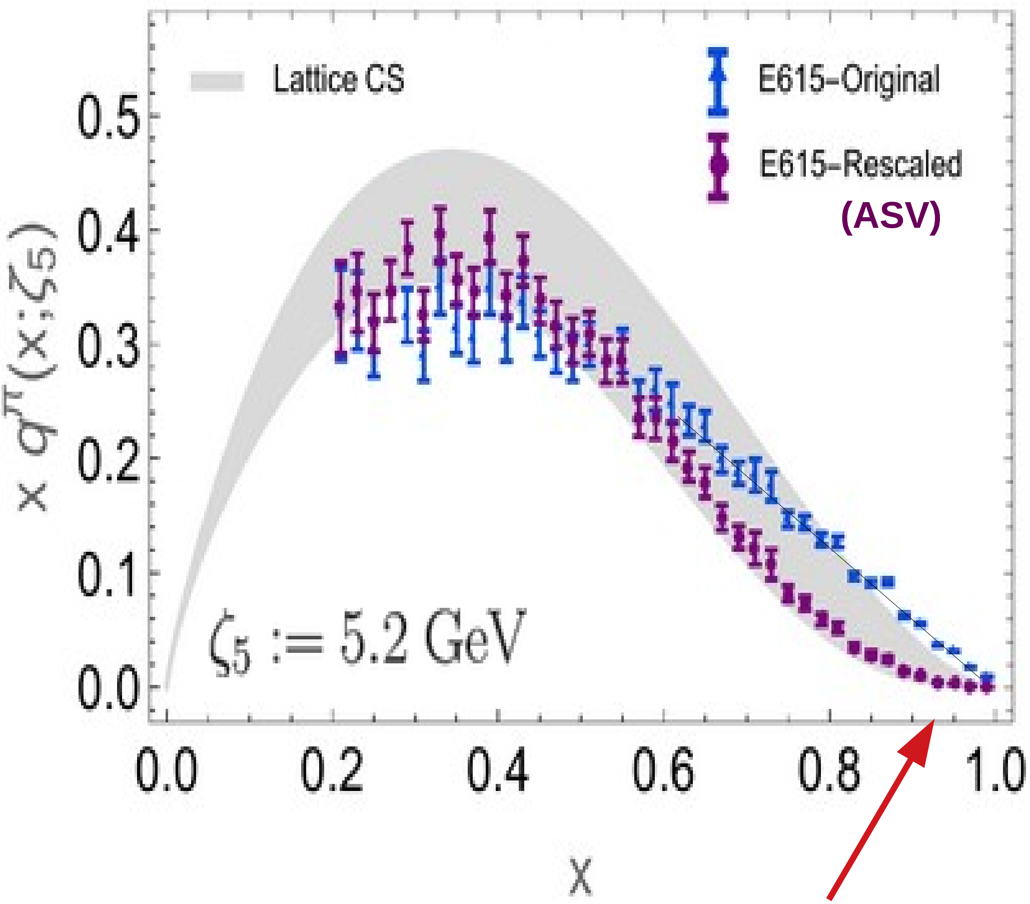


- **Unveiling of glue and sea d.o.f.**

- **Experimental** data is given **here**.
- The interpretation of parton distributions from cross sections demands **special care**.
- In addition, the synergy with **lattice QCD** and phenomenological approaches is welcome.

Parton distributions: energy scales

Conway:1989fs Aicher:2010cb Sufian:2020vzb



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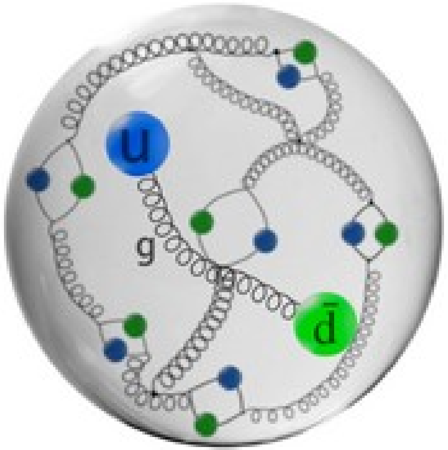

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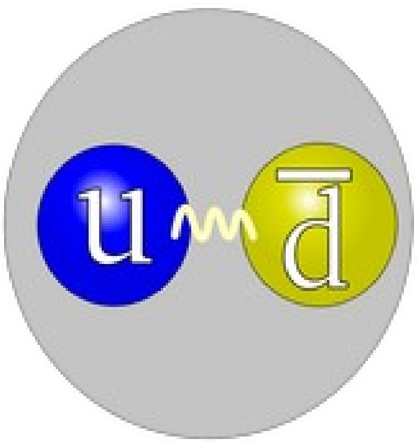
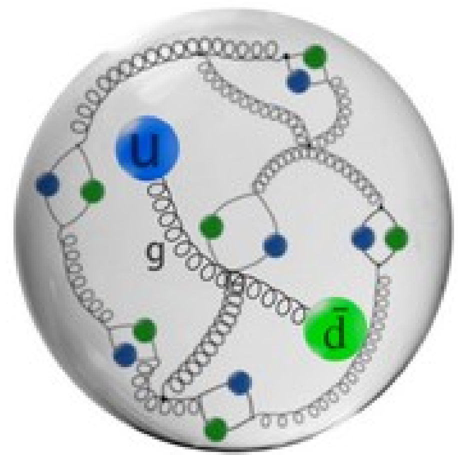


Resolution Scale

Evolution equations

$\zeta > \zeta_H$



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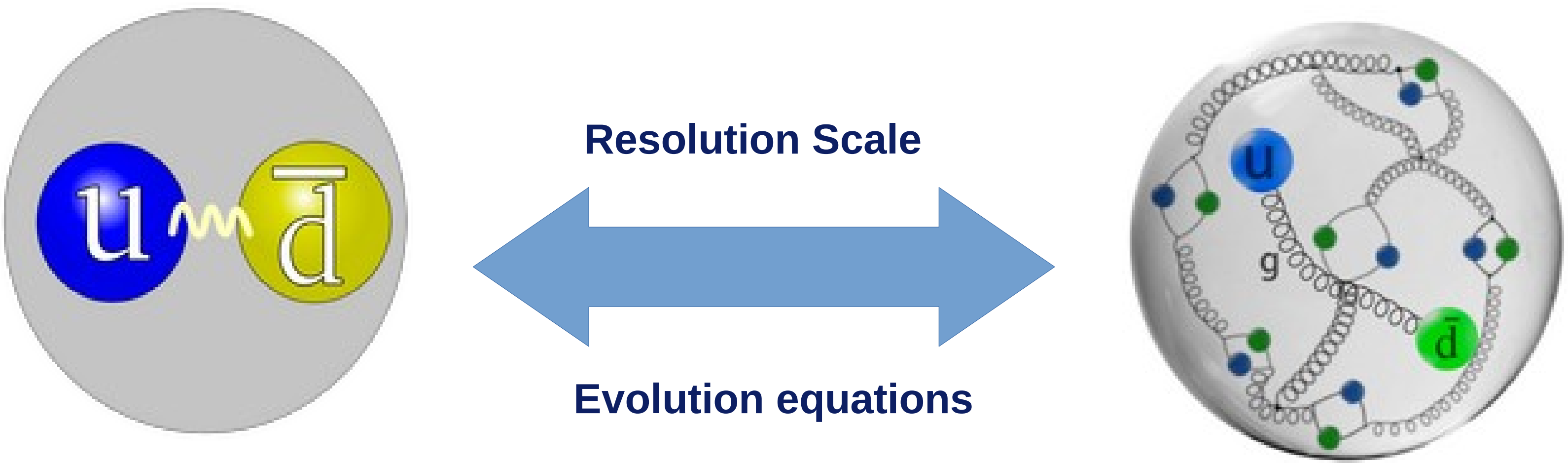

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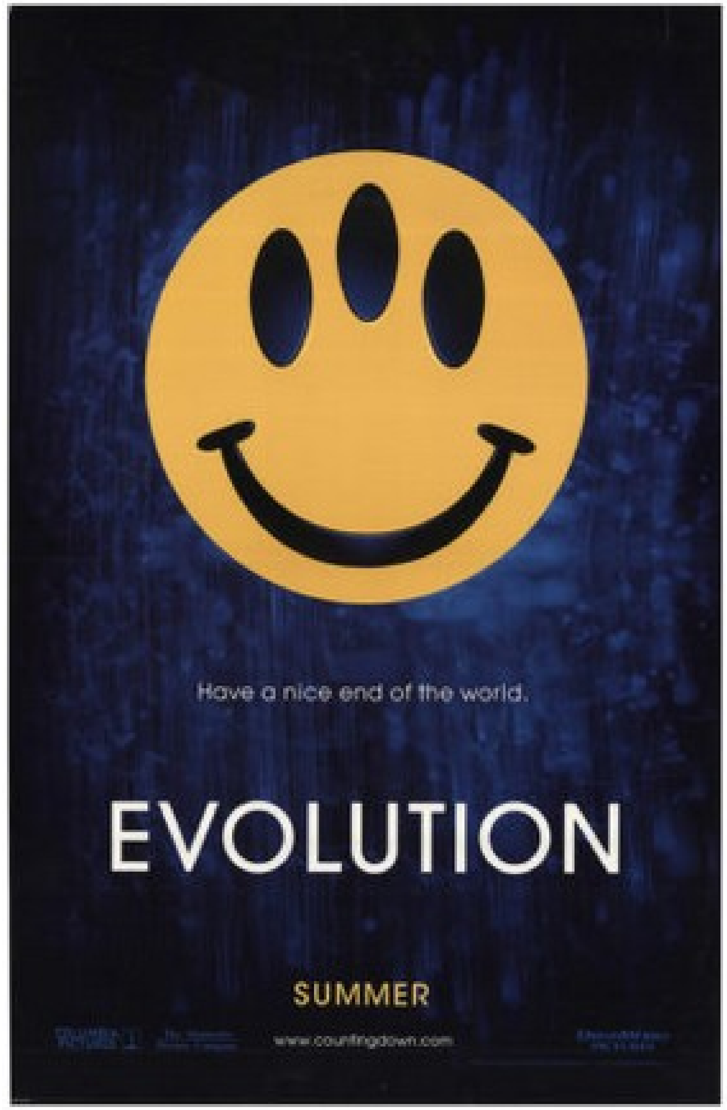
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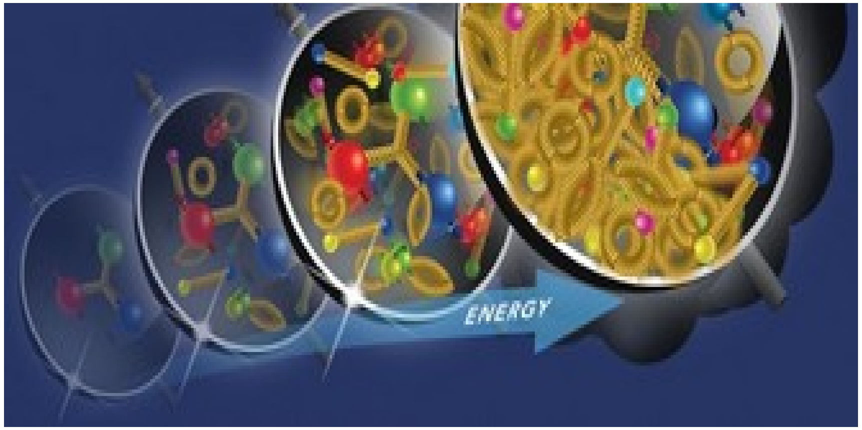
DGLAP: All orders evolution

Raya:2021zrz

Cui:2020tdf

$$\left\{\zeta^2 \frac{d}{d\zeta^2}\int_0^1 dy\delta(y-x) \; - \; \frac{\alpha(\zeta^2)}{4\pi}\int_x^1 \frac{dy}{y}\begin{pmatrix} P_{qq}^{\mathrm{NS}}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\mathrm{S}}\left(\frac{\mathbf{x}}{\mathbf{y}}\right)\end{pmatrix}\right\}\begin{pmatrix} H_\pi^{\mathrm{NS},+}(y,t;\zeta) \\ \mathbf{H}_\pi^{\mathrm{S}}(y,t;\zeta)\end{pmatrix} = 0$$

DGLAP leading-order evolution equations



DGLAP: All orders evolution

Assumption: define an **effective** charge such that

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Starting from fully-dressed **quasiparticles**, at ζ_H

Sea and **Gluon** content unveils, as prescribed by **QCD**

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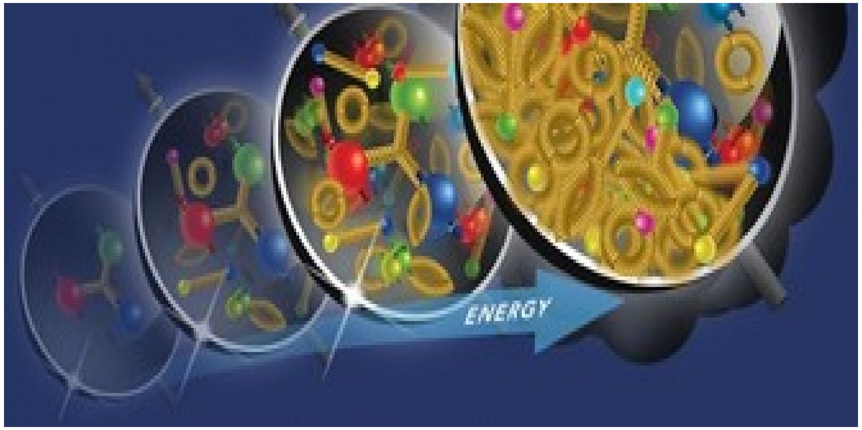

- **Not** the **LO** QCD coupling but an **effective** one.
- Making this equation **exact**.
- Connecting with the **hadron scale**, at which the **fully-dressed** valence-**quarks** express **all** of the hadron's properties.

(thus carrying all the momentum)

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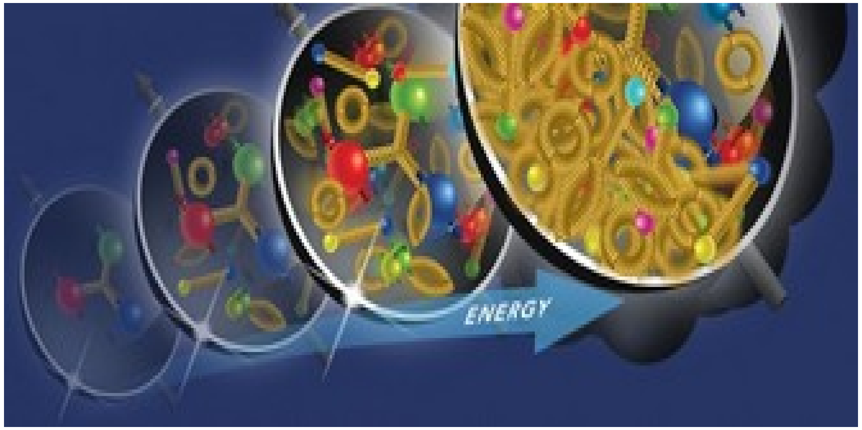
$$\gamma_{AB}^{(n)}=-\int_0^1 dx\, x^n P_{AB}^C(x)$$



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PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\mathsf{H}}(x) = q_{\mathsf{H}}(x) + \bar{q}_{\mathsf{H}}(x)$$

singlet combination

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Valence-quark PDF in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_H}$$

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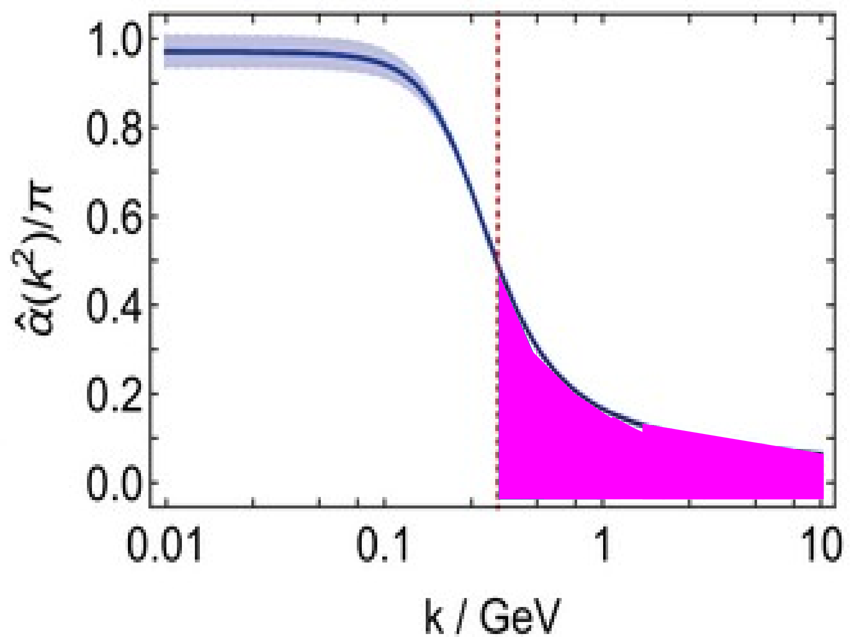

Moments' evolution is controlled by the integrated "strength" of the coupling beyond the hadron scale

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The ratio of lightcone momentum fractions encodes the required information of the charge

DGLAP: All orders evolution

Cui:2020tdf

Implication 1: valence-quark PDF

$$\langle x^n \rangle_{q_H}^{\zeta} = \langle x^n \rangle_{q_H}^{\zeta_H} \exp\left(-\frac{\gamma_{qq}^n}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right) = \langle x^n \rangle_{q_H}^{\zeta_H} \left[\frac{\langle x \rangle_{q_H}^{\zeta}}{\langle x \rangle_{q_H}^{\zeta_H}} \right]^{\gamma_{qq}^n / \gamma_{qq}}$$

This ratio encodes the information of the charge

Direct connection bridging from hadron to experimental scale: only one input is needed to evolve “all” the Mellin moments up and reconstruct the PDF.

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Direct connection bridging from hadron to experimental scale: only one input is needed to evolve "all" the Mellin moments up and reconstruct the PDF.

