Pion transverse-momentum dependent parton distributions at leading- and subleading-twists

Giovanni Salmè

Istituto Nazionale di Fisica Nucleare - Rome
salmeg@roma1.infn.it

Istituto Nazionale di Fisica Nucleare

ECT*
EUROPEAN CENTRE
FOR THEORETICAL STUDIES
IN NUCLEAR PHYSICS AND RELATED AREAS

In collaboration with: Tobias Frederico (ITA), Emanuele Pace (Rome TVG U.), Wayne de Paula (ITA), Michele Viviani (INFN), Abigail Castro (ITA), Aline Noronha (ITA)

Outline

Overview

The *elective* space where QCD is investigated is the Euclidean one, not the physical space, since the indefinite metric of the Minkowski space generates many problems, which could be avoided by replacing the time t by it or the energy E by iE.

The complete equivalence of the quantum field theories, in Minkowski and Euclideean spaces, is granted by the Osterwalder and Schrader theorems, stating necessary and sufficient conditions to be fulfilled by correlation functions in 4D Euclidean space (*Schwinger functions*) for univocally defining the *Wightman correlation functions* in Minkowski spacetime.

Shortly, the O-S theorems show under which conditions the Wick rotation is a well defined isomorphism relating quantum field theories in the two spaces.

The primary tool for investigating QCD in Euclidean space is the lattice (lQCD), but relevant advancements have been achieved with a continuum approach (cQCD), also through a complexification of the Euclidean space.

The continuum approach should be based on the combination of the Bethe-Salpeter equation (BSE) for two and three-body systems (Faddeev-BSE), and, *in principle*, the infinite (!!) set of Dyson-Schwinger equations (DSEs).

This approach should be considered a phenomenological one, since i) a truncation of the infinite tower of DSEs has to be introduced, carefully preserving the symmetries of QCD, as well as ii) a confining interaction suitable for numerical calculations has to be used. Experimental hadron spectra and dynamical observables have been favorably compared with available lattice/continuum calculations.

Summarizing: most of the rigorous results in quantum field theory have been elaborated in the Euclidean space, as well as actual calculations, with unavoidable approximations, which are analyzed and taken under control.

BUT the physical observations are obtained in Minkowski space, and therefore, to improve our confidence in the approximations applied in the Euclidean space, it could be helpful to attempt to replicate in Minkowski space a program analog to the cQCD, already played in the Euclidean space: i.e. combining BSE and a truncated tower of DSEs.

⇒ The approach we are pursuing is based on in-depth analysis of the N-leg amplitudes carried out in the 60' by Noburo Nakanishi, within the Feynman-diagrams framework . One can get rid of the perturbation stigma by using unknown real weight functions, depending upon both compact and non compact variables, on place of distributions (i.e. the typical weights in the Feynman-diagram framework).

The Nakanishi-Integral-Representation way

Nakanishi proposal for a compact and elegant expression of the full N-leg amplitude, $f_N(s)$, written by means of the Feynman parametrization ($\to \vec{\alpha}$), reads: $f_N(s) = \sum_{\mathcal{G}} \; f_{\mathcal{G}}(s)$ ($\mathcal{G} \equiv$ infinite graphs contributing to f_N):

Introducing the identity

$$1 \doteq \prod_h \int_0^1 dz_h \;\; \delta\left(z_h - \frac{\eta_h}{\beta}\right) \int_0^\infty d\gamma \;\; \delta\left(\gamma - \sum_l \frac{\alpha_l m_l^2}{\beta}\right)$$

with $\beta = \sum \eta_i(\vec{\alpha})$ and integrating by parts $n - 2k - 1$ times (n propagators and k integration loops; $\vec{\alpha} =$ Feynman parms)), a graph contribution is

$$f_{\mathcal{G}}(\tilde{s}) \propto \;\; \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \;\frac{\delta(1 - \sum_h z_h)\; \tilde{\phi}_{\mathcal{G}}(\vec{z}, \gamma)}{(\gamma - \sum_h z_h s_h)}$$

$\tilde{\phi}_{\mathcal{G}}(\vec{z}, \gamma) \equiv$ proper combination of distributions, in the analysis á la Feynman, with
$\vec{z} \equiv \{z_1, z_2, \;\ldots\;, z_N\}$, compact real variables, $z_h \in [0, 1]$
$\tilde{s} \equiv \{s_1, s_2, \;\ldots\;, s_N\} \Rightarrow$ N *independent* scalar products, from external momenta of f_N

★ The dependence upon the details of the diagram, $\{n, k\}$, moves from the denominator $\to$ the numerator !! ★

The SAME formal expression for the denominator of ANY diagram $\mathcal{G}$ appears

NIR - II

The full N-leg transition amplitude is the sum of infinite diagrams $\mathcal{G}(n,k)$ and it can be formally written as

$$f_N(\tilde{s}) = \sum_{\mathcal{G}} f_{\mathcal{G}}(\tilde{s}) \propto \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1-\sum_h z_h)\ \phi_N(\vec{z},\gamma)}{(\gamma - \sum_h z_h s_h)}$$

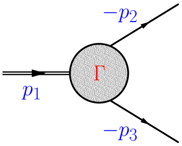
where

$$\phi_N(\vec{z},\gamma) = \sum_{\mathcal{G}} \tilde{\phi}_{\mathcal{G}}(\vec{z},\gamma)$$

is called a Nakanishi weight function and it is REAL (γ is non compact, while $\vec{z}$ is compact).

