The $N - \Delta$ transition GPDs

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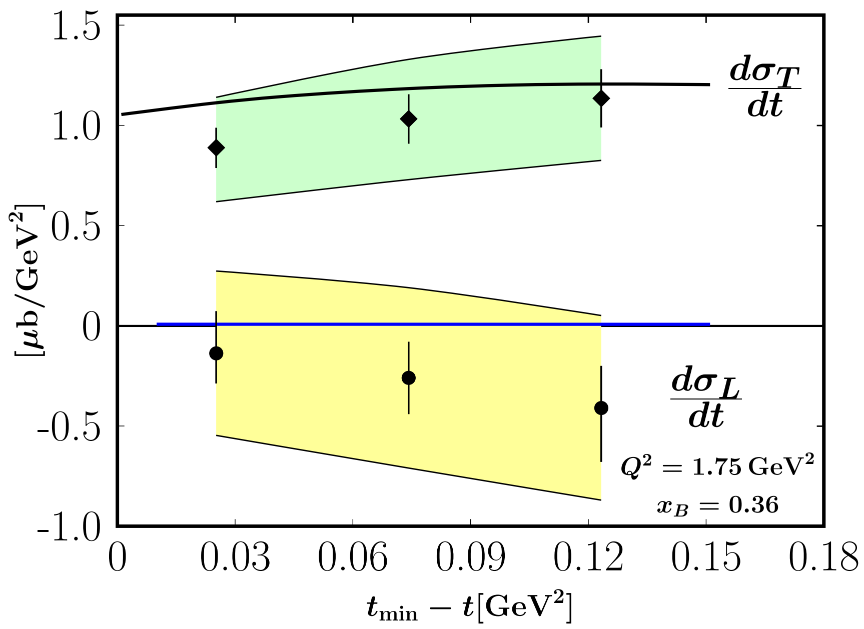
Outline:

- **Lessons from $\gamma^* N \to \pi N$**
- **The chiral-even $N \to \Delta$ GPDs**
- **The transversity $N \to \Delta$ GPDs**
- **The pion pole**
- **GPDs in the large N_C limit**
- **Subprocess amplitudes**
- **Estimates**
- **Summary**

based on K.-Passek-Kumerički arXiv:2211.09474

Lessons from $\gamma^* N \to \pi N$



Hall A collaboration π^0 production

Defurne et al (1608.01003)

(predictions from Goloskokov-K. (11))

$d\sigma_T \gg d\sigma_L$ $(d\sigma_U \simeq d\sigma_T)$

like expectation for $Q^2 \to 0$

to be contrasted with

QCD expectation for $Q^2 \to \infty$: $d\sigma_T \ll d\sigma_L$ ($d\sigma_U \simeq d\sigma_L$)

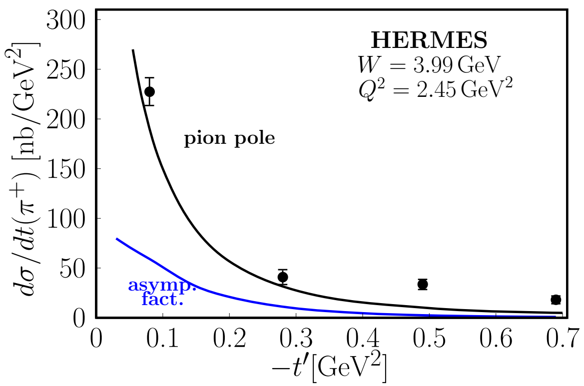
leading twist does not dominate (much larger Q^2 required for it)

Further evidence for large contribution from transverse photons:

$A_{UT}^{\sin\phi_S}(\pi^+)$ HERMES(09); $d\sigma_{TT}/dt(\pi^0)$ CLAS(12) $|d\sigma_{TT}| \leq d\sigma_T$

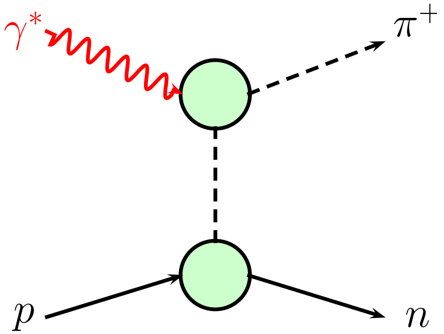
The pion pole

For π^+ production - pion pole: Mankiewicz et al (98), Penttinen et al (99)



