General-relativistic multigroup radiation transport scheme in `Gmunu` and applications

MICRA 2023

14 Sep 2023

Patrick Chi-Kit Cheong
patrick.cheong@berkeley.edu

N3AS postdoc fellow
University of New Hampshire

Network for Neutrinos, Nuclear Astrophysics, and Symmetries
PHYSICS FRONTIER CENTER

University of New Hampshire

Agenda

Patrick Chi-Kit Cheong

Intro

Method

Two-moment radiative transfer

Source terms

Results

Summary

Backup slides

N3AS postdoc fellow
University of New Hampshire

Why do we need ν treatment?

Patrick Chi-Kit Cheong

N3AS postdoc fellow
University of New Hampshire

What do we need to model ν?

MICRA: **M**icrophysics **I**n **C**omputational **R**elativistic **A**strophysics

Minimally, we will need:

- Microphysics

What do we need to model ν?

MICRA: **M**icrophysics **I**n **C**omputational **R**elativistic **A**strophysics

Minimally, we will need:

- Microphysics
  - EOS

What do we need to model ν?

MICRA: **M**icrophysics **I**n **C**omputational **R**elativistic **A**strophysics

Minimally, we will need:

- Microphysics
  - EOS
  - neutrino

What do we need to model ν?

MICRA: **M**icrophysics **In** **C**omputational **R**elativistic **A**strophysics

Minimally, we will need:

- Microphysics
  - EOS
  - neutrino
- Computational methods

What do we need to model ν?

MICRA: **M**icrophysics **I**n **C**omputational **R**elativistic **A**strophysics

Minimally, we will need:

- Microphysics
  - EOS
  - neutrino
- Computational methods
  - accurate and consistant

What do we need to model ν?

MICRA: **M**icrophysics **I**n **C**omputational **R**elativistic **A**strophysics

Minimally, we will need:

- Microphysics
  - EOS
  - neutrino
- Computational methods
  - accurate and consistant
  - affortable/efficient

What do we need to model ν?

MICRA: **M**icrophysics **I**n **C**omputational **R**elativistic **A**strophysics

Minimally, we will need:

- Microphysics
  - EOS
  - neutrino
- Computational methods
  - accurate and consistant
  - affortable/efficient
- (General) Relativistic

What do we need to model ν?

MICRA: **M**icrophysics **In** **C**omputational **R**elativistic **A**strophysics

Minimally, we will need:

- Microphysics
  - EOS
  - neutrino
- Computational methods
  - accurate and consistant
  - affortable/efficient
- (General) Relativistic
  - BH/NS formations

What do we need to model ν?

MICRA: **M**icrophysics **I**n **C**omputational **R**elativistic **A**strophysics

Minimally, we will need:

- Microphysics
  - EOS
  - neutrino
- Computational methods
  - accurate and consistant
  - affortable/efficient
- (General) Relativistic
  - BH/NS formations
  - HMNS is rapidly rotating

What do we need to model ν?

MICRA: **M**icrophysics **I**n **C**omputational **R**elativistic **A**strophysics

Minimally, we will need:

- Microphysics
  - EOS
  - neutrino
- Computational methods
  - accurate and consistant
  - affortable/efficient
- (General) Relativistic
  - BH/NS formations
  - HMNS is rapidly rotating
  - relativistic jets

How do we model ν?

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}, \quad \text{(Einstein equation)}$$

$$\nabla_\mu \left(\rho u^\mu\right) = 0, \quad \text{(cons. rest mass)}$$

$$\nabla_\mu T^{\mu\nu} = 0, \quad \text{(cons. energy/momentum)}$$

$$\nabla_\mu \left(\rho Y_e u^\mu\right) = S_{Y_e}, \quad \text{(composition evolution)}$$

$$p = p\left(\rho, T, Y_\text{e} \cdots\right), \quad \text{(equation of state)}$$

$$\nabla_\mu F^{\mu\nu} = \mathcal{J}^\nu, \quad \nabla_\mu {}^*F^{\mu\nu} = 0, \quad \text{(Maxwell equations)}$$

$$\left(p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu}\right) f = \left(\frac{\partial f}{\partial \tau}\right)_\text{coll} \quad \text{(Boltzmann equation)}$$

$$\vdots$$

$$T^\text{total}_{\mu\nu} = T^\text{fluid}_{\mu\nu} + T^\text{EM}_{\mu\nu} + T^\text{rad}_{\mu\nu} + \cdots$$

Equations needed to be solved

$$\left(p^{\mu} \frac{\partial}{\partial x^{\mu}} - \Gamma^{\mu}_{\alpha\beta} p^{\alpha} p^{\beta} \frac{\partial}{\partial p^{\mu}} \right) f = \left(\frac{\partial f}{\partial \tau} \right)_{\text{coll}} \qquad \text{(Boltzmann equation)}$$

$f(x^i, p^i, t)$, a 7-dimensional problem for each radiation species, so expansive

possible solutions

- (advanced spectral) neutrino leakage + heating schemes

Equations needed to be solved

$$\left(p^{\mu} \frac{\partial}{\partial x^{\mu}} - \Gamma^{\mu}_{\alpha\beta} p^{\alpha} p^{\beta} \frac{\partial}{\partial p^{\mu}} \right) f = \left(\frac{\partial f}{\partial \tau} \right)_{\text{coll}} \qquad \text{(Boltzmann equation)}$$

$f(x^i, p^i, t)$, a 7-dimensional problem for each radiation species, so expansive

possible solutions

- (advanced spectral) neutrino leakage + heating schemes
- Truncated moment schemes (one-, two-, three-,... moment schemes)

N3AS postdoc fellow
University of New Hampshire

Equations needed to be solved

$$\left(p^{\mu}\frac{\partial}{\partial x^{\mu}} - \Gamma^{\mu}_{\alpha\beta}p^{\alpha}p^{\beta}\frac{\partial}{\partial p^{\mu}}\right) f = \left(\frac{\partial f}{\partial \tau}\right)_{\text{coll}} \qquad \text{(Boltzmann equation)}$$

$f(x^i, p^i, t)$, a 7-dimensional problem for each radiation species, so expansive

possible solutions

- (advanced spectral) neutrino leakage + heating schemes
- Truncated moment schemes (one-, two-, three-,... moment schemes)
- ⋮

N3AS postdoc fellow
University of New Hampshire

Equations needed to be solved

$$\left(p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} \right) f = \left(\frac{\partial f}{\partial \tau} \right)_{\text{coll}} \qquad \text{(Boltzmann equation)}$$

$f(x^i, p^i, t)$, a 7-dimensional problem for each radiation species, so expansive

possible solutions

- (advanced spectral) neutrino leakage + heating schemes
- Truncated moment schemes (one-, two-, three-,... moment schemes)
- ⋮
- fully solve it! (see Nagakura-san's talk)

MICRA 2023
Patrick Chi-Kit Cheong

Gmunu (**G**eneral-relativistic **mu**ltigrid **nu**merical solver) [1, 2, 3, 4, 5]

Physics modules

- Consternated-evolution scheme for Einstein equation
  - Conformally flat condition (CFC)
- GRMHD
  - ideal/(**resistive** + dynamo)
  - hyperbolic cleaning
  - constrained transport
  - elliptic cleaning
- Radiative transfer
  - Two-moment scheme
  - grey/multi-group

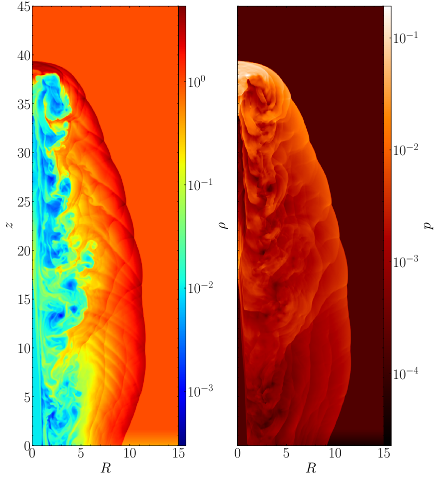
Numerical features

- Block-based Adaptive Mesh Refinement (AMR) (provided by `MPI-AMRVAC`)
- Parallelised with MPI (provided by `MPI-AMRVAC`)
- Multi-dimensional (1-3D)
- Curvilinear geometries
  - Cartesian
  - Cylindrical
  - Spherical

