Partonic structure of hadrons in lattice QCD: From single- to multi-parton distributions

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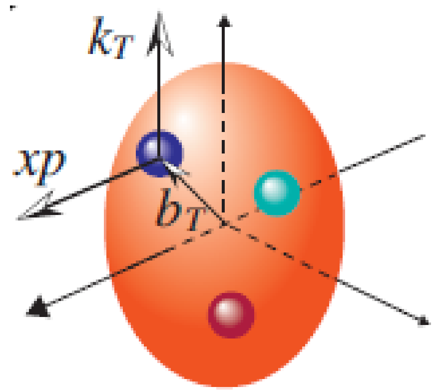
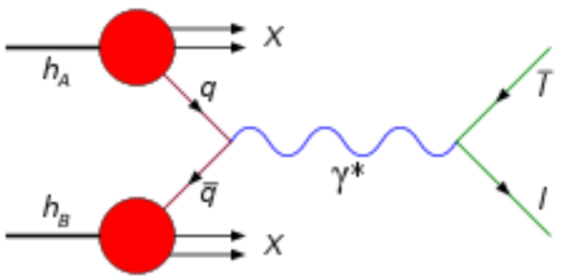
Parton distribution functions at a crossroad, ECT*, Sep 20, 2023

Introduction

- Parton physics plays an important role in mapping out the 3D structure of hadrons and interpreting the experimental data at hadron colliders





Example: Drell-Yan Process

- **Factorization**

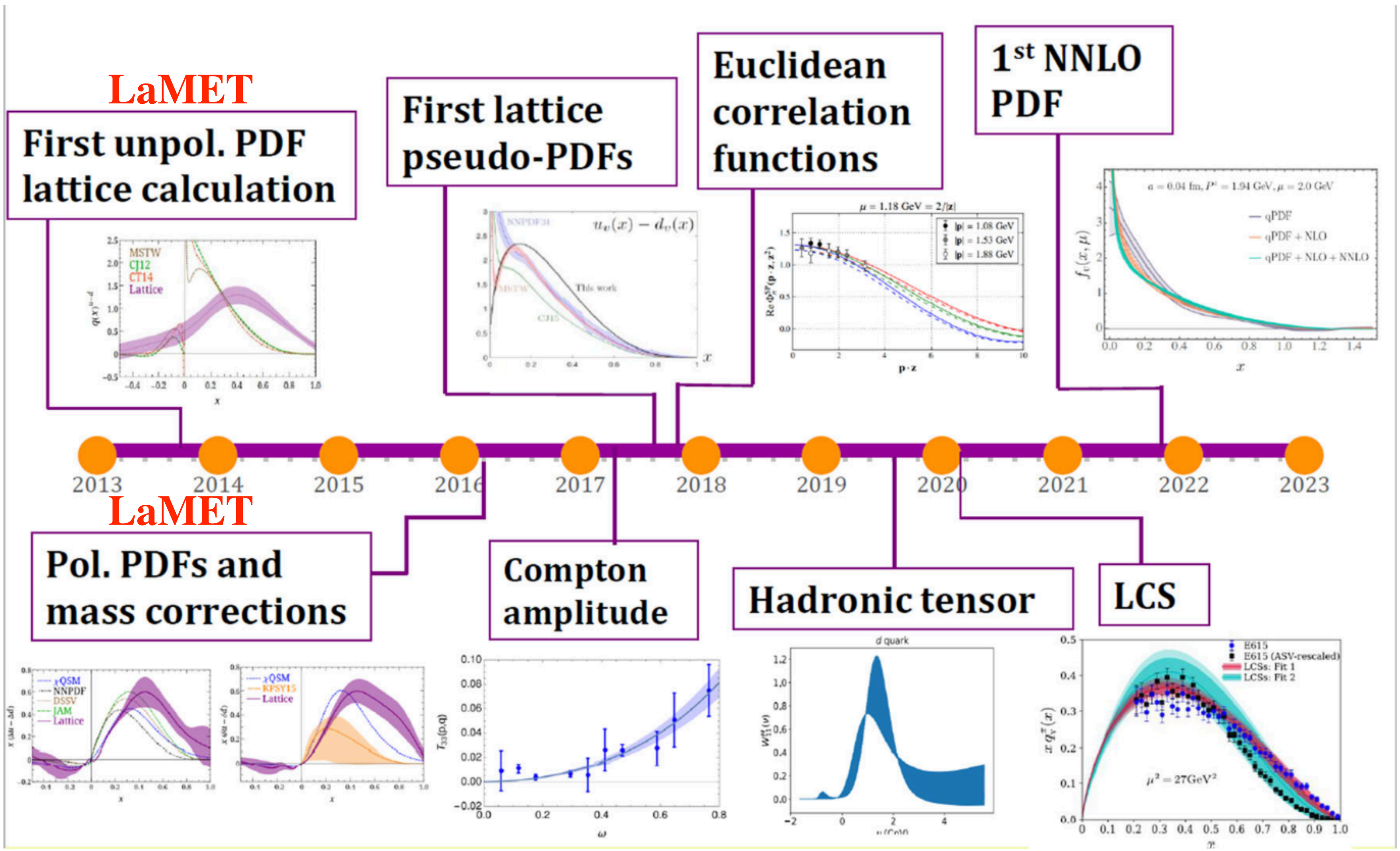
$$\frac{d\sigma}{dQ^2} = \sum_{i,j} \int_0^1 d\xi_a d\xi_b \, f_{i/P_a}(\xi_a) f_{j/P_b}(\xi_b) \frac{d\hat{\sigma}_{ij}(\xi_a, \xi_b)}{dQ^2} \times \left[1 + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{Q}\right)\right] \qquad Q = \sqrt{q^2}$$

$q_T \ll Q$:

$$\frac{d\sigma}{dQ^2 d^2\mathbf{q_T}} = \sum_{i,j} H_{ij}(Q) \int_0^1 d\xi_a d\xi_b \int d^2\mathbf{b_T}\, e^{i\mathbf{b_T}\cdot\mathbf{q_T}} \times f_{i/P}(\xi_a, \mathbf{b_T}) f_{j/P}(\xi_b, \mathbf{b_T}) \times \left[1 + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{Q}, \frac{q_T}{Q}\right)\right]$$

Introduction

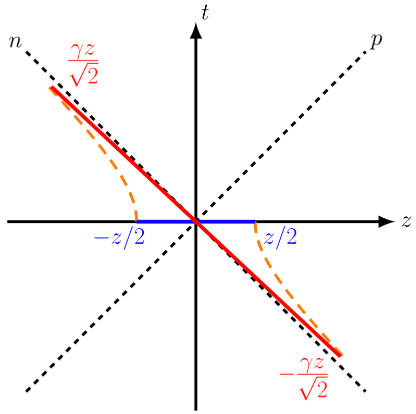
- Tremendous progress has been achieved on calculating the **x-dependent partonic structure** of hadrons from Euclidean lattice



H-W Lin, FBS 23'

Introduction

- **A popular approach: Large-momentum effective theory (LaMET)**
  Ji, PRL 13' & SCPMA 14', Ji, Liu, Liu, JHZ, Zhao, RMP 21'



$$q(x,\mu) = \int \frac{d\lambda}{4\pi} e^{ix\lambda} \langle P | \bar{\psi}(0) n\cdot\gamma L(0,\lambda n)\psi(\lambda n) | P\rangle, \quad n^2 = 0$$

$$\tilde{q}(y,P^z) = N \int \frac{dz}{4\pi} e^{-iyzP^z} \langle P | \bar{\psi}(0)\gamma^0 L(0,z)\psi(z) | P\rangle$$

$$\tilde{q}(y,P^z) = C\left(\frac{y}{x}, \frac{\mu}{xP^z}\right) \otimes q(x,\mu) + \mathcal{O}\left(\frac{\Lambda^2_{QCD}}{(yP^z)^2}, \frac{\Lambda^2_{QCD}}{((1-y)P^z)^2}\right)$$

- Theory studies and lattice calculations available for
  - Collinear PDFs, distribution amplitudes
  - GPDs, TMDPDFs/wave functions
  - Higher-twist distributions

A huge number of references…

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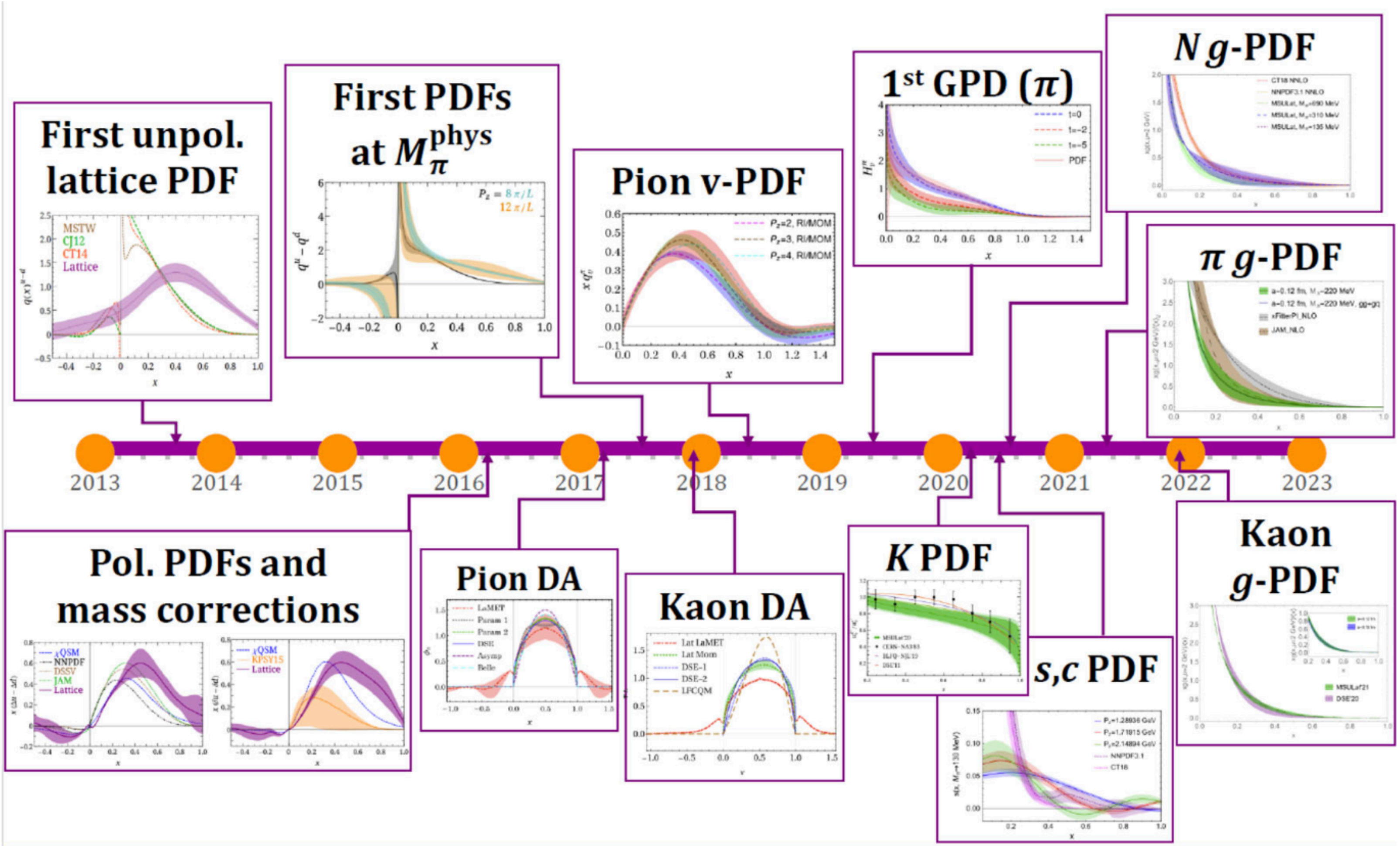