$$\widetilde{E}^u_{\rm pole} = -\widetilde{E}^d_{\rm pole} = \Theta(|x| \leq \xi)\,\frac{m f_\pi g_{\pi NN}}{\sqrt{2}\xi}\,\frac{F_{\pi NN}(t)}{m_\pi^2 - t}\,\Phi_\pi\Big(\frac{x+\xi}{2\xi}\Big)$$

$$\Longrightarrow \frac{d\sigma_L^{\rm pole}}{dt} \sim \frac{-t}{Q^2}\Big[\sqrt{2}e_0 g_{\pi NN}\frac{F_{\pi NN}(t)}{m_\pi^2 - t}Q^2 F_\pi^{\rm pert}(Q^2)\Big]^2$$



understimates c.s.(blue l.) $F_\pi^{\rm pert.} \simeq 0.3-0.5 F_\pi^{\rm exp.}$

(F_π measured in π^+ electroproduction at Jlab)

Goloskokov-K(09): $F_\pi^{\rm pert} \to F_\pi^{\rm exp}$

Expectation:

the same problems will appear in $\gamma^* N \to \pi\Delta$

therefore we will need the transversity transition GPDs

(which in combination with a twist-3 pion DA models amplitudes for γ_T^*)

and the pion-exchange contribution

The helicity-nonflip (chiral-even) GPDs

Belitsky-Radyushkin (05) four even-parity transition GPDs ($A^+ = 0$)

(forerunners: Frankfurt et al (99), Goeke et al (01), Guichon et al (03) incomplete sets)

$$\frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle \Delta^{++}(p',\nu')|\bar{u}(-z/2)\gamma^+ d(z/2)|p(p,\nu)\rangle \Big|_{z^+=0,z_\perp=0} =$$

$$\frac{1}{2P^+}\,\bar{u}_\delta(p',\nu')\left\{\frac{\Delta^\delta n^\mu - \Delta^\mu n^\delta}{m}\left(\gamma_\mu G_1 + \frac{P_\mu}{m}G_2 + \frac{\Delta_\mu}{m}G_3\right) + \frac{\Delta^+\Delta^\delta}{m^2}G_4\right\}\gamma_5 u(p,\nu)$$

and four odd-parity ones

$$\frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle \Delta^{++}(p',\nu')|\bar{u}(-z/2)\gamma^+\gamma^5 d(z/2)|p(p,\nu)\rangle \Big|_{z^+=0,z_\perp=0} =$$

$$\frac{1}{2P^+}\,\bar{u}_\delta(p',\nu')\left\{\frac{\Delta^\delta n^\mu - \Delta^\mu n^\delta}{m}\left(\gamma_\mu \tilde{G}_1 + \frac{P_\mu}{m}\tilde{G}_2\right) + n^\delta \tilde{G}_3 + \frac{\Delta^+\Delta^\delta}{m^2}\tilde{G}_4\right\} u(p,\nu)$$

GPD: $G_i = G_i(x,\xi,t)$, real valued because of time-reversal invariance

$\Delta = p' - p\,,\qquad P = (p+p')/2\,,\qquad \xi = \frac{p^+ - p'^+}{p^+ + p'^+}\,,\qquad n = [0,1,0_\perp]$

Rarita-Schwinger spinor: $p'_\mu u^\mu(p',\nu') = \gamma_\mu u^\mu(p',\nu') = 0$, and satisfies Dirac eq.

$t = t_0 - \frac{\Delta_\perp^2}{1-\xi^2}\,,\qquad t_0 = -\frac{2\xi}{1-\xi^2}\left[(1+\xi)(M^2 - m^2) + 2\xi m^2\right]$

$\Delta(1232)$ assumed to be a stable particle

The $N - \Delta$ transition form factors

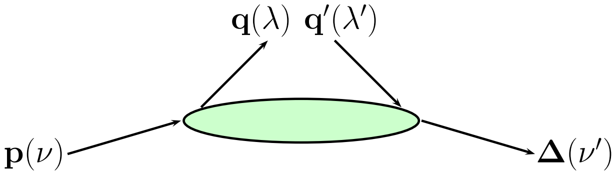
Belitsky-Radyushkin (05)

$$
\begin{aligned}
6mM(M+m)G_M &= ((M+m)(3M+m)-t)mG_1(t) + 2(M^2-m^2)MG_2(t) + 2tMG_3(t) \\
6mM(M+m)G_E &= (M^2-m^2+t)mG_1(t) + 2(M^2-m^2)MG_2(t) + 2tMG_3(t) \\
6mM(M+m)G_C &= 4M^2mG_1(t) + 2(3M^2+m^2-t)MG_2(t) + 2(M^2-m^2+t)MG_3(t)
\end{aligned}
$$