This ratio encodes the information of the charge and use isospin symmetry (pion case)

$$\langle x\rangle_{u_\pi}^{\zeta_H} = \langle x\rangle_{d_\pi}^{\zeta_H} = \frac{1}{2}$$

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$$\langle x \rangle^{\zeta_H}_{u_\pi} = \langle x \rangle^{\zeta_H}_{d_\pi} = \frac{1}{2}$$

Capitalizing on the Mellin moments of asymptotically large order:

$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)} \left(1 + \mathcal{O}(1-x)\right)$$

$$\beta(\zeta) = \beta(\zeta_H) + \frac{3}{2} \ln \frac{\langle x(\zeta_H) \rangle}{\langle x(\zeta) \rangle}$$

DGLAP: All orders evolution

Cui:2020tdf

Implication 1: valence-quark PDF

$$\langle x^n \rangle_{q_\pi}^{\zeta} = \langle x^n \rangle_{q_\pi}^{\zeta_H} \exp\left(-\frac{\gamma_{qq}^n}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right) = \langle x^n \rangle_{q_\pi}^{\zeta_H} \left[\langle 2x \rangle_{q_\pi}^{\zeta} \right]^{\gamma_{qq}^n/\gamma_{qq}}$$

This ratio encodes the information of the charge and use isospin symmetry (pion case)

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Under a sensible assumption at large momentum scale:

$$q(x;\zeta) \underset{x\to 0}{\sim} x^{\alpha(\zeta)} \left(1 + \mathcal{O}(x)\right)$$

$$1 + \alpha(\zeta) = \frac{3}{2} \langle x(\zeta)\rangle \ln \frac{\langle x(\zeta_H)\rangle}{\langle x(\zeta)\rangle} + \beta(\zeta_H)\langle x(\zeta)\rangle + \mathcal{O}\left(\frac{\langle x(\zeta)\rangle}{\ln \frac{\langle x(\zeta_H)\rangle}{\langle x(\zeta)\rangle}} \right)$$

DGLAP: All orders evolution

Cui:2020tdf

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$$\langle x^n \rangle_{q_\pi}^{\zeta} = \langle x^n \rangle_{q_\pi}^{\zeta_H} \exp\left(-\frac{\gamma_{qq}^n}{2\pi}\int_{\zeta_H}^{\zeta}\frac{dz}{z}\alpha(z^2)\right) = \langle x^n \rangle_{q_\pi}^{\zeta_H}\left[\langle 2x \rangle_{q_\pi}^{\zeta}\right]^{\gamma_{qq}^n/\gamma_{qq}}$$

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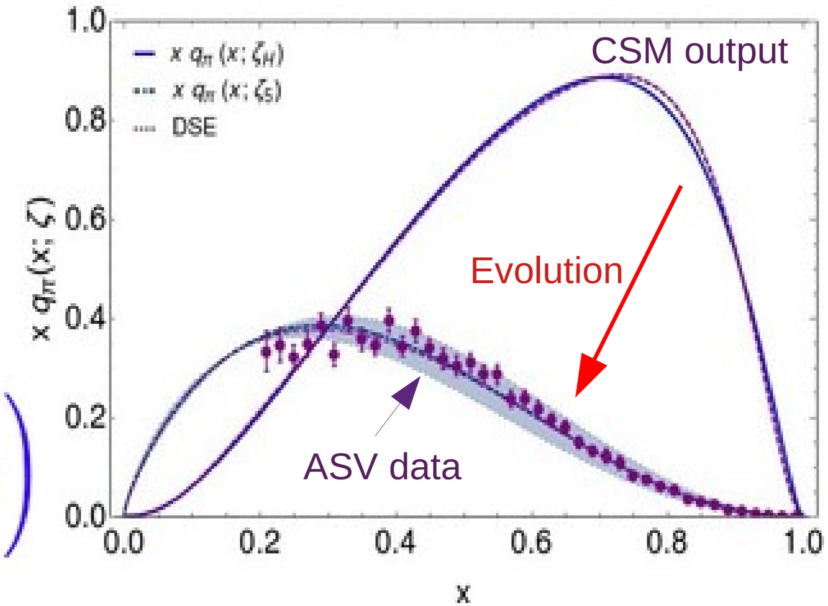
Reconstruction after evolving:



DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1} \rangle_{u_\pi}^{\zeta_H} = \frac{1}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \begin{pmatrix} 2(n+1) \\ j \end{pmatrix} \langle x^j \rangle_{u_\pi}^{\zeta_H}$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{\left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \begin{pmatrix} 2(n+1) \\ j \end{pmatrix} \langle x^j\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^j/\gamma_0^1}.$$

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DGLAP: All orders evolution

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- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{\left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{\gamma_0^{2n+1}/\gamma_0^{1}}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^{j}/\gamma_0^{1}}.$$

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Reported lattice moments agree very well with the recursion formula

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7		0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{(\langle 2x\rangle^{\zeta}_{u_\pi})^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \begin{pmatrix} 2(n+1) \\ j \end{pmatrix} \langle x^j\rangle^{\zeta}_{u_\pi} (\langle 2x\rangle^{\zeta}_{u_\pi})^{-\gamma_0^j/\gamma_0^1}.$$

- Since **isospin symmetry** limit implies:

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Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.

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1	0.230(3)(7)	0.230
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3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7	0.0065(24)	0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{\left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \begin{pmatrix} 2(n+1) \\ j \end{pmatrix} \langle x^j\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^j/\gamma_0^1} .$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

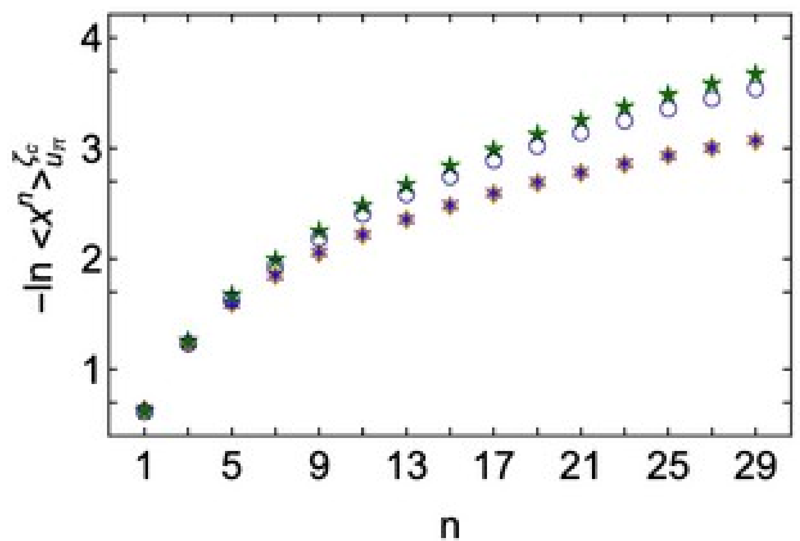
- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.

Moments from global fits can be also compared to the estimated from recursion !

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7	$\boxed{0.0065(24)}$	0.0078

Moments computed from: P. Barry et al., PRL127(2021)232001



[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\langle x^n \rangle^{\zeta}_{u_\pi} (\langle 2x \rangle^{\zeta}_{u_\pi})^{-\gamma_0^n/\gamma_0^1}$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle_{u_\pi}^{\zeta} \left(\langle 2x \rangle_{u_\pi}^{\zeta} \right)^{-\gamma_0^n/\gamma_0^1}$$

$$q(x;\zeta_H) = \delta(x - 1/2)$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.

DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x-1/2) \qquad\qquad q(x;\zeta_H) = 1$$

- Keeping **isospin symmetry,** implying:

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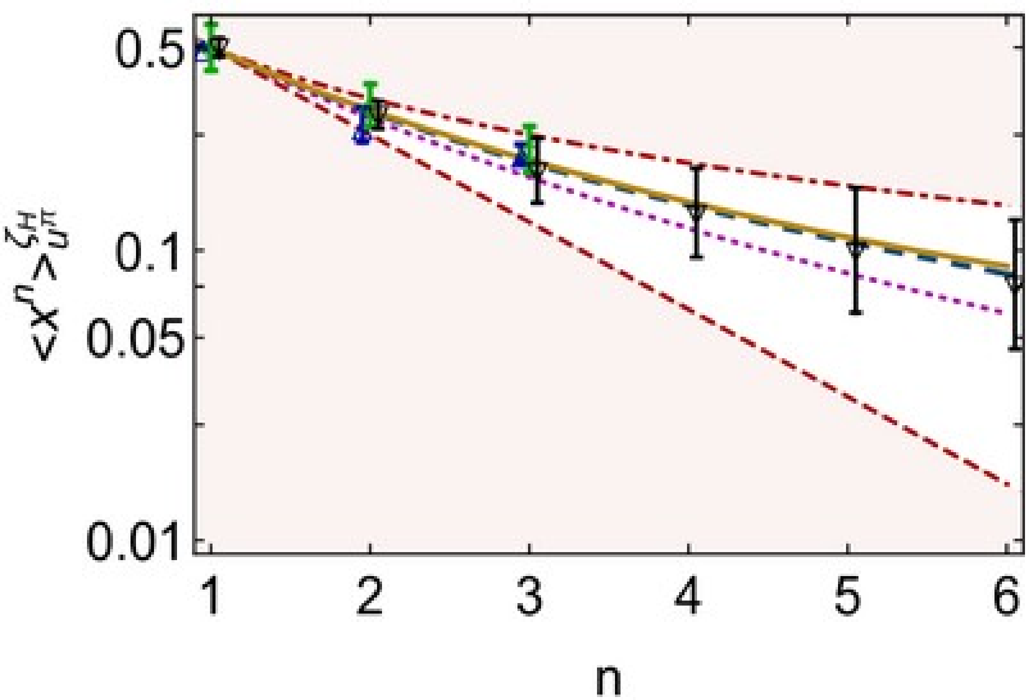
- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.
- **Upper bound** comes out from considering the opposite limit of a weekly interacting system of two (then fully decorrelated) partons: all the momentum fractions are equally probable.

DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x-1/2) \qquad\qquad q(x;\zeta_H) = 1$$



- Keeping **isospin symmetry,** implying:

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Joo:2019bzr Sufian:2019bol Alexandrou:2021mmi

n	[61]	[62]	[63]
1	0.254(03)	0.18(3)	0.23(3)(7)
2	0.094(12)	0.064(10)	0.087(05)(08)
3	0.057(04)	0.030(05)	0.041(05)(09)
4			0.023(05)(06)
5			0.014(04)(05)
6			0.009(03)(03)

Lattice moments verifying the **recurrence relation** too.

DGLAP: All orders evolution

Cui:2020tdf

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\mathsf{H}}(x) = q_{\mathsf{H}}(x) + \bar{q}_{\mathsf{H}}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\mathsf{H}}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + 2P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

Hard-wall threshold

$$\mathcal{P}^{\zeta}_q = \theta(\zeta - M_q)$$

Quark singlet and glue PDFs in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + 2\mathcal{P}^{\zeta}_q \gamma^n_{qg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

DGLAP: All orders evolution

Cui:2020tdf

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$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathrm{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}$$

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$$\mathcal{P}^{\zeta}_q = \theta(\zeta - M_q)$$

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$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

Sea-quark PDF

$$\langle x^n \rangle^{\zeta}_{S^q_H} = \langle x^n \rangle^{\zeta}_{\Sigma^q_H} - \langle x^n \rangle^{\zeta}_{q_H}$$

DGLAP: All orders evolution

Cui:2020tdf

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\rm H}(x) = q_{\rm H}(x) + \bar{q}_{\rm H}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\rm H}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\rm H}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\rm H}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\rm H}(y) + 2P^\zeta_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\rm H}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\rm H}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\rm H}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\rm H}(y) \right\}$$

Hard-wall threshold

$$\mathcal{P}^\zeta_q = \theta(\zeta - M_q)$$

Quark singlet and glue PDFs in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{\Sigma_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^\zeta_{\Sigma_H} + 2n_f \mathcal{P}^\zeta_q \gamma^n_{ug} \langle x^n \rangle^\zeta_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \langle x^n \rangle^\zeta_{\Sigma_H} + \gamma^n_{gg} \langle x^n \rangle^\zeta_{g_H} \right\}$$

Sea-quark PDF

$$\langle x^n \rangle^\zeta_{S^q_H} = \langle x^n \rangle^\zeta_{\Sigma^q_H} - \langle x^n \rangle^\zeta_{q_H}$$

Full singlet and sea

$$\langle x^n \rangle^\zeta_{\Sigma_H} = \sum_q \langle x^n \rangle^\zeta_{\Sigma^q_H} \,,\ \langle x^n \rangle^\zeta_{S_H} = \sum_q \langle x^n \rangle^\zeta_{S^q_H}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H\,,\ \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \left(\begin{array}{c} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{array} \right) = \left(\begin{array}{cc} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{array} \right) \left(\begin{array}{c} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{array} \right)$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

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$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = - \frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = [S(\zeta_H, \zeta)]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H, \; \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \boxed{\begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix}} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr} \left(\Gamma^n \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\Gamma^n \right) - \mathrm{Det} \left(\Gamma^n \right)}$$

$$\beta^n_{\Sigma g} = - \frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H ,\ \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \boxed{\begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix}} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \boxed{\langle x^n \rangle^{\zeta_H}_{\Sigma_H}} \\ 0 \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr} \left(\boxed{\Gamma^n} \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\boxed{\Gamma^n} \right) - \mathrm{Det} \left(\boxed{\Gamma^n} \right)}$$

$$\beta^n_{\Sigma g} = - \frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H \,,\ \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \boxed{\begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix}} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) \end{pmatrix} \boxed{\sum_q \langle x^n \rangle^{\zeta_H}_q}$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = - \frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr} \left(\boxed{\Gamma^n} \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\boxed{\Gamma^n} \right) - \mathrm{Det} \left(\boxed{\Gamma^n} \right)}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H, \ \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) \end{pmatrix} \sum_q \langle x^n \rangle^{\zeta_H}_q$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

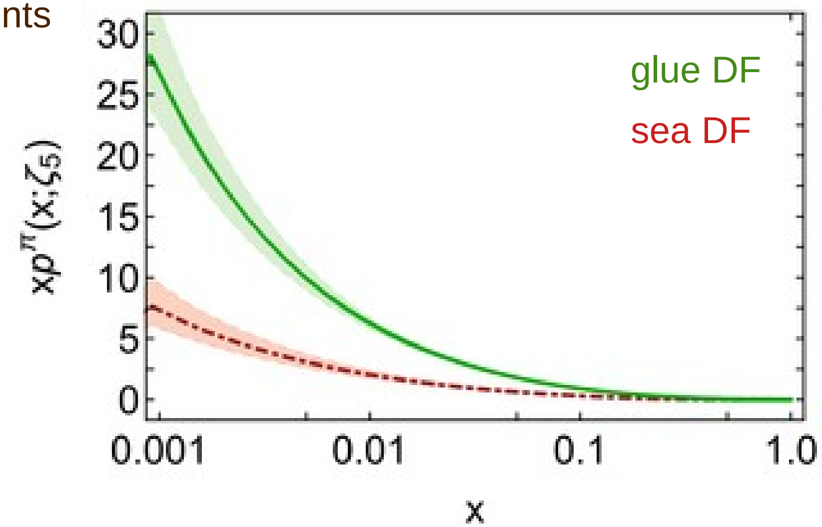
$$\beta^n_{\Sigma g} = -\frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr}\left(\Gamma^n \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\Gamma^n \right) - \mathrm{Det} \left(\Gamma^n \right)}$$

Compute all the moments and reconstruct:



DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H\,,\ \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix} = \begin{pmatrix}\gamma^n_{uu} & 2n_f\gamma^n_{ug}\\ \gamma^n_{gu} & \gamma^n_{gg}\end{pmatrix}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix}$$

$$\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix} = \begin{pmatrix}\alpha^n_+ S^n_- + \alpha^n_- S^n_+\\ \beta^n_{g\Sigma}\left(S^n_- - S^n_+\right)\end{pmatrix}\sum_q \langle x^n\rangle^{\zeta_H}_q$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha^n_\pm = \pm\frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

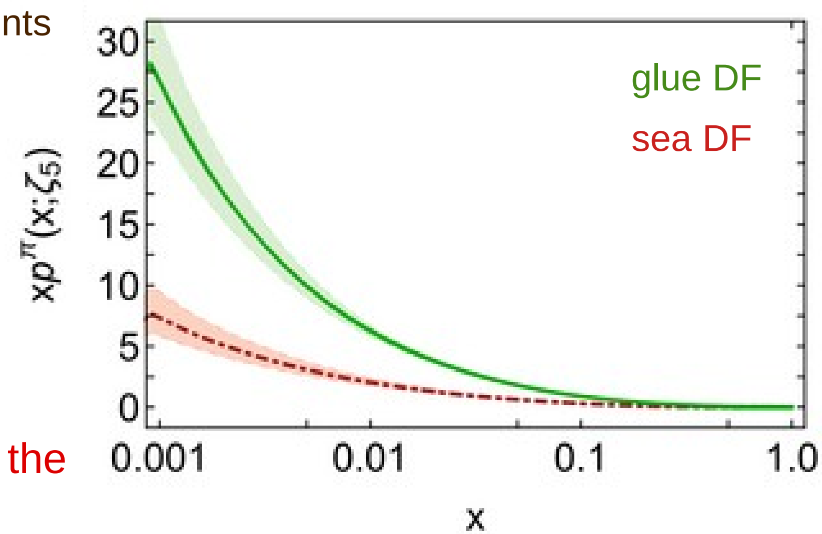
$$\lambda^n_\pm = \frac{1}{2}\mathrm{Tr}\left(\Gamma^n\right) \pm \sqrt{\frac{1}{4}\mathrm{Tr}^2\left(\Gamma^n\right) - \mathrm{Det}\left(\Gamma^n\right)}$$

$$\beta^n_{\Sigma g} = -\frac{2n_f\gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

Compute all the moments and reconstruct:

$$S^n_\pm = \left[S(\zeta_H,\zeta)\right]^{\lambda^n_\pm/\gamma_{uu}} \longrightarrow = \left[\frac{\langle x\rangle^{\zeta}_{q_H}}{\langle x\rangle^{\zeta_H}_{q_H}}\right]^{\lambda^n_\pm/\gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f\gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$



★ The only required input is the the momentum fraction at the probed empirical scale!!