Application: 3-leg transition amplitude $\rightarrow$ vertex function for a scalar theory (N.B. for fermions $\rightarrow$ spinor indexes)

$$f_3(\tilde{s}) = \int_0^1 dz \int_0^\infty d\gamma \frac{\phi_3(z,\gamma)}{\gamma - \frac{p^2}{4} - k^2 - zk\cdot p - i\epsilon}$$



with $p = p_1 + p_2$ and $k = (p_1 - p_2)/2$

Valid at any order in perturbation-theory !
Natural choice as a general trial function for solving the Bethe-Salpeter Eq. , BUT one needs a transition from a distribution $\phi_3(z,\gamma)$ to a smooth function $g(z,\gamma)$

In spite of its apparent simplicity, to determine the Nakanishi weight functions, $g_i(z, \gamma)$, becomes highly non trivial, when the fermionic dof are taken into account in the BSE.

To anticipate: all the scalar functions entering the relevant vertex function describing a bound-system and dependent upon the four-momenta at disposal, are expressed through the Nakanishi Integral Representation (NIIR).

In the Pion, our bed-test, the BS amplitude contains 4 scalar functions, to be expressed by the following NIRs

$$\phi_i(k; P) = \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{g_i(\gamma', z'; \kappa^2)}{[k^2 + z'(P \cdot k) - \gamma' - \kappa^2 + i\epsilon]^3}$$

$P \equiv$ pion 4-momentum ($P^2 = M_\pi^2$)
$k \equiv$ relative 4-momentum
$\kappa^2 \equiv m^2 - M_\pi^2/4$ and m is a fermionic effective mass (to be defined in what follows...)

N.B. The power of the denominator depends on the smoothness we need to implement. This freedom is already in the original Nakanishi analysis: the trade-off was between derivatives of the distributions in the numerator and the power in the denominator.

Pion as a quark-antiquark bound-system

The Bethe-Salpeter Equation for a 0^- system

$$\Phi(k;P) = S\left(k+\tfrac{P}{2}\right)\int \frac{d^4k'}{(2\pi)^4}\, S^{\mu\nu}(q)\, \Gamma_\mu(q)\Phi(k';P)\, \widehat{\Gamma}_\nu(q) S\left(k-\tfrac{P}{2}\right)$$

$$\Phi(k;P) = \mathrm{BS\ amplitude}$$

where we use in the first step: i) bare propagators for quarks and gluons; ii) ladder approximation with massive gluons, iii) an extended quark-gluon vertex

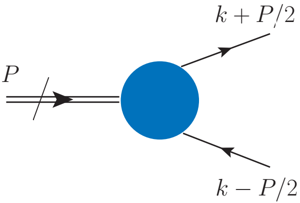
$$S(P) = \frac{i}{\not{P} - m + i\epsilon}\ , \quad S^{\mu\nu}(q) = -i\frac{g^{\mu\nu}}{q^2-\mu^2+i\epsilon}\ , \quad \Gamma^\mu = ig\frac{\mu^2-\Lambda^2}{q^2-\Lambda^2+i\epsilon}\gamma^\mu\ ,$$

$$\widehat{\Gamma}_\nu(q) = C\ \Gamma_\nu(q)\ C^{-1}$$

The quark mass is taken $m = 255$ MeV, while both gluon mass, $\mu = 638$ MeV, and scale parameter, $\Lambda = 306$ MeV, come from the combined analysis of Lattice simulations, the Quark-Gap Equation and Slanov-Taylor identity.
[Oliveira, WP, Frederico, de Melo EPJC 78(7), 553 (2018) & EPJC 79 (2019) 116 & EPJC 80 (2020) 484] N.B. in our most recent calculations, not at pion physical mass, but with running mass quark, the gluon mass is $\mu = 469$ MeV (with $\Lambda = 100$ MeV) [A. Castro et al PLB 845, 138159 (2023)]

NIR for fermion-antifermion 0^- Bound State

In general, the BS amplitude for a quark-antiquark 0^- bound state is decomposed as



$$\Phi(k;P) = \sum_{i=1}^{4} S_i(k;P)\phi_i(k;P)$$

with Dirac structures for a pseudoscalar system given by [N.B. $P^2 = M^2$]

$$S_1(k;P) = \gamma_5,\ S_2(k;P) = \frac{\not{P}}{M}\gamma_5,\ S_3(k;P) = \frac{k \cdot P}{M^3}\not{P}\gamma_5 - \frac{\not{k}}{M}\gamma_5,\ S_4(k;P) = \frac{i}{M^2}\sigma^{\mu\nu}P_\mu k_\nu \gamma_5$$

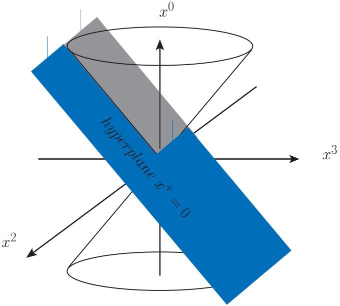
Using the NIR for each scalar functions $\Rightarrow$ System of coupled integral equations

$$\int_{-1}^{1} dz' \int_0^\infty d\gamma' \frac{g_i(\gamma', z)}{\left[k^2 + z' k \cdot P - \gamma' - (m^2 - M^2/4) + i\epsilon\right]^3} = \sum_j \int_{-1}^{1} dz' \int_0^\infty d\gamma' \mathcal{K}_{ij}(\alpha_S, P, k, \gamma', z')\, g_j(\gamma', z')$$

N.B. It is a general expression not dependent on the approximations, and singularities are present on both sides.