H-W Lin, FBS 23'

Lattice results

- **A popular approach: Large-momentum effective theory (LaMET)**
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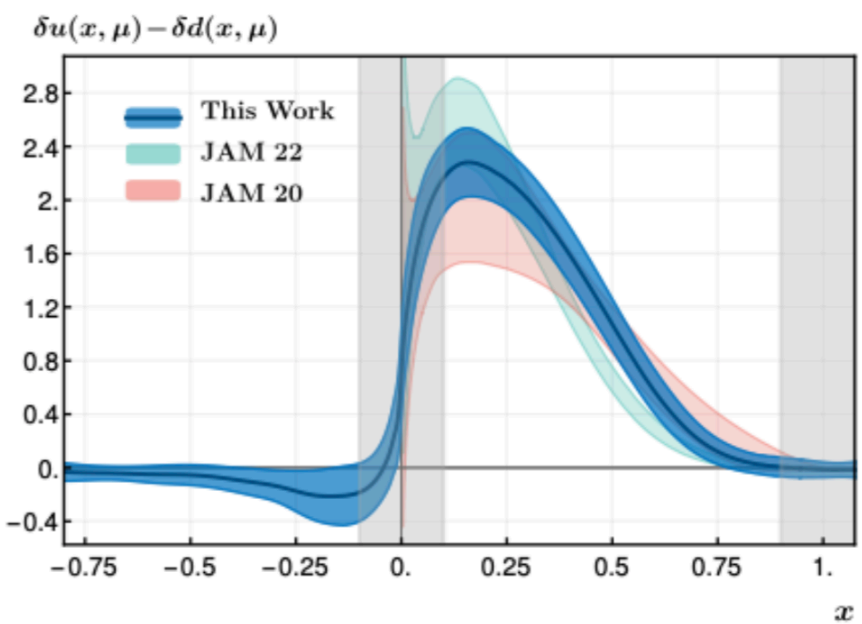
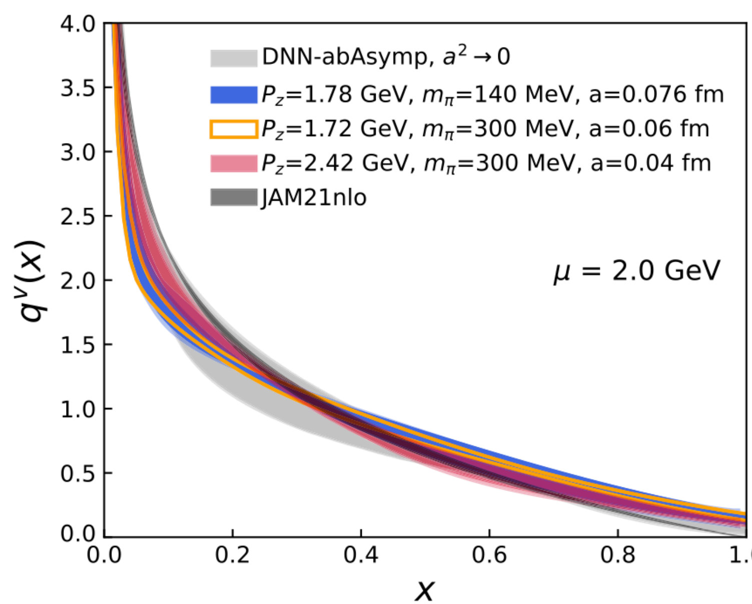
- Examples of the state-of-the-art:

Nucleon quark transversity



Yao, JHZ et al (LPC) 22’

Pion valence quark PDF



Gao et al, PRD 22’

Lattice results

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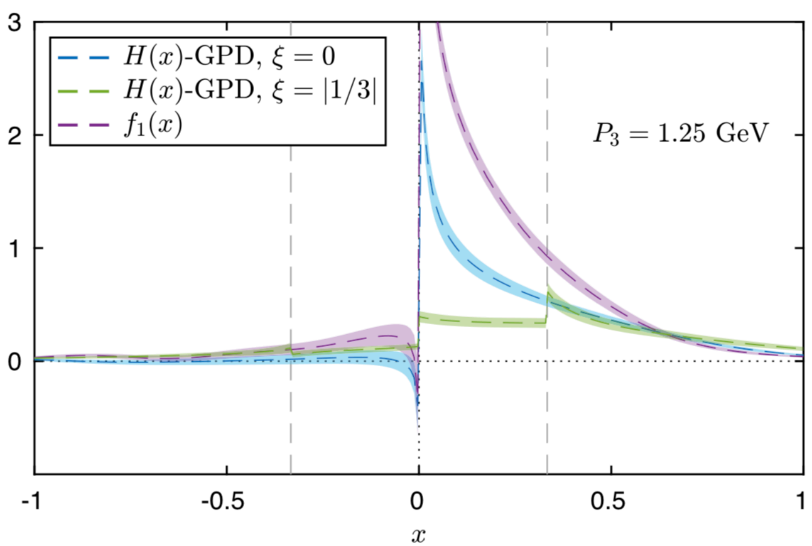
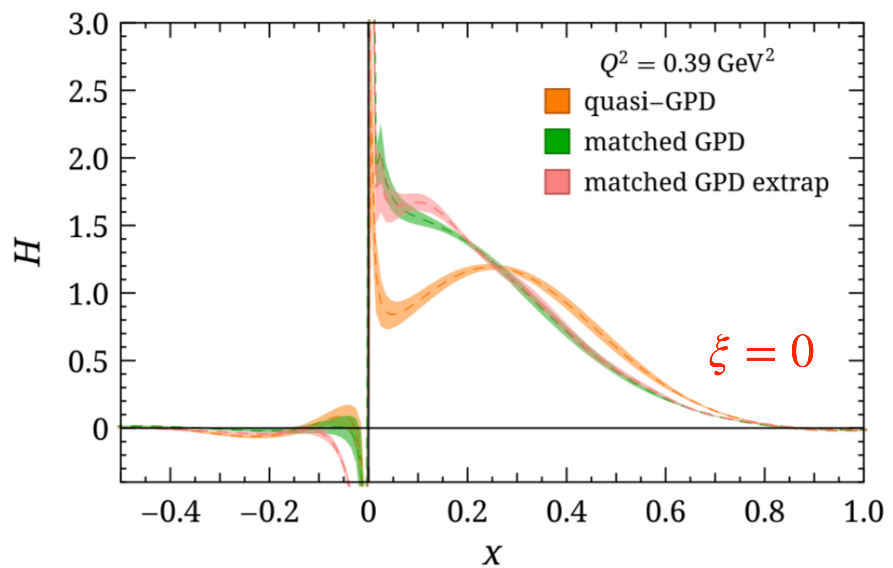
- Examples of the state-of-the-art:

Nucleon quark unpolarized GPD



Alexandrou et al, PRL 20'



Lin, PRL 21'

Lattice results

- **A popular approach: Large-momentum effective theory (LaMET)**
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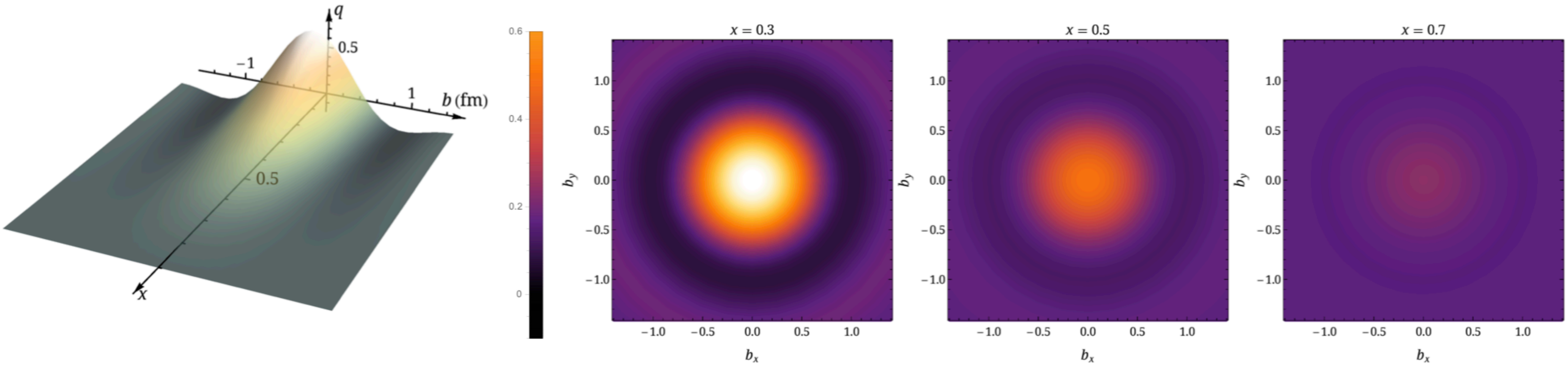
- Examples of the state-of-the-art:

Impact parameter distribution

$$\mathsf{q}(x,b)=\int\frac{d\mathbf{q}}{(2\pi)^2}H(x,\xi=0,t=-\mathbf{q}^2)e^{i\mathbf{q}\cdot\mathbf{b}}$$



Lin, PRL 21'

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Impact parameter distribution

$$\mathrm{q}(x,b) = \int \frac{d\mathbf{q}}{\cdots} H(x,\xi=0,t=-\mathbf{q}^2)e^{i\mathbf{q}\cdot\mathbf{b}}$$

- Potential improvement:
  - ratio-hybrid renormalization of lattice matrix elements
  - Perturbative matching to be updated
    - **Y. Ji, Yao, JHZ, 2212.14415** (complete perturbative manual for all collinear leading-twist quantities)
  - Control of power corrections

Lin, PRL 21'

Lattice results

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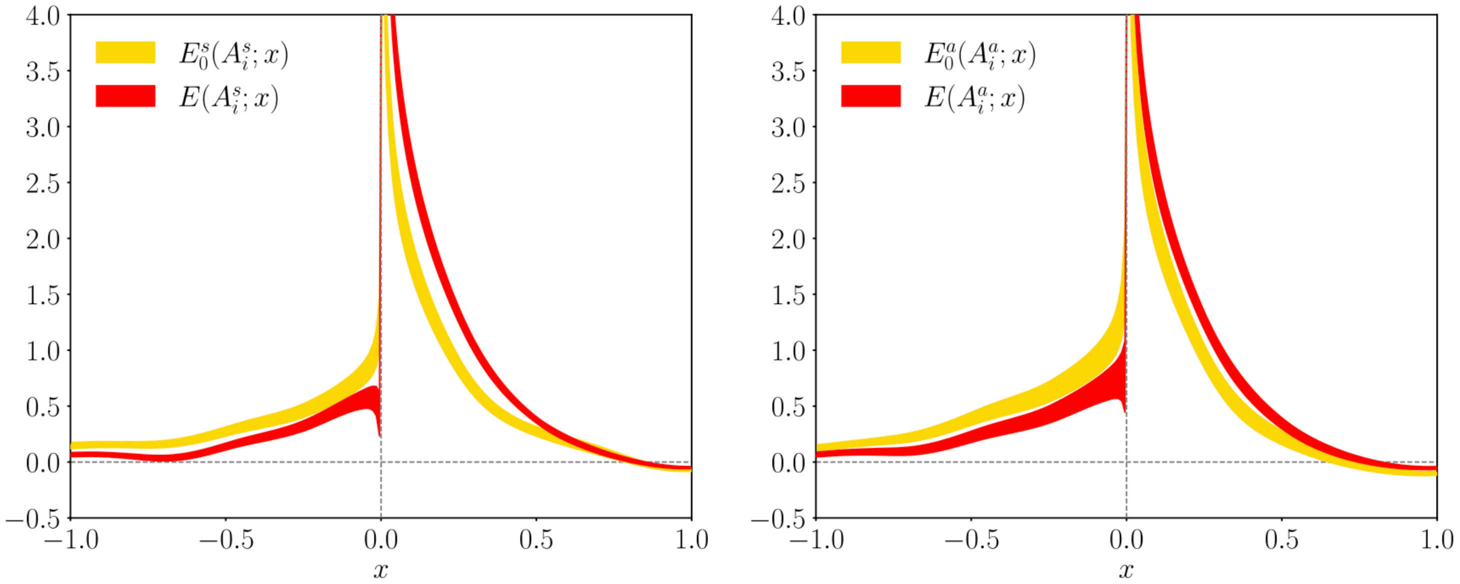
- Examples of the state-of-the-art:

Nucleon quark GPD in asymmetric frame

Bhattacharya et al, PRD 22'



- Leading-twist violates translational invariance, which can be restored by including kinematic higher-twist contributions **Braun, 23'**

Lattice results

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- Examples of the state-of-the-art:

Transverse-momentum-dependent PDF

$$\Phi^{[\gamma^+]} = f_1(x, \mathbf{k}_\perp^2) - \frac{\epsilon_\perp^{ij} k_{\perp i} S_{\perp j}}{M} f_{1T}^\perp(x, \mathbf{k}_\perp^2)$$

$$\Phi^{[\gamma^+\gamma_5]} = \lambda_N g_{1L}(x, \mathbf{k}_\perp^2) - \frac{\mathbf{k}_\perp \cdot \mathbf{S}_\perp}{M} g_{1T}^\perp(x, \mathbf{k}_\perp^2)$$

$$\Phi^{[i\sigma^{i+}\gamma_5]} = S_\perp^i h_1(x, \mathbf{k}_\perp^2) + \frac{\lambda_N}{M} k_\perp^i h_{1L}^\perp(x, \mathbf{k}_\perp^2) + \frac{1}{M^2}(\frac{1}{2} g_\perp^{ij} \mathbf{k}_\perp^2 - k_\perp^i k_\perp^j) S_{\perp j} h_{1T}^\perp(x, \mathbf{k}_\perp^2) - \frac{\epsilon_\perp^{ij} k_{\perp j}}{M} h_1^\perp(x, \mathbf{k}_\perp^2)$$

Lattice results

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Unpol. **Sivers** → **T-odd** ← **Boer-Mulders**

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Unpolarized quark isovector TMDPDF has been calculated
He, JHZ et al, LPC 22'

T-odd Sivers and Boer-Mulders functions in progress

Lattice results

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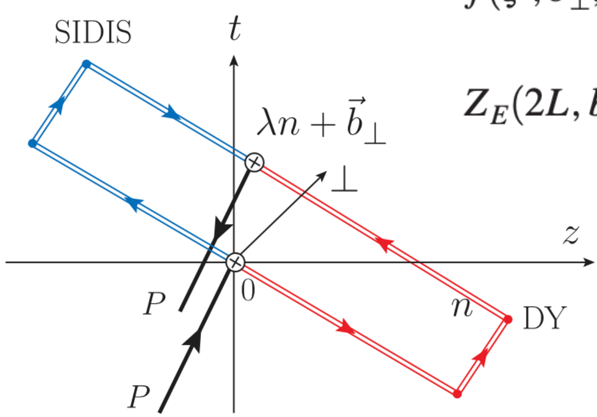
  Ji, PRL 13’ & SCPMA 14’, Ji, Liu, Liu, JHZ, Zhao, RMP 21’

- Examples of the state-of-the-art:

Transverse-momentum-dependent PDF

$$\tilde{f}(\xi^z, b_\perp, \mu, \zeta_z) = \lim_{L\to\infty} \frac{\langle P|\bar{\psi}(\xi^z n_z/2 + \vec{b}_\perp)\gamma^z \mathcal{W}_z(\xi^z n_z/2 + \vec{b}_\perp; L)\psi(-\xi^z n_z/2)|P\rangle}{\sqrt{Z_E(2L, b_\perp, \mu)}}$$

$$Z_E(2L, b_\perp, \mu) = \frac{1}{N_c}\mathrm{Tr}\langle 0|W_\perp \mathcal{W}_z(\vec{b}_\perp; 2L)|0\rangle$$



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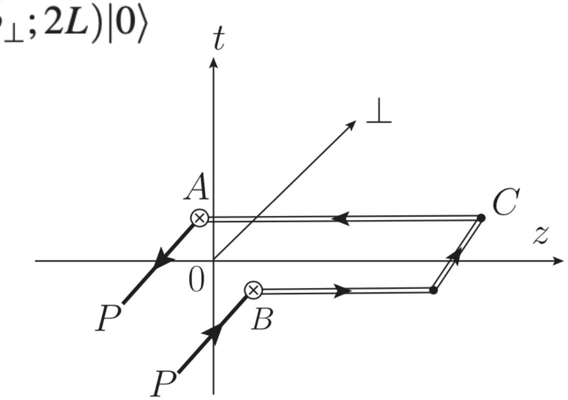
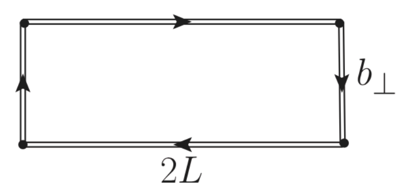


$$f^{\mathrm{TMD}}(x, b_\perp, \mu, \zeta) = H\left(\frac{\zeta_z}{\mu^2}\right) e^{-\ln(\zeta_z/\zeta)K(b_\perp,\mu)} \tilde{f}(x, b_\perp, \mu, \zeta_z) S_r^{1/2}(b_\perp, \mu) + \cdots$$



Ji et al, NPB 20’, PLB 20’, Ebert et al JHEP 19’, Zhang, JHZ et al, PRL 22’

Lattice results

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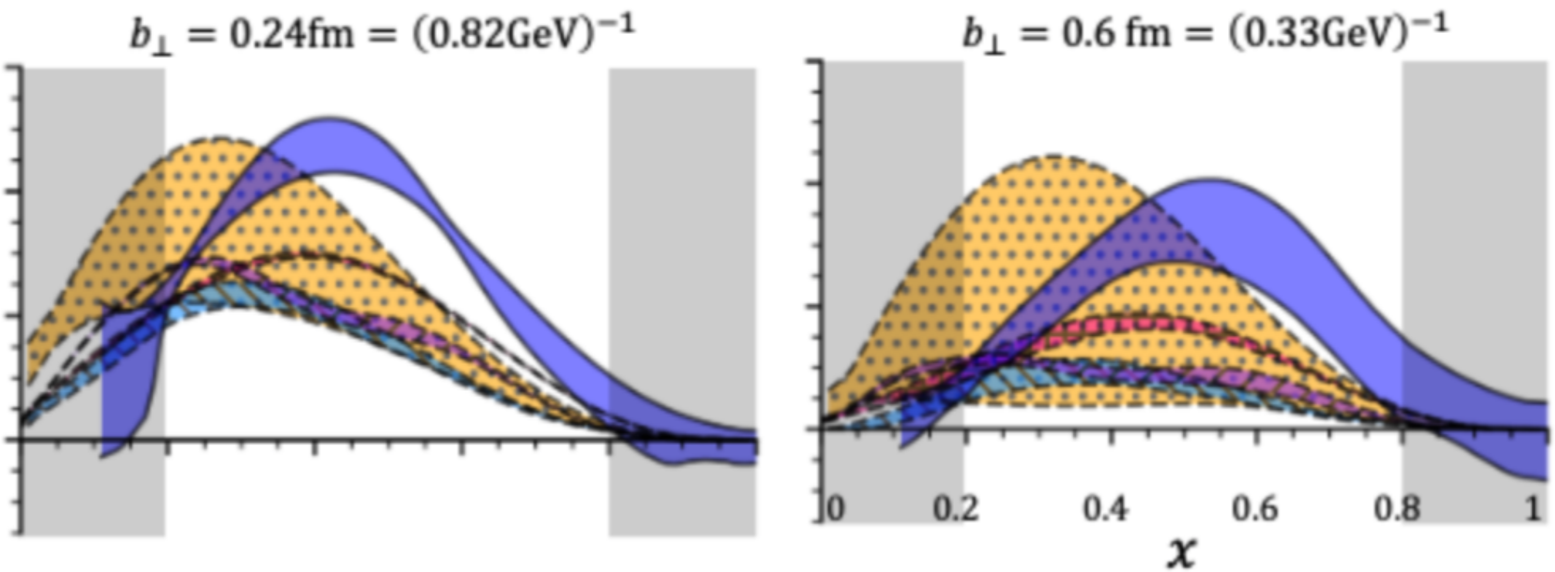
- Examples of the state-of-the-art:

Transverse-momentum-dependent PDF



He, JHZ et al, LPC 22'

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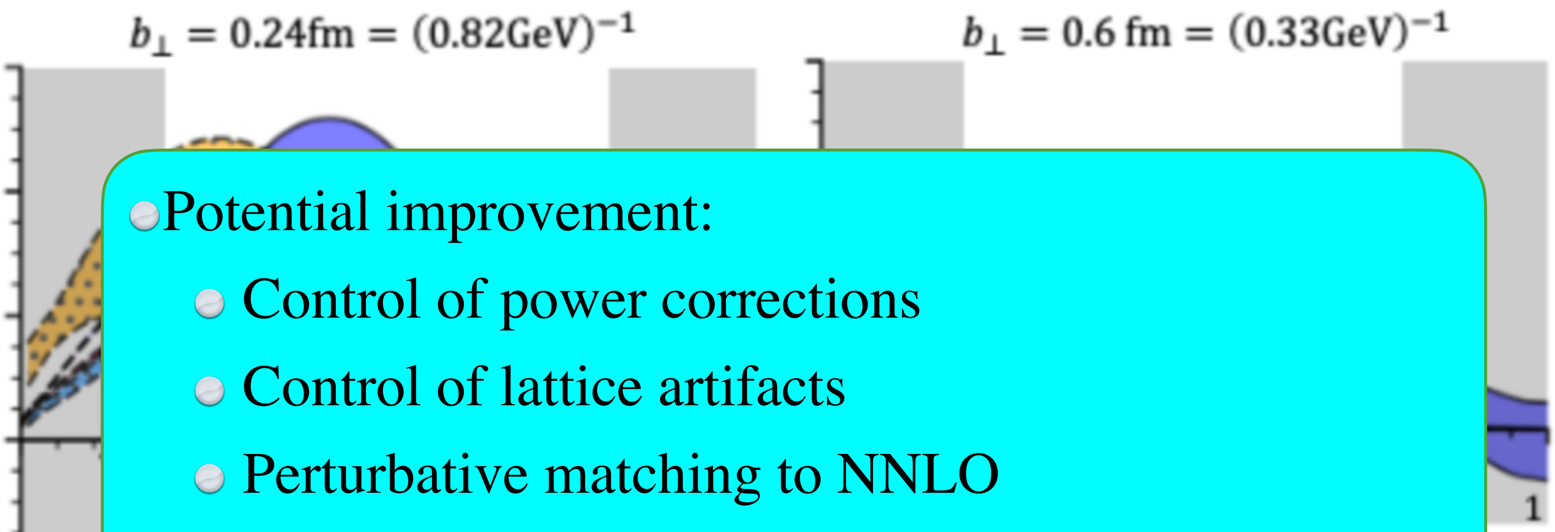