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H,\ \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix} = \begin{pmatrix}\gamma^n_{uu} & 2n_f\gamma^n_{ug}\\ \gamma^n_{gu} & \gamma^n_{gg}\end{pmatrix}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix}$$

$$\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix} = \begin{pmatrix}\alpha^n_+ S^n_- + \alpha^n_- S^n_+\\ \beta^n_{g\Sigma}\left(S^n_- - S^n_+\right)\end{pmatrix}\sum_q \langle x^n\rangle^{\zeta_H}_q$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha^n_\pm = \pm\frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = -\frac{2n_f\gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = \left[S(\zeta_H,\zeta)\right]^{\lambda^n_\pm/\gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f\gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$

$$\lambda^n_\pm = \frac{1}{2}\mathrm{Tr}\left(\Gamma^n\right) \pm \sqrt{\frac{1}{4}\mathrm{Tr}^2\left(\Gamma^n\right) - \mathrm{Det}\left(\Gamma^n\right)}$$

$\boxed{\text{n=1 case} \quad n_f = 4}$

$$\langle x\rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x\rangle^{\zeta}_{q_H} + \langle x\rangle^{\zeta}_{S_H} = \frac{3}{7} + \frac{4}{7}\left[S(\zeta_H,\zeta)\right]^{7/4}$$

$$\langle x\rangle^{\zeta}_{g_H} = \frac{4}{7}\left(1 - \left[S(\zeta_H,\zeta)\right]^{7/4}\right)$$

★ The only required input is the the momentum fraction at the probed empirical scale!!

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H, \ \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix} = \begin{pmatrix}\gamma^n_{uu} & 2n_f\gamma^n_{ug}\\ \gamma^n_{gu} & \gamma^n_{gg}\end{pmatrix}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix}$$

$$\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix} = \begin{pmatrix}\alpha^n_+ S^n_- + \alpha^n_- S^n_+\\ \beta^n_{g\Sigma}\left(S^n_- - S^n_+\right)\end{pmatrix}\sum_q \langle x^n\rangle^{\zeta_H}_q$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha^n_\pm = \pm\frac{\lambda^n_\mp - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2}\mathrm{Tr}\left(\Gamma^n\right) \pm \sqrt{\frac{1}{4}\mathrm{Tr}^2\left(\Gamma^n\right) - \mathrm{Det}\left(\Gamma^n\right)}$$

$$\beta^n_{\Sigma g} = -\frac{2n_f\gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

n=1 case

$n_f = 4$

$$\langle x\rangle^{\zeta}_{\Sigma_\pi} = \langle 2x\rangle^{\zeta}_{q_\pi} + \langle x\rangle^{\zeta}_{S_\pi} = \frac{3}{7} + \frac{4}{7}\left[S(\zeta_H,\zeta)\right]^{7/4}$$

$$\langle x\rangle^{\zeta}_{g_\pi} = \frac{4}{7}\left(1 - \left[S(\zeta_H,\zeta)\right]^{7/4}\right)$$

$$S^n_\pm = \left[S(\zeta_H,\zeta)\right]^{\lambda^n_\pm/\gamma_{uu}} = \left[\langle 2x\rangle^{\zeta}_{q_\pi}\right]^{\gamma^n_{qq}/\gamma_{qq}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f\gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$

ζ_5	$\langle 2x\rangle^{\pi}_{q}$	$\langle x\rangle^{\pi}_{g}$	$\langle x\rangle^{\pi}_{\text{sea}}$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

★ The only required input is the the momentum fraction at the probed empirical scale!!

Z-F. Cui et al., arXiv:2006.1465

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H,\ \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \gamma_{uu}^n & 2n_f \gamma_{ug}^n \\ \gamma_{gu}^n & \gamma_{gg}^n \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \alpha_+^n S_-^n + \alpha_-^n S_+^n \\ \beta_{g\Sigma}^n \left(S_-^n - S_+^n \right) \end{pmatrix} \sum_q \langle x^n \rangle_q^{\zeta_H}$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha_\pm^n = \pm \frac{\lambda_\pm^n - \gamma_{uu}^n}{\lambda_+^n - \lambda_-^n}$$

$$\beta_{\Sigma g}^n = -\frac{2n_f \gamma_{ug}^n}{\lambda_+^n - \lambda_-^n}$$

$$S_\pm^n = \left[S(\zeta_H, \zeta) \right]^{\lambda_\pm^n / \gamma_{uu}} = \left[\langle 2x \rangle_{q_\pi}^{\zeta} \right]^{\gamma_{qq}^n / \gamma_{qq}}$$

$$\beta_{g\Sigma}^n = \frac{(\lambda_+^n - \gamma_{uu}^n)(\lambda_-^n - \gamma_{uu}^n)}{2n_f \gamma_{ug}^n (\lambda_+^n - \lambda_-^n)}$$

$$\lambda_\pm^n = \frac{1}{2} \mathrm{Tr}\left(\Gamma^n\right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2\left(\Gamma^n\right) - \mathrm{Det}\left(\Gamma^n\right)}$$

n=1 case

$n_f = 4$

$$\langle x \rangle_{\Sigma_\pi}^{\zeta} = \langle 2x \rangle_{q_\pi}^{\zeta} + \langle x \rangle_{S_\pi}^{\zeta} = \frac{3}{7} + \frac{4}{7} \left[S(\zeta_H, \zeta) \right]^{7/4}$$

$$\langle x \rangle_{g_\pi}^{\zeta} = \frac{4}{7} \left(1 - \left[S(\zeta_H, \zeta) \right]^{7/4} \right)$$

ζ_5	$\langle 2x \rangle_q^\pi$	$\langle x \rangle_g^\pi$	$\langle x \rangle_{\text{sea}}^\pi$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

★ The only required input is the the momentum fraction at the probed empirical scale!!

Z-F. Cui et al., arXiv:2006.1465

R.S. Sufian et al., arXiv:2001.04960

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H \,, \ \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \boxed{\begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix}} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) \end{pmatrix} \boxed{\sum_q \langle x^n \rangle^{\zeta_H}_q}$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr} \left(\Gamma^n \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\Gamma^n \right) - \mathrm{Det} \left(\Gamma^n \right)}$$

$$\beta^n_{\Sigma g} = - \frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

n=1 case

$n_f = 4$

$$\langle x \rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x \rangle^{\zeta}_{q_H} + \langle x \rangle^{\zeta}_{S_H} = \frac{3}{7} + \frac{4}{7} \left[S(\zeta_H, \zeta) \right]^{7/4}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}} = \left[\langle 2x \rangle^{\zeta}_{q_\pi} \right]^{\gamma^n_{qq} / \gamma_{qq}}$$

$$\langle x \rangle^{\zeta}_{g_H} = \frac{4}{7} \left(1 - \left[S(\zeta_H, \zeta) \right]^{7/4} \right)$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

★ The only required input is the the pion momentum fraction at the probed empirical scale (assuming charge universality)!!

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H, \ \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix} = \begin{pmatrix}\gamma^n_{uu} & 2n_f\gamma^n_{ug}\\ \gamma^n_{gu} & \gamma^n_{gg}\end{pmatrix}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix}$$

$$\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix} = \begin{pmatrix}\alpha^n_+ S^n_- + \alpha^n_- S^n_+\\ \beta^n_{g\Sigma}\left(S^n_- - S^n_+\right)\end{pmatrix}\sum_q \langle x^n\rangle^{\zeta_H}_q$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha^n_\pm = \pm\frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = -\frac{2n_f\gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = \left[S(\zeta_H,\zeta)\right]^{\lambda^n_\pm/\gamma_{uu}} = \left[\langle 2x\rangle^{\zeta}_{q_\pi}\right]^{\gamma^n_{qq}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f\gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$

$$\lambda^n_\pm = \frac{1}{2}\mathrm{Tr}\left(\Gamma^n\right) \pm \sqrt{\frac{1}{4}\mathrm{Tr}^2\left(\Gamma^n\right) - \mathrm{Det}\left(\Gamma^n\right)}$$

n=1 case

$n_f = 4$

$$\langle x\rangle^{\zeta}_{\Sigma_H} \underset{\zeta^2\to\infty}{=} \langle x\rangle^{\zeta}_{S_H} \underset{\zeta^2\to\infty}{=} \frac{3}{7}$$

$$\langle x\rangle^{\zeta}_{g_H} \underset{\zeta^2\to\infty}{=} \frac{4}{7}$$

Asymptotic limit: G. Altarelli, Phys. Rep. 81, 1 (1982)

★ The only required input is the the pion momentum fraction at the probed empirical scale (assuming charge universality)!!

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H,\ \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \gamma_{uu}^n & 2n_f \gamma_{ug}^n \\ \gamma_{gu}^n & \gamma_{gg}^n \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \alpha_+^n S_-^n + \alpha_-^n S_+^n \\ \beta_{g\Sigma}^n \left(S_-^n - S_+^n \right) \end{pmatrix} \sum_q \langle x^n \rangle_q^{\zeta_H}$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha_\pm^n = \pm \frac{\lambda_\pm^n - \gamma_{uu}^n}{\lambda_+^n - \lambda_-^n}$$

$$\beta_{\Sigma g}^n = -\frac{2n_f \gamma_{ug}^n}{\lambda_+^n - \lambda_-^n}$$

$$S_\pm^n = \left[S(\zeta_H, \zeta) \right]^{\lambda_\pm^n / \gamma_{uu}} = \left[\langle 2x \rangle_{q_\pi}^{\zeta} \right]^{\gamma_{qq}^n}$$

$$\beta_{g\Sigma}^n = \frac{(\lambda_+^n - \gamma_{uu}^n)(\lambda_-^n - \gamma_{uu}^n)}{2n_f \gamma_{ug}^n (\lambda_+^n - \lambda_-^n)}$$

$$\lambda_\pm^n = \frac{1}{2} \mathrm{Tr}\left(\Gamma^n\right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2\left(\Gamma^n\right) - \mathrm{Det}\left(\Gamma^n\right)}$$

n=1 case

$n_f = 4$

$$\langle x \rangle_{\Sigma_H}^{\zeta} \underset{\zeta^2 \to \infty}{=} \langle x \rangle_{S_H}^{\zeta} \underset{\zeta^2 \to \infty}{=} \frac{3}{7}$$

$$\langle x \rangle_{g_H}^{\zeta} \underset{\zeta^2 \to \infty}{=} \frac{4}{7}$$

Asymptotic limit: G. Altarelli, Phys. Rep. 81, 1 (1982)

$$\langle x^n \rangle_{\Sigma_H}^{\zeta} \underset{\zeta^2 \to \infty}{=} \langle x^n \rangle_{S_H}^{\zeta} \underset{\zeta^2 \to \infty}{=} \langle x^n \rangle_{g_H}^{\zeta} \underset{\zeta^2 \to \infty}{=} 0\,, \quad \text{for } n > 1$$

owing to $\lambda_\pm^n > 0$

★ The only required input is the the pion momentum fraction at the probed empirical scale (assuming charge universality)!!

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H \,,\ \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) \end{pmatrix} \sum_q \langle x^n \rangle^{\zeta_H}_q$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = - \frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2}\mathrm{Tr}\left(\Gamma^n\right) \pm \sqrt{\frac{1}{4}\mathrm{Tr}^2\left(\Gamma^n\right) - \mathrm{Det}\left(\Gamma^n\right)}$$

n=1 case

$n_f = 4$

$$S^n_\pm = \left[S(\zeta_H, \zeta)\right]^{\lambda^n_\pm/\gamma_{uu}} \longrightarrow = \left[\langle 2x \rangle^{\zeta}_{q_\pi}\right]^{\gamma^n_{qq}/\gamma_{qq}}$$

$$\Sigma_H(x) \underset{\zeta^2 \to \infty}{=} \frac{3}{7}\frac{\delta(x)}{x}$$

$$g_H(x) \underset{\zeta^2 \to \infty}{=} \frac{4}{7}\frac{\delta(x)}{x}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$

★ The only required input is the the pion momentum fraction at the probed empirical scale (assuming charge universality)!!

DGLAP: All orders evolution

Implication 5: correlating glue and sea

$$M_q = \zeta_H , \ \forall q$$

All quarks active

$$\left(\begin{array}{c} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{array} \right) = \left(\begin{array}{cc} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{array} \right) \left(\begin{array}{c} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{array} \right)$$

DGLAP: All orders evolution

Implication 5: correlating glue and sea

$$M_q = \zeta_H\,,\ \forall q$$

All quarks active

$$\begin{pmatrix} \alpha_+^n \left[S_-^n\right]^{-1} + \alpha_-^n \left[S_+^n\right]^{-1} & \beta_{\Sigma g}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) \\ \beta_{g\Sigma}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) & \alpha_-^n \left[S_-^n\right]^{-1} + \alpha_+^n \left[S_+^n\right]^{-1} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta_H} \\ \langle x^n \rangle_{g_H}^{\zeta_H} \end{pmatrix}$$

The equation can be easily inverted

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The equation can be easily inverted and, relying on the hadronic scale definition, delivers a constraint for all Mellin moments of glue and sea at any experimental scale:

$$\frac{\langle x^n \rangle_{\Sigma_\pi}^{\zeta}}{\langle x^n \rangle_{g_\pi}^{\zeta}} = \frac{\langle x^n \rangle_{\mathcal{S}_\pi}^{\zeta} + \langle 2x^n \rangle_{u_\pi}^{\zeta}}{\langle x^n \rangle_{g_\pi}^{\zeta}} = \frac{\alpha_-^n S_+^n + \alpha_+^n S_-^n}{\beta_{g\Sigma}^n \left(S_-^n - S_+^n\right)}$$

DGLAP: All orders evolution

Implication 5: correlating glue and sea

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$$\begin{pmatrix} \alpha_+^n \left[S_-^n\right]^{-1} + \alpha_-^n \left[S_+^n\right]^{-1} & \beta_{\Sigma g}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) \\ \beta_{g\Sigma}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) & \alpha_-^n \left[S_-^n\right]^{-1} + \alpha_+^n \left[S_+^n\right]^{-1} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta_H} \\ 0 \end{pmatrix}$$

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$$\frac{\langle x \rangle_{\Sigma_\pi}^{\zeta}}{\langle x \rangle_{g_\pi}^{\zeta}} = \frac{\langle x \rangle_{\mathcal{S}_\pi}^{\zeta} + \langle 2x \rangle_{u_\pi}^{\zeta}}{\langle x \rangle_{g_\pi}^{\zeta}} = \frac{\frac{3}{4} + \left(\langle 2x \rangle_{u_\pi}^{\zeta}\right)^{7/4}}{1 - \left(\langle 2x \rangle_{u_\pi}^{\zeta}\right)^{7/4}}$$

	$\langle 2x \rangle_{u_\pi}^{\zeta}$	$\langle x \rangle_{\mathcal{S}_\pi}^{\zeta}$	$\langle x \rangle_{g_\pi}^{\zeta}$	$n_f = 4$ $\langle x \rangle_{\Sigma_\pi}^{\zeta_H}$	$\langle x \rangle_{g_\pi}^{\zeta_H}$
NLO	0.53(2)	0.14(4)	0.34(6)	1.15(14)	-0.14(13)
NLL-Cos	0.47(2)	0.14(5)	0.39(6)	1.11(16)	-0.11(16)
NLL-Exp	0.46(2)	0.16(5)	0.38(6)	1.15(12)	-0.14(13)
NLL-dM	0.46(3)	0.15(7)	0.40(5)	1.12(22)	-0.11(18)

All-orders DGLAP: hard-wall thresholds

Let us now solve generally the hard-wall model of **massless partons** with **hard-wall thresholds** for each flavor activation, that can be analytically solved!

$$\gamma^n_{qq} = \gamma^n_{uu},\ \gamma^n_{gq} = \gamma^n_{gu},\ \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s$$

$$
\begin{aligned}
\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{q_H} &= -\frac{\alpha(\zeta^2)}{4\pi}\gamma^n_{uu}\langle x^n\rangle^{\zeta}_{q_H} \\
\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{\Sigma^q_H} &= -\frac{\alpha(\zeta^2)}{4\pi}\left\{\gamma^n_{uu}\langle x^n\rangle^{\zeta}_{\Sigma^q_H} + 2\theta(\zeta - M_q)\gamma^n_{ug}\langle x^n\rangle^{\zeta}_{g_H}\right\} \\
\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{g_H} &= -\frac{\alpha(\zeta^2)}{4\pi}\left\{\gamma^n_{gu}\sum_q \langle x^n\rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg}\langle x^n\rangle^{\zeta}_{g_H}\right\}
\end{aligned}
$$

Consider, for the sake of simplicity, three flavors and $\zeta \le M_s$

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$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{uu} \langle x^n \rangle^{\zeta}_{q_H}$$

$$\gamma^n_{qq} = \gamma^n_{uu},\ \gamma^n_{gq} = \gamma^n_{gu},\ \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^s_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{uu} \langle x^n \rangle^{\zeta}_{\Sigma^s_H}$$

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma^{u+d}_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma^n_{uu} & 4\gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma^{u+d}_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} - \frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} 0 \\ \gamma^n_{gu} \langle x^n \rangle^{\zeta}_{\Sigma^s_H} \end{pmatrix}$$

Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$, such that the singlet combinations can be rearranged and the strange decoupled from the light flavors.

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Let us now solve generally the hard-wall model of **massless partons** with **hard-wall thresholds** for each flavor activation, that can be analytically solved!

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi}\gamma^n_{uu}\langle x^n\rangle^{\zeta}_{q_H}$$

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$$\zeta^2 \frac{d}{d\zeta^2}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma^{u+d}_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi}\begin{pmatrix}\gamma^n_{uu} & 4\gamma^n_{ug}\\ \gamma^n_{gu} & \gamma^n_{gg}\end{pmatrix}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma^{u+d}_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix}$$

Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$, such that the singlet combinations can be rearranged and the strange decoupled from the light flavors. Specializing for the averaged momentum fraction.