G_M, G_E, G_C multipole form factors   Devenish-Eisenschitz-Koerner (76)

$G_i(t) = \int_{-1}^{1} dx G_i(x,\xi,t)$ lowest moments of the GPDs

Long. amplitudes for π production



matrix element for emission and absorption of partons with definite light-cone helicity Diehl(01)

$$O_{\pm\pm} = \frac{1}{4}\bar{u}(-z/2)\gamma^{+}(1\pm\gamma_5)d(z/2)$$

$$A_{\nu'\lambda',\nu\lambda} = \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle \Delta^{++}(p'\nu'|O_{\lambda\lambda}|p(p\nu)\rangle|_{z^+=0,z_\perp=0}$$

leading-twist amplitude for $\gamma^* p \to \pi^- \Delta^{++}$ Frankfurt et al (99)

$$\begin{aligned} M^{tw2}_{0\nu'0\nu} &= e_0 \int_{-1}^{1} dx \sum_{\lambda} \mathcal{H}^{\pi^-}_{0\lambda 0\lambda} A_{\nu'\lambda\nu\lambda} \\ &= e_0 \int_{-1}^{1} dx \mathcal{H}^{\pi^-}_{0+0+}\left[A_{\nu'+\nu+} - A_{\nu'-\nu-}\right] \end{aligned}$$

with the help of parity symmetry ($\mathcal{H}$ ampl. for $\gamma^*_L q \to \pi^- q$)

$$\mathcal{H}^{\pi}_{0-\lambda'-\mu-\lambda} = -(-1)^{\mu-\lambda+\lambda'}\mathcal{H}^{\pi}_{0\lambda'\mu\lambda}$$

For other Δ-states (isospin symmetry):

$$G^{ud}_{p\Delta^{++}} = -\frac{\sqrt{3}}{2}(G^{uu}_{p\Delta^+} - G^{dd}_{p\Delta^+}) = -\sqrt{3}G^{du}_{p\Delta^0}$$

Transversity (chiral-odd) $N - \Delta$ transition GPDs

$$\frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle \Delta^{++}(p',\nu')|\bar{u}(-z/2)i\sigma^{+j}d(z/2)|p(p,\nu)\rangle \Big|_{z^+=0,z_\perp=0}$$

$$= \bar{u}_\delta(p',\nu')\Gamma^{\delta+j}\gamma_5 u(p,\nu)$$

$\sigma^{+j} = i/2[\gamma^+,\gamma^j]$

γ_5 required to match spin and parity of the $\Delta(1232)$

Γ^{δ}_{+j} tensor being antisymmetric in Lorentz label $+$ and the transverse one j

at disposal for construction: $P, \Delta, n, \gamma^\alpha, \sigma^{\alpha\beta}$ and Rarita-Schwinger spinor

$$\frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \, \langle \Delta^{++}(p',\nu')|\bar{u}(-z/2) i\sigma^{+j} d(z/2)|p(p,\nu)\rangle \Big|_{z^+=0, z_\perp=0} =$$

$$\frac{1}{2P^+}\,\bar{u}_\delta(p',\nu')\left[G_{T1}\frac{p^\delta}{m} i\sigma^{+j} + G_{T2}\,p^\delta\,\frac{P^+\Delta^j - \Delta^+ P^j}{m^3} + G_{T3}\,p^\delta\,\frac{\gamma^+\Delta^j - \Delta^+\gamma^j}{2m^2}\right.$$

$$\left. + G_{T4}\,p^\delta\,\frac{\gamma^+ P^j - P^+\gamma^j}{m^2} + G_{T5}\,(n^\delta\gamma^j - \gamma^\delta n^j) + G_{T6}\,\frac{n^\delta\Delta^j - \Delta^\delta n^j}{m}\right]\gamma_5\,u(p,\nu)$$

$$\frac{1}{2P^+}\left[\quad G_{T7}\,(\bar{u}^+(p',\nu')\gamma^j - \bar{u}^j(p',\nu')\gamma^+) + G_{T8}\,\frac{\bar{u}^+(p',\nu')\Delta^j - \bar{u}^j(p',\nu')\Delta^+}{m}\right]\gamma_5\,u(p,\nu)$$

any other antisymmetric tensor can be expressed as a linear combination of the above eight tensors with the help of the Dirac eq. and the generalized Gordon identities:

$$\begin{aligned}
\bar{u}_\delta(p') i\sigma^{\alpha\beta}(p'\mp p)_\beta u(p) &= (M\pm m)\bar{u}_\delta(p')\gamma^\alpha u(p) - \bar{u}_\delta(p')(p'\pm p)^\alpha u(p) \\
\bar{u}_\delta(p') i\sigma^{\alpha\beta}(p'\pm p)_\beta \gamma_5 u(p) &= (M\pm m)\bar{u}_\delta(p')\gamma^\alpha\gamma_5 u(p) - \bar{u}_\delta(p')(p'\mp p)^\alpha \gamma_5 u(p)
\end{aligned}$$