- Potential improvement:
  - Control of power corrections
  - Control of lattice artifacts
  - Perturbative matching to NNLO
    - **Del Rio et al, 2304.14440, Ji et al, JHEP 23'**

He, JHZ et al, LPC 22'

Towards precision calculations

- Lattice calculations of partonic structure of hadrons (to the leading-power accuracy) now reach the stage of **precision control**
  - Higher-order perturbative correction
    - Unpol. quark PDF@NNLO **Li et al, PRL 21', Chen et al, PRL 21'**
    - Quark TMDPDF@NNLO **Del Rio et al, 2304.14440, Ji et al, JHEP 23'**
  - RG resummation **Su, JHZ et al, NPB 23'**
  - Threshold resummation **Gao et al, PRD 21', Ji et al, JHEP 23'**
  - Power correction, renormalon ambiguity
    Braun, JHZ et al, PRD 19', Liu et al, PRD 21', Zhang et al, PLB 23'
  - Control of lattice artifacts
    - ANL-BNL, ETMC, LPC, MSU…

Towards precision calculations

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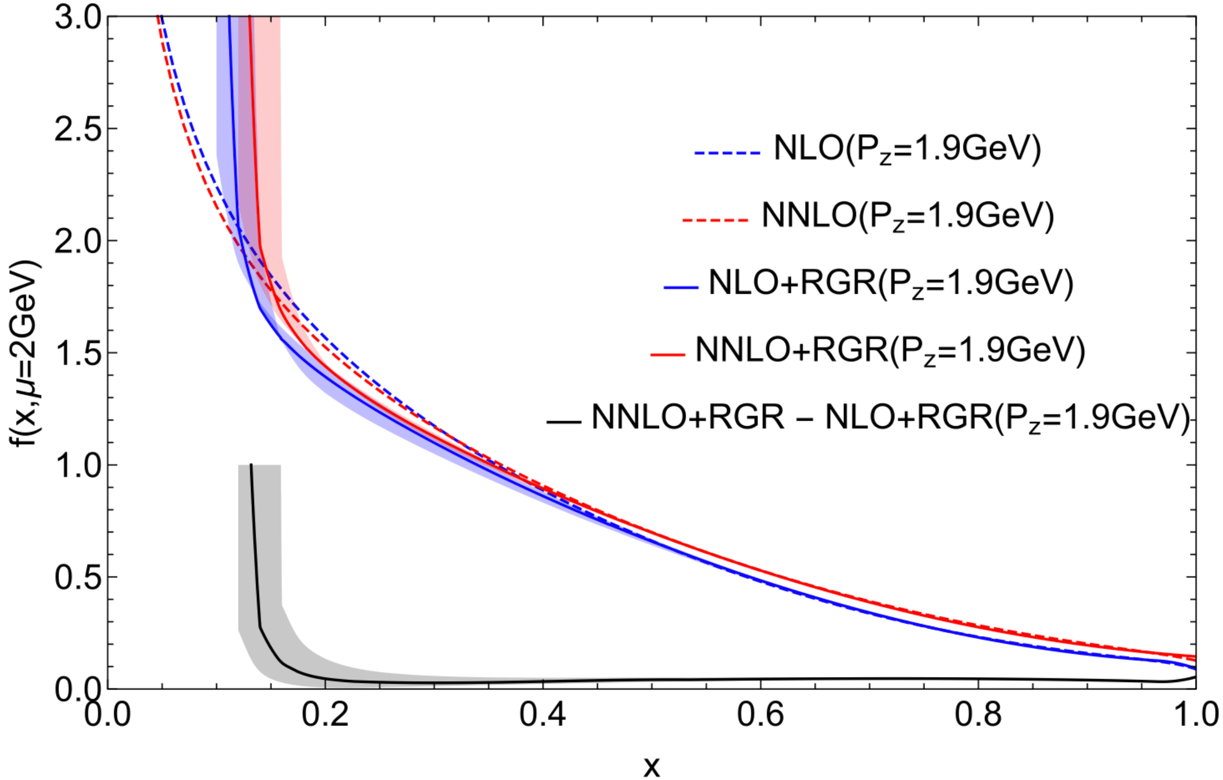


Intrinsic scale is the parton momentum

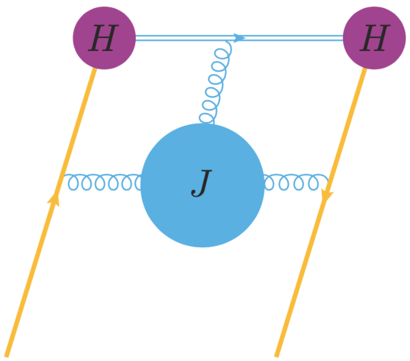
RGE to resum large logarithms $\ln \mu^2/(2xP_z)^2$

Numerical impact of the NNLO correction is small Su, JHZ et al, NPB 23'

RG resummation improves the accuracy at intermediate x, and exhibits the limitation of perturbation theory at small x

Towards precision calculations

- Lattice calculations of partonic structure of hadrons (to the leading-power accuracy) now reach the stage of **precision control**



RGE to resum large threshold logarithms $\ln^n|1-\xi|/(1-\xi)$

Factorization in the threshold limit Ji et al, JHEP 23'

$$\mathcal{C}\left(\xi, \frac{p^z}{\mu}, \alpha(\mu)\right)\bigg|_{\xi\to 1} = H\left(\ln\frac{4p_z^2}{\mu^2}, \alpha(\mu)\right) p^z J_f\left((1-\xi)p^z, \ln\frac{4p_z^2}{\mu^2}, \alpha(\mu)\right)$$

Threshold resummation improves the behavior at large x

Towards precision calculations

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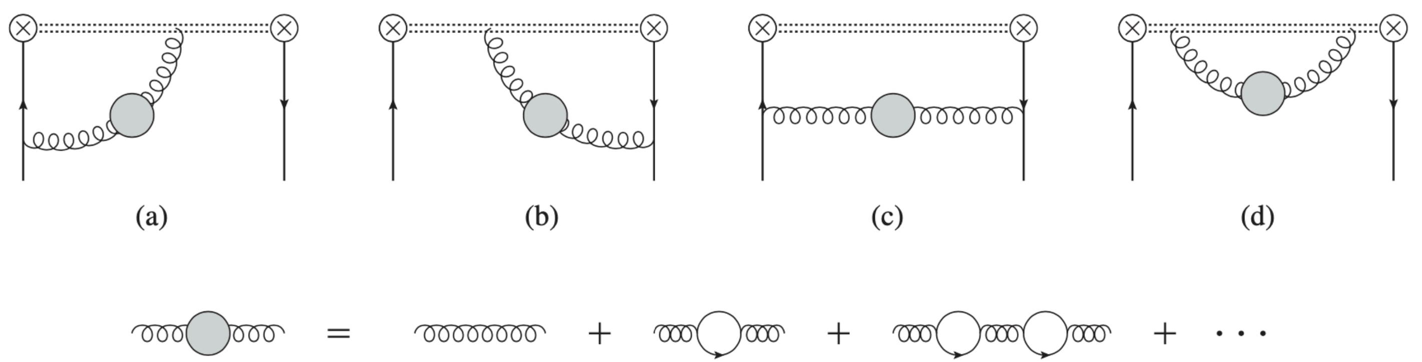
Estimate of twist-4 contribution Braun, JHZ et al, PRD 19'



Large-β_0 approximation

Quasi-PDF

$$\mathcal{Q}(x,p) = q(x)\left\{1 + \mathcal{O}\left(\frac{\Lambda^2}{p^2}\cdot\frac{1}{x^2(1-x)}\right)\right\}$$

Pseudo-PDF

$$\mathcal{P}(x,z) = q(x)\{1 + \mathcal{O}(z^2\Lambda^2(1-x))\}$$

Zero-momentum matrix element helps to suppress power corrections at $x \to 1$

Towards precision calculations

- Lattice calculations of partonic structure of hadrons (to the leading-power accuracy) now reach the stage of **precision control**

Removal of twist-3 contribution from Wilson line renormalization
Zhang et al, PLB 23'

$$h^R(z, P_z) = h^B(z, P_z) e^{(\delta m - m_0) z}$$

introduces an intrinsic ambiguity of $\mathcal{O}(z\Lambda_{QCD})$

Modified OPE

$$\begin{aligned} & h^R(z, P_z, \mu, \tau) \\ &= \left(1 - m_0(\tau) z\right) \sum_{k=0}^{\infty} C_k\left(\alpha_s(\mu), \mu^2 z^2\right) \lambda^k a_{k+1}(\mu) + \mathcal{O}(z^2) \\ &= \sum_{k=0}^{\infty} \left[C_k\left(\alpha_s(\mu), \mu^2 z^2\right) - z m_0(\tau)\right] \lambda^k a_{k+1}(\mu) + \mathcal{O}(z\alpha_s, z^2), \end{aligned}$$

$m_0(\tau)$ **can be determined by fitting to zero-momentum matrix element**

Perturbative kernel improved by resumming leading renormalons from static quark potential

Towards precision calculations

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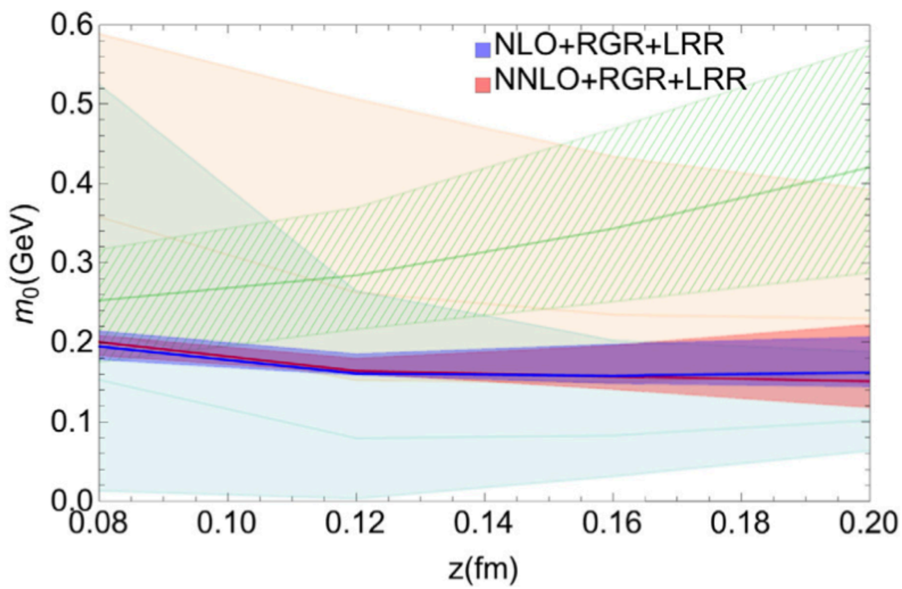
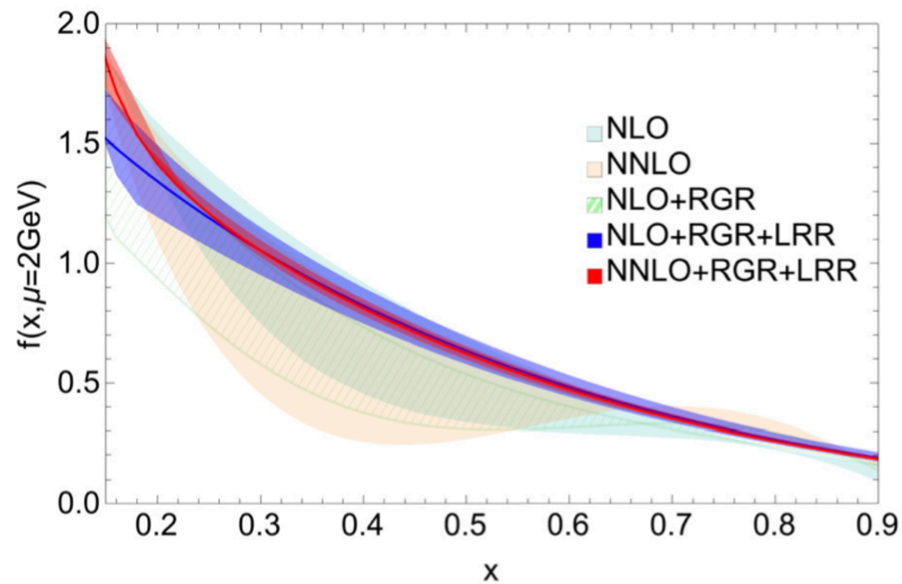
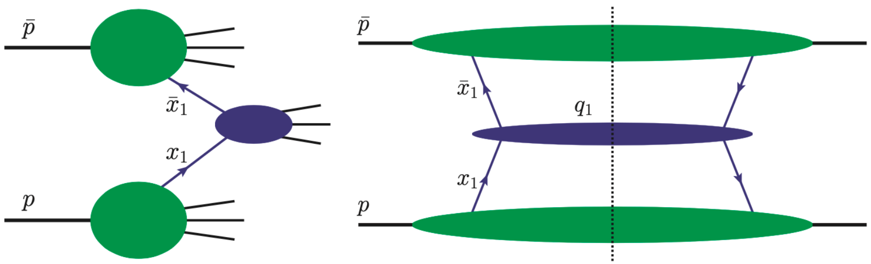