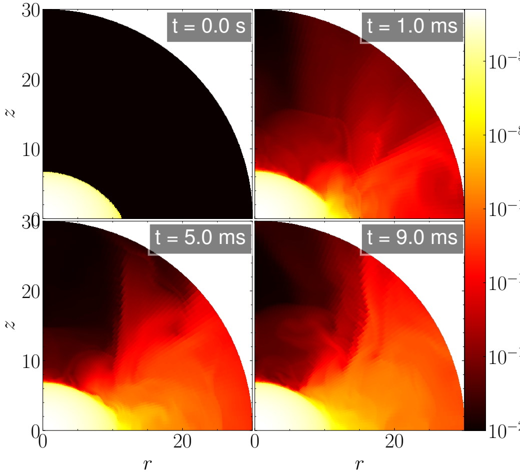
Skip Ad ▶|

N3AS postdoc fellow
University of New Hampshire

Examples

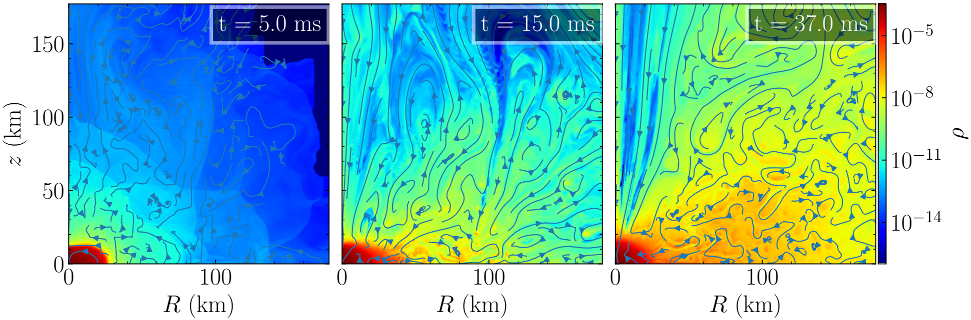


Cylindrical

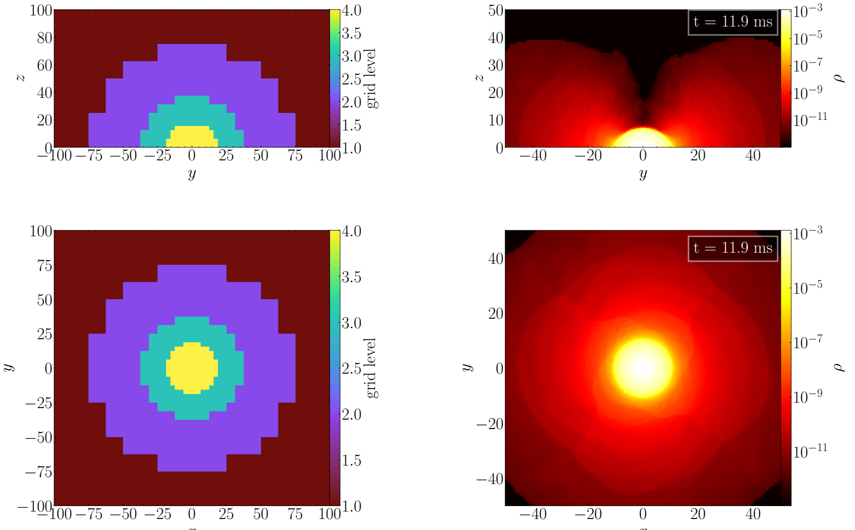


Spherical

Examples


t = 5.0 ms
t = 15.0 ms
t = 37.0 ms
150
100
50
0
z (km)
0
100
R (km)
10^{-5}
10^{-8}
10^{-11}
10^{-14}
ρ

Examples



Radiative transfer

The *comoving-frame* zeroth-, first-, second- and third-order moments are defined as

$$\begin{aligned}
\mathcal{J}\left(x^{\mu},\varepsilon\right) &\equiv \frac{\varepsilon}{4\pi}\int f\left(x^{\mu},\varepsilon,\Omega\right)\mathrm{d}\Omega\,, \\
\mathcal{H}^{\alpha}\left(x^{\mu},\varepsilon\right) &\equiv \frac{\varepsilon}{4\pi}\int \ell^{\alpha} f\left(x^{\mu},\varepsilon,\Omega\right)\mathrm{d}\Omega\,, \\
\mathcal{K}^{\alpha\beta}\left(x^{\mu},\varepsilon\right) &\equiv \frac{\varepsilon}{4\pi}\int \ell^{\alpha}\ell^{\beta} f\left(x^{\mu},\varepsilon,\Omega\right)\mathrm{d}\Omega\,, \\
\mathcal{L}^{\alpha\beta\gamma}\left(x^{\mu},\varepsilon\right) &\equiv \frac{\varepsilon}{4\pi}\int \ell^{\alpha}\ell^{\beta}\ell^{\gamma} f\left(x^{\mu},\varepsilon,\Omega\right)\mathrm{d}\Omega\,, \\
&\vdots
\end{aligned} \tag{1}$$

ε is the radiation energy observed in the comoving frame while $\mathrm{d}\Omega$ is the solid angle in the comoving frame.
This is now (`ndir+1`)-d problem

N3AS postdoc fellow
University of New Hampshire

Radiative transfer

The monochromatic energy-momentum tensor $\mathcal{T}^{\mu\nu}$ and the third-rank momentum moment $\mathcal{U}^{\mu\nu\rho}$ can be *Lagrangian* decomposed with respect to the comoving observer with four-velocity u^μ as follows

$$\mathcal{T}^{\mu\nu} = \mathcal{J} u^\mu u^\nu + \mathcal{H}^\mu u^\nu + u^\mu \mathcal{H}^\nu + \mathcal{K}^{\mu\nu}, \tag{2}$$

$$\begin{aligned}\mathcal{U}^{\mu\nu\rho} =& \varepsilon\Big(\mathcal{J} u^\mu u^\nu u^\rho + \mathcal{H}^\mu u^\nu u^\rho + u^\mu \mathcal{H}^\nu u^\rho + u^\mu u^\nu \mathcal{H}^\rho \\ &+ \mathcal{K}^{\mu\nu} u^\rho + \mathcal{K}^{\nu\rho} u^\mu + \mathcal{K}^{\rho\mu} u^\nu + \mathcal{L}^{\mu\nu\rho}\Big),\end{aligned} \tag{3}$$

in case you want to collect all the energy-space

$$T^{\mu\nu}_{\text{rad}} = \int_0^\infty 4\pi\varepsilon^2 \mathcal{T}^{\mu\nu}\, \mathrm{d}\varepsilon = \int_0^\infty \mathcal{T}^{\mu\nu}\, \mathrm{d}V_\varepsilon\,, \tag{4}$$

Radiative transfer

Patrick Chi-Kit Cheong

However, we are working on Eulerian code, so:

$$\begin{aligned} \mathcal{T}^{\mu\nu} =& \mathcal{E} n^\mu n^\nu + \mathcal{F}^\mu n^\nu + n^\mu \mathcal{F}^\nu + \mathcal{P}^{\mu\nu}, \\ \mathcal{U}^{\mu\nu\rho} =& \varepsilon \Big(\mathcal{Z} n^\mu n^\nu n^\rho + \mathcal{Y}^\mu n^\nu n^\rho + n^\mu \mathcal{Y}^\nu n^\rho + n^\mu n^\nu \mathcal{Y}^\rho \\ & + \mathcal{X}^{\mu\nu} n^\rho + \mathcal{X}^{\nu\rho} n^\mu + \mathcal{X}^{\rho\mu} n^\nu + \mathcal{W}^{\mu\nu\rho} \Big), \end{aligned} \tag{5}$$

The evolution equation is

$$\boxed{\nabla_\nu \mathcal{T}^{\mu\nu} - \frac{1}{\varepsilon^2} \frac{\partial}{\partial \varepsilon} \left(\varepsilon^2 \mathcal{U}^{\mu\nu\rho} \nabla_\rho u_\nu \right) = \mathcal{S}^\mu}$$

recall that $T^{\text{total}}_{\mu\nu} = T^{\text{fluid}}_{\mu\nu} + T^{\text{EM}}_{\mu\nu} + T^{\text{rad}}_{\mu\nu} + \cdots$, the radiation four-force $\mathcal{S}^\mu$, describes the interaction between the radiation and the fluid (anything else).