G_{Ti} real valued fcts. of x, ξ, t

Transversal amplitudes for π production

matrix elements for emission and reabsorption of partons with definite helicity Diehl(01)

$$A_{\nu'\mp\nu\pm} = \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle \Delta^{++}(p',\nu')|\mp\frac{i}{4}\bar{u}(-z/2)\Big(\sigma^{+1}\mp i\sigma^{+2}\Big)d(z/2)|p(p,\nu)\rangle|_{z^+=0,z_\perp=0}$$

describes, in region $\xi < x < 1$, the emission of an on-shell quark with helicity λ and reabsorption of a quark with helicity $-\lambda$

A s are suitable for calculation of amplitudes for any meson-Δ(1232) final state

for $\gamma_T^* p \to \pi^-\Delta^{++}$ - (twist 3) only G_{Ti} contribute

$$M^{tw3}_{0\nu'\mu\nu} = e_0 \int_{-1}^{1} dx \Big[\mathcal{H}^{\pi^-}_{0-\mu+} A_{\nu'-\nu+} + \mathcal{H}^{\pi^-}_{0+\mu-} A_{\nu'+\nu-}\Big]$$

Properties of A:

$$A_{-\nu'-\lambda'-\nu-\lambda} = (-1)^{(\nu'-\lambda'-\nu+\lambda)} A_{\nu'\lambda'\nu\lambda}$$

and for $t' \to 0$: $\quad A_{\nu'\lambda'\nu\lambda} \sim \sqrt{-t'}^{|\nu'-\lambda'-\nu+\lambda|} (*)$

as for c.m.s. helicity amplitudes Wong (66)

reflects conservation of spin-3 component in the collinear situation at $t' = 0$

for $t \to 0$ subprocess amplitudes behave as

$$\mathcal{H}_{0\lambda'\mu\lambda} \sim \sqrt{-t}^{|-\lambda'-\mu+\lambda|}$$

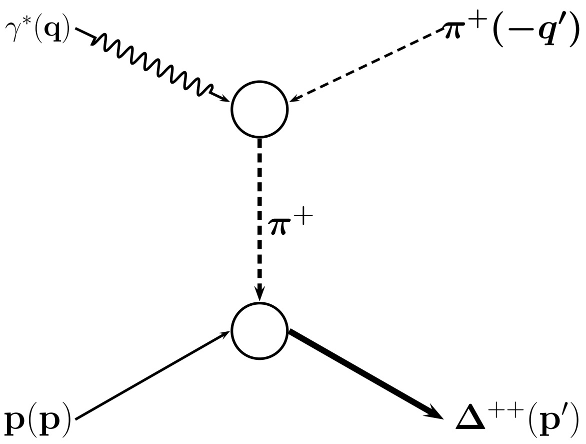
In generalized Bjorken limit where $-t \ll Q^2$, dominant contribution comes from helicity non-flip subprocess amplitude

$$\Longrightarrow \qquad \mu = \lambda - \lambda'$$

Hence, (*) provides correct behavior of M

$$M_{0\nu'\mu\nu} \sim \sqrt{-t'}^{|\nu'+\mu-\nu|}$$

The pion pole



$$M^{pole}_{0\nu'\mu\nu} = e_0 \frac{\rho_{\pi\Delta}}{t - m_\pi^2}\, \bar{u}_\delta(p',\nu') \frac{\Delta^\delta}{m} u(p,\nu)(q-2q')_\rho \epsilon^\rho(\mu)$$

$$\rho_{\pi\Delta} = \sqrt{2} g_{\pi\Delta^{++}p} F_{\pi\Delta p}(t) F_\pi(Q^2)$$

$g_{\pi\Delta^{++}p} = 14.8$ Ellis-Tang (98)