Projecting BSE onto the LF hyper-plane $x^+ = 0$

Light-Front variables: $x^\mu = (x^+, x^-, \vec{x}_\perp)$



LF-time $x^+ = x^0 + x^3$

$$x^- = x^0 - x^3$$

$$\vec{x}_\perp = (x^1, x^2)$$

Within the LF framework, one introduces LF-projected amplitudes for each $\phi_i(k, P)$ by integrating on k^- ($\Rightarrow$ s.t. $x^+ = 0$, with x^+ relative LF-time), and can analytically master the singularities

$$\psi_i(\gamma, \xi) = \int \frac{dk^-}{2\pi}\ \phi_i(k, p) = -\frac{i}{M}\int_0^\infty d\gamma' \frac{g_i(\gamma', z; \kappa^2)}{[\gamma + \gamma' + m^2 z^2 + (1 - z^2)\kappa^2]^2}$$

where $\kappa^2 = m^2 - M^2/4$. By LF-projecting both sides of BSE (after applying the

suitable traces on Dirac indexes) one gets a coupled integral-equation system for the Nakanishi weight functions g_i. Once they are known, also $\Phi(k, P)$ is known.

The coupled integral-equation system (see also NIR+covariant LF, Carbonell and Karmanov JPA 2010) in ladder approximation, reads (cf. W. de Paula,et al, PRD **94**, 071901 (2016) & EPJC **77**, 764 (2017))

$$\int_0^\infty \frac{d\gamma' \, g_i(\gamma', z; \kappa^2)}{[\gamma + \gamma' + m^2 z^2 + (1 - z^2)\kappa^2]^2} = \alpha \sum_j \int_0^\infty d\gamma' \int_{-1}^1 dz' \, \tilde{\mathcal{L}}_{ij}(\gamma, z; \gamma', z') \, g_j(\gamma', z'; \kappa^2)$$

In ladder approximation, the Nakanishi Kernel, $\tilde{\mathcal{L}}_{ij}$, has an analytical expression and contains terms that come from singular contributions in k^- regularized 'a la Yan (Chang and Yan, Quantum field theories in the infinite momentum frame. II. PRD **7**, 1147 (1973)).

Numerical solutions are obtained by discretizing the system using a polynomial basis, given by the Cartesian product of Laguerre(γ) $\times$ Gegenbauer(z). One remains with a Generalized eigenvalue problem (GEPV), where a non-symmetric matrix and a symmetric one are present

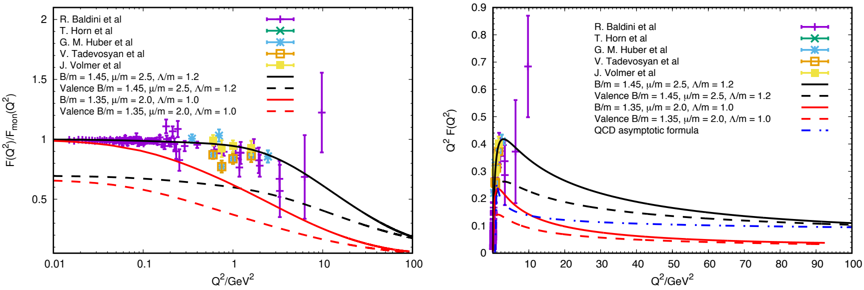
$$A \, \vec{c} = \lambda \, B \, \vec{c}$$

N.B. the eigenvector $\vec{c}$ contains the coefficients of the expansion of the Nakanishi weight functions $g_i(z, \gamma; \kappa^2)$.

If one gets $\boxed{\text{real solutions of the GEVP}}$, then one can $\boxed{\text{validate}}$ the NIR approach.

Pion em form factor in ladder approximation

From E. Ydrefors et al., PLB **820** (2021) 136494.



Black solid curve: pion FF, obtained from the solution of the BSE in ladder approximation, with $m_q = 255$ MeV, $m_g = 637.5$ MeV and $\Lambda = 306$ MeV, that controls the extended quark-gluon vertex. With those values, inspired by LQCD calculations, the experimental value of the decay constant $f_\pi^{PDG} = 130.50(1)(3)(13)$ MeV is reproduced.

Dashed line: LF-valence contribution (LF-valence probability $= 0.70$), from the above solution, after integrating $\bar{u}\ \gamma^+\Phi(k,P)\gamma^+\ v$ on k^- ($\Leftarrow$ pion Fock expansion).

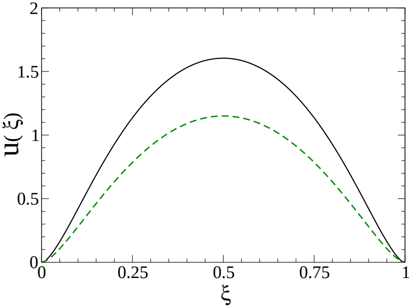
Right Panel: Dash-dotted line; asymptotic expression from Brodsky-Lepage PRD **22** (1980): $Q^2 F_{\rm asy}(Q^2) = 8\pi\alpha_s(Q^2) f_\pi^2$.