Leading-renormlaon resummation helps to improve the accuracy of m_0 determination

From single- to multi-parton distributions

- The computational effort so far has been mainly focused on **single parton distributions** (relevant for single parton scattering)



$$\sigma = \sum_{ij} \int dx_1 \int d\bar{x}_1 f_i(x_1) f_j(\bar{x}_1) \hat{\sigma}_{ij}.$$

Single parton distributions

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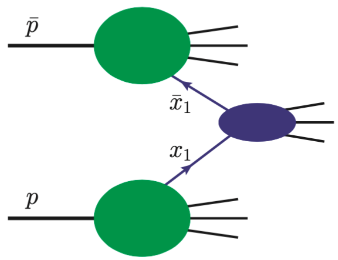

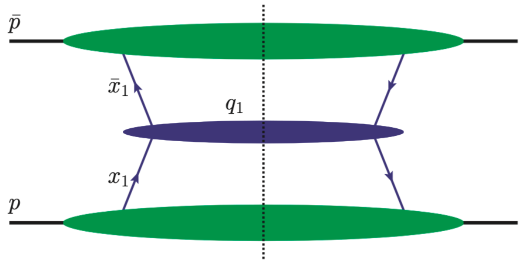

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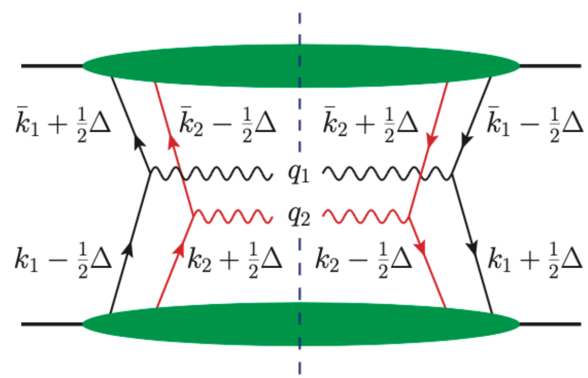
Single parton distributions

- With the increasing energy of hadron colliders, **multiparton scattering** (e.g., double parton scattering) processes become increasingly important



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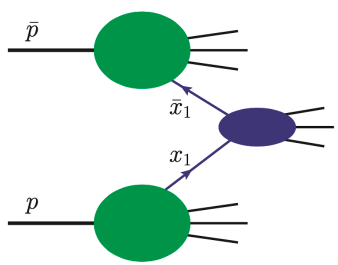

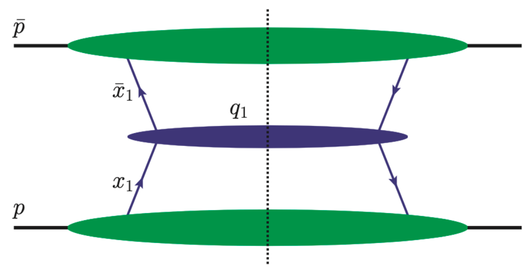

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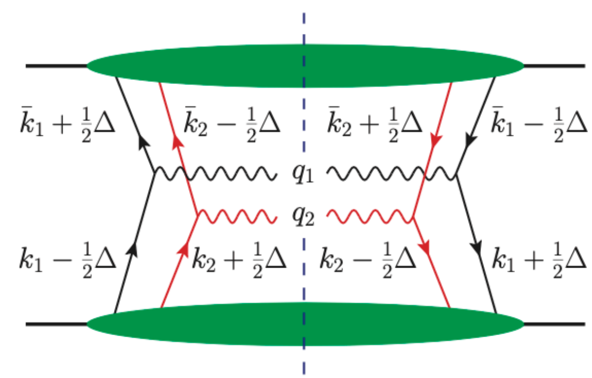
Single parton distributions

- With the increasing energy of hadron colliders, **multiparton scattering** (e.g., double parton scattering) processes become increasingly important

It can compete with single parton scattering in certain situations

Longitudinal parton momenta fixed by final state kinematics

Transverse parton momenta can differ by Δ, conjugate to transverse separation of partons



From single- to multi-parton distributions

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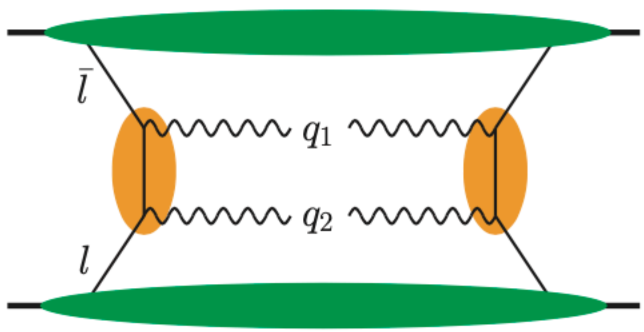
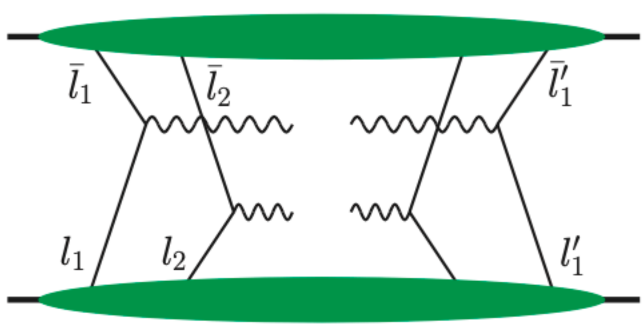
Double parton scattering power counting

		$\dfrac{s\,d\sigma}{\prod_{i=1}^{2} dx_i\, d\bar{x}_i\, d^2\boldsymbol{q}_i}$	$\dfrac{s\,d\sigma}{\prod_{i=1}^{2} dx_i\, d\bar{x}_i}$
a		$\dfrac{1}{\Lambda^2 Q^2}$	1
b		$\dfrac{1}{\Lambda^2 Q^2}$	$\dfrac{\Lambda^2}{Q^2}$

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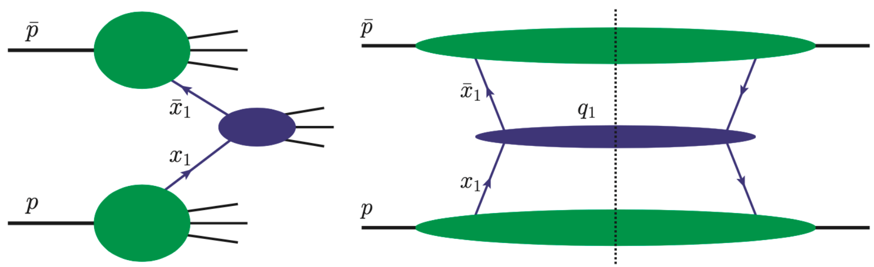

$$\sigma = \sum_{ij} \int dx_1 \int d\bar{x}_1 f_i(x_1) f_j(\bar{x}_1) \hat{\sigma}_{ij}.$$

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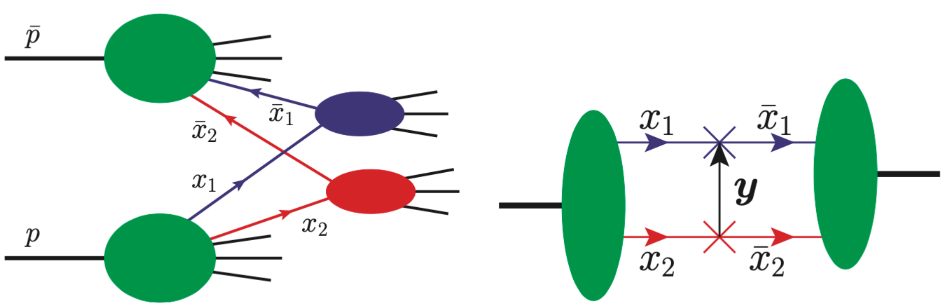


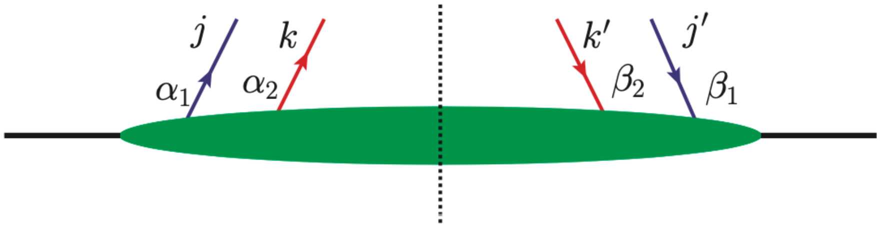
(a) (b)

Double parton distributions (the two partons can have transverse separations)

$$\sigma_{DPS} \sim \sum_{ijkl} \int dx_1 dx_2 \int d\bar{x}_1 d\bar{x}_2 \int d^2\boldsymbol{y} f_{ij}(x_1, x_2, \boldsymbol{y}) f_{kl}(\bar{x}_1, \bar{x}_2, \boldsymbol{y}) \hat{\sigma}_{ik} \hat{\sigma}_{jl}.$$

From single- to multi-parton distributions

- Double parton distributions (DPDs) **Diehl, Ostermeier, Schaefer, JHEP 12', Diehl, Gaunt, 17'**



- Two-quark correlation

$$\Phi_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'}(k_1,k_2,r) = \int \frac{d^4 z_1}{(2\pi)^4} \mathrm{e}^{i z_1 k_1} \frac{d^4 z_2}{(2\pi)^4} \mathrm{e}^{i z_2 k_2} \frac{d^4 y}{(2\pi)^4} \mathrm{e}^{-i y r}$$
$$\times \langle p | \bar{T}\left[\bar{\psi}_{\Sigma_1'}(y - \tfrac{1}{2}z_1)\bar{\psi}_{\Sigma_2'}(-\tfrac{1}{2}z_2)\right] T\left[\psi_{\Sigma_2}(\tfrac{1}{2}z_2)\psi_{\Sigma_1}(y + \tfrac{1}{2}z_1)\right] | p \rangle$$