N3AS postdoc fellow
University of New Hampshire

Two moment scheme

$$\mathcal{T}^{\mu\nu} = \mathcal{E} n^\mu n^\nu + \mathcal{F}^\mu n^\nu + n^\mu \mathcal{F}^\nu + \mathcal{P}^{\mu\nu}$$
$$\nabla_\nu \mathcal{T}^{\mu\nu} - \frac{1}{\varepsilon^2}\frac{\partial}{\partial\varepsilon}\left(\varepsilon^2 \mathcal{U}^{\mu\nu\rho}\nabla_\rho u_\nu\right) = \mathcal{S}^\mu \quad (6)$$

The equations here are exact.
In two-moment schemes, we evolve the first two moments,

$$\frac{\partial}{\partial t}\left[\sqrt{\gamma/\hat{\gamma}}\mathcal{E}\right] + \hat{\nabla}_i\left[\sqrt{\gamma/\hat{\gamma}}\left(\alpha\mathcal{F}^i - \mathcal{E}\beta^i\right)\right] - \alpha\sqrt{\gamma/\hat{\gamma}}\frac{1}{\varepsilon^2}\frac{\partial}{\partial\varepsilon}\left[-\varepsilon^2 n_\mu \mathcal{U}^{\mu\nu\rho}\nabla_\rho u_\nu\right]$$
$$= \sqrt{\gamma/\hat{\gamma}}\left[-\mathcal{F}^j\partial_j\alpha + \mathcal{P}^{ij}K_{ij}\right] - \alpha\sqrt{\gamma/\hat{\gamma}}\mathcal{S}^\mu n_\mu, \quad (7)$$

$$\frac{\partial}{\partial t}\left[\sqrt{\gamma/\hat{\gamma}}\mathcal{F}_i\right] + \hat{\nabla}_i\left[\sqrt{\gamma/\hat{\gamma}}\left(\alpha\mathcal{P}^i_j - \mathcal{F}_j\beta^i\right)\right] - \alpha\sqrt{\gamma/\hat{\gamma}}\frac{1}{\varepsilon^2}\frac{\partial}{\partial\varepsilon}\left[\varepsilon^2 \gamma_{i\mu}\mathcal{U}^{\mu\nu\rho}\nabla_\rho u_\nu\right]$$
$$= \sqrt{\gamma/\hat{\gamma}}\left[-\mathcal{E}\partial_i\alpha + \mathcal{F}_k\hat{\nabla}_i\beta^k + \frac{1}{2}\alpha\mathcal{P}^{jk}\hat{\nabla}_i\gamma_{jk}\right] + \alpha\sqrt{\gamma/\hat{\gamma}}\mathcal{S}^\mu\gamma_{i\mu}, \quad (8)$$

With a (analytic) closure $\mathcal{P}^{\mu\nu} = \mathcal{P}^{\mu\nu}\left(\mathcal{E}, \mathcal{F}^\mu\right)$.

Two moment scheme

$$\frac{\partial}{\partial t}\left[\sqrt{\gamma/\hat{\gamma}}\mathcal{E}\right] + \hat{\nabla}_i\left[\sqrt{\gamma/\hat{\gamma}}\left(\alpha\mathcal{F}^i - \mathcal{E}\beta^i\right)\right] - \alpha\sqrt{\gamma/\hat{\gamma}}\frac{1}{\varepsilon^2}\frac{\partial}{\partial\varepsilon}\left[-\varepsilon^2 n_\mu \mathcal{U}^{\mu\nu\rho}\nabla_\rho u_\nu\right] = \sqrt{\gamma/\hat{\gamma}}\left[-\mathcal{F}^j\partial_j\alpha + \mathcal{P}^{ij}K_{ij}\right] - \alpha\sqrt{\gamma/\hat{\gamma}}\mathcal{S}^\mu n_\mu, \tag{9}$$

$$\frac{\partial}{\partial t}\left[\sqrt{\gamma/\hat{\gamma}}\mathcal{F}_i\right] + \hat{\nabla}_i\left[\sqrt{\gamma/\hat{\gamma}}\left(\alpha\mathcal{P}^i{}_j - \mathcal{F}_j\beta^i\right)\right] - \alpha\sqrt{\gamma/\hat{\gamma}}\frac{1}{\varepsilon^2}\frac{\partial}{\partial\varepsilon}\left[\varepsilon^2 \gamma_{i\mu} \mathcal{U}^{\mu\nu\rho}\nabla_\rho u_\nu\right] = \sqrt{\gamma/\hat{\gamma}}\left[-\mathcal{E}\partial_i\alpha + \mathcal{F}_k\hat{\nabla}_i\beta^k + \frac{1}{2}\alpha\mathcal{P}^{jk}\hat{\nabla}_i\gamma_{jk}\right] + \alpha\sqrt{\gamma/\hat{\gamma}}\mathcal{S}^\mu\gamma_{i\mu}, \tag{10}$$

$$\partial_t \boldsymbol{q} + \frac{1}{\sqrt{\hat{\gamma}}}\partial_j\left[\sqrt{\hat{\gamma}}\boldsymbol{f}^j\right] + \frac{1}{\varepsilon^2}\partial_\varepsilon\left[\varepsilon^2\boldsymbol{f}_\varepsilon\right] = \boldsymbol{s}_{\text{grav}} + \boldsymbol{s}_{\text{geom}} + \boldsymbol{s}_{\text{int}}$$

Two moment scheme

Evolution of lowest two moments

$$\partial_t \boldsymbol{q} + \frac{1}{\sqrt{\hat{\gamma}}}\partial_j\left[\sqrt{\hat{\gamma}}\boldsymbol{f}^j\right] + \frac{1}{\varepsilon^2}\partial_\varepsilon\left[\varepsilon^2\boldsymbol{f_\varepsilon}\right] = \boldsymbol{s}_{\text{grav}} + \boldsymbol{s}_{\text{geom}} + \boldsymbol{s}_{\text{int}}$$

- explicit in spatial space
- explicit in energy space
- implicit, due to its stiffness, in general in both spatial and energy spaces

This is already a huge problem, i.e. at each grid point, resolve energy space $(N_x \times N_y \times N_z \times N_\varepsilon)$. This indeed even larger when deal with source terms properly.

Radiative transfer

$$\nabla_\nu \mathcal{T}^{\mu\nu} - \frac{1}{\varepsilon^2}\frac{\partial}{\partial \varepsilon}\left(\varepsilon^2 \mathcal{U}^{\mu\nu\rho}\nabla_\rho u_\nu\right) = \mathcal{S}^\mu$$

If we energy-integrated it, due to the energy conservation

$$\Rightarrow \nabla_\nu T^{\mu\nu}_{\text{rad}} = \int_0^\infty \mathcal{S}^\mu \, \mathrm{d}V_\varepsilon$$

usually refer as energy-integrated scheme or grey scheme.

- Grey two-moment scheme is getting popular in NS mergers
- Largely reduce the problem size, computational much cheaper
- Not an option for CCSNe, where the cross sections of ν ($\propto \varepsilon^2$)
- spectra of ν is quite non-thermal in CCSNe

Radiative transfer

The $\frac{1}{\varepsilon^2}\frac{\partial}{\partial \varepsilon}\left(\varepsilon^2 \mathcal{U}^{\mu\nu\rho}\nabla_\rho u_\nu\right)$ terms describe the coupling of the radiation in energy space.

These capture the gravitational and Doppler redshifts, e.g.

$$f_{\varepsilon\,\mathcal{E}} = \alpha\psi^6\sqrt{\bar{\gamma}/\hat{\gamma}}\left[n_\mu \mathcal{U}^{\mu\nu\rho}\nabla_\rho u_\nu\right] \tag{11}$$

$$= \psi^6\sqrt{\bar{\gamma}/\hat{\gamma}}\,\varepsilon\Bigg\{W\Bigg[\left(\mathcal{Z}v^i - \mathcal{Y}^i\right)\partial_i\alpha - \mathcal{Y}_k v^i \partial_i \beta^k \tag{12}$$

$$- \alpha\mathcal{X}^{ki}\left(\frac{1}{2}v^m\partial_m\gamma_{ki} - K_{ki}\right)\Bigg] \tag{13}$$

$$+ \Bigg[\left[\mathcal{Z}\partial_t W - \mathcal{Y}_k\partial_t\left(Wv^k\right)\right] + \left[\alpha\mathcal{Y}^i - \mathcal{Z}\beta^i\right]\partial_i W \tag{14}$$

$$- \left[\alpha\mathcal{X}_k{}^i - \mathcal{Y}_k\beta^i\right]\partial_i\left(Wv^k\right)\Bigg]\Bigg\}, \tag{15}$$

Two moment scheme

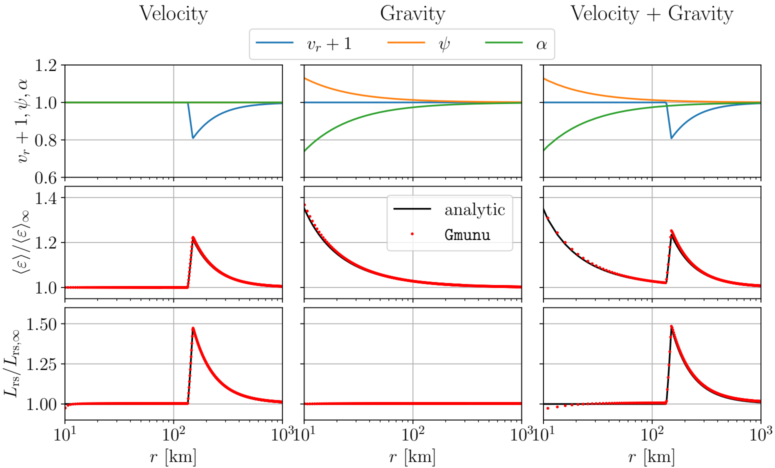


Figure: also see Bernuzzi's talk

source terms

Depends on the physics considered, source terms can be very different. e.g.