Pion Parton distribution function

W. de Paula et al., PRD **105**, L071505 (2022).

From the charge-symmetric (anti-symmetric) expression for the leading-twist transverse-momentum dependent PDF (in short TMD) $f_1^{S(AS)}(\gamma,\xi)$, one gets the *longitudinal* PDF at the initial scale $u(\xi)$

$$f_1^{S(AS)}(\gamma,\xi) = \frac{f_1^q(\gamma,\xi) \pm f_1^{\bar{q}}(\gamma,1-\xi)}{2} \quad \Rightarrow \quad u(\xi) = \int_0^\infty d\gamma \ f_1^S(\gamma,\xi).$$



Solid line: from the full calculation of the BSE at the model scale (norm $=1$)
Dashed line: The LF valence contribution (norm $=0.7$, once the Fock expansion for the pion state is assumed) .

At the initial scale, for $\xi \to 1$, the exponent of $(1-\xi)^{\eta_0}$ is $\eta_0 = 1.4$. What about LQCD predictions?

Pion Parton distribution function II

Low order Mellin moments at scales $Q = 2.0$ GeV and $Q = 5.2$ GeV.

	BSE_2	$LQCD_2$	BSE_5	$LQCD_5$
$\langle x \rangle$	0.259	0.261 ± 0.007	0.221	0.229 ± 0.008
$\langle x^2 \rangle$	0.105	0.110 ± 0.014	0.082	0.087 ± 0.009
$\langle x^3 \rangle$	0.052	0.024 ± 0.018	0.039	0.042 ± 0.010
$\langle x^4 \rangle$	0.029		0.021	0.023 ± 0.009
$\langle x^5 \rangle$	0.018		0.012	0.014 ± 0.007
$\langle x^6 \rangle$	0.012		0.008	0.009 ± 0.005

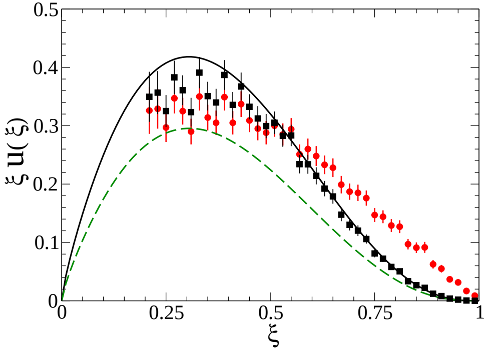
LQCD, at $Q = 2.0$ GeV and $5, 2$ GeV scales from: Alexandrou et al PRD 103, 014508 (2021) and PRD 104, 054504 (2021)

N.B. For evolving our pion PDF we followed Cui et al EPJC 2020 80 1064: lowest order DGLAP equation + effective running charge.
The choice of the initial scale of our dynamical approach is not conceptually plagued by the Landau pole at low scales, once we adopt an effective running charge.

From Cui et al $\Rightarrow$ the hadronic scale is suggested by the inflection point of the effective running charge i.e. :$Q_0 = 0.330 \pm 0.030$ GeV
Within the error bar, we choose $Q_0 = 0.360$ GeV to reproduce the LQCD $\langle x \rangle$.

Pion Parton distribution function III

Comparison with the data at 5.2 GeV scale



Solid line: full calculation of the BSE evolved from the initial scale $Q_0 = 0.360$ GeV to $Q = 5.2$ GeV

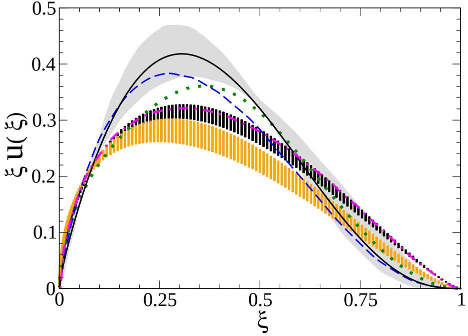
Dashed line: The evolved LF valence contribution

Full dots: original experimental data from E615

Full squares: reanalyzed experimental data from Aicher et al PRL 105, 252003 (2010). evolved to $Q = 5.2$ GeV

Pion Parton distribution function IV

Comparison with other theoretical calculations



Solid line: full calculation of the BSE evolved from the initial scale $Q_0 = 0.360$ GeV to $Q = 5.2$ GeV

Dashed line: DSE calculation from Cui et al, Eur. Phys. J. A 58, 10 (2022)

Dash-dotted line: DSE calculation with dressed quark-photon vertex from Bednar et al PRL 124, 042002 (2020)

Dotted line: basis light front quantization (BLFQ) Coll., PLB 825, 136890 (2022)

Gray area: LQCD results from C. Alexandrou et al (2021)

Black and Orange vertical lines from JAM collaboration, private communication.