In pion's (proton's) case $\langle x\rangle^{\zeta_H}_{s_\pi} = 0$

$$\langle x\rangle^{\zeta}_{\Sigma^s_\pi} \equiv 0$$

$$\begin{pmatrix}\langle x\rangle^{\zeta}_{\Sigma^{u+d}_\pi}\\ \langle x\rangle^{\zeta}_{g_\pi}\end{pmatrix} = \begin{pmatrix}\frac{3}{11} + \frac{8}{11}\left[S(\zeta_H,\zeta\right]^{11/8}\\ \frac{8}{11}\left(1 - \left[S(\zeta_H,\zeta\right]^{11/8}\right)\end{pmatrix}$$

$$S(\zeta_H,\zeta) = \exp\left(-\frac{\gamma_{uu}}{2\pi}\int_{\zeta_H}^{\zeta}\frac{dz}{z}\alpha(z^2)\right)$$

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$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{q_H}^{\zeta} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma_{uu}^n \langle x^n \rangle_{q_H}^{\zeta}$$

$$\gamma_{qq}^n = \gamma_{uu}^n,\ \gamma_{gq}^n = \gamma_{gu}^n,\ \gamma_{qg}^n = \gamma_{ug}^n \qquad q = u, d, s$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{\Sigma_H^s}^{\zeta} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma_{uu}^n \langle x^n \rangle_{\Sigma_H^s}^{\zeta}$$

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Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$, such that the singlet combinations can be rearranged and the strange decoupled from the light flavors. Specializing for the averaged momentum fraction.

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$$\begin{pmatrix} \langle x \rangle_{\Sigma_\pi^{u+d}}^{\zeta} \\ \langle x \rangle_{g_\pi}^{\zeta} \end{pmatrix} = \begin{pmatrix} \frac{3}{11} + \frac{8}{11} [S(\zeta_H, \zeta]^{11/8} \\ \frac{8}{11} \left(1 - [S(\zeta_H, \zeta]^{11/8}\right) \end{pmatrix}$$

In kaon's case (after some algebra) $\langle x \rangle_{s_K}^{\zeta_H} = s_0$

$$\langle x \rangle_{\Sigma_K^s}^{\zeta} = s_0\, S(\zeta_H, \zeta)$$

$$\begin{pmatrix} \langle x \rangle_{\Sigma_K^{u+d}}^{\zeta} \\ \langle x \rangle_{g_K}^{\zeta} \end{pmatrix} = \begin{pmatrix} \frac{3}{11} + \frac{8}{11} [S(\zeta_H, \zeta)]^{11/8} - \langle x \rangle_{\Sigma_K^s}^{\zeta} \\ \frac{8}{11} \left(1 - [S(\zeta_H, \zeta)]^{11/8}\right) \end{pmatrix}$$

$$S(\zeta_H, \zeta) = \exp\left(-\frac{\gamma_{uu}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

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Let us now solve generally the hard-wall model of **massless partons** with **hard-wall thresholds** for each flavor activation, that can be analytically solved!

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{uu} \langle x^n \rangle^{\zeta}_{q_H}$$

$$\gamma^n_{qq} = \gamma^n_{uu}, \; \gamma^n_{gq} = \gamma^n_{gu}, \; \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^s_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{uu} \langle x^n \rangle^{\zeta}_{\Sigma^s_H}$$

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma^{u+d}_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma^n_{uu} & 4\gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma^{u+d}_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} - \frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} 0 \\ \gamma^n_{gu} \langle x^n \rangle^{\zeta}_{\Sigma^s_H} \end{pmatrix}$$

Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$, such that the singlet combinations can be rearranged and the strange decoupled from the light flavors. Specializing for the averaged momentum fraction.

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In kaon's case (after some algebra) $\langle x \rangle^{\zeta_H}_{s_K} = s_0$

$$\langle x \rangle^{\zeta}_{\Sigma^s_K} = s_0 \, S(\zeta_H, \zeta)$$

$$\begin{pmatrix} \langle x \rangle^{\zeta}_{\Sigma_K} \\ \langle x \rangle^{\zeta}_{g_K} \end{pmatrix} = \begin{pmatrix} \frac{3}{11} + \frac{8}{11} [S(\zeta_H, \zeta)]^{11/8} \\ \frac{8}{11} \left(1 - [S(\zeta_H, \zeta)]^{11/8}\right) \end{pmatrix}$$

$$S(\zeta_H, \zeta) = \exp\left(-\frac{\gamma_{uu}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2)\right)$$

$$\langle x^n \rangle^{\zeta}_{\Sigma_H} = \sum_{q=u,d,s,c} \langle x^n \rangle^{\zeta}_{\Sigma^q_H}$$

All-orders DGLAP: hard-wall thresholds

Let us now solve generally the hard-wall model of **massless partons** with **hard-wall thresholds** for each flavor activation, that can be analytically solved!

$$\gamma_{qq}^n = \gamma_{uu}^n,\ \gamma_{gq}^n = \gamma_{gu}^n,\ \gamma_{qg}^n = \gamma_{ug}^n$$

$$q = u, d, s, c$$

$$\begin{aligned}
\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{q_H}^{\zeta} &= -\frac{\alpha(\zeta^2)}{4\pi} \gamma_{uu}^n \langle x^n \rangle_{q_H}^{\zeta} \\
\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{\Sigma_H^q}^{\zeta} &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{uu}^n \langle x^n \rangle_{\Sigma_H^q}^{\zeta} + 2\theta(\zeta - M_q) \gamma_{ug}^n \langle x^n \rangle_{g_H}^{\zeta} \right\} \\
\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{g_H}^{\zeta} &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{gu}^n \sum_q \langle x^n \rangle_{\Sigma_H^q}^{\zeta} + \gamma_{gg}^n \langle x^n \rangle_{g_H}^{\zeta} \right\}
\end{aligned}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

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\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{\Sigma_H^q}^{\zeta} &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{uu}^n \langle x^n \rangle_{\Sigma_H^q}^{\zeta} + 2\theta(\zeta - M_q) \gamma_{ug}^n \langle x^n \rangle_{g_H}^{\zeta} \right\} \\
\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{g_H}^{\zeta} &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{gu}^n \sum_q \langle x^n \rangle_{\Sigma_H^q}^{\zeta} + \gamma_{gg}^n \langle x^n \rangle_{g_H}^{\zeta} \right\}
\end{aligned}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

$$\langle x \rangle_{q_H}^{\zeta} = \langle x \rangle_{q_H}^{\zeta_H} S(\zeta_H, \zeta) \qquad \langle x \rangle_{g_H}^{\zeta} = \frac{4}{7} - \tau(M_s, M_c) \left[\langle 2x \rangle_{u_\pi}^{\zeta} \right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175} \left[\langle 2x \rangle_{u_\pi}^{M_c} \right]^{-7/4} - \frac{24}{275} \left[\langle 2x \rangle_{u_\pi}^{M_c} \right]^{-3/16} \left[\langle 2x \rangle_{u_\pi}^{M_s} \right]^{-25/16} + \frac{8}{11} \left[\langle 2x \rangle_{u_\pi}^{M_c} \langle 2x \rangle_{u_\pi}^{M_s} \right]^{-3/16}$$

Capitalizing on the universality of the effective charge, **all hadrons'** momentum fraction averages can be expressed in terms of **pion's** ones.

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\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_H} &= -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{uu} \langle x^n \rangle^{\zeta}_{q_H} \\
\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + 2\theta(\zeta - M_q) \gamma^n_{ug} \langle x^n \rangle^{\zeta}_{g_H} \right\} \\
\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \sum_q \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}
\end{aligned}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

$$\langle x \rangle^{\zeta}_{q_H} = \langle x \rangle^{\zeta_H}_{q_H} S(\zeta_H, \zeta) \qquad \langle x \rangle^{\zeta}_{g_H} = \frac{4}{7} - \tau(M_s, M_c) \left[\langle 2x \rangle^{\zeta}_{u_\pi} \right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-7/4} - \frac{24}{275} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-3/16} \left[\langle 2x \rangle^{M_s}_{u_\pi} \right]^{-25/16} + \frac{8}{11} \left[\langle 2x \rangle^{M_c}_{u_\pi} \langle 2x \rangle^{M_s}_{u_\pi} \right]^{-3/16}$$

$$\tau(\zeta_H, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u} \right]^{-7/4} + \frac{16}{25} \left[\langle 2x \rangle^{M_c}_{u} \right]^{-3/16}$$

3 (always) active flavors

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$$\gamma_{qq}^n = \gamma_{uu}^n,\ \gamma_{gq}^n = \gamma_{gu}^n,\ \gamma_{qg}^n = \gamma_{ug}^n$$

$$q = u, d, s, c$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{q_H}^{\zeta} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma_{uu}^n \langle x^n \rangle_{q_H}^{\zeta}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{\Sigma_H^q}^{\zeta} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{uu}^n \langle x^n \rangle_{\Sigma_H^q}^{\zeta} + 2\theta(\zeta - M_q) \gamma_{ug}^n \langle x^n \rangle_{g_H}^{\zeta} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{g_H}^{\zeta} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{gu}^n \sum_q \langle x^n \rangle_{\Sigma_H^q}^{\zeta} + \gamma_{gg}^n \langle x^n \rangle_{g_H}^{\zeta} \right\}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

$$\langle x \rangle_{q_H}^{\zeta} = \langle x \rangle_{q_H}^{\zeta_H} S(\zeta_H, \zeta)$$

$$\langle x \rangle_{g_H}^{\zeta} = \frac{4}{7} - \tau(M_s, M_c) \left[\langle 2x \rangle_{u_\pi}^{\zeta} \right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175} \left[\langle 2x \rangle_{u_\pi}^{M_c} \right]^{-7/4} - \frac{24}{275} \left[\langle 2x \rangle_{u_\pi}^{M_c} \right]^{-3/16} \left[\langle 2x \rangle_{u_\pi}^{M_s} \right]^{-25/16} + \frac{8}{11} \left[\langle 2x \rangle_{u_\pi}^{M_c} \langle 2x \rangle_{u_\pi}^{M_s} \right]^{-3/16}$$

3 (always) active flavors

$$\tau(\zeta_H, M_c) = -\frac{12}{175} \left[\langle 2x \rangle_{u}^{M_c} \right]^{-7/4} + \frac{16}{25} \left[\langle 2x \rangle_{u}^{M_c} \right]^{-3/16}$$

4 (always) active flavors

$$\tau(\zeta_H, \zeta_H) = \frac{4}{7}$$

Thus recovering the previous result!

All-orders DGLAP: hard-wall thresholds

Let us now solve generally the hard-wall model of **massless partons** with **hard-wall thresholds** for each flavor activation, that can be analytically solved!

$$\gamma^n_{qq} = \gamma^n_{uu},\ \gamma^n_{gq} = \gamma^n_{gu},\ \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s, c$$

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi}\gamma^n_{uu}\langle x^n\rangle^{\zeta}_{q_H}$$

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi}\left\{\gamma^n_{uu}\langle x^n\rangle^{\zeta}_{\Sigma^q_H} + 2\theta(\zeta - M_q)\gamma^n_{ug}\langle x^n\rangle^{\zeta}_{g_H}\right\}$$

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi}\left\{\gamma^n_{gu}\sum_q \langle x^n\rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg}\langle x^n\rangle^{\zeta}_{g_H}\right\}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

$$\langle x\rangle^{\zeta}_{q_H} = \langle x\rangle^{\zeta_H}_{q_H} S(\zeta_H, \zeta)$$

$$\langle x\rangle^{\zeta}_{g_H} = \frac{4}{7} - \tau(M_s, M_c)\left[\langle 2x\rangle^{\zeta}_{u_\pi}\right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175}\left[\langle 2x\rangle^{M_c}_{u_\pi}\right]^{-7/4} - \frac{24}{275}\left[\langle 2x\rangle^{M_c}_{u_\pi}\right]^{-3/16}\left[\langle 2x\rangle^{M_s}_{u_\pi}\right]^{-25/16} + \frac{8}{11}\left[\langle 2x\rangle^{M_c}_{u_\pi}\langle 2x\rangle^{M_s}_{u_\pi}\right]^{-3/16}$$

$$\langle x\rangle^{\zeta}_{S^q_H} = \langle x\rangle^{\zeta}_{\Sigma^q_H} - \langle x\rangle^{\zeta}_{q_H} = \theta(\zeta - M_q)\frac{1}{3\pi}\int_{M_q}^{\zeta}\frac{dz}{z}\alpha(z^2)\langle x\rangle^{z}_{g_H} S(z, \zeta)$$

All-orders DGLAP: hard-wall thresholds

Let us now solve generally the hard-wall model of **massless partons** with **hard-wall thresholds** for each flavor activation, that can be analytically solved!

$$\gamma^n_{qq} = \gamma^n_{uu},\ \gamma^n_{gq} = \gamma^n_{gu},\ \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s, c$$

$$\begin{aligned}
\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{q_H} &= -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{uu} \langle x^n \rangle^\zeta_{q_H} \\
\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{\Sigma^q_H} &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^\zeta_{\Sigma^q_H} + 2\theta(\zeta - M_q) \gamma^n_{ug} \langle x^n \rangle^\zeta_{g_H} \right\} \\
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$$\langle x \rangle^\zeta_{S^q_H} = \langle x \rangle^\zeta_{\Sigma^q_H} - \langle x \rangle^\zeta_{q_H} = \theta(\zeta - M_q) \frac{1}{3\pi} \int_{M_q}^{\zeta} \frac{dz}{z} \alpha(z^2) \langle x \rangle^z_{g_H} S(z, \zeta)$$

$$\sum_q \langle x \rangle^\zeta_{S^q_H} = \frac{3}{7} + \tau(M_s, M_c) \left[\langle 2x \rangle^\zeta_{u_\pi} \right]^{7/4} - \sum_q \langle x \rangle^\zeta_{q_H}$$

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Momentum conservation

All-orders DGLAP: Pauli blocking

Cui:2020tdf

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions

$$\begin{aligned}
\zeta^2 \frac{d}{d\zeta^2} q_{\mathrm{H}}(x) &= \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\mathrm{H}}(y) \\
\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\mathrm{H}}(x) &= \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathrm{H}}(y) + 2 P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\} \\
\zeta^2 \frac{d}{d\zeta^2} g_{\mathrm{H}}(x) &= \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathrm{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}
\end{aligned}$$

Modeling the Pauli-blocking contribution:

$$P^{\zeta}_{q\leftarrow g}(z) = \left[P_{q\leftarrow g}(z) + \delta_q \sqrt{3}(1-2z)\mathcal{D}\left(\frac{\zeta}{\zeta_H}\right)\right] \theta(\zeta - M_q)$$

$$\mathcal{D}(t) = \frac{1}{1+(t-1)^2}$$

All-orders DGLAP: Pauli blocking

Cui:2020tdf

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi}\gamma^n_{qq}\langle x^n\rangle^{\zeta}_{q_\pi}$$

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{\Sigma^q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi}\left\{\gamma^n_{qq}\langle x^n\rangle^{\zeta}_{\Sigma^q_\pi} + 2\theta(\zeta - M_q)\left[\gamma^n_{qg} + \delta_q a_n \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right)\right]\langle x^n\rangle^{\zeta}_{g_\pi}\right\}$$

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{g_\pi} = -\frac{\alpha(\zeta^2)}{4\pi}\left\{\sum_q \gamma^n_{gq}\langle x^n\rangle^{\zeta}_{\Sigma^q_\pi} + \gamma^n_{gg}\langle x^n\rangle^{\zeta}_{g_\pi}\right\} \; ;$$

Modeling the Pauli-blocking contribution:

$$a_n = \frac{\sqrt{3}n}{2+3n+n^2}$$

$$\mathcal{D}(t) = \frac{1}{1+(t-1)^2}$$

Momentum conservation

$$\gamma_{qq} + \gamma_{gq} = 2\sum_q \gamma_{qg} + \gamma_{gg} = \sum_q \delta_q = 0$$

All-orders DGLAP: Pauli blocking

Cui:2020tdf

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space

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$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} + 2\theta(\zeta - M_q) \left[\gamma^n_{qg} + \delta_q a_n \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x^n \rangle^{\zeta}_{g_\pi} \right\}$$

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Momentum conservation

$$\gamma_{qq} + \gamma_{gq} = 2\sum_q \gamma_{qg} + \gamma_{gg} = \sum_q \delta_q = 0$$

Equations and solutions for $\sum_q \langle x \rangle^{\zeta}_{S^q_H}$ and $\langle x \rangle^{\zeta}_{g_H}$ remain the same, while:

$$\langle x^n \rangle^{\zeta}_{S^q_H} = -\theta(\zeta - M_q)\frac{1}{\pi}\int_{M_q}^{\zeta} \frac{dz}{z}\alpha(z^2)\left[\gamma^n_{ug} + \delta^{\zeta}_q a_n \mathcal{D}\left(\frac{z}{\zeta_H}\right)\right] \langle x^n \rangle^{z}_{g_H} \left[S(z,\zeta)\right]^{\gamma^n_{uu}/\gamma_{uu}}$$

Illustration: lightcone momentum fractions

$$
\begin{aligned}
\langle x\rangle^{\zeta}_{S_H^{u/d}} =& \frac{1}{4}\left(1-\langle x\rangle^{\zeta}_{g_H}\right)+\frac{1}{12}\left(1-\langle x\rangle^{M_c}_{g_H}\right)S(M_c,\zeta)+\frac{1}{6}\left(1-\langle x\rangle^{M_s}_{g_H}\right)S(M_s,\zeta) \\
&-\frac{1}{2}S(\zeta_H,\zeta)-\Delta P_{u/d}(\zeta) \\
\langle x\rangle^{\zeta}_{S_H^{s}} =& \frac{1}{4}\left(1-\langle x\rangle^{\zeta}_{g_H}\right)+\frac{1}{12}\left(1-\langle x\rangle^{M_c}_{g_H}\right)S(M_c,\zeta)-\frac{1}{3}\left(1-\langle x\rangle^{M_s}_{g_H}\right)S(M_s,\zeta)-\Delta P_s(\zeta) \\
\langle x\rangle^{\zeta}_{S_H^{c}} =& \frac{1}{4}\left(1-\langle x\rangle^{\zeta}_{g_H}\right)-\frac{1}{4}\left(1-\langle x\rangle^{M_c}_{g_H}\right)S(M_c,\zeta)-\Delta P_c(\zeta)
\end{aligned}
$$

where: $\Delta P_q(\zeta) = \frac{1}{2\sqrt{3}\pi}\int_{M_q}^{\zeta}\frac{dz}{z}\delta_q^{\zeta}\alpha(z^2)\mathcal{D}\left(\frac{z}{\zeta_H}\right)\langle x\rangle^{z}_{g_H}S(z,\zeta)$