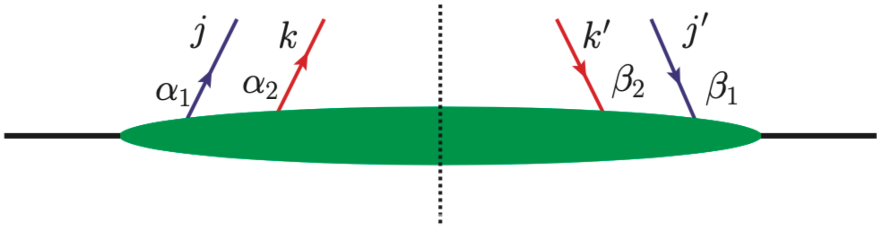
- Fourier transform to transverse position space

$$F_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'}(x_1,x_2,\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{y}) = 2p^+ \int \frac{dz_1^-}{2\pi}\frac{dz_2^-}{2\pi}\, dy^-\; e^{i x_1 z_1^- p^+ + i x_2 z_2^- p^+}$$
$$\times \langle p | \bar{\psi}_{\Sigma_2'}(-\tfrac{1}{2}z_2)\psi_{\Sigma_2}(\tfrac{1}{2}z_2)\bar{\psi}_{\Sigma_1'}(y - \tfrac{1}{2}z_1)\psi_{\Sigma_1}(y + \tfrac{1}{2}z_1) | p \rangle \Big|_{z_i^+ = y^+ = 0}.$$

- Σ_i denotes collectively the spin, color and flavor of the corresponding quark

From single- to multi-parton distributions

- Double parton distributions (DPDs) **Diehl, Ostermeier, Schaefer, JHEP 12', Diehl, Gaunt, 17'**



- Two-quark correlation

$$\Phi_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'}(k_1,k_2,r) = \int \frac{d^4z_1}{(2\pi)^4}\mathrm{e}^{iz_1k_1}\frac{d^4z_2}{(2\pi)^4}\mathrm{e}^{iz_2k_2}\frac{d^4y}{(2\pi)^4}\mathrm{e}^{-iyr}$$
$$\times \langle p|\,\bar{T}\left[\bar{\psi}_{\Sigma_1'}(y-\tfrac{1}{2}z_1)\bar{\psi}_{\Sigma_2'}(-\tfrac{1}{2}z_2)\right]T\left[\psi_{\Sigma_2}(\tfrac{1}{2}z_2)\psi_{\Sigma_1}(y+\tfrac{1}{2}z_1)\right]|p\rangle$$

- Leading-power spin projection

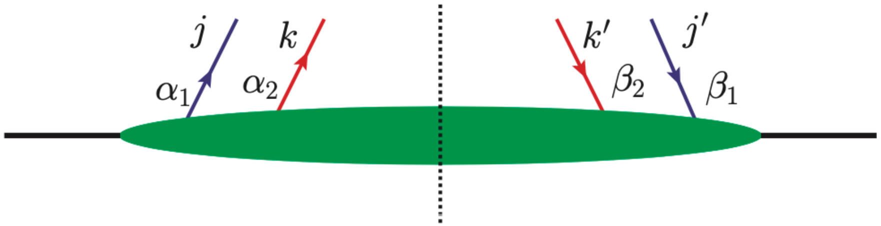
$$F_{a_1a_2}(x_1,x_2,\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{y}) = 2p^+\int \frac{dz_1^-}{2\pi}\,\frac{dz_2^-}{2\pi}\,dy^-\;e^{ix_1z_1^-p^+ + ix_2z_2^-p^+}$$
$$\times \langle p|\,\mathcal{O}_{a_2}(0,z_2)\,\mathcal{O}_{a_1}(y,z_1)\,|p\rangle\;,$$

$$\mathcal{O}_{a_i}(y,z_i) = \bar{q}_i(y-\tfrac{1}{2}z_i)\,\Gamma_{a_i}q_i(y+\tfrac{1}{2}z_i)\Big|_{z_i^+=y^+=0}$$

$$\Gamma_q = \tfrac{1}{2}\gamma^+\,, \qquad \Gamma_{\Delta q} = \tfrac{1}{2}\gamma^+\gamma_5\,, \qquad \Gamma_{\delta q}^j = \tfrac{1}{2}i\sigma^{j+}\gamma_5$$

From single- to multi-parton distributions

- Double parton distributions (DPDs) **Diehl, Ostermeier, Schaefer, JHEP 12', Diehl, Gaunt, 17'**



- Two-quark correlation

$$\Phi_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'}(k_1,k_2,r) = \int \frac{d^4 z_1}{(2\pi)^4} \mathrm{e}^{iz_1k_1} \frac{d^4 z_2}{(2\pi)^4} \mathrm{e}^{iz_2k_2} \frac{d^4 y}{(2\pi)^4} \mathrm{e}^{-iyr} \times \langle p| \bar{T}\left[\bar{\psi}_{\Sigma_1'}(y-\tfrac{1}{2}z_1)\bar{\psi}_{\Sigma_2'}(-\tfrac{1}{2}z_2)\right] T\left[\psi_{\Sigma_2}(\tfrac{1}{2}z_2)\psi_{\Sigma_1}(y+\tfrac{1}{2}z_1)\right]|p\rangle$$

- Color structure

$$F_{jj',kk'} = \frac{1}{N_c^2}\left({}^1F\delta_{jj'}\delta_{kk'} + \frac{N_c}{\sqrt{N_c^2-1}}\,{}^8F t^a_{jj'}t^a_{kk'}\right)$$

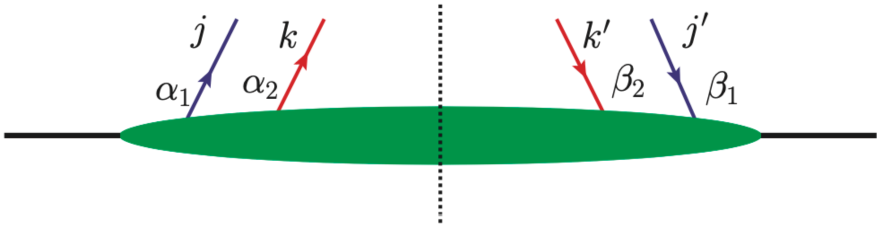
yielding ($c = \{1,8\}$)

$${}^cF_{a_1a_2}(x_1,x_2,\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{y}) = 2p^+\int \frac{dz_1^-}{2\pi}\frac{dz_2^-}{2\pi}\,dy^-\; e^{ix_1z_1^-p^+ + ix_2z_2^-p^+} \times \langle p|\,{}^c\mathcal{O}_{a_2}(0,z_2)\,{}^c\mathcal{O}_{a_1}(y,z_1)\,|p\rangle\ ,$$

$${}^1\mathcal{O}(y,z_i) = \bar{q}_{j'}(y-\tfrac{1}{2}z_i)\,\delta_{jj'}q_j(y+\tfrac{1}{2}z_i)\Big|_{z_i^+=y^+=0} \quad {}^8\mathcal{O}(y,z_i) = \bar{q}_{j'}(y-\tfrac{1}{2}z_i)\,t^a_{jj'}q_j(y+\tfrac{1}{2}z_i)\Big|_{z_i^+=y^+=0}$$

From single- to multi-parton distributions

- Double parton distributions (DPDs) **Diehl, Ostermeier, Schaefer, JHEP 12', Diehl, Gaunt, 17'**



- Two-quark correlation

$$\Phi_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'}(k_1,k_2,r) = \int \frac{d^4 z_1}{(2\pi)^4} e^{iz_1k_1} \frac{d^4 z_2}{(2\pi)^4} e^{iz_2k_2} \frac{d^4 y}{(2\pi)^4} e^{-iyr}$$
$$\times \langle p| \bar{T}\left[\bar{\psi}_{\Sigma_1'}(y-\tfrac{1}{2}z_1)\bar{\psi}_{\Sigma_2'}(-\tfrac{1}{2}z_2)\right] T\left[\psi_{\Sigma_2}(\tfrac{1}{2}z_2)\psi_{\Sigma_1}(y+\tfrac{1}{2}z_1)\right] |p\rangle$$

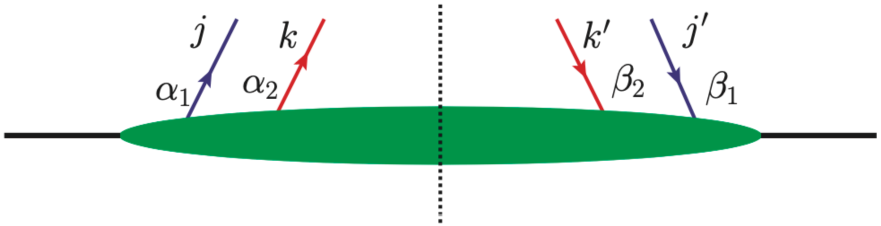
- Flavor structure

$$F^I_{a_1a_2}(x_1,x_2,\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{y}) = 2p^+ \int \frac{dz_1^-}{2\pi}\frac{dz_2^-}{2\pi}\, dy^-\; e^{ix_1z_1^-p^+ + ix_2z_2^-p^+}$$
$$\times \langle p|\, \mathcal{O}^I_{a_2}(0,z_2)\, \mathcal{O}^I_{a_1}(y,z_1)\, |p\rangle$$

$$\mathcal{O}^I_{a_2}(0,z_2)\,\mathcal{O}^I_{a_1}(y,z_1) = \bar{q}_1(-\tfrac{1}{2}z_2)\,\Gamma_{a_2}\, q_2(\tfrac{1}{2}z_2)\; \bar{q}_2(y-\tfrac{1}{2}z_1)\,\Gamma_{a_1}\, q_1(y+\tfrac{1}{2}z_1)\Big|_{\substack{z_1^+=z_2^+=0\\ y^+=0}}$$

From single- to multi-parton distributions

- Double parton distributions (DPDs) **Diehl, Ostermeier, Schaefer, JHEP 12', Diehl, Gaunt, 17'**



- Two-quark correlation

$$\Phi_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'}(k_1,k_2,r) = \int \frac{d^4 z_1}{(2\pi)^4} e^{iz_1k_1} \frac{d^4 z_2}{(2\pi)^4} e^{iz_2k_2} \frac{d^4 y}{(2\pi)^4} e^{-iyr}$$
$$\times \langle p| \bar{T} \left[\bar{\psi}_{\Sigma_1'}(y - \tfrac{1}{2}z_1)\bar{\psi}_{\Sigma_2'}(-\tfrac{1}{2}z_2)\right] T \left[\psi_{\Sigma_2}(\tfrac{1}{2}z_2)\psi_{\Sigma_1}(y + \tfrac{1}{2}z_1)\right] |p\rangle$$