For the evolved $\xi\, u(\xi)$, the exponent of $(1-\xi)^{\eta_5}$ is $\eta_5 = 2.94$, when $\xi \to 1$,

LQCD: Alexandrou et al PRD 104, 054504 (2021) obtained 2.20 ± 0.64

Cui et al EPJA 58, 10 (2022) obtained 2.81 ± 0.08

Pion Transverse Momentum-Dependent PDFs (TMDs)

The unpolarized T-even leading and subleading quark TMDs (uTMDs) are defined from the decomposition of the pion correlator [Mulders and Tangerman, Nucl. Phys. B 461, 197 (1996)].

twist-2 uTMD (in LC gauge, $A^+ = 0$):

$$f_1^q(\gamma,\xi) = \frac{N_c}{4} \int d\phi_{\hat{\mathbf{k}}_\perp} \int_{-\infty}^{\infty} \frac{dy^- d\mathbf{y}_\perp}{2(2\pi)^3}\, e^{i[\tilde{k}\cdot\tilde{y}]}\, \langle P|\bar{\psi}_q(-\frac{y}{2})\, \gamma^+\, \psi_q(\frac{y}{2})|P\rangle\Big|_{y^+=0}$$

twist-3 uTMD (in LC gauge, $A^+ = 0$):

$$\frac{M}{P^+}\, e^q(\gamma,\xi) = \frac{N_c}{4} \int d\phi_{\hat{\mathbf{k}}_\perp} \int_{-\infty}^{\infty} \frac{dy^- d\mathbf{y}_\perp}{2(2\pi)^3}\, e^{i[\tilde{k}\cdot\tilde{y}]}\, \langle P|\bar{\psi}_q(-\frac{y}{2})\, \mathbb{1}\, \psi_q(\frac{y}{2})|P\rangle\Big|_{y^+=0}$$

and

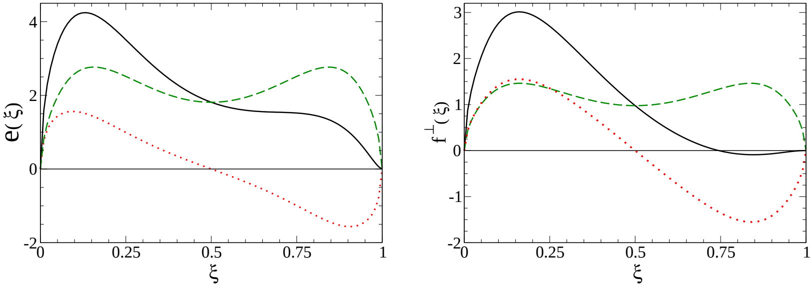
$$\frac{M}{P^+}\, f^{\perp q}(\gamma,\xi) = \frac{N_c M}{4|\mathbf{k}_\perp|^2} \int d\phi_{\hat{\mathbf{k}}_\perp} \int_{-\infty}^{\infty} \frac{dy^- d\mathbf{y}_\perp}{2(2\pi)^3}\, e^{i[\tilde{k}\cdot\tilde{y}]}\, \langle P|\bar{\psi}_q(-\frac{y}{2})\, \mathbf{k}_\perp\cdot\boldsymbol{\gamma}_\perp\, \psi_q(\frac{y}{2})|P\rangle\Big|_{y^+=0}$$

with $\tilde{k}\cdot\tilde{y} = \xi P^+ y^-/2 - \mathbf{k}_\perp\cdot\mathbf{y}_\perp$.

The corresponding symmetric and antisymmetric collinear PDFs are:

$$e^{S(AS)}(\xi) = \int_0^\infty d\gamma\; e^{S(AS)}(\gamma,\xi)\;, \quad f^{\perp S(AS)}(\xi) = \int_0^\infty d\gamma\; f^{\perp S(AS)}(\gamma,\xi)$$

Pion Transverse Momentum-Dependent PDFs II

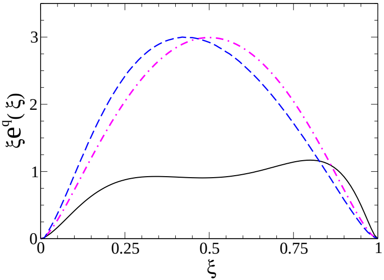


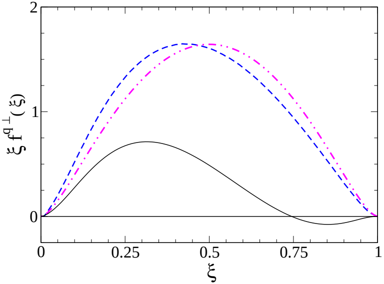
Solid line: quark twist-3 uTMD $e(\xi)$
Dashed line: symmetric twist-3 uTMD $e^S(\xi)$
Dotted: anti-symmetric twist-3 uTMD $e^{AS}(\xi)$

Solid line: quark twist-3 uTMD $f^{\perp}(\xi)$
Dashed line: symmetric twist-3 uTMD $f^{\perp S}(\xi)$
Dotted: anti-symmetric twist-3 uTMD $f^{\perp AS}(\xi)$