$$\mathcal{S}^{\mu} = \mathcal{S}^{\mu}_{\mathrm{e/a}} + \mathcal{S}^{\mu}_{\mathrm{elastic}} + \mathcal{S}^{\mu}_{\mathrm{inelastic}} + \mathcal{S}^{\mu}_{\mathrm{pair}} + \cdots , \tag{16}$$

Beta processes	Neutrino-pair processes
$\nu_e + n \leftrightarrow p + e^-$	$e^- + e^+ \leftrightarrow \nu + \bar{\nu}$
$\bar{\nu}_e + p \leftrightarrow n + e^+$	$N + N \leftrightarrow N + N + \nu + \bar{\nu}$
$\nu_e + (A, Z-1) \leftrightarrow (A, Z) + e^-$	
Elastic scattering	**Inelastic scattering**
$\nu + N \leftrightarrow \nu + N$	$\nu + e^- \leftrightarrow \nu + e^-$
$\nu + (A, Z) \leftrightarrow \nu + (A, Z)$	
$\nu + \alpha \leftrightarrow \nu + \alpha$	

copled to `NuLib`. see also `Weakhub`, a recent paper arXiv:2309.03526

source terms

The mean-free-path of these interactions are usually small in the interested regions, $\tau_{\text{interaction}} \gg \tau_{\text{fluid}}$, super expansive if we evolve it explicitly

$$\boldsymbol{q}^{\mathtt{n+1}} = \boldsymbol{q}^{\mathtt{n},(k)} + \Delta t \boldsymbol{s}_{\text{stiff}} \left(\boldsymbol{q}^{\mathtt{n+1}} \right) . \tag{17}$$

We need to solve these implicitly. We adpot IMEX schemes

source terms

$$\mathcal{S}^{\mu} = \mathcal{S}^{\mu}\left(\rho, T, Y_e, \varepsilon, \varepsilon', \nu_{\text{species}}, \nu'_{\text{species}}, \cdots\right) \tag{18}$$

- Consider $N_{\varepsilon} = 18$ energy-bins, $N_{\text{species}} = 3$ species of neutrino $(\nu_e, \bar{\nu}_e, \nu_x)$, in $N_{\text{dim}} = 3$, coupled with energy-momentum equations of hydro (with $N_{\text{hydro}} = (1 + N_{\text{dim}})$ variables)
- $(N_{\text{dim}} + 1) \times N_{\varepsilon} \times N_{\text{species}} + N_{\text{hydro}} = 220$
- calculate and invert (non-analytical) Jacobian of dimension 220^2
- tabulate/caculated these source terms at every iteration
- in GR cases, conserved to primitive transformation, and need to call tabulated EoS
- at each grid point, at each (sub)timestep

source terms

by default, we assume ρ, T, Y_e are fixed, and solve the radiation part only, by:

$$\boldsymbol{q}_{\rm rad}^{\tt n+1} = \boldsymbol{q}_{\rm rad}^{{\tt n},(k)} + \Delta t \boldsymbol{s}_{\rm stiff}\left(\boldsymbol{q}_{\rm rad}^{\tt n+1}\right) . \tag{19}$$

and

$$q_\tau \rightarrow q_\tau - \Delta t \sum_{\rm species} \int s_{{\rm rad}\,\mathcal{E}} \, {\rm d}V_\nu \, , \tag{20}$$

$$q_{S_i} \rightarrow q_{S_i} - \Delta t \sum_{\rm species} \int s_{{\rm rad}\,\mathcal{F}_i} \, {\rm d}V_\nu \, , \tag{21}$$

$$q_{DY_e} \rightarrow q_{DY_e} + \Delta t m_{\rm u} \int \frac{{\rm d}V_{\nu'}}{\nu'} \left[s^\mu_{{\rm rad},\nu_{\rm e}}\left(\nu'\right) - s^\mu_{{\rm rad},\bar{\nu}_{\rm e}}\left(\nu'\right) \right] u_\mu , \tag{22}$$

source terms

by default, we assume ρ, T, Y_e are fixed, and solve the radiation part only, by:

$$\boldsymbol{q}_{\rm rad}^{\tt n+1} = \boldsymbol{q}_{\rm rad}^{{\tt n},(k)} + \Delta t \boldsymbol{s}_{\rm stiff}\left(\boldsymbol{q}_{\rm rad}^{\tt n+1}\right). \tag{23}$$

- *multi-species multi-group* (MSMG): $(N_{\rm dim}+1) \times N_\varepsilon \times N_{\rm species}$

source terms

by default, we assume ρ, T, Y_e are fixed, and solve the radiation part only, by:

$$\boldsymbol{q}_{\mathrm{rad}}^{\mathtt{n+1}} = \boldsymbol{q}_{\mathrm{rad}}^{\mathtt{n},(k)} + \Delta t \boldsymbol{s}_{\mathrm{stiff}} \left(\boldsymbol{q}_{\mathrm{rad}}^{\mathtt{n+1}} \right) . \tag{23}$$

- *multi-species multi-group* (MSMG): $(N_{\mathrm{dim}} + 1) \times N_{\varepsilon} \times N_{\mathrm{species}}$
- *single-species multi-group* (SSMG): N_{species} non-linear systems of $(N_{\mathrm{dim}} + 1) \times N_{\varepsilon}$.

source terms

by default, we assume ρ, T, Y_e are fixed, and solve the radiation part only, by:

$$\boldsymbol{q}_{\mathrm{rad}}^{\mathtt{n+1}} = \boldsymbol{q}_{\mathrm{rad}}^{\mathtt{n},(k)} + \Delta t \boldsymbol{s}_{\mathrm{stiff}}\left(\boldsymbol{q}_{\mathrm{rad}}^{\mathtt{n+1}}\right). \tag{23}$$

- *multi-species multi-group* (MSMG): $(N_{\mathrm{dim}} + 1) \times N_{\varepsilon} \times N_{\mathrm{species}}$
- *single-species multi-group* (SSMG): N_{species} non-linear systems of $(N_{\mathrm{dim}} + 1) \times N_{\varepsilon}$.
- *single-species single-group* (SSSG): $N_{\mathrm{species}} \times N_{\varepsilon}$ non-linear systems of dimensions $(N_{\mathrm{dim}} + 1)$.

source terms

However...

$$q_{\tau} \to q_{\tau} - \Delta t \sum_{\text{species}} \int s_{\text{rad}\,\mathcal{E}} \, \mathrm{d}V_{\nu} \, , \tag{24}$$

$$q_{S_i} \to q_{S_i} - \Delta t \sum_{\text{species}} \int s_{\text{rad}\,\mathcal{F}_i} \, \mathrm{d}V_{\nu} \, , \tag{25}$$

$$q_{DY_e} \to q_{DY_e} + \Delta t m_{\mathrm{u}} \int \frac{\mathrm{d}V_{\nu'}}{\nu'} \left[s^{\mu}_{\text{rad},\nu_{\mathrm{e}}} \left(\nu'\right) - s^{\mu}_{\text{rad},\bar{\nu}_{\mathrm{e}}} \left(\nu'\right) \right] u_{\mu} , \tag{26}$$

- still need to limit the timestep, especially because of Y_e

However...

$$q_{\tau} \rightarrow q_{\tau} - \Delta t \sum_{\text{species}} \int s_{\text{rad}\,\mathcal{E}} \, \mathrm{d}V_{\nu} \,, \tag{24}$$

$$q_{S_i} \rightarrow q_{S_i} - \Delta t \sum_{\text{species}} \int s_{\text{rad}\,\mathcal{F}_i} \, \mathrm{d}V_{\nu} \,, \tag{25}$$

$$q_{DY_e} \rightarrow q_{DY_e} + \Delta t m_{\text{u}} \int \frac{\mathrm{d}V_{\nu'}}{\nu'} \left[s^{\mu}_{\text{rad},\nu_{\text{e}}}\left(\nu'\right) - s^{\mu}_{\text{rad},\bar{\nu}_{\text{e}}}\left(\nu'\right) \right] u_{\mu}, \tag{26}$$

- still need to limit the timestep, especially because of Y_e
- could break the lepton conservation

source terms

However...