$\delta_u^{\zeta}=\delta_d^{\zeta}=-\delta_s^{\zeta}/2=\delta=0.093$

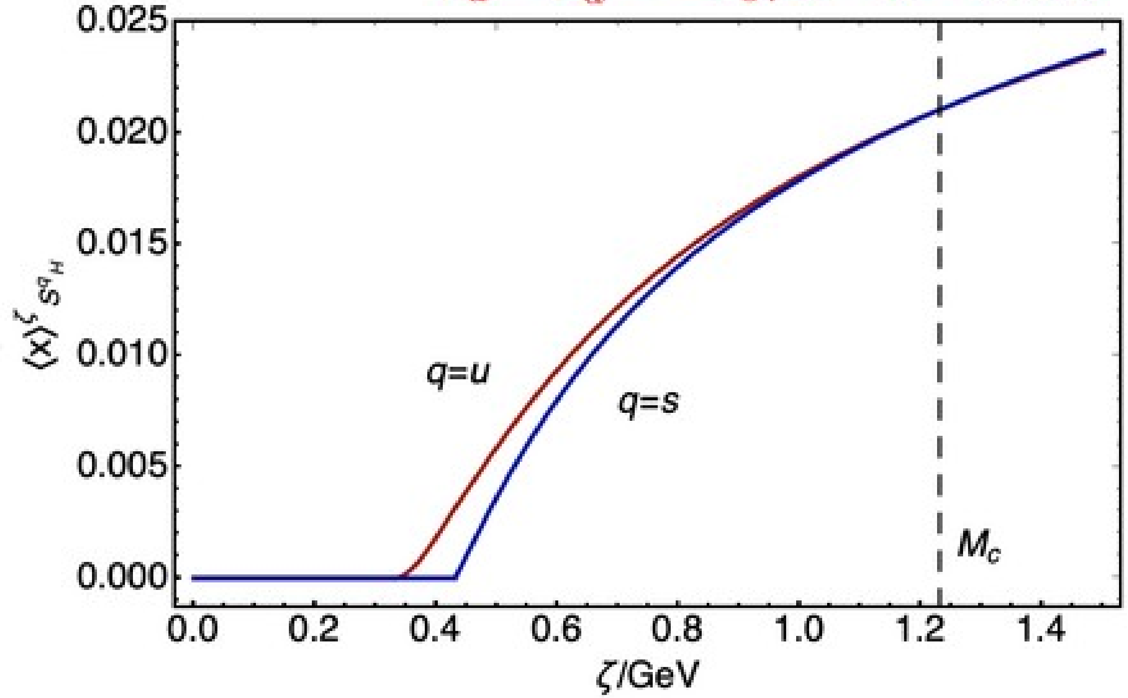

Illustration: lightcone momentum fractions

$$\langle x\rangle^{\zeta}_{S_H^{u/d}} = \frac{1}{4}\left(1-\langle x\rangle^{\zeta}_{g_H}\right) + \frac{1}{12}\left(1-\langle x\rangle^{M_c}_{g_H}\right) S(M_c,\zeta) + \frac{1}{6}\left(1-\langle x\rangle^{M_s}_{g_H}\right) S(M_s,\zeta) - \frac{1}{2} S(\zeta_H,\zeta) - \Delta P_{u/d}(\zeta)$$

$$\langle x\rangle^{\zeta}_{S_H^{s}} = \frac{1}{4}\left(1-\langle x\rangle^{\zeta}_{g_H}\right) + \frac{1}{12}\left(1-\langle x\rangle^{M_c}_{g_H}\right) S(M_c,\zeta) - \frac{1}{3}\left(1-\langle x\rangle^{M_s}_{g_H}\right) S(M_s,\zeta) - \Delta P_s(\zeta)$$

$$\langle x\rangle^{\zeta}_{S_H^{c}} = \frac{1}{4}\left(1-\langle x\rangle^{\zeta}_{g_H}\right) - \frac{1}{4}\left(1-\langle x\rangle^{M_c}_{g_H}\right) S(M_c,\zeta) - \Delta P_c(\zeta)$$

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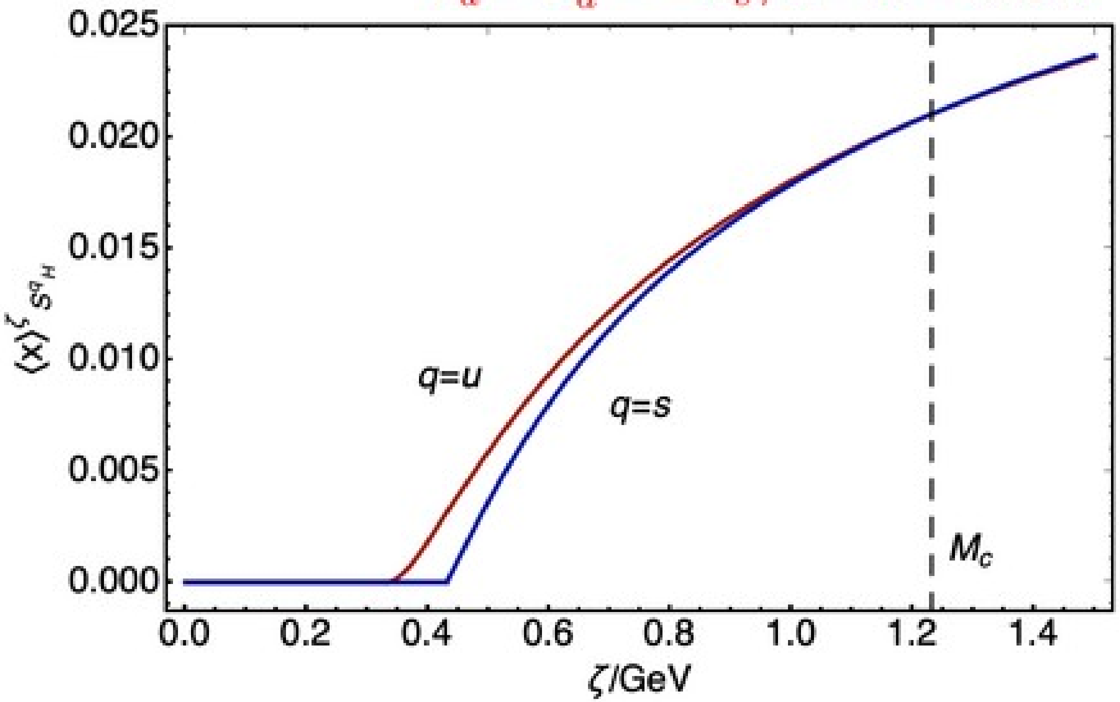
$\delta_u^{\zeta} = \delta_d^{\zeta} = -\delta_s^{\zeta}/2 = \delta = 0.093$

and $\delta = 0.17(5)$ implies:

$$\langle x\rangle^{\epsilon_c}_{S_\pi^s} = 1.29(11)\langle x\rangle^{\epsilon_c}_{S_\pi^u} = 1.29(11)\langle x\rangle^{\epsilon_c}_{S_\pi^d}$$

In good agreement with SeaQuest data!!!

Chang, Gao, Roberts,
Phys. Lett. B829(2022)137078



All-orders DGLAP: polarized PDFs

PDFs DGLAP evolutions equations, in the polarized case and specialized for 0-th Mellin moments, read

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H}$$

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$$a^{\zeta}_{0H} = \boxed{\langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H}} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \,,$$

$$\Delta G^{\zeta}_H = \boxed{\langle x^0 \rangle^{\zeta}_{\tilde{g}^q_H}} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \left\{ \frac{12}{29} \left([S(\zeta_H, M_s)]^{-87/32} - 1 \right) [S(M_s, M_c)]^{-81/32} [S(M_c, \zeta)]^{-75/32} \right.$$
$$\left. + \frac{4}{9} \left([S(M_s, M_c)]^{-81/32} - 1 \right) [S(M_c, \zeta)]^{-75/32} + \frac{12}{25} \left([S(M_c, \zeta)]^{-75/32} - 1 \right) \right\}$$

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No threshold $n_f = 4$

Non-abelian anomaly corrected:

$$\tilde{a}^{\zeta}_{0p} = a^{\zeta}_{0p} - n_f \frac{\hat{\alpha}(\zeta)}{2\pi} \Delta G^{\zeta}_p$$

	A	B	C	D
	CT18	Ref. [47]	Ref. [47]	Ref. [47]
$\langle x \rangle^{\zeta}_{\mathcal{V}_p}$	0.42(01)	0.49(02)	0.49(02)	0.49(02)
	Eq. (30)	Eq. (30)	Ref. [58]	Ref. [58]
a^{ζ}_{0p}	0.74(11)	0.74(11)	0.65(02)	0.65(02)
	Eq. (29b)	Eq. (29b)	Eq. (29b)	Eq. (32)
ΔG^{ζ}_p	2.27(30)	1.50(25)	1.33(15)	1.41(16)
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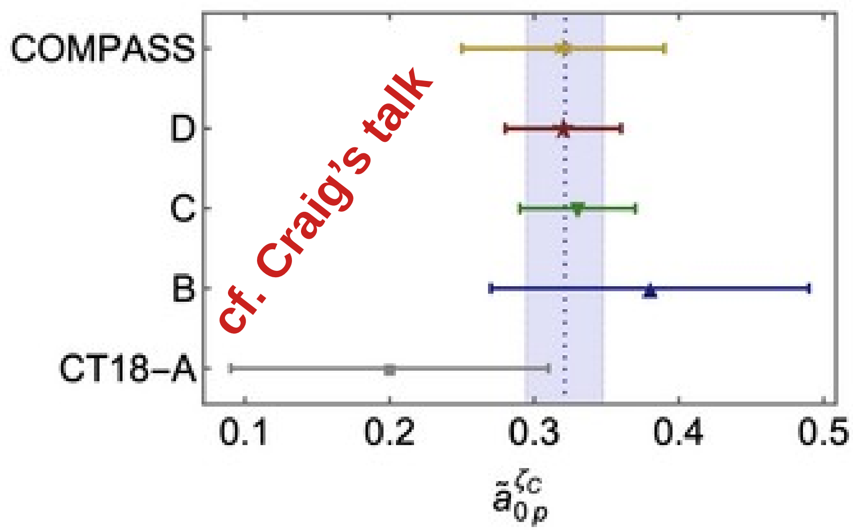
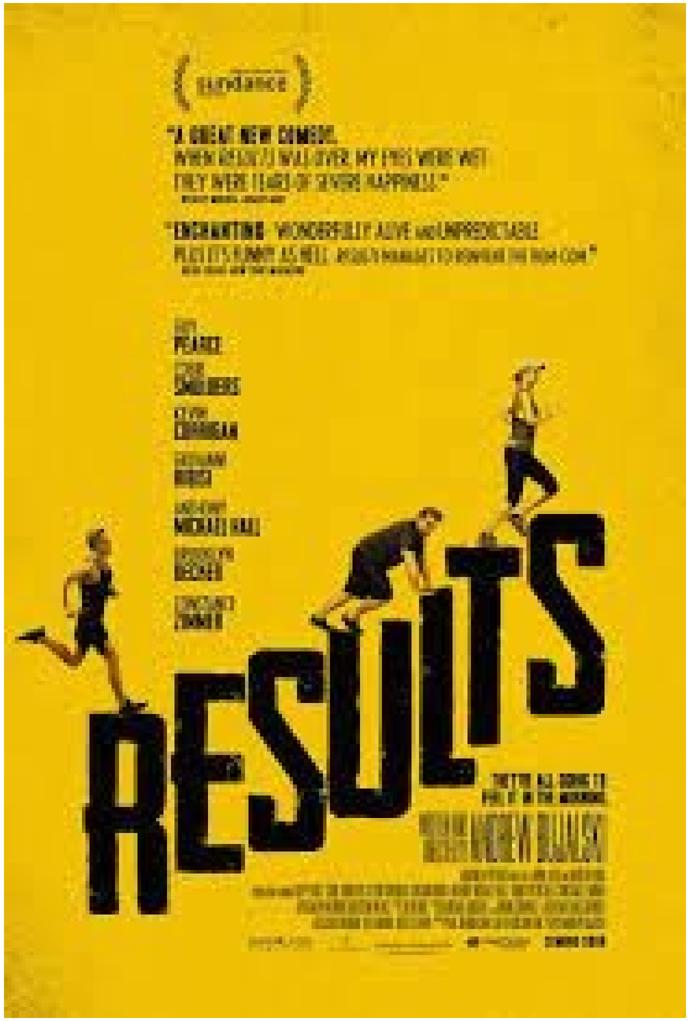


More



Pion PDF: from CSM (DSEs) to the experiment

Symmetry-preserving DSE computation of the valence-quark PDF:

[L. Chang et al., Phys.Lett.B737(2014)23]

[M. Ding et al., Phys.Rev.D101(2020)054014

$$q^{\pi}(x;\zeta) = N_c \mathrm{tr} \int_{dk} \delta_n^x(k_\eta) \Gamma_\pi^P(k_{\bar\eta\eta};\zeta)\, S(k_{\bar\eta};\zeta) \times \{n \cdot \frac{\partial}{\partial k_\eta} [\Gamma_\pi^{-P}(k_{\eta\bar\eta};\zeta) S(k_\eta;\zeta)]\}.$$

$$q_{\text{O}}^{\pi}(x;\zeta_H) = 213.32\, x^2(1-x)^2 \times [1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)]$$

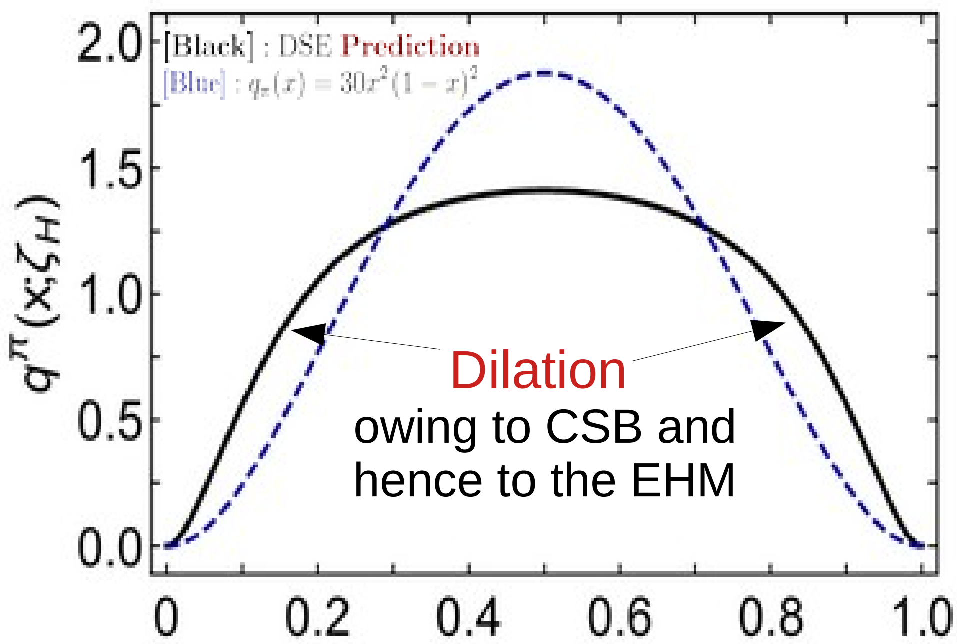
$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)}\,(1+\mathcal{O}(1-x))$$

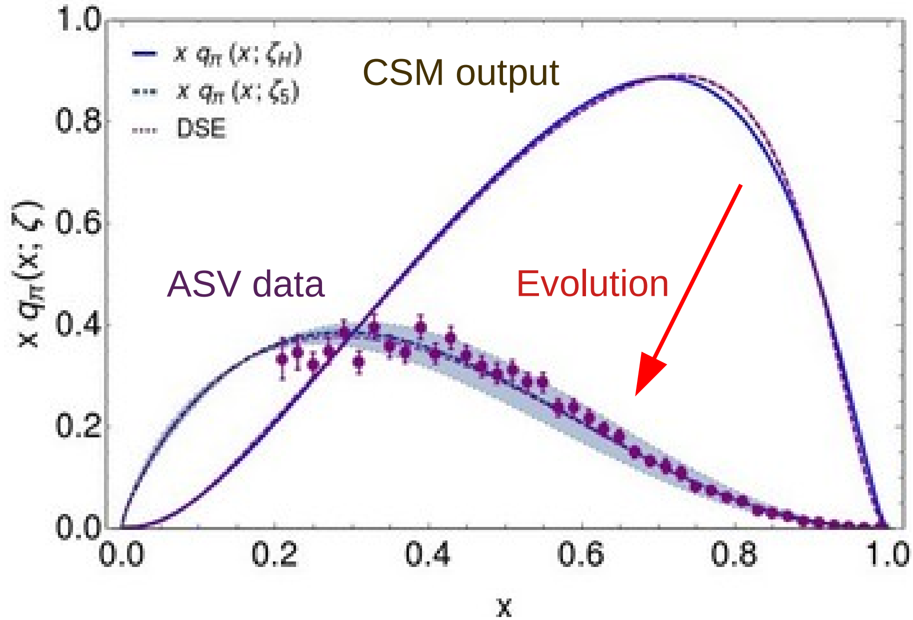
$$\beta(\zeta_H) = 2$$

Farrar, Jackson, Phys.Rev.Lett 35(1975)1416

Berger, Brodsky, Phys.Rev.Lett 42(1979)940

- The EHM-triggered broadening shortens the extent of the domain of convexity lying on the neighborhood of the endpoints, induced too by the QCD dynamics
- It cannot however spoil the asymptotic QCD behaviour at large-x (and, owing to isospin symmetry, at low-x)





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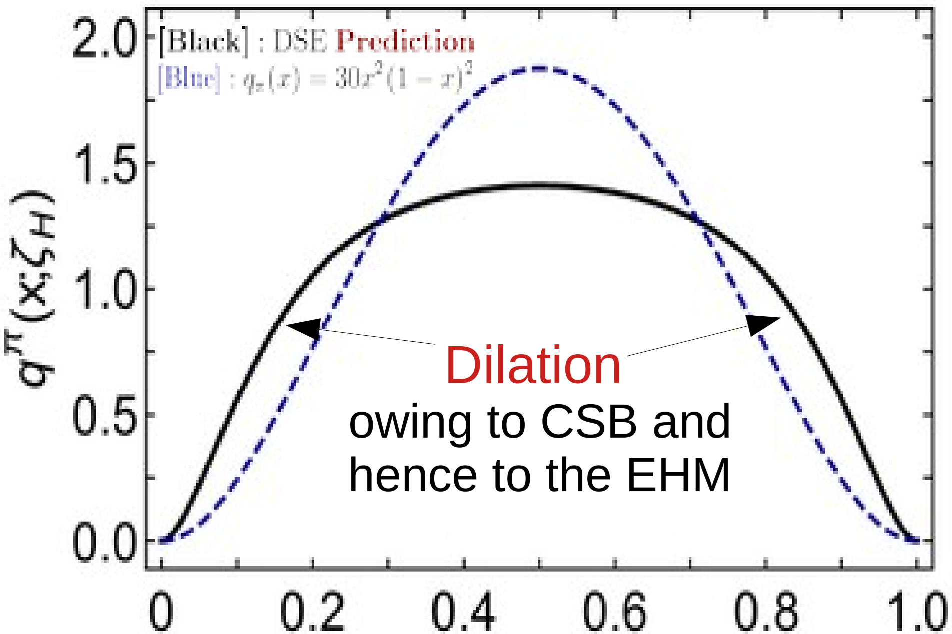
$$q_{\mathrm{O}}^\pi(x;\zeta_H) = 213.32\, x^2(1-x)^2 \times [1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)]$$