Pion Transverse Momentum-Dependent PDFs II

Comparison with a LF Constituent QM

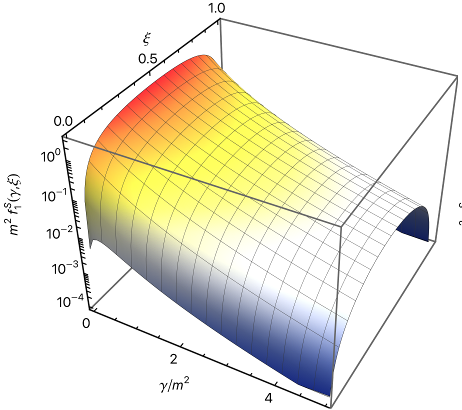




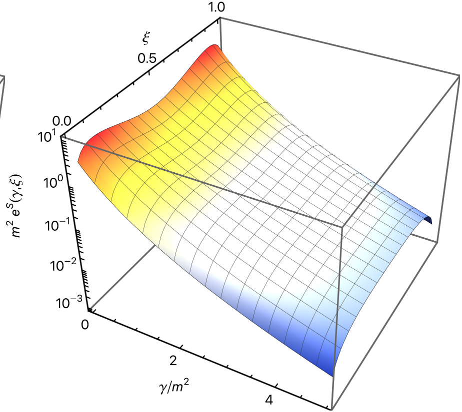
Quark unpolarized collinear PDF, $\xi \ e^q(\xi)$.
Solid line: full calculation.
Dashed line: $m/M \ u^q(\xi)$, as suggested by Lorcé et al, EPJC 76, 415 (2016), but with our PDF.
Double-dot-dashed line: the same as the dashed line, but using the valence approximation of $u^q(\xi)$ with norm $= 1$ and not 0.7.

Quark unpolarized collinear PDF, $\xi \, f^{q\perp}(\xi)$.
Solid line: full calculation.
Dashed line: $\xi \, f^{q\perp}(\xi) = u^q(\xi)$, as suggested by Lorcé et al, EPJC 76, 415 (2016), but with our PDF.
Double-dot-dashed line: the same as the dashed line, but using the valence approximation of $u^q(\xi)$ with norm$= 1$ and not 0.7

3D Pion Transverse Momentum-Dependent PDFs I

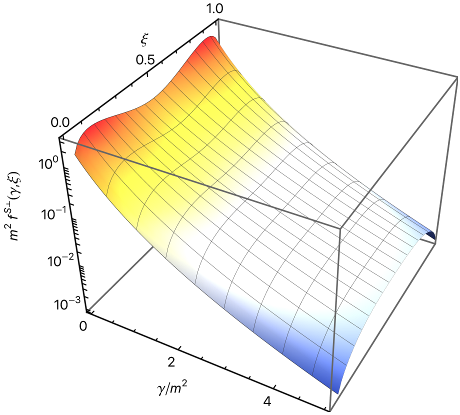


Twist-2 uTMD $f_1^S(\gamma, \xi)$



Twist-3 uTMD $e^S(\gamma, \xi)$

3D Pion Transverse Momentum-Dependent PDFs II



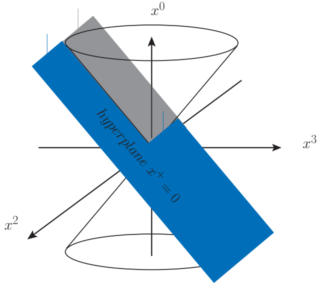
Twist-3 uTMD $f^{\perp S}(\gamma, \xi)$

An iconic view of the pion from the light-cone

W. de Paula,et al, PRD 103, 014002 (2021)

The probability distribution of the quarks inside the pion, sitting on the the hyperplane $x^+ = 0$, tangent to the light-cone, is evaluated in the space given by the Cartesian product of the *Ioffe-time* and the plane spanned by the transverse coordinates $\mathbf{b}_\perp$.

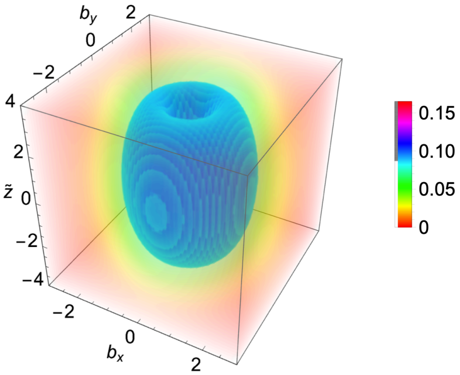
Why? In addition to the standard infinite-momentum frame, one can study the deep-inelastic scattering processes in the target frame, adopting the configuration space. Thus, a more detailed investigation of the space-time structure of the hadrons can be performed. The *Ioffe-time* is useful for studying the relative importance of short and long light-like distances.



Covariant definition of the Ioffe-time: $\tilde{z} = x \cdot P_{target}$, where $x^\mu \equiv (t, \mathbf{x})$. On the hyperplane $x^+ = 0$ (recall x^+ is the LF-time), thee Ioffe-time becomes $\tilde{z} = x^- P^+_{target}/2$

The pion on the light-cone

Density plot of $|\mathbf{b}_\perp|^2 \; |\psi(\tilde{z}, b_x, b_y)|^2$, with $\psi(\tilde{z}, b_x, b_y)$ obtained from our solutions of the ladder Bethe-Salpeter equation [W. de Paula et al PRD 103, (2021) 014002]



$\tilde{z} \equiv$ Ioffe-time

$\{b_x, b_y\} \equiv$ transverse coordinates

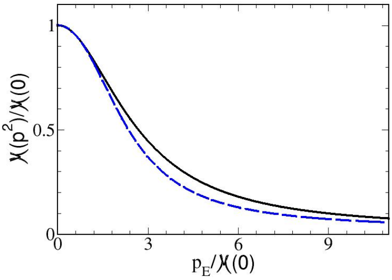
What next?