$F_{\pi\Delta p} = \frac{\Lambda_N^2 - m_\pi^2}{\Lambda_N^2 - t}$ $(\Lambda_N = 0.44\,\mathrm{GeV})$

asymptotically (leading twist): pion pole contributes to $\tilde{G}_4$

$$\tilde{G}_{4pole} = \Theta(|x| \le \xi)\Phi_\pi\left(\frac{x+\xi}{2\xi}\right) \frac{m f_\pi}{\sqrt{2}\xi} \frac{g_{\pi\Delta^{++}p} F_{\pi\Delta p}}{t - m_\pi^2}$$

but $F_\pi(Q^2) \to F_\pi^{pert}(Q^2)$

GPDs in the large-N_C limit

$N - \Delta$ GPDs are unknown as yet - experimental data are lacking

for numerical estimates of $N - \Delta$ - take recourse to large-N_C results

provides relations between matrix elements of quark field operators

$$\sqrt{2}\langle p_{\uparrow(\downarrow)}|O^{uu} - O^{dd}|p_\uparrow\rangle = \langle \Delta^+_{\uparrow(\downarrow)}|O^{uu} - O^{dd}|p_\uparrow\rangle$$

leads to relations between p-p GPDs and $p - \Delta^+$ ones Frankfurt et al (00)

and isospin symmetry $p - \Delta^+ \to p - \Delta^{++}$ Belitsky-Radyushkin (05)

$$\tilde{G}_3 = \frac{3}{2}\Big(\tilde{H}^u - \tilde{H}^d\Big) \qquad \tilde{G}_4 = \frac{3}{8}\Big(\tilde{E}^u - \tilde{E}^d\Big)$$

and more complicated relations for transversity GPDs simplest

$$G_{T5} + \frac{1}{2}G_{T7} = -\frac{3}{2}\Big(H_T^u - H_T^d\Big)$$

assumption for numerical estimates:

all GPDs are zero except $\tilde{G}_3$, $\tilde{G}_4$ and either G_{T5} or G_{T7}

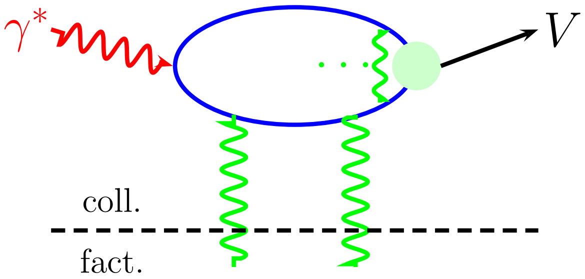
Subprocess amplitudes

same as for $\gamma^* N \to \pi N$ Goloskokov-K (10,11)

mod. pert. approach - quark trans. momenta in subprocess

(emission and absorption of partons from proton collinear to proton momenta)

transverse separation of color sources $\Longrightarrow$ gluon radiation



Sudakov factor Sterman et al(93)

$$S(\tau, \mathbf{b}, Q^2) \propto \ln \frac{\ln(\tau Q/\sqrt{2}\Lambda_{\rm QCD})}{-\ln(b\Lambda_{\rm QCD})} + \text{NLL}$$

resummed gluon radiation $\Rightarrow \exp[-S]$

LO pQCD

$+$ quark trans. mom.

$+$ Sudakov supp.

$\Rightarrow$ asymp. fact. formula (lead. twist) for $Q^2 \to \infty$

$$\mathcal{H}^M_{0\lambda,0\lambda} = \int d\tau d^2b\, \hat{\Psi}_M(\tau, -\mathbf{b})\, e^{-S} \hat{\mathcal{F}}_{0\lambda,0\lambda}(\bar{x}, \xi, \tau, Q^2, \mathbf{b})$$

$\hat{\Psi}_M \sim \exp[\tau\bar{\tau}b^2/4a_M^2]$ LC wave fct of meson

$\hat{\mathcal{F}}$ FT of hard scattering kernel

Twist-3

approach extended to transverse photon amplitudes $\mathcal{H}_{0-++}$
twist-3 pion distr. amplitudes $\Phi_P \equiv 1$ in WW approximation
Sudakov factor and $k_\perp$ regularize end-point singularity

$$\mathcal{H}_{0-++} \sim \mu_\pi/Q \qquad\qquad \mu_\pi = m_\pi^2/(m_u+m_d) \simeq 2\mathrm{GeV}$$