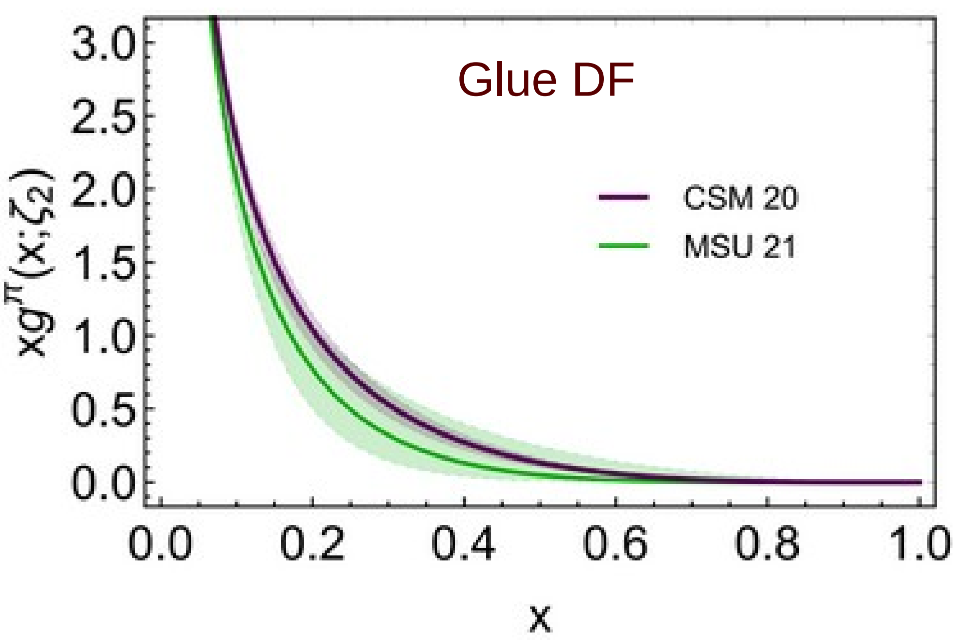
$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)}\,(1 + \mathcal{O}(1-x))$$

$$\beta(\zeta_H) = 2$$

Farrar, Jackson, Phys.Rev.Lett 35(1975)1416
Berger, Brodsky, Phys.Rev.Lett 42(1979)940

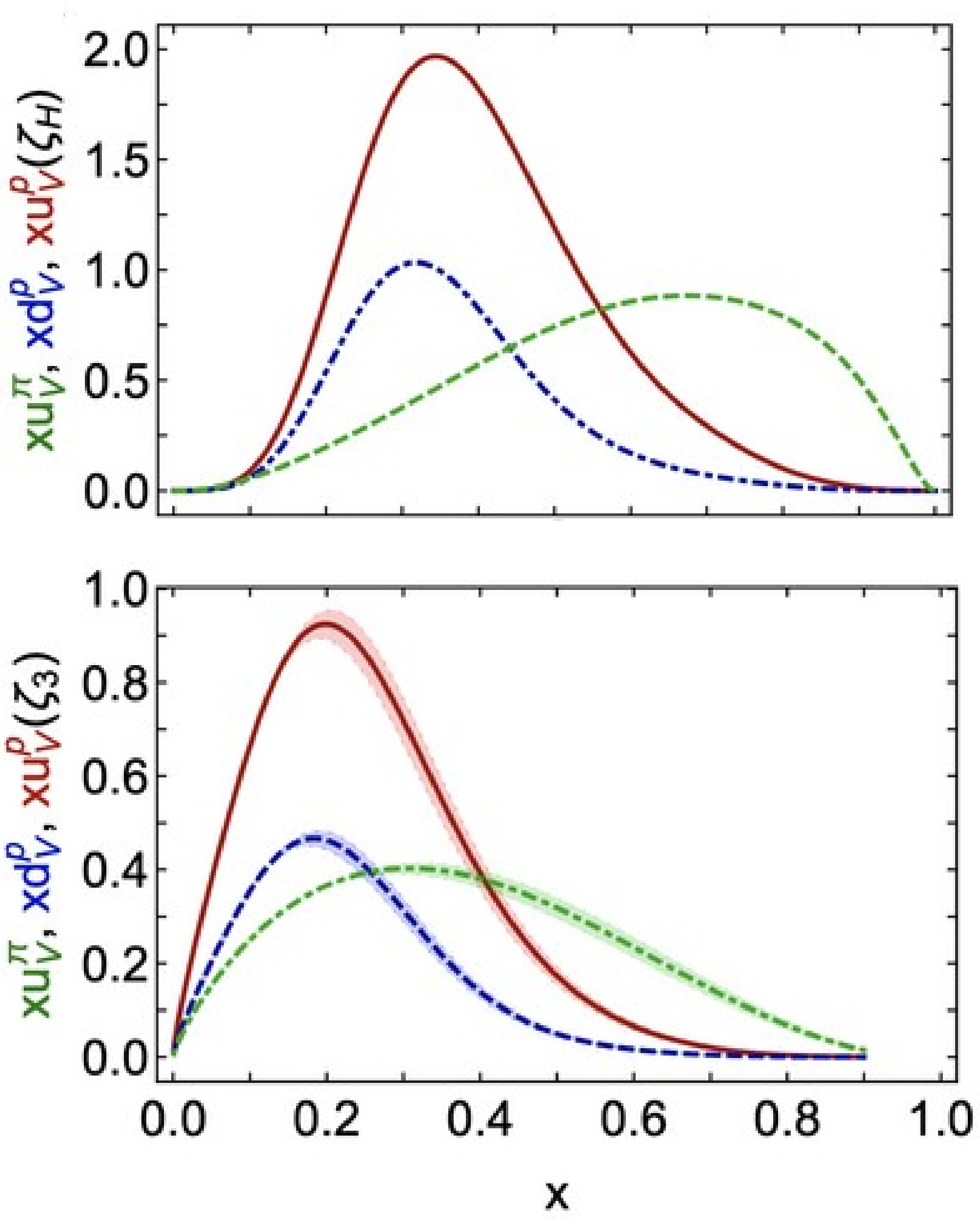
- The EHM-triggered broadening shortens the extent of the domain of convexity lying on the neighborhood of the endpoints, induced too by the QCD dynamics
- It cannot however spoil the asymptotic QCD behaviour at large-x (and, owing to isospin symmetry, at low-x)





Proton PDF: from CSM (DSEs) to the experiment

An analogous symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach:
[L. Chang et al., Phys.Lett.B, arXiv:2201.07870]



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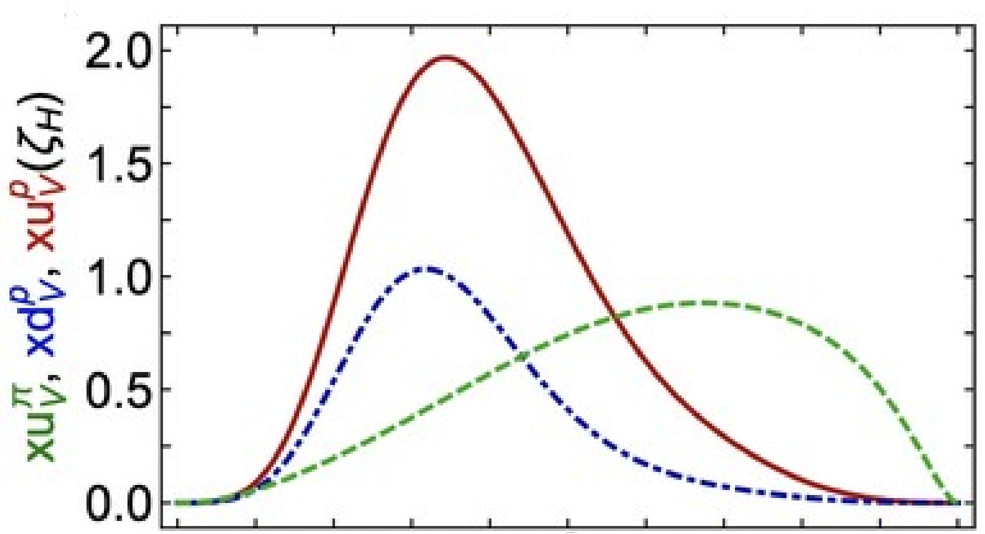
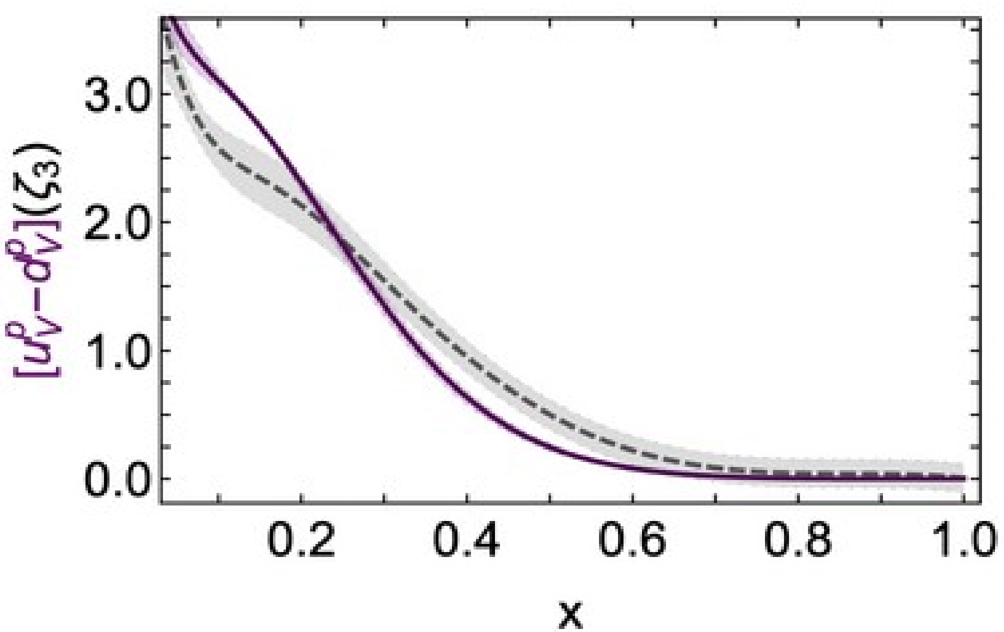
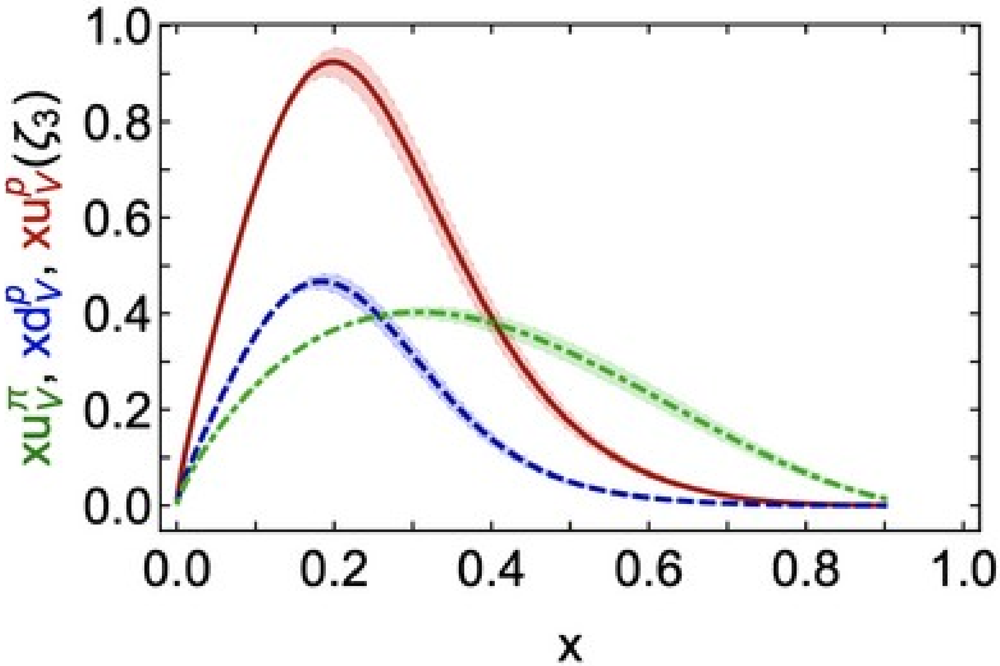
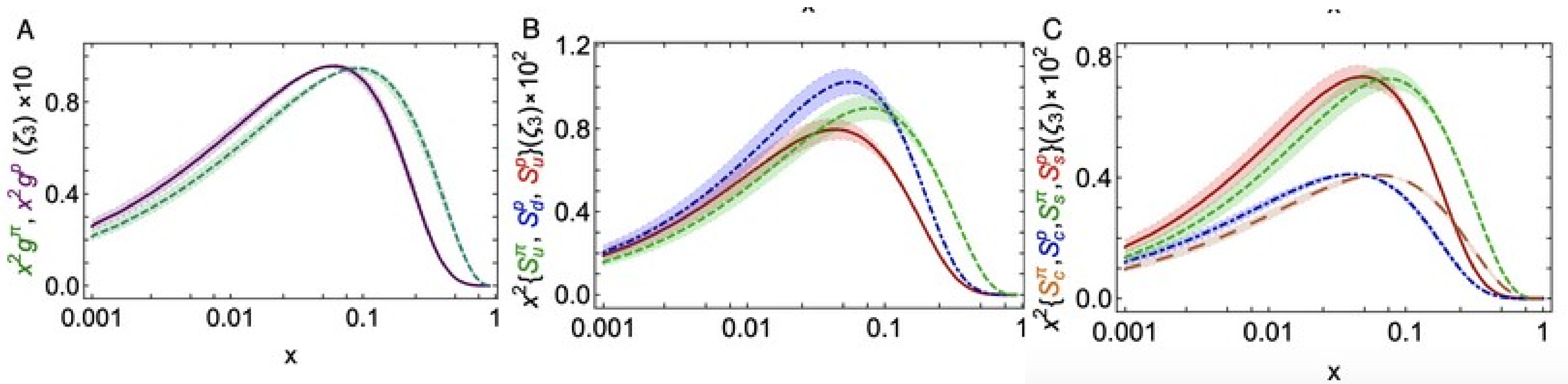


Producing an isovector distribution in fair agreement with lattice results
[H-W. Lin et al., arXiv:2011.14791]





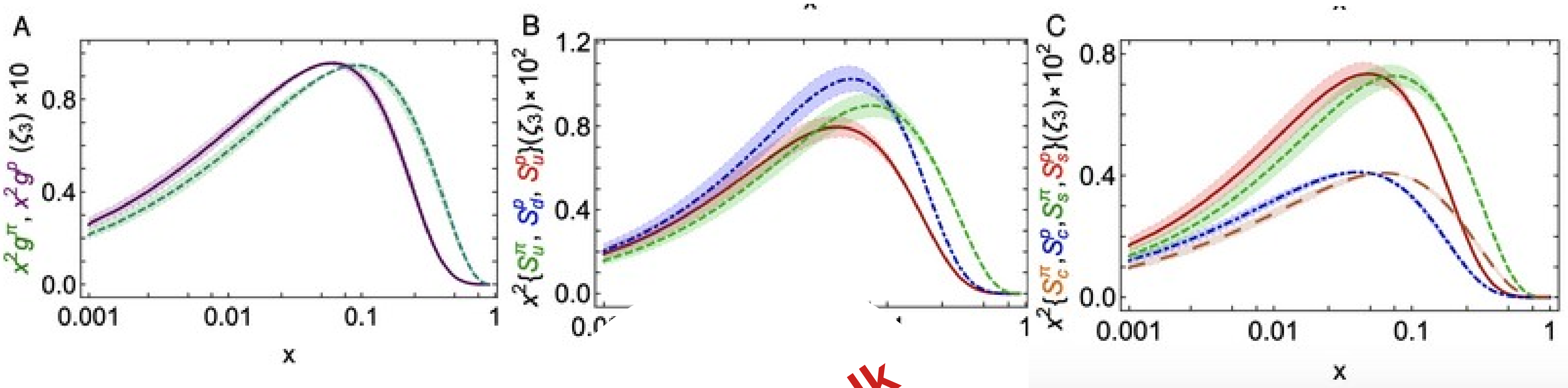
Proton PDF: pion and proton in counterpoint



pion	u^π	$\bar{d}^\pi$	g^π	S_π^u	$S_\pi^{\bar{d}}$	S_π^s	S_π^c
$\langle x\rangle^{\zeta_2}$	24.0(1.1)	24.0(1.1)	41.0(1.2)	3.3(3)	3.3(3)	2.65(22)	1.33(5)
$\langle x^2\rangle^{\zeta_2}$	9.5(7)	9.5(7)	3.7(1)	0.27(1)	0.27(1)	0.21(1)	0.092(2)
$\langle x^3\rangle^{\zeta_2}$	4.7(4)	4.7(4)	0.92(6)	0.057(1)	0.057(1)	0.044(0)	0.018(1)
$\langle x\rangle^{\zeta_3}$	22.1(1.0)	22.1(1.0)	42.9(1.0)	3.7(3)	3.7(3)	3.0(2)	1.83(6)
$\langle x^2\rangle^{\zeta_3}$	8.4(6)	8.4(6)	3.5(1)	0.27(1)	0.27(1)	0.22(1)	0.120(3)
$\langle x^3\rangle^{\zeta_3}$	4.0(3)	4.0(3)	0.82(5)	0.056(0)	0.056(0)	0.044(0)	0.022(1)

proton	u^p	d^p	g^p	S_p^u	S_p^d	S_p^s	S_p^c
$\langle x\rangle^{\zeta_2}$	32.9(1.4)	15.0(0.7)	40.9(1.1)	2.9(2)	3.7(3)	2.64(22)	1.32(5)
$\langle x^2\rangle^{\zeta_2}$	8.7(6)	3.6(2)	2.4(1)	0.14(1)	0.21(1)	0.13(0)	0.059(2)
$\langle x^3\rangle^{\zeta_2}$	2.9(3)	1.1(1)	0.39(2)	0.019(0)	0.030(1)	0.019(0)	0.008(0)
$\langle x\rangle^{\zeta_3}$	30.4(1.3)	13.8(0.6)	42.8(1.0)	3.3(3)	4.1(3)	3.0(2)	1.82(6)
$\langle x^2\rangle^{\zeta_3}$	7.7(5)	3.2(2)	2.2(1)	0.15(1)	0.21(1)	0.14(0)	0.075(2)
$\langle x^3\rangle^{\zeta_3}$	2.5(2)	0.9(1)	0.35(2)	0.019(0)	0.028(0)	0.019(0)	0.010(1)