We are going to address the running-mass case, pointing to the other face of the medal....Dynamical Chiral-Symmetry Breaking [First results in A. Castro et al, PLB 845, 138159 (2023)]

The dressed quark propagator we adopted is [see also Mello et al PLB 766 (2017) 86]

$$S(p) = i\, Z(p^2)\, \frac{\not p + \mathcal{M}(p^2)}{p^2 - \mathcal{M}^2(p^2)}\,, \text{ with } Z(p^2) = 1 \text{ and } \mathcal{M}(p_E^2) = m_0 - \frac{m^3}{(p_E^2 - \lambda^2)}$$

where $m_0 = 0.008\,\text{GeV}$, $m = 0.648\,\text{GeV}$ and $\lambda = 0.9\,\text{GeV}$ adjusted to LQCD calculations by O. Oliveira, et al, PRD **99** (2019) 094506.



The quark running-mass, $\mathcal{M}(p^2)$, as a function of the Euclidean momentum $p_E = \sqrt{-p^2}$, in units of the IR mass $\mathcal{M}(0) = 0.344\,\text{GeV}$.
Solid line: our model.
Dashed line: accurate fit of the LQCD calculations by O. Oliveira, et al

One also writes

$$S(p) = S^V(p^2)\not{p} + S^S(p^2) = i\int_0^\infty ds \; \frac{\rho^V(s)\,\not{p} + \rho^S(s)}{p^2 - s + i\epsilon} \;,$$

where it is shown the decomposition in terms of allowed Dirac structures and corresponding scalar functions, while in the rightmost part one has the dispersive representation *á la Källen-Lehman representation*, with two weights to be phenomenologically parametrized.
It is understood that in QCD the weights do not satisfy the familiar positivity constraints: $\rho^V(s) \geq 0$ and $\sqrt{s}\,\rho^V(s) - \rho^S(s) \geq 0$.

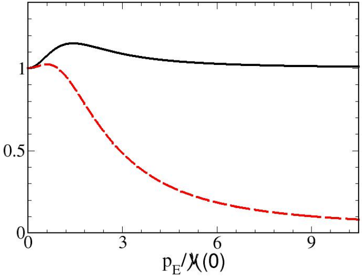
From the adopted running-mass function $\mathcal{M}(p_E^2)$, it turns out that

$$\rho^{S(V)}(s) = \sum_{a=1}^{3} R_a^{S(V)}\delta(s - m_a^2) \;\; \text{with} \;\; R_a^V = \frac{(\lambda^2 - m_a^2)^2}{(m_a^2 - m_b^2)(m_a^2 - m_c^2)} \;, \;\; R_a^S = R_a^V\, \mathcal{M}(m_a^2),$$

The three masses, m_a, are the poles in the quark propagator and are given by the solution of the cubic equation: $m_i(m_i^2 - \lambda^2) = \pm[m_0(m_i^2 - \lambda^2) - m^3]$.

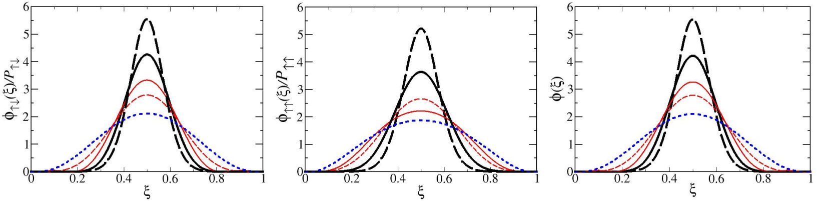
$m_1 = 0.469$ GeV, $m_2 = 0.573$ GeV and $m_3 = 1.035$ GeV $\Rightarrow$ $\rho^V(s)$ is negative !

Scalar functions $S^{V(S)}(p^2)$, normalized to the corresponding free-case quantities

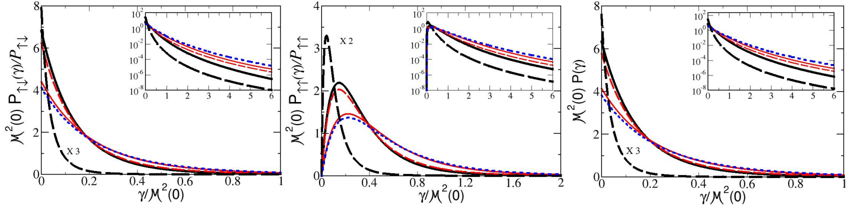


Solid line: $S^V(p^2)\ [p^2 - \mathcal{M}^2(0)]$.
Dashed line: $S^S(p^2)\ [p^2 - \mathcal{M}^2(0)]/\mathcal{M}(0)$
Notice the asymptotic behavior: $\Rightarrow\ \not{p}/p^2$

Taking fixed the gluon mass, now $\mu = 0.469$ GeV, as well as the same phenomenological structure of the quark-gluon vertex, but with $\Lambda = 0.100$ GeV, we have explored the BSE solutions for a 0^- bound state in the range $3m_\pi \leq M \leq 5m_\pi$.



LF-valence longitudinal momentum distributions with gluon mass $\mu = 0.469$ GeV and vertex parameter $\Lambda = 0.1$ GeV. Solid lines: *running quark-mass*, thick lines $M = 5m_\pi$ GeV, thin lines $M = 3m_\pi$ GeV. Dashed Lines: *fixed quark-mass* equal to $\mathcal{M}(0) = 0.344$ GeV. Dotted line: fixed-mass, but for $M = m_\pi = 0.141$ GeV.