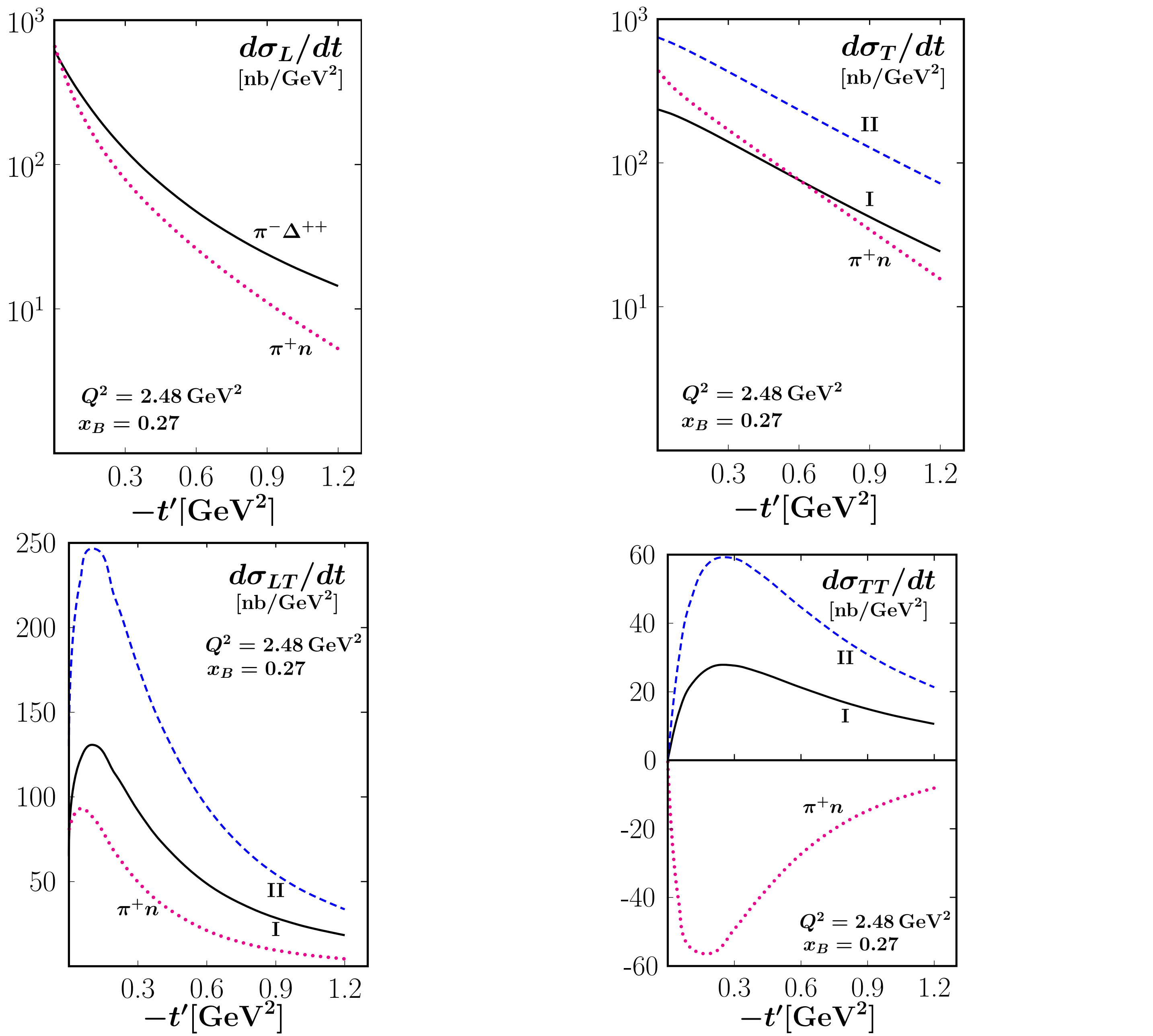
μ_π enhanced by chiral condensate - large, strong twist-3 contribution

full twist-3 sub. amplitudes in K.-Passek-Kumerički(21)
$\Phi_P \neq 1$ and 3-body $(q\bar{q}g)$ contribution

application to $\gamma^* N \to \pi N'$ in progress Duplančić-K.-Passek-K.-Szymanowski proton-proton GPDs

$\widetilde{H}$, $\tilde{E}$ and H_T for valence quarks from

Goloskokov-K (09), (11)



I: $G_{T5} = -3/2 H_T$ II: $G_{T7} = -3H_T$ (E_T)

