

Evolution effects in GPDs

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GPD definition

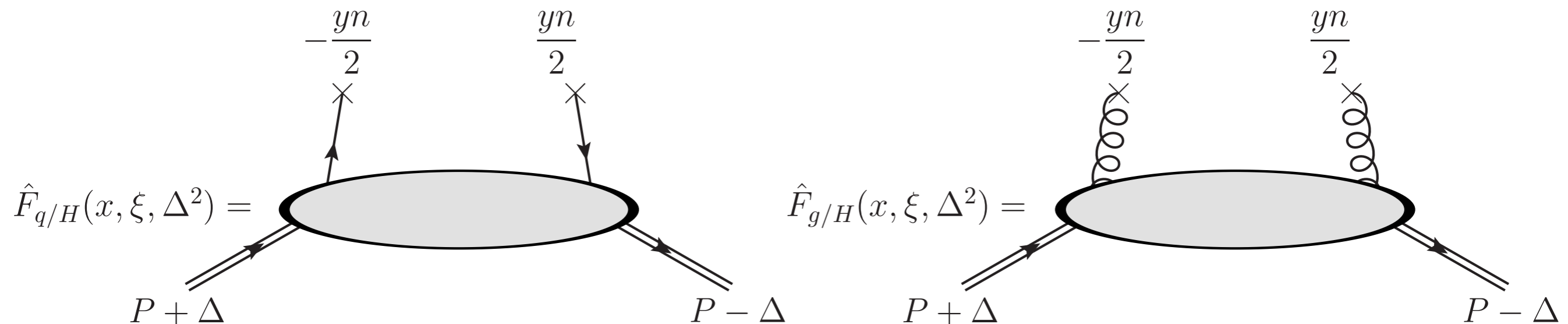
- Generalised parton distributions (GPDs) are a “byproduct” of factorisation of *amplitudes* for **exclusive** processes such as deeply-virtual Compton scattering.

[Collins, Freund, *Phys.Rev.D* 59 (1999) 074009] [Ji, *Phys.Rev.D* 55 (1997) 7114-7125]

- An operator definition of the GPDs in the **light-cone gauge** ($n \cdot A = 0$) reads:

$$\hat{F}_{q/H}^{ij}(x, \xi, \Delta^2) = \int \frac{dy}{2\pi} e^{-ix(n \cdot P)y} \left\langle P - \Delta \left| \bar{\psi}_q \left(\frac{yn}{2} \right) \psi_q \left(-\frac{yn}{2} \right) \right| P + \Delta \right\rangle \quad \xi = \frac{\Delta^+}{P^+}$$

$$\hat{F}_{g/H}^{\mu\nu}(x, \xi, \Delta^2) = \frac{n_\alpha n_\beta}{x(n \cdot P)} \int \frac{dy}{2\pi} e^{-ix(n \cdot P)y} \left\langle P - \Delta \left| F_a^{\mu\alpha} \left(\frac{yn}{2} \right) F_a^{\nu\beta} \left(-\frac{yn}{2} \right) \right| P + \Delta \right\rangle$$



GPD definition

🍏 GPD correlators are obtained by projection:

$$\hat{F}_{q/H}^{[\Gamma]}(x, \xi, \Delta^2) = \frac{1}{2} \Gamma_q^{ij} \hat{F}_{q/H}^{ij}(x, \xi, \Delta^2)$$

$$\hat{F}_{g/H}^{[\Gamma]}(x, \xi, \Delta^2) = \Gamma_{g,\mu\nu} \hat{F}_{g/H}^{\mu\nu}(x, \xi, \Delta^2)$$

🍏 Projectors are parameterised in terms of a basis of four four-vectors:

🍏 n and \bar{n} parameterise the **longitudinal** directions,

🍏 R and L parameterise the **transverse** directions,

🍏 all scalar products are zero except: $(n\bar{n}) = -(RL) = 1$.

🍏 A typical realisation in Sudakov decomposition is:

$$n^\mu = (0, 1, \mathbf{0}_T), \quad \bar{n}^\mu = (1, 0, \mathbf{0}_T), \quad R^\mu = \left(0, 0, -\frac{1}{\sqrt{2}}(1, i)\right), \quad L^\mu = \left(0, 0, -\frac{1}{\sqrt{2}}(1, -i)\right)$$

🍏 The relevant **twist-2** projectors are:

$$\Gamma_q \in \{\not{n}, \not{n}\gamma_5, i\sigma^{\alpha+}\gamma_5\}$$

$$\Gamma_g^{\mu\nu} \in \left\{-g_T^{\mu\nu} \equiv -g^{\mu\nu} + n^\mu \bar{n}^\nu + \bar{n}^\mu n^\nu, -i\epsilon_T^{\mu\nu} \equiv -i\epsilon^{\alpha\beta\mu\nu} \bar{n}_\alpha n_\beta, -R^\mu R^\nu - L^\mu L^\nu\right\}_3$$

GPD definition

🍏 GPD correlators are typically parameterised in terms of **eight** independent GPDs for quarks ($i = q$) and as many for gluons ($i = g$):

🍏 labelling $\Gamma_{q/g} \in \{U, L, T\}$.

$$\hat{F}_{i/H}^{[U]}(x, \xi, \Delta^2) = \frac{1}{n \cdot P} \bar{u}(P - \Delta) \left[\hat{H}_{i/H}(x, \xi, \Delta^2) \frac{\not{n}}{2} + \hat{E}_{i/H}(x, \xi, \Delta^2) \frac{i\sigma^{\mu\nu} n_\mu \Delta_\nu}{4M} \right] u(P + \Delta)$$