Proton PDF: pion and proton in counterpoint

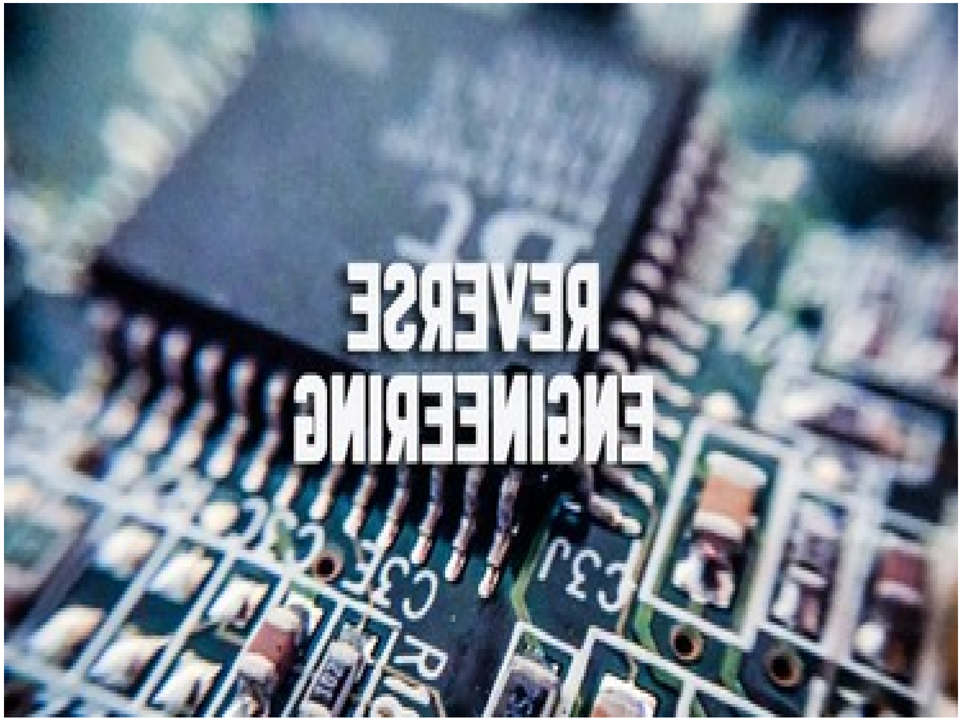


cf. Craig's talk

pion	u^π	$\bar{d}^\pi$			$S_\pi^{\bar{d}}$	S_π^s	S_π^c
$\langle x\rangle^{\zeta_2}$	24.0(1.1)	24.0(1.1)			3.3(3)	2.65(22)	1.33(5)
$\langle x^2\rangle^{\zeta_2}$	9.5(7)	9.5(7)			0.27(1)	0.21(1)	0.092(2)
$\langle x^3\rangle^{\zeta_2}$	4.7(4)	4.7(4)			0.057(1)	0.044(0)	0.018(1)
$\langle x\rangle^{\zeta_3}$	22.1(1.0)	22.1(1.0)	[illegible]		3.7(3)	3.0(2)	1.83(6)
$\langle x^2\rangle^{\zeta_3}$	8.4(6)	8.4(6)	3.[illegible]		0.27(1)	0.22(1)	0.120(3)
$\langle x^3\rangle^{\zeta_3}$	4.0(3)	4.0(3)	0.82([illegible])	[illegible]6(0)	0.056(0)	0.044(0)	0.022(1)

proton	u^p	d^p	g^p	S_p^u	S_p^d	S_p^s	S_p^c
$\langle x\rangle^{\zeta_2}$	32.9(1.4)	15.0(0.7)	40.9(1.1)	2.9(2)	3.7(3)	2.64(22)	1.32(5)
$\langle x^2\rangle^{\zeta_2}$	8.7(6)	3.6(2)	2.4(1)	0.14(1)	0.21(1)	0.13(0)	0.059(2)
$\langle x^3\rangle^{\zeta_2}$	2.9(3)	1.1(1)	0.39(2)	0.019(0)	0.030(1)	0.019(0)	0.008(0)
$\langle x\rangle^{\zeta_3}$	30.4(1.3)	13.8(0.6)	42.8(1.0)	3.3(3)	4.1(3)	3.0(2)	1.82(6)
$\langle x^2\rangle^{\zeta_3}$	7.7(5)	3.2(2)	2.2(1)	0.15(1)	0.21(1)	0.14(0)	0.075(2)
$\langle x^3\rangle^{\zeta_3}$	2.5(2)	0.9(1)	0.35(2)	0.019(0)	0.028(0)	0.019(0)	0.010(1)

Reverse engineering the PDF data



Pion PDF

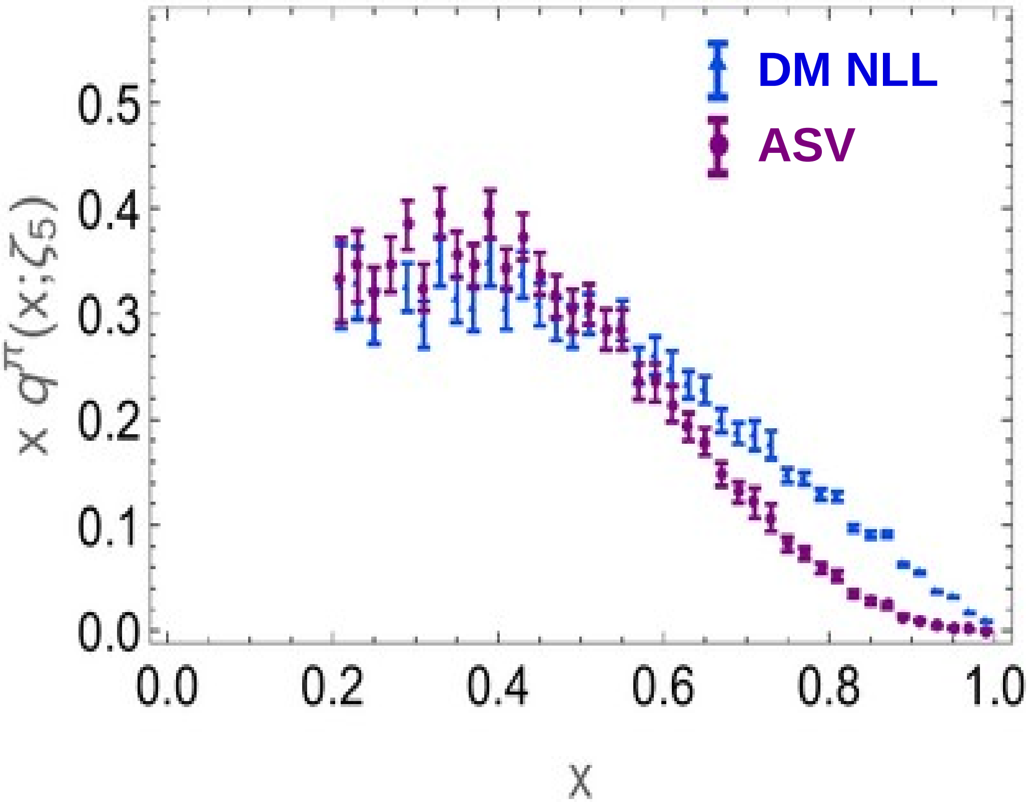
- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^\pi(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

Normalization

$$\{\alpha_i^\zeta | i = 1,2,3\}$$

Free parameters



- Then, we proceed as follows:

1) **Determine** the **best values α_i** via least-squares fit to the data.

2) **Generate** new **values α_i,** distributed randomly around the best fit.

3) **Using** the latter set, evaluate:

$$\chi^2 = \sum_{l=1}^{N} \frac{(u^\pi(x_l;[\alpha_i];\zeta_5) - u_j)^2}{\delta_l^2}$$

Data point with error

4) **Accept** a replica with probability:

$$\mathcal{P} = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}\,,\quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

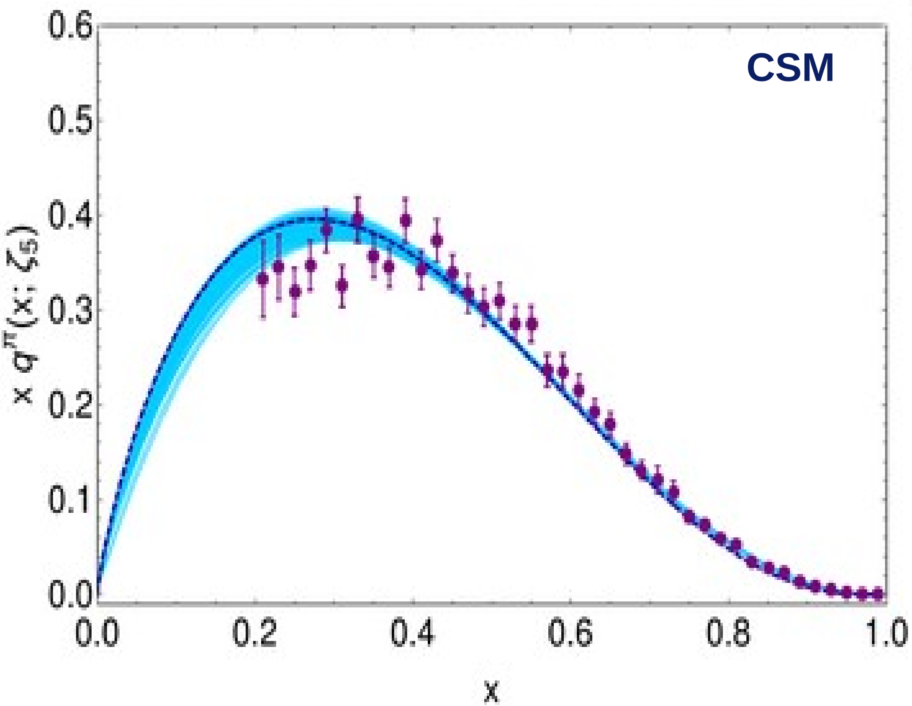
5) **Evolve** back to ζ_H

Repeat (2-5).

Pion PDF: ASV analysis of E615 data

➢ Applying this algorithm to the **ASV data** yields:

(average)



Mean values (of moments) and errors

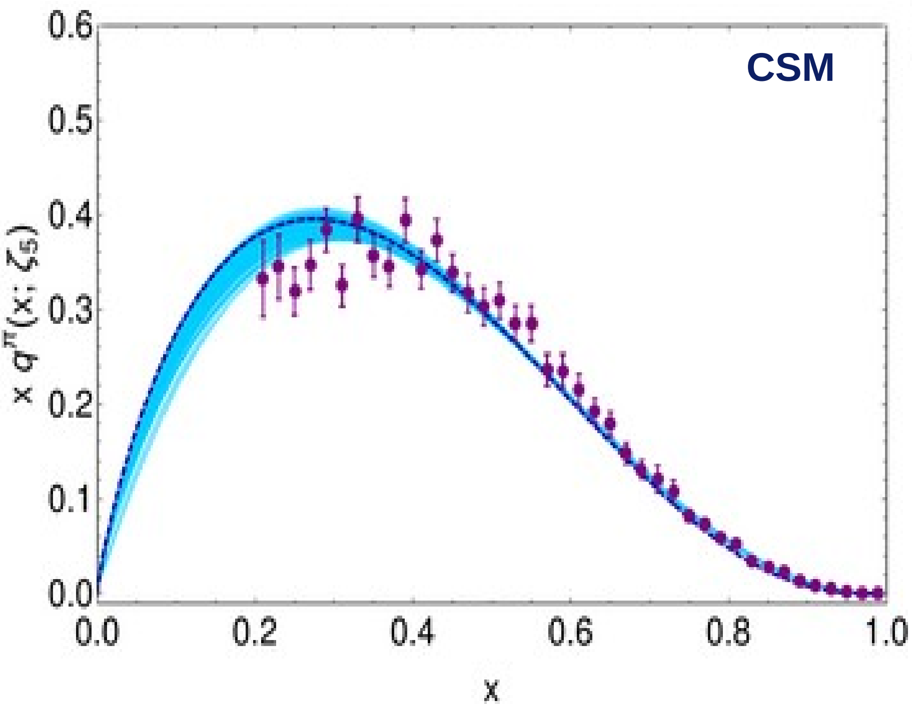
$\{\{0.5, 2.75144\times10^{-17}\}, \{0.299833, 0.00647045\}, \{0.199987, 0.00735448\}, \{0.142895, 0.0068623\}, \{0.107274, 0.00608759\}, \{0.0835168, 0.00532834\}, \{0.0668711, 0.0046596\}, \{0.0547511, 0.00409028\}, \{0.0456496, 0.00361041\}, \{0.0386394, 0.00320609\}\}$

✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

✔ It seems it favors a **soft end-point** behavior just like the **CSM result**.

Pion PDF: ASV analysis of E615 data

➢ Applying this algorithm to the **ASV data** yields:

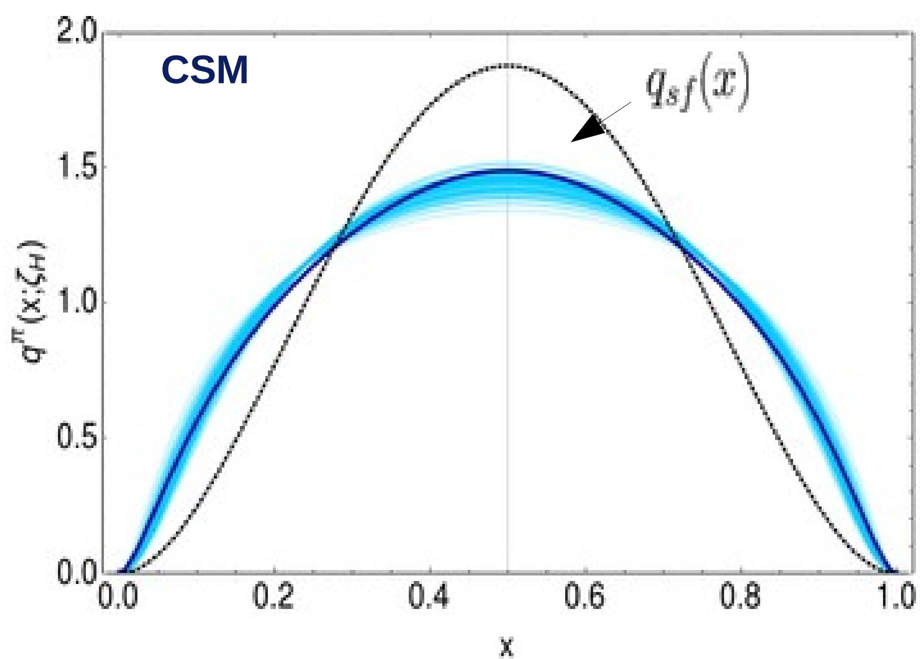


Mean values (of moments) and errors

{{0.5, 2.75144 × 10^-17}, {0.299833, 0.00647045}, {0.199987, 0.00735448}, {0.142895, 0.0068623}, {0.107274, 0.00608759}, {0.0835168, 0.00532834}, {0.0668711, 0.0046596}, {0.0547511, 0.00409028}, {0.0456496, 0.00361041}, {0.0386394, 0.00320609}}

✔ Then, we can **reconstruct** the moments produced by each replica, using the **single-parameter Ansatz**:

$$u^{\pi}(x;\zeta_{\mathcal{H}}) = n_0 \ln(1 + x^2(1-x)^2/\rho^2)$$

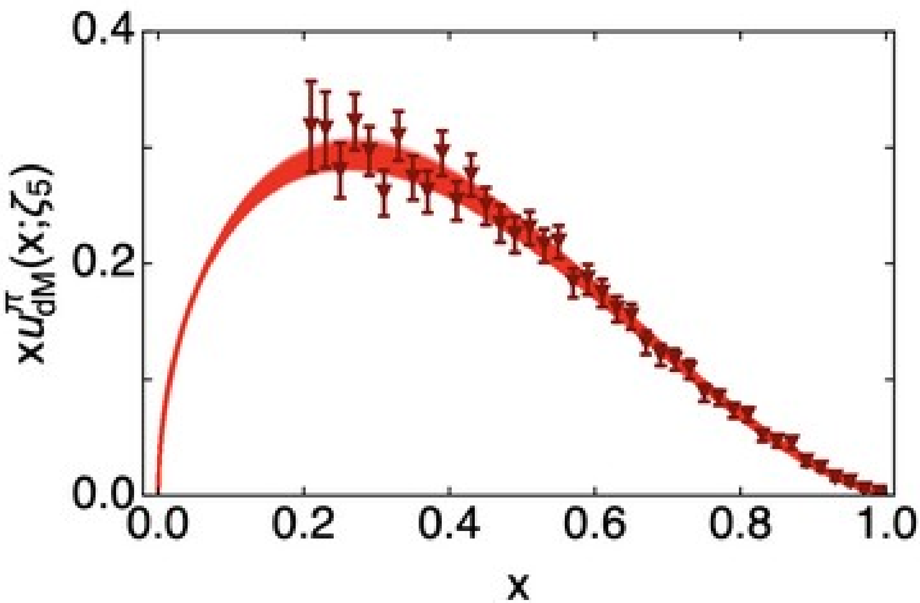


✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

✔ It seems it favors a **soft end-point** behavior… just like the **CSM result**.