Summary

We are ready for analyzing $p - \Delta$ data

$$
\begin{aligned}
\mathcal{M}^{tw2}_{03,0+} &= \frac{e_0}{\sqrt{2}}\,\frac{\sqrt{-t'}}{m^2}\,\frac{1}{1-\xi}\left\{(1-\xi^2)\,m\,\langle\tilde{G}_1\rangle + \frac{\kappa_+}{2}\langle\tilde{G}_2\rangle - \xi\kappa_+\langle\tilde{G}_4\rangle\right\}\,,\\
\mathcal{M}^{tw2}_{0-3,0+} &= -\frac{e_0}{\sqrt{2}}\,\frac{t'}{m^2}\sqrt{\frac{1+\xi}{1-\xi}}\left\{\frac{1}{2}\langle\tilde{G}_2\rangle - \xi\langle\tilde{G}_4\rangle\right\}\,,\\
\mathcal{M}^{tw2}_{01,0+} &= \frac{e_0}{\sqrt{6}}\,\frac{1}{m^2 M}\,\frac{1}{(1-\xi^2)^{3/2}}\ \left\{(1-\xi^2)m\left[(1-\xi^2)t' + 2\xi M\kappa_+\right]\langle\tilde{G}_1\rangle\right.\\
&+ \frac{1}{2}\left[\left(\xi(1-\xi)^2m^2 + \xi(1+\xi)(3-\xi)M^2 + (1-\xi^2)t'\right)\kappa_+\right.\\
&+ \left.(1+\xi)(1-\xi^2)Mt'\right]\langle\tilde{G}_2\rangle + (1-\xi^2)^2(1-\xi)m^2\kappa_+\langle\tilde{G}_3\rangle\\
&- \left.\xi\left[(1-\xi^2)t'(\kappa_+ + (1+\xi)M) + \kappa_-\kappa_+^2\right]\langle\tilde{G}_4\rangle\right\}\,,\\
\mathcal{M}^{tw2}_{0-1,0+} &= \frac{e_0}{\sqrt{6}}\,\frac{\sqrt{-t'}}{m^2 M}\,\frac{1}{1-\xi^2}\ \left\{-(1-\xi^2)m\left[2\xi M + (1-\xi)m\right]\langle\tilde{G}_1\rangle\right.\\
&- \frac{1}{2}\left[(1-\xi^2)t' + \xi(1-\xi)^2m^2 + (1+\xi)M\kappa_+ + \xi(1+\xi)(3-\xi)M^2\right]\langle\tilde{G}_2\rangle\\
&- \left.(1-\xi)(1-\xi^2)m^2\langle\tilde{G}_3\rangle + \xi\left[(1-\xi^2)t' + \kappa_-\kappa_+ + (1+\xi)M\kappa_+\right]\langle\tilde{G}_4\rangle\right\}
\end{aligned}
$$

$$
\langle\tilde{G}_i\rangle \;=\; \int_{-1}^{1} dx\,\mathcal{H}^{\pi^-}_{0+0+}\tilde{G}_i \qquad \kappa_\pm \;=\; (1+\xi)M \pm (1-\xi)m
$$