$$q_\tau \to q_\tau - \Delta t \sum_{\text{species}} \int s_{\text{rad}\,\mathcal{E}} \, \mathrm{d}V_\nu \,, \tag{24}$$

$$q_{S_i} \to q_{S_i} - \Delta t \sum_{\text{species}} \int s_{\text{rad}\,\mathcal{F}_i} \, \mathrm{d}V_\nu \,, \tag{25}$$

$$q_{DY_e} \to q_{DY_e} + \Delta t m_{\mathrm{u}} \int \frac{\mathrm{d}V_{\nu'}}{\nu'} \left[s^\mu_{\text{rad},\nu_\mathrm{e}} \left(\nu'\right) - s^\mu_{\text{rad},\bar{\nu}_\mathrm{e}} \left(\nu'\right) \right] u_\mu , \tag{26}$$

- still need to limit the timestep, especially because of Y_e
- could break the lepton conservation
- see Peter's and Eirik's talk, and [6]

However...

$$q_\tau \to q_\tau - \Delta t \sum_{\text{species}} \int s_{\text{rad}\,\mathcal{E}} \, \mathrm{d}V_\nu \,, \tag{24}$$

$$q_{S_i} \to q_{S_i} - \Delta t \sum_{\text{species}} \int s_{\text{rad}\,\mathcal{F}_i} \, \mathrm{d}V_\nu \,, \tag{25}$$

$$q_{DY_e} \to q_{DY_e} + \Delta t m_{\mathrm{u}} \int \frac{\mathrm{d}V_{\nu'}}{\nu'} \left[s^\mu_{\text{rad},\nu_e}(\nu') - s^\mu_{\text{rad},\bar{\nu}_e}(\nu') \right] u_\mu , \tag{26}$$

- still need to limit the timestep, especially because of Y_e
- could break the lepton conservation
- see Peter's and Eirik's talk, and [6]
- more efficient and robust solver are still under development

However...

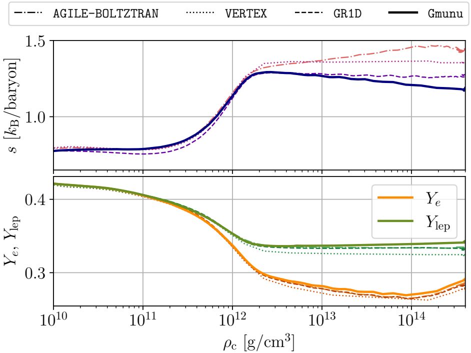
$$q_\tau \to q_\tau - \Delta t \sum_{\text{species}} \int s_{\text{rad}\,\mathcal{E}} \, \mathrm{d}V_\nu \,, \tag{24}$$

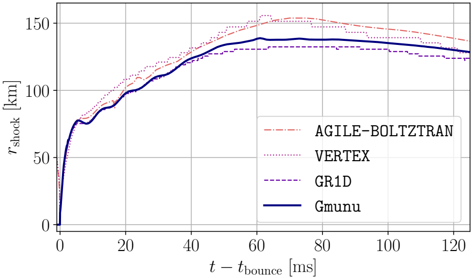
$$q_{S_i} \to q_{S_i} - \Delta t \sum_{\text{species}} \int s_{\text{rad}\,\mathcal{F}_i} \, \mathrm{d}V_\nu \,, \tag{25}$$

$$q_{DY_e} \to q_{DY_e} + \Delta t m_{\text{u}} \int \frac{\mathrm{d}V_{\nu'}}{\nu'} \left[s^\mu_{\text{rad},\nu_{\text{e}}}(\nu') - s^\mu_{\text{rad},\bar{\nu}_{\text{e}}}(\nu') \right] u_\mu, \tag{26}$$

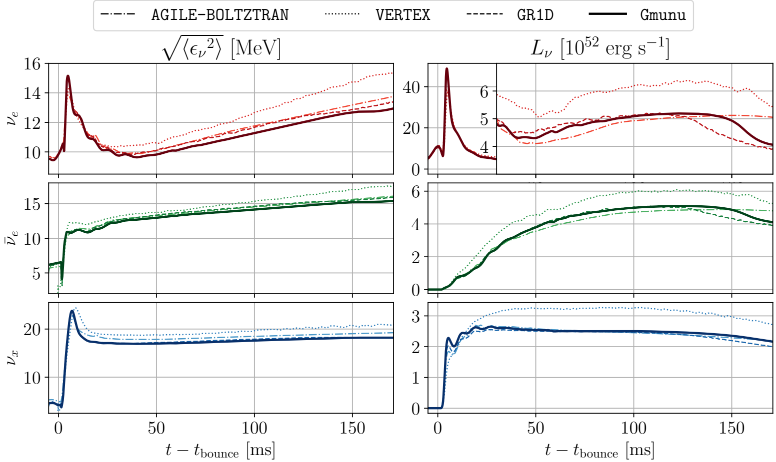
- still need to limit the timestep, especially because of Y_e
- could break the lepton conservation
- see Peter's and Eirik's talk, and [6]
- more efficient and robust solver are still under development
- load-balancing can also be an issue

15 $M_\odot$ CCSN

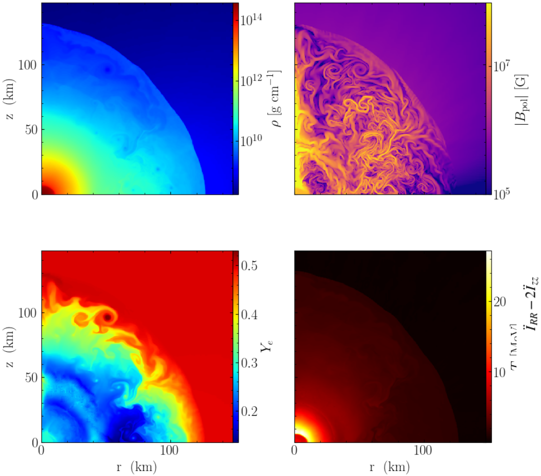


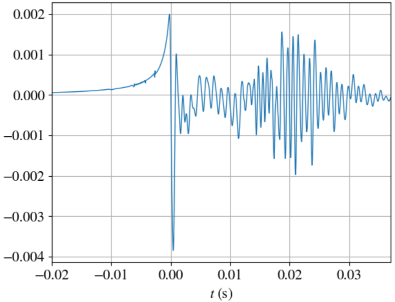


15 $M_\odot$ CCSN



Applications





Summary

- Presented the neutrino transport in `Gmunu`
  - General relativistic
  - Fully include energy-advection term
  - Fully include velocities dependents
  - discussed some ways to implicitly handle the radiation source

References I

[1] P. C.-K. Cheong, L.-M. Lin, and T. G. F. Li, "Gmunu: toward multigrid based Einstein field equations solver for general-relativistic hydrodynamics simulations," *Classical and Quantum Gravity*, vol. 37, p. 145015, July 2020.

[2] P. C.-K. Cheong, A. T.-L. Lam, H. H.-Y. Ng, and T. G. F. Li, "Gmunu: paralleled, grid-adaptive, general-relativistic magnetohydrodynamics in curvilinear geometries in dynamical space-times," *MNRAS*, vol. 508, pp. 2279–2301, Dec. 2021.

[3] P. C.-K. Cheong, D. Y. T. Pong, A. K. L. Yip, and T. G. F. Li, "An Extension of Gmunu: General-relativistic Resistive Magnetohydrodynamics Based on Staggered-meshed Constrained Transport with Elliptic Cleaning," *ApJS*, vol. 261, p. 22, Aug. 2022.

[4] P. C.-K. Cheong, H. H.-Y. Ng, A. T.-L. Lam, and T. G. F. Li, "General-relativistic Radiation Transport Scheme in Gmunu. I. Implementation of Two-moment-based Multifrequency Radiative Transfer and Code Tests," *ApJS*, vol. 267, p. 38, Aug. 2023.

[5] H. H.-Y. Ng, P. C.-K. Cheong, A. T.-L. Lam, and T. G. F. Li, "General-relativistic radiation transport scheme in gmunu ii: Implementation of novel microphysical library for neutrino radiation – weakhub," 2023.

References II

[6] M. P. Laiu, E. Endeve, R. Chu, J. A. Harris, and O. E. B. Messer, “A DG-IMEX Method for Two-moment Neutrino Transport: Nonlinear Solvers for Neutrino-Matter Coupling,” *ApJS*, vol. 253, p. 52, Apr. 2021.

[7] I. Cordero-Carrión, P. Cerdá-Durán, H. Dimmelmeier, J. L. Jaramillo, J. Novak, and E. Gourgoulhon, “Improved constrained scheme for the Einstein equations: An approach to the uniqueness issue,” *Phys. Rev. D*, vol. 79, p. 024017, Jan. 2009.