- Gauge links shall be included beyond LO analysis to ensure gauge invariance

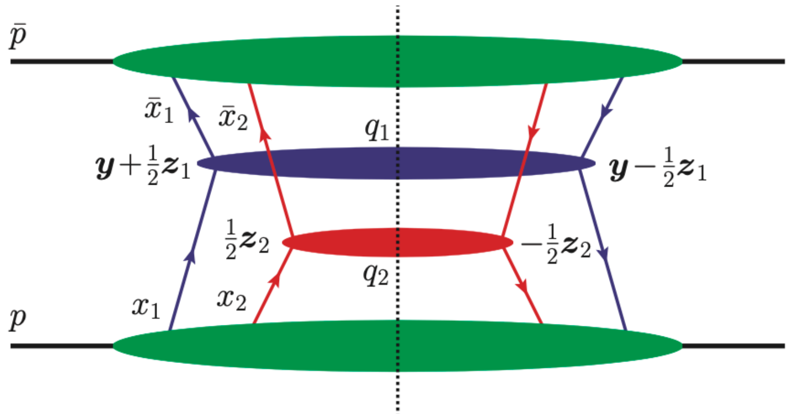
$$q_j(z) \to [W(z,v)]_{jk}\, q_k(z)\,, \qquad \bar{q}_j(z) \to \bar{q}_k(z) \left[W^\dagger(z,v)\right]_{kj}$$

$$W(z,v) = P \exp\left[ig \int_0^\infty d\lambda v A^a(z - \lambda v) t^a\right]$$

- Rapidity divergences can appear even in collinear double parton distributions (color-correlated ones)

From single- to multi-parton distributions

- Double parton scattering in terms of DPDs



- Factorization

$$\frac{d\sigma}{\prod_{i=1}^{2} dx_i\, d\bar{x}_i\, d^2\boldsymbol{q}_i} = \frac{1}{C} \sum_{a_1 a_2 a_3 a_4} \int \frac{d^2\boldsymbol{z}_1}{(2\pi)^2} \frac{d^2\boldsymbol{z}_2}{(2\pi)^2}\, e^{-i\boldsymbol{z}_1\boldsymbol{q}_1 - i\boldsymbol{z}_2\boldsymbol{q}_2} \int d^2\boldsymbol{y}$$
$$\times \Big\{ d\hat{\sigma}_{a_1\bar{a}_3}\, d\hat{\sigma}_{a_2\bar{a}_4} \left[{}^1F_{a_1 a_2}\, {}^1\bar{F}_{\bar{a}_3\bar{a}_4} + c_8\, {}^8F_{a_1 a_2}\, {}^8\bar{F}_{\bar{a}_3\bar{a}_4} \right]$$
$$+ d\hat{\sigma}^I_{a_1\bar{a}_3}\, d\hat{\sigma}^I_{a_2\bar{a}_4} \left[{}^1F^I_{a_1 a_2}\, {}^1\bar{F}^I_{\bar{a}_3\bar{a}_4} + c_8\, {}^8F^I_{a_1 a_2}\, {}^8\bar{F}^I_{\bar{a}_3\bar{a}_4} \right] \Big\}\,.$$

DPDs in phenomenology

- Simplified modeling often ignores the correlation between partons in analyzing double parton scattering

- DPDs in terms of single parton impact parameter distributions

$$F_{a_1 a_2}(x_1, x_2, \boldsymbol{y}) \stackrel{?}{=} \int d^2\boldsymbol{b}\, f_{a_1}(x_1, \boldsymbol{b}+\boldsymbol{y})\, f_{a_2}(x_2, \boldsymbol{b})$$

- Such a factorization has been investigated on the lattice for the pion, significant differences have been observed between l.h.s. and r.h.s.

- DPDs in a completely factorized form

$$F_{a_1 a_2}(x_1, x_2, \boldsymbol{y}) \stackrel{?}{=} f_{a_1}(x_1)\, f_{a_2}(x_2)\, G(\boldsymbol{y})$$

- Double parton scattering X-sec is given by

$$\sigma_{\mathrm{DPS},ij} = \frac{1}{C}\frac{\sigma_{\mathrm{SPS},i}\ \sigma_{\mathrm{SPS},j}}{\sigma_{\mathrm{eff}}}$$

DPDs from lattice

- DPDs involve fields at lightlike distances and thus are not well-suited for lattice calculations
- What can be studied on the lattice are two-current correlations at space like separations, e.g., **Bali et al, JHEP 21'**

$$M^{\mu_1\cdots\mu_2\cdots}_{q_1q_2,i_1i_2}(p,y) = \langle h(p)|\, J^{\mu_1\cdots}_{q_1,i_1}(y)\, J^{\mu_2\cdots}_{q_2,i_2}(0)\,|h(p)\rangle$$

- Take the color singlet quark DPDs as an example:
- The correlations with currents

$$J^{\mu}_{q,V}(y) = \bar{q}(y)\gamma^{\mu} q(y)\,, \qquad J^{\mu}_{q,A}(y) = \bar{q}(y)\gamma^{\mu}\gamma_5\, q(y)\,, \qquad J^{\mu\nu}_{q,T}(y) = \bar{q}(y)\sigma^{\mu\nu} q(y)$$

are related to the lowest double Mellin moment of the DPDs, e.g.,

$$\int_{-\infty}^{\infty} dy^-\, M^{++}_{q_1q_2,VV}(p,y)\Big|_{y^+=0,\,\boldsymbol{p}=\boldsymbol{0}} = 2p^+\, I_{q_1q_2}(y^2)$$

$$I_{a_1a_2}(y^2) = \int_{-1}^{1} dx_1 \int_{-1}^{1} dx_2\, f_{a_1a_2}(x_1,x_2,y^2)$$

DPDs from lattice

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- Take the color singlet quark DPDs as an example:
- On the other hand

$$\frac{1}{2}[M^{\mu\nu}_{q_1q_2,VV}(p,y) + M^{\nu\mu}_{q_1q_2,VV}(p,y)] = t^{\mu\nu}_{VV,A}\, A_{q_1q_2} + t^{\mu\nu}_{VV,B}\, m^2 B_{q_1q_2} + t^{\mu\nu}_{VV,C}\, m^4 C_{q_1q_2} + t^{\mu\nu}_{VV,D}\, m^2 D_{q_1q_2}$$

$$t^{\mu\nu}_{VV,A} = 2p^\mu p^\nu - \frac{1}{2} g^{\mu\nu} p^2\,, \qquad t^{\mu\nu}_{VV,C} = 2y^\mu y^\nu - \frac{1}{2} g^{\mu\nu} y^2\,,$$

$$t^{\mu\nu}_{VV,B} = p^\mu y^\nu + p^\nu y^\mu - \frac{1}{2} g^{\mu\nu} py\,, \qquad t^{\mu\nu}_{VV,D} = g^{\mu\nu}\,,$$

$$\Downarrow$$

$$A_{q_1q_2} = \frac{1}{8N^2}\left\{3(y^2)^2\, t^{\mu\nu}_{VV,A} - 6y^2 py\, t^{\mu\nu}_{VV,B} + \left[p^2y^2 + 2(py)^2\right] t^{\mu\nu}_{VV,C}\right\} \left[M_{q_1q_2,VV}\right]_{\mu\nu}$$

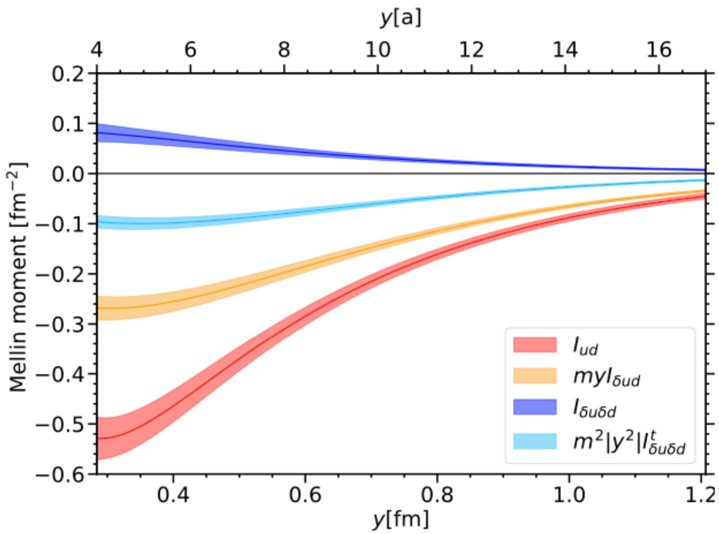
$$I_{a_1a_2}(y^2) = \int_{-\infty}^{\infty} d(py)\, A_{a_1a_2}(py, y^2)$$

DPDs from lattice

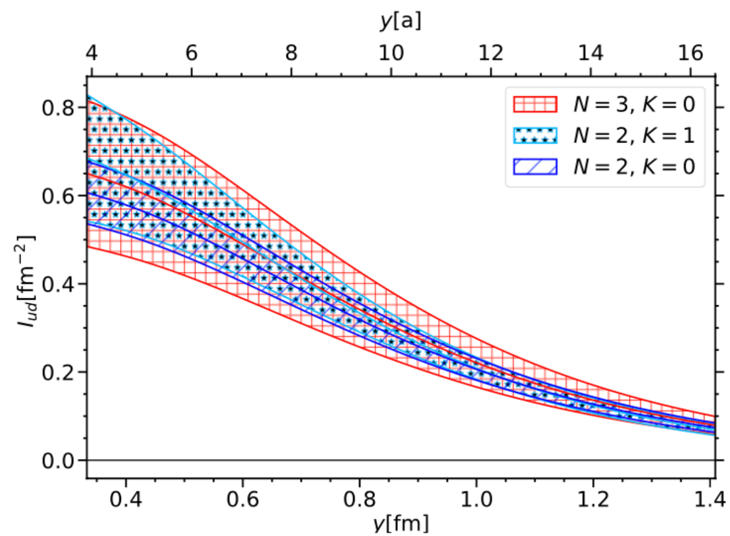
- DPDs involve fields at lightlike distances and thus are not well-suited for lattice calculations
- What can be studied on the lattice are two-current correlations at space like separations, e.g., **Bali et al, JHEP 21'**

$$M^{\mu_1\cdots\mu_2\cdots}_{q_1q_2,i_1i_2}(p,y) = \langle h(p)|\, J^{\mu_1\cdots}_{q_1,i_1}(y)\, J^{\mu_2\cdots}_{q_2,i_2}(0)\,|h(p)\rangle$$

- Take the color singlet quark DPDs as an example:



(a) Mellin moments $I(y^2, \zeta = 0)$



(a) fit comparison $I_{ud}(\zeta = 0, y^2)$

- Can we access the full DPD rather than the lowest double Mellin moments from lattice?