$$
\begin{aligned}
\mathcal{M}^{tw3}_{03++} &= -\frac{e_0}{4\sqrt{2}}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{t'}{m^2}\,\left[\frac{\kappa_-}{m}\,\langle G_{T2}\rangle + (1+\xi)\,(\langle G_{T3}\rangle - \langle G_{T4}\rangle)\right], \\
\mathcal{M}^{tw3}_{03-+} &= -\frac{e_0}{4\sqrt{2}}\sqrt{\frac{1+\xi}{1-\xi}}\,\left[\,\frac{t'}{m^2}\left(\frac{\kappa_-}{m}\,\langle G_{T2}\rangle + (1-\xi)(\langle G_{T3}\rangle + \langle G_{T4}\rangle)\right)\right. \\
&\qquad\qquad\left. +4(1-\xi)\langle G_{T7}\rangle - 4\,\frac{\xi}{1+\xi}\,\frac{\kappa_-}{m}\,\langle G_{T8}\rangle\right], \\
\mathcal{M}^{tw3}_{0-3++} &= -\frac{e_0}{4\sqrt{2}}\,\frac{\sqrt{-t'}}{m}\left[\,-4(1+\xi)\,\langle G_{T1}\rangle + (1+\xi)\,\frac{t'}{m^2}\,\langle G_{T2}\rangle\right. \\
&\qquad\qquad\left. +2\,\frac{\kappa_+}{m}\,\frac{\xi\langle G_{T3}\rangle - \langle G_{T4}\rangle}{1-\xi} - 4\xi\langle G_{T8}\rangle\right], \\
\mathcal{M}^{tw3}_{0-3-+} &= \frac{e_0}{4\sqrt{2}}\,\frac{(-t')^{3/2}}{m^3}(1+\xi)\,\langle G_{T2}\rangle\,, \\
\mathcal{M}^{tw3}_{01++} &= \frac{e_0}{2\sqrt{6}}\,\frac{\sqrt{-t'}}{m}\,\frac{1}{1-\xi}\left\{\quad -2(1+\xi)\,\langle G_{T1}\rangle\right. \\
&+ \quad \frac{\kappa_-^2\kappa_+ + (1-\xi^2)t'(\kappa_- + (1+\xi)M)}{2(1+\xi)m^2M}\,\langle G_{T2}\rangle \\
&+ \quad \frac{\kappa_+\kappa_- + (1-\xi^2)t'}{2mM}\left(\langle G_{T3}\rangle - \langle G_{T4}\rangle\right) + \frac{\kappa_+}{m}\left(\xi\langle G_{T3}\rangle - \langle G_{T4}\rangle\right) \\
&- \quad \left.(1-\xi^2)(1-\xi)\,\frac{m}{M}\,\langle G_{T5}\rangle - (1-\xi)^2\,\frac{\kappa_-}{M}\,\langle G_{T6}\rangle - (1-\xi)\,\frac{\kappa_-}{M}\,\langle G_{T8}\rangle\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}^{tw3}_{01-+} &= \frac{e_0}{4\sqrt{6}}\,\frac{\sqrt{-t'}}{m}\,\Bigg\{ \frac{\kappa_-^2\kappa_+ + (1-\xi^2)t'(\kappa_- + (1+\xi)M)}{(1-\xi^2)m^2M}\,\langle G_{T2}\rangle \\
&+ \frac{\kappa_-\kappa_+ + (1-\xi^2)t'}{(1+\xi)mM}\,\Big(\langle G_{T3}\rangle + \langle G_{T4}\rangle\Big) + 2(1-\xi)^2\,\frac{m}{M}\,\langle G_{T5}\rangle \\
&- 2(1-\xi)\Big(\frac{\kappa_-}{M}\,\langle G_{T6}\rangle - 2\,\frac{m}{M}\,\langle G_{T7}\rangle\Big) - 2\,\frac{\kappa_- + 2\xi M}{M}\,\langle G_{T8}\rangle\Bigg\}\,, \\
\mathcal{M}^{tw3}_{0-1++} &= \frac{e_0}{4\sqrt{6}}\,\frac{1}{\sqrt{1-\xi^2}}\,\Bigg\{-4\,\frac{\kappa_-\kappa_+ + (1-\xi^2)t'}{mM}\,\langle G_{T1}\rangle \\
&+ \frac{t'}{m^2}\,\frac{\kappa_-(\kappa_+ + (1+\xi)M) + (1-\xi^2)t'}{mM}\,\langle G_{T2}\rangle \\
&+ (1+\xi)^2\,\frac{t'}{m^2}\,\Big(\langle G_{T3}\rangle - \langle G_{T4}\rangle\Big) \\
&+ \frac{2\kappa_+(\kappa_-\kappa_+ + (1-\xi^2)t')}{(1-\xi^2)m^2M}\,\Big(\xi\langle G_{T3}\rangle - \langle G_{T4}\rangle\Big) \\
&- 4(1-\xi)\,\frac{\kappa_+}{m}\,\langle G_{T5}\rangle - 2(1-\xi)(1-\xi^2)\,\frac{t'}{Mm}\,\langle G_{T6}\rangle \\
&- 4(1-\xi)^2\,\frac{m}{M}\,\langle G_{T7}\rangle - 2\,\frac{2\xi\kappa_- m + (1-\xi^2)t'}{mM}\,\langle G_{T8}\rangle\Bigg\}\,, \\
\mathcal{M}^{tw3}_{0-1-+} &= \frac{e_0}{4\sqrt{6}}\,\frac{1}{\sqrt{1-\xi^2}}\,\frac{t'}{m^2}\,\Bigg\{\frac{\kappa_-(\kappa_+ + (1+\xi)M) + (1-\xi^2)t'}{mM}\,\langle G_{T2}\rangle \\
&+ (1-\xi^2)\Big(\langle G_{T3}\rangle + \langle G_{T4}\rangle\Big) - 2(1-\xi^2)\,\frac{m}{M}\,\Big((1-\xi)\,\langle G_{T6}\rangle + \langle G_{T8}\rangle\Big)\Bigg\}
\end{aligned}
$$