[8] N. Bucciantini and L. Del Zanna, “A fully covariant mean-field dynamo closure for numerical 3 + 1 resistive GRMHD,” *MNRAS*, vol. 428, pp. 71–85, Jan. 2013.

Thank you for your attention.

Q & A

Network for Neutrinos, Nuclear Astrophysics, and Symmetries

PHYSICS FRONTIER CENTER

University of New Hampshire

Einstein solver: xCFC scheme

By following [7]

$$\tilde{\Delta} X^i + \frac{1}{3}\tilde{\nabla}^i\left(\tilde{\nabla}_j X^j\right) = 8\pi \tilde{S}^i$$

$$\tilde{A}^{ij} \approx \tilde{\nabla}^i X^j + \tilde{\nabla}^j X^i - \frac{2}{3}\tilde{\nabla}_k X^k f^{ij}$$

$$\tilde{\Delta}\psi = -2\pi \tilde{E}\psi^{-1} - \frac{1}{8} f_{ik} f_{jl} \tilde{A}^{kl}\tilde{A}^{ij}\psi^{-7}$$

$$\tilde{\Delta}(\alpha\psi) = (\alpha\psi)\left[2\pi\left(\tilde{E} + 2\tilde{S}\right)\psi^{-2} + \frac{7}{8} f_{ik} f_{jl}\tilde{A}^{kl}\tilde{A}^{ij}\psi^{-8}\right]$$

$$\tilde{\Delta}\beta^i + \frac{1}{3}\tilde{\nabla}^i\left(\tilde{\nabla}_j \beta^j\right) = 16\pi\alpha\psi^{-6} f^{ij}\tilde{S}_i + 2\tilde{A}^{ij}\tilde{\nabla}_j\left(\alpha\psi^{-6}\right)$$

Robin B. C.

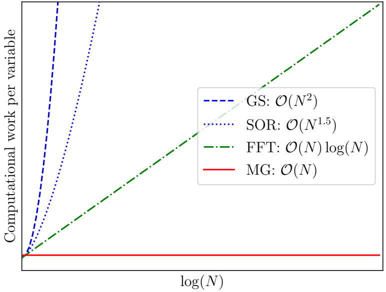
$$\left.\frac{\partial\psi}{\partial r}\right|_{r_{\max}} = \frac{1-\psi}{r},$$

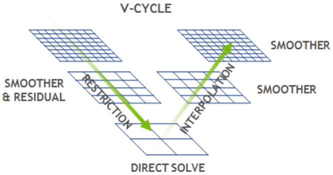
$$\left.\frac{\partial\alpha}{\partial r}\right|_{r_{\max}} = \frac{1-\alpha}{r},$$

$$\left.\beta^i\right|_{r_{\max}} = 0$$

Einstein solver: Elliptic solvers

Method	Complexity (accuracy
Gauss-Seidel iteration	$\mathcal{O}(N^2)\log(\epsilon)$
SOR	$\mathcal{O}(N^{3/2})\log(\epsilon)$
Conjugate Gradient	$\mathcal{O}(N^{3/2})\log(\epsilon)$
ADI	$\mathcal{O}(N\log(N))\log(\epsilon)$
FFT	$\mathcal{O}(N\log(N))$
Multigrid (FMG)	$\mathcal{O}(N)$





- fast and efficient
- suitable for non-linear problems

Conservative form of GRMHD equations

Conservative formulation + Reference metric

$$\partial_t \boldsymbol{q} + \frac{1}{\sqrt{\hat{\gamma}}}\partial_j \left[\sqrt{\hat{\gamma}}\boldsymbol{f}^j\right] = \boldsymbol{s}_{\text{grav}} + \boldsymbol{s}_{\text{geom}}$$

where $\hat{\gamma}_{ij}$ is a *time-independent* reference metric.

$$\boldsymbol{q} \propto \begin{bmatrix} D & \text{(Conserved Density)} \\ S_j & \text{(Conserved Momentum)} \\ \tau & \text{(Energy density)} \\ B^i & \text{(Magnetic field)} \\ E^i & \text{(Electric field)} \end{bmatrix}$$

- No need to tailor make for different coordinates

Conservative form of GRMHD equations

Conservative formulation + Reference metric

$$\partial_t \boldsymbol{q} + \frac{1}{\sqrt{\hat{\gamma}}}\partial_j \left[\sqrt{\hat{\gamma}}\boldsymbol{f}^j\right] = \boldsymbol{s}_{\text{grav}} + \boldsymbol{s}_{\text{geom}}$$

where $\hat{\gamma}_{ij}$ is a *time-independent* reference metric.

$$\boldsymbol{q} \propto \begin{bmatrix} D & \text{(Conserved Density)} \\ S_j & \text{(Conserved Momentum)} \\ \tau & \text{(Energy density)} \\ B^i & \text{(Magnetic field)} \\ E^i & \text{(Electric field)} \end{bmatrix}$$

- No need to tailor make for different coordinates
- well suited for finite-volume method for *curvilinear* coordinates, see [2]

Conservative form of GRMHD equations

Conservative formulation + Reference metric

$$\partial_t \boldsymbol{q} + \frac{1}{\sqrt{\hat{\gamma}}}\partial_j \left[\sqrt{\hat{\gamma}}\boldsymbol{f}^j\right] = \boldsymbol{s}_{\text{grav}} + \boldsymbol{s}_{\text{geom}}$$

where $\hat{\gamma}_{ij}$ is a *time-independent* reference metric.

$$\boldsymbol{q} \propto \begin{bmatrix} D & \text{(Conserved Density)} \\ S_j & \text{(Conserved Momentum)} \\ \tau & \text{(Energy density)} \\ B^i & \text{(Magnetic field)} \\ E^i & \text{(Electric field)} \end{bmatrix}$$

- No need to tailor make for different coordinates
- well suited for finite-volume method for *curvilinear* coordinates, see [2]
- Momentum conserved $\sim \mathcal{O}$(machine precision)

Conservative form of GRMHD equations

Conservative formulation + Reference metric

$$\partial_t \boldsymbol{q} + \frac{1}{\sqrt{\hat{\gamma}}}\partial_j \left[\sqrt{\hat{\gamma}}\boldsymbol{f}^j\right] = \boldsymbol{s}_{\text{grav}} + \boldsymbol{s}_{\text{geom}}$$

where $\hat{\gamma}_{ij}$ is a *time-independent* reference metric.

$$\boldsymbol{q} \propto \begin{bmatrix} D & \text{(Conserved Density)} \\ S_j & \text{(Conserved Momentum)} \\ \tau & \text{(Energy density)} \\ B^i & \text{(Magnetic field)} \\ E^i & \text{(Electric field)} \end{bmatrix}$$

- No need to tailor make for different coordinates
- well suited for finite-volume method for *curvilinear* coordinates, see [2]
- Momentum conserved $\sim \mathcal{O}$(machine precision)
- Solved with *high-resolution shock-capturing (HRSC)* method

Equations needed to be solved

$$\nabla_\mu \left(\rho u^\mu\right) = 0, \qquad \text{(cons. rest mass)}$$

$$\nabla_\mu T^{\mu\nu} = 0, \qquad \text{(cons. energy/momentum)}$$

$$p = p\left(\rho, \epsilon, Y_{\rm e} \cdots\right), \qquad \text{(equation of state)}$$

$$\nabla_\mu F^{\mu\nu} = \mathcal{J}^\nu, \quad \nabla_\mu {}^*F^{\mu\nu} = 0, \qquad \text{(Maxwell equations)}$$

$$T^{\rm total}_{\mu\nu} = T^{\rm fluid}_{\mu\nu} + T^{\rm EM}_{\mu\nu}$$

GRMHD

$F^{\mu\nu}$ is the Maxwell tensor and ${}^*F^{\mu\nu}$ the Faraday tensor.

$$T^{\mu\nu}_{\text{EM}} = F^{\mu\alpha}F^{\mu}_{\ \alpha} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