DPDs from lattice

- Start from the simplest DPD: unpolarized color singlet case **JHZ, 23'**

$$f_{q_1q_2}(x_1,x_2,y^2) = 2P^+ \int dy^- \int \frac{dz_1^-}{2\pi}\frac{dz_2^-}{2\pi} e^{i(x_1 z_1^- + x_2 z_2^-)P^+} h_0(y,z_1,z_2,P)$$
$$= 2\int d\lambda \int \frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi} e^{i(x_1\lambda_1 + x_2\lambda_2)} h(\lambda,\lambda_1,\lambda_2,y^2),$$

with

$$h_0(y,z_1,z_2,P) = \langle P|O_{q_1}(y,z_1)O_{q_2}(0,z_2)|P\rangle,$$
$$h(\lambda,\lambda_1,\lambda_2,y^2) = \frac{1}{(P^+)^2} h_0(y,z_1,z_2,P),$$
$$O_q(y,z) = \bar{\psi}_q\big(y-\frac{z}{2}\big)\frac{\gamma^+}{2}W\big(y-\frac{z}{2};y+\frac{z}{2}\big)\psi_q\big(y+\frac{z}{2}\big),$$
$$\lambda = P\cdot y, \quad \lambda_1 = P\cdot z_1, \quad \lambda_2 = P\cdot z_2,$$

- Double Mellin moments are given by

$$M_{q_1q_2}^{n_1 n_2}(y^2) = \int_{-1}^{1} dx_1 dx_2\, x_1^{n_1-1} x_2^{n_2-1} f_{q_1q_2}(x_1,x_2,y^2)$$
$$= \frac{(P^+)^{1-n_1-n_2}}{2}\int dy^- \langle P|\mathcal{O}_{q_1}^{+\cdots+}(y)\mathcal{O}_{q_2}^{+\cdots+}(0)|P\rangle$$

$$\mathcal{O}_q^{\mu_1\cdots\mu_n}(y) = \bar{\psi}_q(y)\gamma^{\{\mu_1} i\overleftrightarrow{D}^{\mu_2}(y)\cdots i\overleftrightarrow{D}^{\mu_n\}}(y)\psi_q(y),$$

DPDs from lattice

- Start from the simplest DPD: unpolarized color singlet case **JHZ, 23'**

$$f_{q_1q_2}(x_1,x_2,y^2) = 2P^+\int dy^-\int \frac{dz_1^-}{2\pi}\frac{dz_2^-}{2\pi}e^{i(x_1z_1^-+x_2z_2^-)P^+}h_0(y,z_1,z_2,P)$$
$$= 2\int d\lambda\int \frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi}e^{i(x_1\lambda_1+x_2\lambda_2)}h(\lambda,\lambda_1,\lambda_2,y^2),$$

with

$$h_0(y,z_1,z_2,P) = \langle P|O_{q_1}(y,z_1)O_{q_2}(0,z_2)|P\rangle,$$
$$h(\lambda,\lambda_1,\lambda_2,y^2) = \frac{1}{(P^+)^2}h_0(y,z_1,z_2,P),$$
$$O_q(y,z) = \bar\psi_q\big(y-\frac{z}{2}\big)\frac{\gamma^+}{2}W\big(y-\frac{z}{2};y+\frac{z}{2}\big)\psi_q\big(y+\frac{z}{2}\big),$$
$$\lambda = P\cdot y,\quad \lambda_1 = P\cdot z_1,\quad \lambda_2 = P\cdot z_2,$$

- Double Mellin moments can be turned into a manifestly covariant form **Diehl, Ostermeier, Schaefer, JHEP 12'**

$$\langle P|\mathcal{O}_{q_1}^{\mu_1\cdots\mu_{n_1}}(y)\mathcal{O}_{q_2}^{\nu_1\cdots\nu_{n_2}}(0)|P\rangle =$$
$$2P^{\mu_1}\cdots P^{\mu_{n_1}}P^{\nu_1}\cdots P^{\nu_{n_2}}\langle\mathcal{O}_{q_1}^{n_1}\mathcal{O}_{q_2}^{n_2}\rangle(\lambda,y^2)+\cdots,$$

$$M_{q_1q_2}^{n_1n_2}(y^2) = \int d\lambda\,\langle\mathcal{O}_{q_1}^{n_1}\mathcal{O}_{q_2}^{n_2}\rangle(\lambda,y^2).$$

DPDs from lattice

- Consider the correlation of equal-time nonlocal operators following the spirit of LaMET **Ji, PRL 13' & SCPMA 14', Ji, Liu, Liu, JHZ, Zhao, RMP 21'**

$$\tilde{h}(z_1, z_2, y, P) = \frac{1}{N}\langle P|O_{q_1}(y, z_1)O_{q_2}(0, z_2)|P\rangle,$$

$$y^\mu = (0, \vec{y}_\perp, y^z), \quad z_i^\mu = (0, \vec{0}_\perp, z_i)$$

- From OPE **Izubuchi et al, PRD 18'**

$$\tilde{h}(z_i, \mu_i, y, P) = \frac{1}{4N}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{(-iz_1)^{n_1-1}}{(n_1-1)!}\frac{(-iz_2)^{n_2-1}}{(n_2-1)!} \times C_{q_1}^{(n_1-1)}(\mu_1^2 z_1^2)C_{q_2}^{(n_2-1)}(\mu_2^2 z_2^2)\tilde{\mathcal{M}}_{q_1 q_2}^{n_1 n_2}(\mu_i, y, P) + \cdots,$$

$$\begin{aligned}\tilde{\mathcal{M}}_{q_1 q_2}^{n_1 n_2}(\mu_i, y, P) &= n_{\mu_1}\cdots n_{\mu_{n_1}} n_{\nu_1}\cdots n_{\nu_{n_2}} \\ &\times \langle P|\mathcal{O}_{q_1}^{\mu_1\cdots\mu_{n_1}}(y, \mu_1)\mathcal{O}_{q_2}^{\nu_1\cdots\nu_{n_2}}(0, \mu_2)|P\rangle \\ &= 2(n\cdot P)^{n_1+n_2}\langle\mathcal{O}_{q_1}^{n_1}\mathcal{O}_{q_2}^{n_2}\rangle(\mu_i, \lambda, y^2) + \cdots,\end{aligned}$$

- The same Lorentz invariant reduced matrix element appears both in correlations of lightcone and Euclidean nonlocal operators

DPDs from lattice

- Factorization

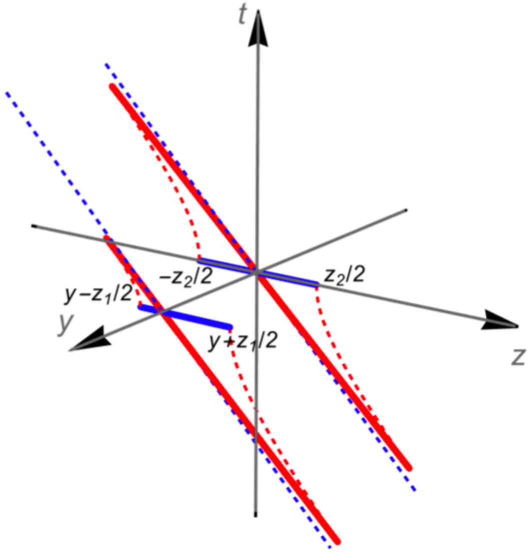
$$\tilde{H}(\lambda_i,\mu_i,z_i^2,y^2)=\int du_1du_2\,\mathcal{C}_{q_1}(u_1,\mu_1^2z_1^2)\mathcal{C}_{q_2}(u_2,\mu_2^2z_2^2)H(u_i\lambda_i,\mu_i,y^2)+\cdots$$

$$\tilde{H}(\lambda_i,\mu_i,z_i^2,y^2)=\int d\lambda\,\tilde{h}(\lambda,\lambda_i,\mu_i,z_i^2,y^2),$$

$$H(\lambda_i,\mu_i,y^2)=\int d\lambda\,h(\lambda,\lambda_i,\mu_i,y^2).$$

- FT w.r.t. z_i with P fixed

$$\tilde{f}(x_1,x_2,\mu_i,y^2)=2\int\frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi}e^{i(x_1\lambda_1+x_2\lambda_2)}$$
$$\times\,\tilde{H}\Big(\lambda_i,\mu_i,-\frac{\lambda_i^2}{(P^z)^2},y^2\Big)$$
$$=\int\frac{dx_1'}{|x_1'|}\frac{dx_2'}{|x_2'|}C_{q_1}\Big(\frac{x_1}{x_1'},\frac{\mu_1^2}{(x_1'P^z)^2}\Big)C_{q_2}\Big(\frac{x_2}{x_2'},\frac{\mu_2^2}{(x_2'P^z)^2}\Big)$$
$$\times\,f(x_i',\mu_i^2,y^2)+\cdots\ ,$$



JHZ, 23'

DPDs from lattice

- Factorization

$$\tilde{H}(\lambda_i, \mu_i, z_i^2, y^2) = \int du_1 du_2\, \mathcal{C}_{q_1}(u_1, \mu_1^2 z_1^2)\mathcal{C}_{q_2}(u_2, \mu_2^2 z_2^2) H(u_i\lambda_i, \mu_i, y^2) + \cdots$$

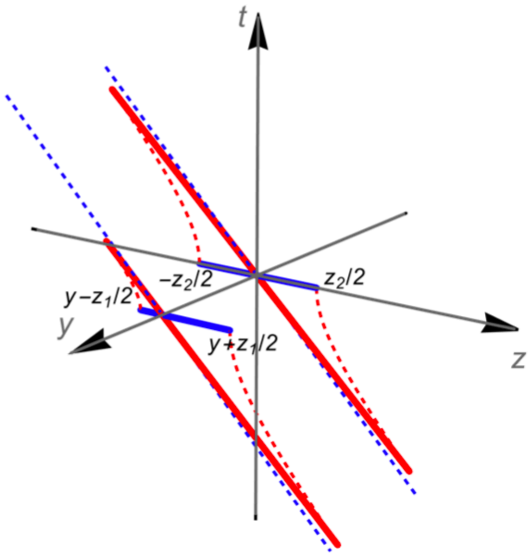
$$\tilde{H}(\lambda_i, \mu_i, z_i^2, y^2) = \int d\lambda\, \tilde{h}(\lambda, \lambda_i, \mu_i, z_i^2, y^2),$$

$$H(\lambda_i, \mu_i, y^2) = \int d\lambda\, h(\lambda, \lambda_i, \mu_i, y^2).$$

- FT w.r.t. λ_i with z_i^2 fixed

$$\mathcal{D}(x_i, \mu_i, z_i^2, y^2) = 2\int \frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi} e^{i(x_1\lambda_1 + x_2\lambda_2)}\tilde{H}(\lambda_i, \mu_i, z_i^2, y^2)$$

$$= \int \frac{dx_1'}{|x_1'|}\frac{dx_2'}{|x_2'|}\mathcal{C}_{q_1}(\frac{x_1}{x_1'}, \mu_1^2 z_1^2)\mathcal{C}_{q_2}(\frac{x_2}{x_2'}, \mu_2^2 z_2^2) f(x_i', \mu_i, y^2) + \cdots,$$



JHZ, 23'

DPDs from lattice

- Color-correlated DPD **Jaarsma, Rahn, Waalewijn, 23', JHZ, in preparation**

$$
\begin{aligned}
&{}^{R_1R_2}\tilde{F}^{\mathrm{NS}}_{q_1q_2}(x_1, x_2, b_\perp, \mu, \tilde{\zeta}_p, \tilde{P}^z) \\
&= \sum_{R_1',R_2'}\sum_{q_1',q_2'}\int_0^1 \frac{\mathrm{d}x_1'}{x_1'}\frac{\mathrm{d}x_2'}{x_2'}\, {}^{R_1R_1'}C_{q_1q_1'}\left(\frac{x_1}{x_1'}, x_1'\tilde{P}^z, \mu\right) {}^{R_2R_2'}C_{q_2q_2'}\left(\frac{x_2}{x_2'}, x_2'\tilde{P}^z, \mu\right) \\
&\qquad \times \exp\left[\frac{1}{2}{}^{R_{1/2}}J(b_\perp, \mu)\ln\left(\frac{\tilde{\zeta}_p}{\zeta_p}\right)\right] {}^{R_1'R_2'}F^{\mathrm{NS}}_{q_1'q_2'}(x_1', x_2', b_\perp, \mu, \zeta_p)\,.
\end{aligned}
$$

- Rapidity divergences show up in the collinear DPDs, and introduce rapidity scale dependence

- Checked by an explicit one-loop calculation, consistent with RG and rapidity evolution of DPDs

Summary and outlook

- Lattice calculations of single parton distributions have reached a stage of precision control

- Multiparton distributions are important both for collider phenomenology and for understanding the correlated partonic structure of hadrons

- Very limited knowledge even on the simplest case (DPDs)

- The full DPD can be accessed following the spirit of LaMET, lattice calculations can provide important inputs for phenomenological analyses

- The same development in single parton distributions (PDFs, TMDs, GPDs…) can be extended to DPDs, and generalized to multiparton distributions, a lot more to be explored…