Pion PDF: dM NLL analysis of E615 data

- Applying this algorithm to the original data yields:



(average)

lean values (of moments) and errors, ζ_H

{{0.5, 2.52187 × 10^{-17}}, {0.331527, 0.00803273}, {0.247615, 0.0110893},
{0.19784, 0.0121977}, {0.165066, 0.0124911}, {0.141928, 0.0124198},
{0.124755, 0.0121811}, {0.111521, 0.0118683}, {0.101021, 0.0115275},
{0.0924926, 0.0111824}, {0.085431, 0.010845}, {0.0794897, 0.0105214},
{0.0744232, 0.0102142}, {0.0700521, 0.00992435}, {0.0662432, 0.00965182}}

(SCI)

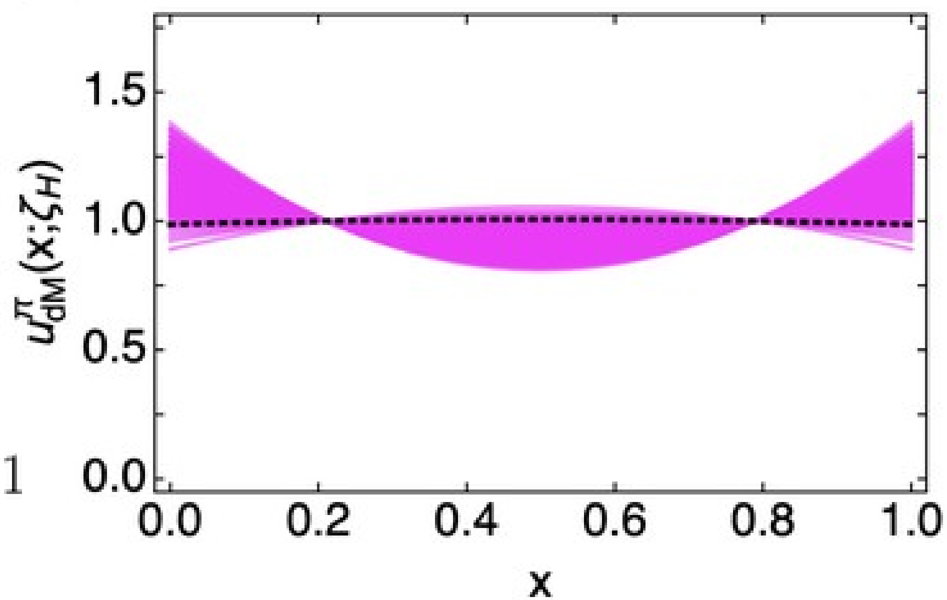
loments from SCI, ζ_H

0.5, 0.332885, 0.249327, 0.199231, 0.165865, 0.142056, 0.124215, 0.11035,
0.0992657, 0.090203, 0.0826552, 0.0762721, 0.0708035, 0.0660661, 0.0619225}

✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

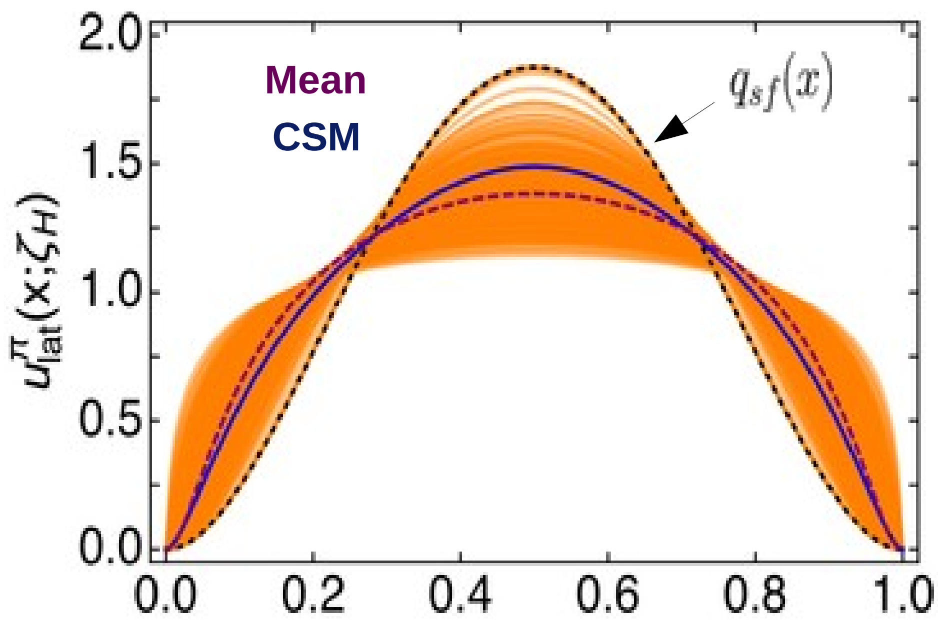
✗ But also exhibit agreement with the **SCI results**.

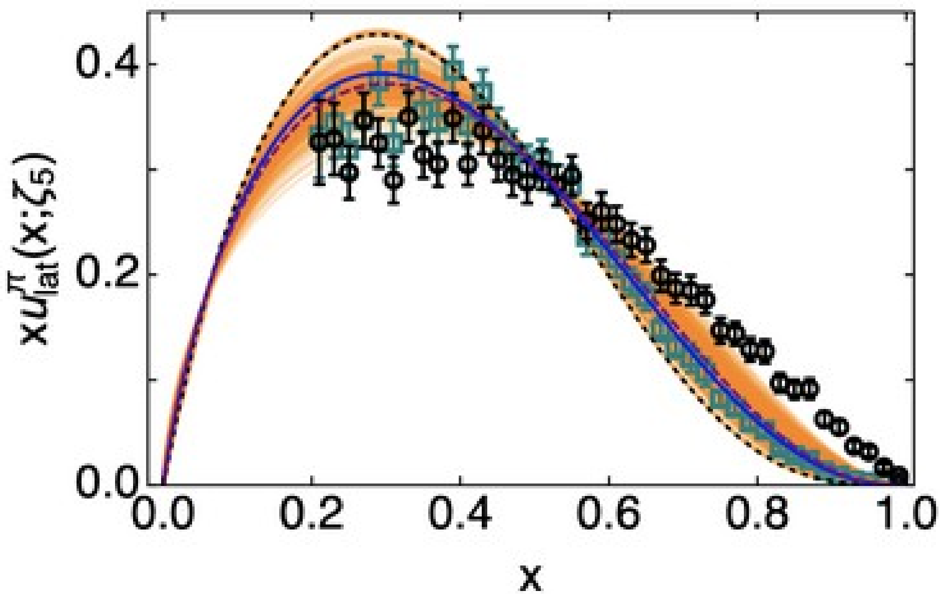
$$q_{\rm SCI}(x;\zeta_H) \approx 1$$



Pion PDF: lattice data

- An analagous procedure, similarly based on the all-orders evolution, can be applied to the **lattice data** for Mellin moments. Here, the moments obtained at the lattice scale are evolved down to the hadronic scale and up to the **experimental** one.
- Both (**ASV**) **experimental** and **lattice** data yield hadronic scale **PDFs** exhibiting soft end-point behavior and **EHM-induced broadening**.
- The results are **compatible**, although current precision of the lattice moments still leaves us with a somewhat **wide band** of **uncertainty**.
- **Lattice** results, analyzed on the basis of all-orders evolution, are clearly **inconsistent** with those resulting from the **dM NLL analysis** of E615 data.





Pion PDF: lattice data

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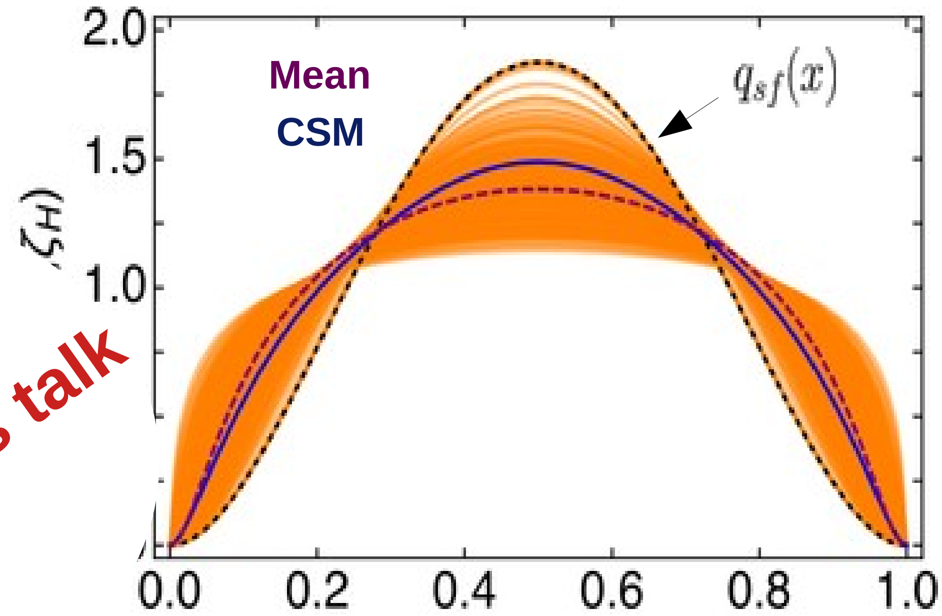
one.

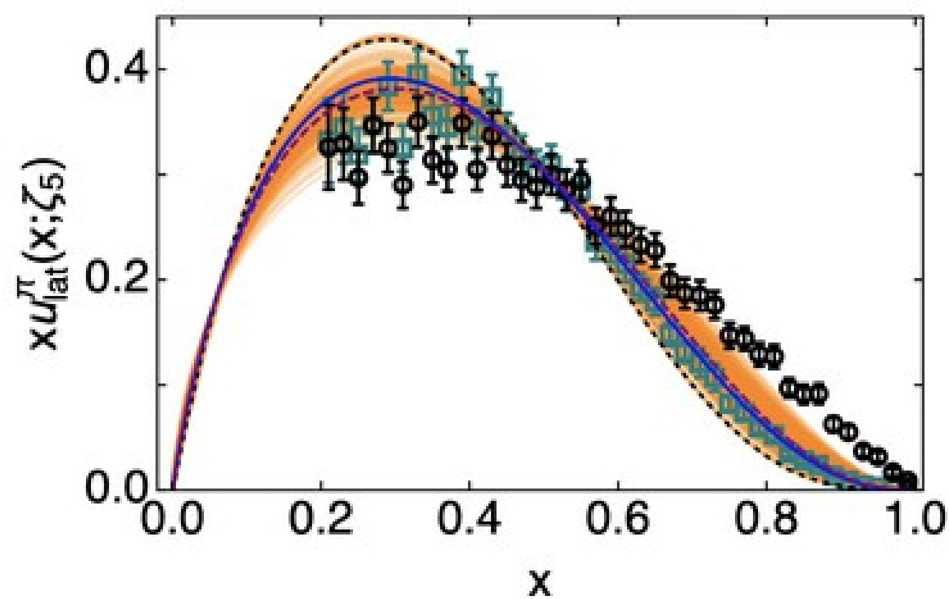
- Both (**ASV**) **experimental** an
hadronic scale **PDFs** exhibit
behavior and **EHM-induced br**

cf. Daniele's talk

- The results are **compatible**, although current precision of the lattice moments still leaves us with a somewhat **wide band** of **uncertainty**.

- **Lattice** results, analyzed on the basis of all-orders evolution, are clearly **inconsistent** with those resulting from the **dM NLL analysis** of E615 data.

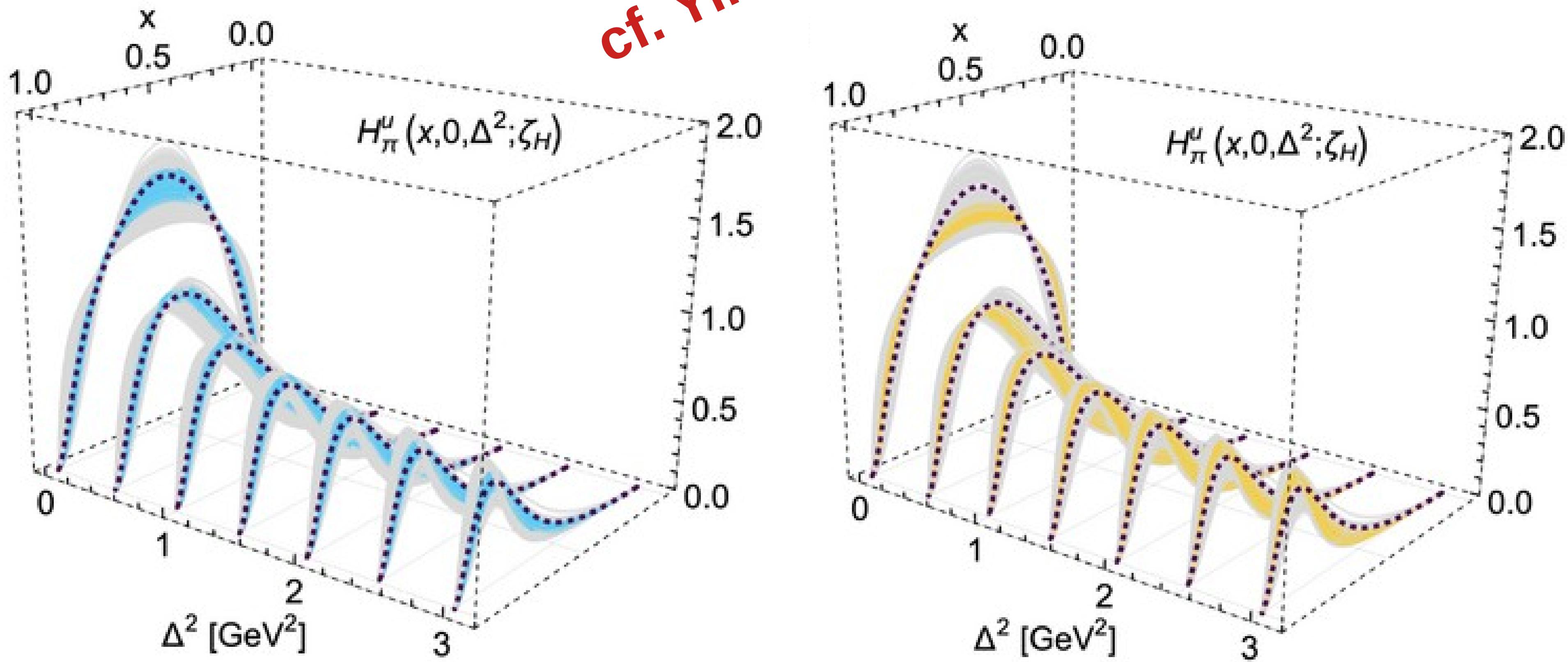




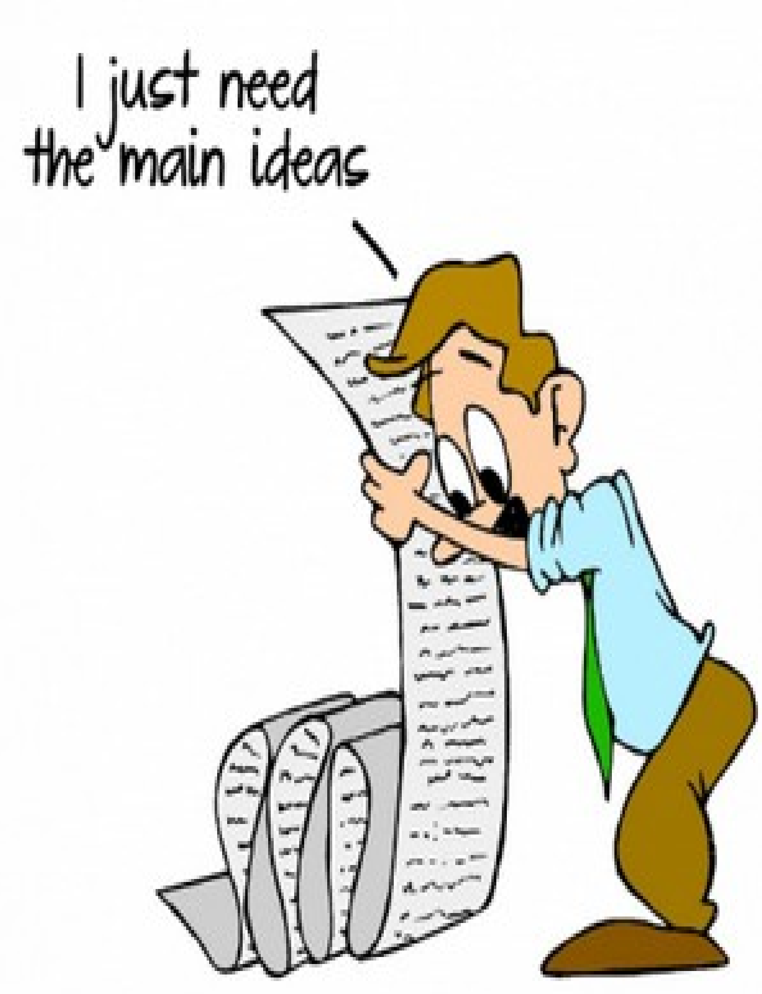
Pion GPDs

GPDs can be also derived from empirical information relying on the all-orders evolution, reverse engineering and factorization.

cf. Yin-Zhen's talk



Summary



Summary

- The **EHM** is argued to be intimately connected to a **PI effective** charge which enters a conformal regime, below a given momentum scale, where gluons acquiring a dynamical mass decouple from interaction.

- Capitalizing on the latter, two main ideas emerge: (I) the identification of that decoupling with a **hadronic scale** at which the structure of hadrons can be expressed only in terms of valence dressed partons; and (ii) the reliability of an **all-orders** evolution scheme to describe the splitting of valence into more partons, generating thus the glue and sea, when the resolution scale decreases.

- Key implications stemming from both ideas have been derived and tested for the pion PDFs. Grounding on them, **Lattice QCD** and experimental data have been shown to confirm **CSM** results.

- The robustness of the approach based on **all-orders** evolution from **hadronic** to experimental scale has been proved with its application to the pion and proton case. A model featuring massless evolution for quark flavors activated after a hard-wall threshold and accounrting ofer Pauli blocking has been solved analytically, and seen to expose some of the main results implied by the approach.

To be continued...