Define

$$E^\mu := F^{\mu\nu}n_\nu, \qquad B^\mu := {}^*F^{\mu\nu}n_\nu.$$

and

$$e^\mu := F^{\mu\nu}u_\nu \qquad b^\mu := {}^*F^{\mu\nu}u_\nu$$

GRMHD

The $3+1$ components of the total energy-momentum tensor $T_{\mu\nu}$ can be expressed as

$$U = \rho h W^2 - p + \frac{1}{2}\left(E^2 + B^2\right),$$
$$S_i = \rho h W^2 v_i + \epsilon_{ijk} E^j B^k,$$
$$S^{ij} = \rho h W^2 v^i v^j - E^i E^j - B^i B^j + \left[p + \frac{1}{2}\left(E^2 + B^2\right)\right]\gamma^{ij}.$$

just the GRHD, but with EM additional terms

GRMHD

$$\nabla_\mu F^{\mu\nu} = -\mathcal{J}^\nu, \qquad \nabla_\mu {}^*F^{\mu\nu} = 0,$$

which give

$$\frac{1}{\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma}E^i\right) = \rho_e$$

$$\partial_t\left(\sqrt{\gamma}E^j\right) + \partial_i\left(-\sqrt{\gamma}\epsilon^{jik}\hat{B}_k\right) = -\sqrt{\gamma}\left(\alpha J^j - \rho_e\beta^j\right),$$

$$\frac{1}{\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma}B^i\right) = 0$$

$$\partial_t\left(\sqrt{\gamma}B^j\right) + \partial_i\left(\sqrt{\gamma}\epsilon^{jik}\hat{E}_k\right) = 0$$

where we have defined

$$\hat{E}_i := \alpha E_i + \epsilon_{ijk}\beta^j B^k, \qquad \hat{B}_i := \alpha B_i - \epsilon_{ijk}\beta^j E^k$$

Resistive MHD and Ohm's law

Closer look at Ampére's law

$$\partial_t \left(\sqrt{\gamma} E^j\right) + \partial_i \left(-\sqrt{\gamma}\epsilon^{jik}\hat{B}_k\right) = -\sqrt{\gamma}\left(\alpha J^j - \rho_e \beta^j\right)$$

Ohm's law is needed to describe the coupling between EM fields and the fluid

- *ideal plasma*: $\boxed{e^\mu = 0}$ (only b^μ needed to be solved in this case)

Resistive MHD and Ohm's law

Closer look at Ampére's law

$$\partial_t \left(\sqrt{\gamma} E^j\right) + \partial_i \left(-\sqrt{\gamma}\epsilon^{jik}\hat{B}_k\right) = -\sqrt{\gamma}\left(\alpha J^j - \rho_e \beta^j\right)$$

Ohm's law is needed to describe the coupling between EM fields and the fluid

- *ideal plasma*: $\boxed{e^\mu = 0}$ (only b^μ needed to be solved in this case)
- *resistive plasma*: $\boxed{e^\mu = \eta j^\mu}$, where $\sigma_c := 1/\eta$.

Resistive MHD and Ohm's law

Closer look at Ampére's law

$$\partial_t \left(\sqrt{\gamma} E^j \right) + \partial_i \left(-\sqrt{\gamma} \epsilon^{jik} \hat{B}_k \right) = -\sqrt{\gamma} \left(\alpha J^j - \rho_e \beta^j \right)$$

Ohm's law is needed to describe the coupling between EM fields and the fluid

- *ideal plasma*: $\boxed{e^\mu = 0}$ (only b^μ needed to be solved in this case)
- *resistive plasma*: $\boxed{e^\mu = \eta j^\mu}$, where $\sigma_c := 1/\eta$.
- *resistive plasma + dynamo*: $\boxed{e^\mu = \eta j^\mu + \xi b^\mu}$ (see [8])

Resistive MHD and Ohm's law

Closer look at Ampére's law

$$\partial_t\left(\sqrt{\gamma}E^j\right) + \partial_i\left(-\sqrt{\gamma}\epsilon^{jik}\hat{B}_k\right) = -\sqrt{\gamma}\left(\alpha J^j - \rho_e\beta^j\right)$$

Ohm's law is needed to describe the coupling between EM fields and the fluid

- *ideal plasma*: $\boxed{e^\mu = 0}$ (only b^μ needed to be solved in this case)
- *resistive plasma*: $\boxed{e^\mu = \eta j^\mu}$, where $\sigma_c := 1/\eta$.
- *resistive plasma + dynamo*: $\boxed{e^\mu = \eta j^\mu + \xi b^\mu}$ (see [8])
- other techniques are required, such as Implicit-Explicit (IMEX) time integrator approaches, see [3].

IMEX scheme

Implicit-explicit (IMEX) Runge-Kutta schemes for

$$\partial_t \boldsymbol{q} = \mathcal{L}(\boldsymbol{q}) + \frac{1}{\epsilon}\mathcal{R}(\boldsymbol{q}), \tag{27}$$

where $\mathcal{L}(\boldsymbol{q})$ is non-stiff,
while $\mathcal{R}(\boldsymbol{q})/\epsilon$ is the *stiff* term with a *relaxation parameter* ϵ.

$$\partial_t \boldsymbol{q} = \mathcal{L}_{\text{non-stiff}}(\boldsymbol{q}) + \mathcal{L}_{\text{stiff}}(\boldsymbol{q}) \tag{28}$$

- explicit
- implicit

IMEX scheme

Split

$$J^i = \rho_e v^i + \frac{W}{\eta}\left\{ \left[E^i + \epsilon^{ijk} v_j B_k - \left(E^j v_j\right) v^i\right] - \xi\left[B^i - \epsilon^{ijk} v_j E_k - \left(B^j v_j\right) v^i\right] \right\} \tag{29}$$

into

$$J^i_{\text{non-stiff}} := \rho_e v^i \tag{30}$$

$$J^i_{\text{stiff}} := \frac{W}{\eta}\left\{ \left[E^i + \epsilon^{ijk} v_j B_k - \left(E^j v_j\right) v^i\right] - \xi\left[B^i - \epsilon^{ijk} v_j E_k - \left(B^j v_j\right) v^i\right] \right\}. \tag{31}$$

IMEX scheme

$$\frac{\left(\sqrt{\gamma}\left(E^j\right)^{\mathtt{n+1}}\right) - \left(\sqrt{\gamma}\left(E^j\right)^{\mathtt{n}}\right)}{\Delta t} = -\sqrt{\gamma}\left[\alpha J^j\left(\left(E^j\right)^{\mathtt{n+1}}\right)\right]$$

Need to solve iteratively

IMEX scheme

Rewrite as

$$\boldsymbol{q}^{\mathtt{n+1}} = \boldsymbol{q}^{\mathtt{n},(k)} + \Delta t \boldsymbol{s}_{\text{stiff}}\left(\boldsymbol{q}^{\mathtt{n+1}}\right)$$

Define

$$\boldsymbol{f}\left(\boldsymbol{q}\right) \equiv -\boldsymbol{q} + \boldsymbol{q}^{\mathtt{n},(k)} + \Delta t \boldsymbol{s}_{\text{stiff}}\left(\boldsymbol{q}\right)$$

$$J_{ij} \equiv \frac{\partial f_i}{\partial q_j} = -\delta_i^j + \Delta t \frac{\partial \left[s_{\text{stiff}}\right]_i}{\partial q_j}$$

$$\Rightarrow q_j^{(m+1)} = q_j^{(m)} - \left[J_{ij}^{(m)}\right]^{-1} f_i^{(m)}$$

- Newton-Raphson method
- Broyden method
- analytic/finite difference Jacobian
- good initial guess helps a lot!

Divergenceless problem

However...

$$\frac{1}{\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma}B^i\right) = 0$$

$$\partial_t\left(\sqrt{\gamma}B^j\right) + \partial_i\left(\sqrt{\gamma}\epsilon^{jik}\hat{E}_k\right) = 0$$

"magnetic monopoles" are introduced if no treatment is used!

- hyperbolic cleaning (widely used)
- constrained transport
- elliptic cleaning (unlike to see)
- vector potential (not available)

Hyperbolic cleaning

Also known as generalised Lagrange multiplier (GLM)

$$\nabla_\nu \left(F^{\mu\nu} - \Psi g^{\mu\nu} \right) = \mathcal{J}^\mu - \kappa_E n^\mu \Psi, \tag{32}$$

$$\nabla_\nu \left({}^*F^{\mu\nu} - \Phi g^{\mu\nu} \right) = -\kappa_B n^\mu \Phi, \tag{33}$$

where κ_E and κ_B are parameters.
This reduces to usual Maxwell equations if $\Psi = 0 = \Phi$. The scalar fields Ψ and Φ are being damped exponentially if $\kappa > 0$.