LF-valence transverse momentum distributions. The legend of the lines is the same as in the longitudinal distributions case. $\gamma = k_\perp^2$

Fermion-scalar bound-system in the chiral limit

A. Noronha et al, PRD **107**, 096019 (2023)

The homogeneous BSE of a $(1/2)^+$ bound-system, with both fermionic and bosonic degrees of freedom (dubbed a mock nucleon in J. Nogueira et al [PRD 100, 016021 (2019)]), is studied in Minkowski space, in order to analyze the chiral limit in covariant gauges. The BS amplitude reads

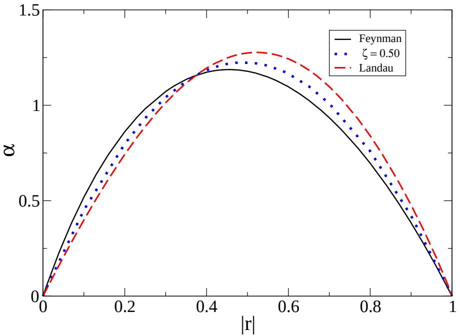
$$\Phi\left(k, P; \frac{1}{2}^{+}\right) = \left[\phi_1(k,P) + \frac{\not{k}}{M}\, \phi_2(k,P)\right]\, U(P,\sigma)$$

where $P^2 = M^2$ and $U(P,\sigma)$ is the spinor with spin projection σ.

The chiral limit induces a scale invariance of the model and consequently generates a wealth of striking features:

- it reduces the number of nontrivial Nakanishi weight functions, from two to one: only $\phi_2(k,P)$ survives;
- the form of the surviving Nakanishi weight function, $g_2(\gamma, z, ; \kappa^2)$, has a factorized dependence on the two relevant variables, compact, $\gamma = k_\perp^2$, and non-compact, $z \in [-1, 1]$;
- the coupling constant becomes an explicit function of the real exponent governing the power-law fall-off in γ of $g_2(\gamma, z)$, and in turn) the LF valence wave function.

Fermion-scalar bound system in the chiral limit I

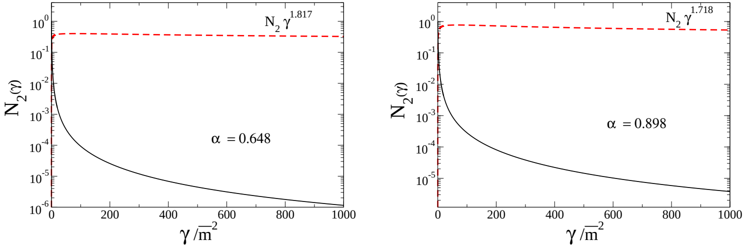


Coupling constant of the fermion-boson system in the chiral limit vs. the power r ($g_2(\gamma, z) = \gamma^r f_{2,r}(z)$). Solid line: Feynman gauge ($\zeta = 1$). Dotted line: $\zeta = 0.5$ gauge. Dashed line: Landau gauge ($\zeta = 0$).

The coupling constant α is a well-determined function of the power r, such that $g_2(\gamma, z) = \gamma^r \ f(z)$.

$$\alpha(r, \zeta) = 2\pi \left[\frac{1 + 2|r|}{|r|(1 - r^2)} + (1 - \zeta) \frac{-3\,|r|^3 + 17\,r^2 - 22\,|r| + 6}{2\,|r|\,(1 - r^2)\,(2 - |r|)(3 - |r|)} \right]^{-1} . \tag{1}$$

Fermion-scalar bound system in the chiral limit II



Mock-nucleon LF amplitude $N_2(\gamma) = \psi_2(\gamma, \xi_0 = 0.5)/\psi_2(0, \xi_0 = 0.5)$ vs transverse-momentum square $\gamma = |\vec{k}_\perp|^2$ in the Feynman gauge,($m_s/m_f = 2$ and $M/\bar{m} = 1.9$, with $M = .938$ GeV and $\bar{m} = (m_s + m_f)/2$), and two values of the gluon mass

Left panel: $\mu/\bar{m} = 0.15$ and $\alpha = 0.648$. Solid line: $N_2(\gamma)$. Dashed line: $N_2(\gamma) \times \gamma^{1.817}$.
Right panel: $\mu/\bar{m} = 0.5$ and $\alpha = 0.898$. Solid line: $N_2(\gamma)$. Dashed line: $N_2(\gamma) \times \gamma^{1.718}$.

N.B. $N_2(\gamma)$ is obtained by solving the full BSE, with M and $m_{f(s)}$ above, i.e. without the assumption of a factorized form of $g_i(\gamma, z)$.

Summary

- The near future will offer an innovative view of the dynamics inside the hadrons, thanks to the experimental activity planned at the Electron-ion colliders, and plenty of measurements pointing to the 3D tomography of hadrons will become available.
- For the pion, many results, em form factor, PDF, TMDs, Ioffe-time$\times$ transverse plane distribution, have been obtained by using the ladder-approximation of the $q\bar{q}$-BSE.
- The pion has an important role, given its dual nature: $q\bar{q}$ bound-system and Goldstone boson, i.e. the golden gate to address the DCSB and the emergent-mass phenomena. Our aim is to implement a framework analogous to the one already developed in Euclidean space. An initial investigation with running quark-mass has been shown.
- Phenomenological investigations based on solutions in Minkowski space, once the approach composed by BSE, gap-equations, realistic quark-gluon vertex and possibly confining interaction will be fully available, could offer fresh insights in hadron dynamics. Not forgetting the synergy with well-established lattice- and continuous-QCD communities.

Thank you very much for the attention!!!