Hyperbolic cleaning

$$\frac{1}{\alpha\sqrt{\gamma}}\left\{\partial_t\left(\sqrt{\gamma}\Psi\right)-\partial_i\left[\sqrt{\gamma}\left(\alpha E^i-\Psi\beta^i\right)\right]\right\} = -\kappa_E\Psi+E^i\partial_i\alpha/\alpha+\Psi\gamma^{ij}K_{ij}+\rho_e, \quad (34)$$

$$\frac{1}{\alpha\sqrt{\gamma}}\left\{\partial_t\left(\sqrt{\gamma}\Phi\right)-\partial_i\left[\sqrt{\gamma}\left(\alpha B^i-\Phi\beta^i\right)\right]\right\} = -\kappa_B\Phi+B^i\partial_i\alpha/\alpha+\Phi\gamma^{ij}K_{ij}. \quad (35)$$

and introduced additional source terms for EM fields:

$$s_{E^i} := \psi^6\sqrt{\bar{\gamma}/\hat{\gamma}}\left(\rho_e\beta^i-\alpha J^i-\alpha\gamma^{ij}\partial_j\Psi\right), \quad (36)$$

$$s_{B^i} := \psi^6\sqrt{\bar{\gamma}/\hat{\gamma}}\left(-\alpha\gamma^{ij}\partial_j\Phi\right). \quad (37)$$

Note that the modified Faraday equations are not hyperbolic since the existence of $\partial_j\Psi$ and $\partial_j\Phi$.

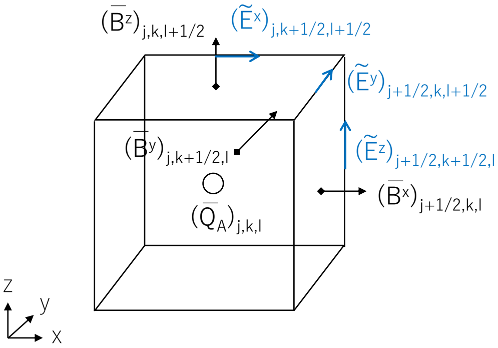
Constrained transport

By integrating the induction equation for each surface of the cell, together with Stokes' theorem:

$$\frac{\partial}{\partial t}\int_{\partial V\left(x^1_{\mathtt{i}\pm 1/2,\mathtt{j},\mathtt{k}}\right)} \sqrt{\gamma} B^1_{\mathtt{i}\pm 1/2,\mathtt{j},\mathtt{k}} dx^2 dx^3 = \oint_{\partial A\left(x^1_{\mathtt{i}\pm 1/2,\mathtt{j},\mathtt{k}}\right)} \hat{E}_k dx^k,$$

which can be written as

$$\frac{d}{dt}\Phi_{\mathtt{i}\pm 1/2,\mathtt{j},\mathtt{k}} = \mathcal{E}_{\mathtt{i}\pm 1/2,\mathtt{j},\mathtt{k}}. \quad (38)$$



Constrained transport

Numerically obtain RHS, update LHS, so that B fields is updated.
How do we obtain the E fields at the cell edges?

$$\frac{d}{dt}\Phi_{\mathtt{i\pm1/2,j,k}} = \mathcal{E}_{\mathtt{i\pm1/2,j,k}}. \quad (39)$$

Arithmetic averaging

- NOT upwind (numerically unstable)

UCT-contact

- upwind
- cost is low
- automatically bounded to the Riemann solver used
- seems stable

UCT-HLL

- upwind
- costy (need to apply limiter many times)
- suffer from overshoot problem
- bounded to HLL by construction (can be fixed, seems non-trivial)

Elliptic cleaning

Solve the following Poisson's equation, and update the B fields.

$$\hat{\nabla}^2 \Phi = \hat{\nabla}_i q_{B^i}^{\text{old}}, \tag{40}$$

$$q_{B^i}^{\text{new}} = q_{B^i}^{\text{old}} - \left(\hat{\nabla}\Phi\right)^i . \tag{41}$$

Note that this Φ is not the one in GLM method.

- requires elliptic solver (e.g. multigrid solver)
- expansive
- boundary sensitive
- might be acausal
- useful in initialisation

Summary of divergenceless handling

	GLM	fluxCT	CT	elliptic
pros	1. easy to implement 2. can start with not-so-good ID	1. not so difficult to implement 2. divB can be suppressed low	1. divB $\lesssim \epsilon_{\rm DP}$ 2. favour block-based AMR 3. can be upwind	1. can be used for initialisation 2. can be mixed with CT
cons	1. divB still quite large 2. introduced free parameters 3. not hyperbolic	1. not favour AMR 2. not upwind	1. implementation is complex	1. slow 2. maybe acasual 3. divB still quite large

ideal GRMHD

- consider the conductivity of the fluid $\sigma_c \to \infty$ (or the resistivity $\sigma_c^{-1} \to 0$).
- the electric field in fluid-frame are required to be vanished, $e^\mu := F^{\mu\nu} u_\nu = 0$.
- In the observer frame, this relation reads $\hat{E}_i = -\epsilon^{ijk} \hat{v}_j B_k$.
- Electric fields are determinted, no need to evolve.
- the induction equation becomes

$$\partial_t \left(\sqrt{\gamma} B^j \right) + \partial_i \left[\sqrt{\gamma} \left(\hat{v}^i B^j - \hat{v}^j B^i \right) \right] = 0 \quad (42)$$

ideal GRMHD

$$\partial_t \left(\sqrt{\gamma}\boldsymbol{Q}\right) + \partial_i \left(\sqrt{\gamma}\boldsymbol{F}^i\right) = \sqrt{\gamma}\boldsymbol{S}, \tag{43}$$

$$\boldsymbol{Q} = \begin{bmatrix} D \\ S_j \\ \tau \\ B^i \end{bmatrix} = \begin{bmatrix} \rho W \\ \rho h^* W^2 v_i - \alpha b^0 b_i \\ \rho h^* W^2 - p^* - \left(\alpha b^0\right)^2 - D \\ B^i \end{bmatrix}, \tag{44}$$

$$\boldsymbol{F}^i = \begin{bmatrix} D\left(\alpha v^i - \beta^i\right) \\ S_j \hat{v}^i + \delta^i_j \alpha p^* - \alpha b_j B^i / W \\ \tau \hat{v}^i + \alpha p^* v^i - \alpha^2 b^0 B^i / W \\ \hat{v}^i B^j - \hat{v}^j B^i \end{bmatrix}, \qquad \boldsymbol{S} = \begin{bmatrix} 0 \\ \frac{1}{2}\alpha S^{ik}\partial_j \gamma_{ik} + S_i \partial_j \beta^i - U \partial_j \alpha \\ \alpha S^{ik} K_{ik} - S^i \partial_i \alpha \\ 0 \end{bmatrix}. \tag{45}$$

Two moment scheme

and close the problem with a closure,

$$\mathcal{P}^{\mu\nu} = \mathcal{P}^{\mu\nu}\left(\mathcal{E}, \mathcal{F}^{\mu}\right). \tag{46}$$

- Many ways to obtain this closure relation

Two moment scheme

Patrick Chi-Kit Cheong

and close the problem with a closure,

$$\mathcal{P}^{\mu\nu} = \mathcal{P}^{\mu\nu}\left(\mathcal{E}, \mathcal{F}^{\mu}\right). \tag{46}$$

- Many ways to obtain this closure relation
- different closure could affect the accuracy

N3AS postdoc fellow
University of New Hampshire

Two moment scheme

and close the problem with a closure,

$$\mathcal{P}^{\mu\nu} = \mathcal{P}^{\mu\nu}\left(\mathcal{E}, \mathcal{F}^{\mu}\right). \tag{46}$$

- Many ways to obtain this closure relation
- different closure could affect the accuracy
- still under debate

Two moment scheme

We adopt an *approximated* analytic closure which combines the optically thin and optically thick limits

$$\mathcal{P}^{\mu\nu} = d_{\text{thin}} \mathcal{P}^{\mu\nu}_{\text{thin}} + d_{\text{thick}} \mathcal{P}^{\mu\nu}_{\text{thick}}, \tag{47}$$

where

$$d_{\text{thin}} \equiv \frac{1}{2}(3\chi - 1)\,; \quad d_{\text{thick}} \equiv \frac{3}{2}(1 - \chi)\,, \tag{48}$$

where $\chi \in \left[\frac{1}{3}, 1\right]$ is the Eddington factor.

Closure relation

Minerbo closure (also known as maximum-entropy closure)

$$\chi(\zeta) = \frac{1}{3} + \zeta^2 \frac{2}{15}\left(3 - \zeta + 3\zeta^2\right), \quad (49)$$

where the flux factor ζ is defined as $\zeta \equiv \sqrt{\mathcal{H}^\mu \mathcal{H}_\mu / \mathcal{J}^2}$.

- Optically thin: $\zeta \approx 1$ and thus $\chi \approx 1$.
- Optically thick: $\zeta \approx 0$ and thus $\chi \approx 1/3$.
- We find $\zeta(\mathcal{E}, \mathcal{F}_\mu)$ by solving $f(\zeta) = 0 = \left(\zeta^2 \mathcal{J}^2 - \mathcal{H}^\mu \mathcal{H}_\mu\right)/\mathcal{E}^2$.
- M1 scheme? M1 closure?