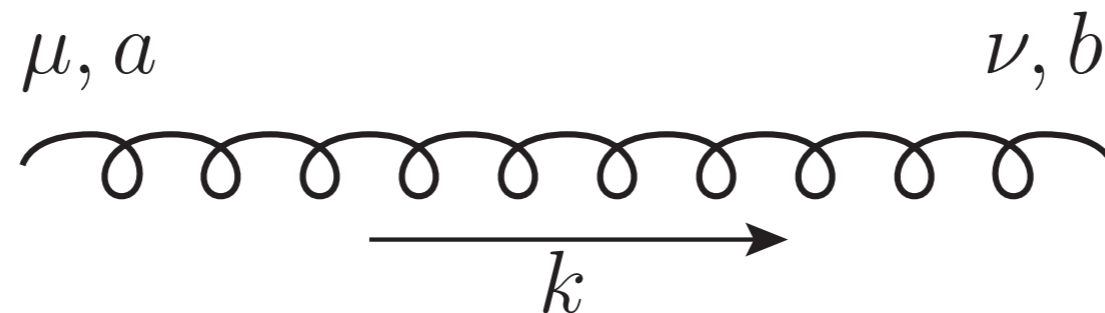
$$\hat{F}_{i/H}^{[L]}(x, \xi, \Delta^2) = \frac{1}{n \cdot P} \bar{u}(P - \Delta) \left[\hat{\tilde{H}}_{i/H}(x, \xi, \Delta^2) \frac{\not{n}\gamma^5}{2} + \hat{\tilde{E}}_{i/H}(x, \xi, \Delta^2) \frac{n^\mu \Delta_\mu \gamma^5}{4M} \right] u(P + \Delta)$$

$$\begin{aligned} \hat{F}_{i/H}^{[T]}(x, \xi, \Delta^2) &= \frac{1}{n \cdot P} \bar{u}(P - \Delta) \left[\hat{H}_{i/H}^{[T]}(x, \xi, \Delta^2) \frac{n_\mu \sigma^{\mu j} \gamma^5}{2} + \hat{\tilde{H}}_{i/H}^{[T]}(x, \xi, \Delta^2) \frac{n_\mu \epsilon^{\mu j \alpha \beta} \Delta_\alpha P_\beta}{2M^2} \right. \\ &\quad \left. + \hat{E}_{i/H}^{[T]}(x, \xi, \Delta^2) \frac{n_\mu \epsilon^{\mu j \alpha \beta} \Delta_\alpha \gamma_\beta}{4M} + \hat{\tilde{E}}_{i/H}^{[T]}(x, \xi, \Delta^2) \frac{n_\mu \epsilon^{\mu j \alpha \beta} P_\alpha \gamma_\beta}{4M} \right] u(P + \Delta) \end{aligned}$$

🍏 All the GPDs having the same polarisation label evolve in the same way.

GPD definition

- 🍏 The use of light-cone gauge implies:
 - 🍏 the **absence of the Wilson line**,
 - 🍏 a simpler gluon GPD written in terms of the **gluon field** and not the field strength,
 - 🍏 the **absence of ghosts** in perturbative calculations,
 - 🍏 **more complicated** gluon propagator:



$$\mathcal{D}_{ab}^{\mu\nu}(k) = \frac{i\delta_{ab}d^{\mu\nu}(k)}{k^2 + i0}, \quad d^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{(nk)_{\text{Reg}}}.$$

GPD evolution

- Using dimensional regularisation in $4 - 2\varepsilon$ dimensions, the **UV** renormalisation of GPDs can be implemented in a multiplicative fashion:

$$F_{i/H}^{[\Gamma]}(x, \xi, \Delta^2; \mu) = \lim_{\varepsilon \rightarrow 0} \sum_{j=q,g} \int_{-1}^1 \frac{dy}{|y|} Z_{ij}^{[\Gamma]} \left(\frac{x}{y}, \frac{\xi}{x}, \alpha_s(\mu), \varepsilon \right) \hat{F}_{j/H}^{[\Gamma]}(y, \xi, \Delta^2; \varepsilon, \mu^{-\varepsilon})$$

- In the $\overline{\text{MS}}$ scheme, renormalisation constants have the following structure:

$$Z_{ij}^{[\Gamma]}(z, \kappa, \alpha_s, \varepsilon) = \delta_{ij} \delta(1-z) + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^n \sum_{p=1}^n \frac{1}{\varepsilon^p} Z_{ij}^{[\Gamma],[n,p]}(z, \kappa)$$

- with:

$$\frac{1}{\varepsilon} = \frac{S_\varepsilon}{\varepsilon} = \frac{1}{\varepsilon} + \ln 4\pi - \gamma_E + \mathcal{O}(\varepsilon)$$

- Exploiting the independence of the bare GPDs on μ (for $\varepsilon \rightarrow 0$), one can derive a **RGE** governing the evolution of renormalised GPDs w.r.t. μ :

$$\frac{dF_{i/H}^{[\Gamma]}(x, \xi, \Delta^2; \mu)}{d \ln \mu^2} = \sum_{k=q,g} \int_{-1}^1 \frac{dz}{|z|} \mathcal{P}_{ik}^{[\Gamma]} \left(\frac{x}{z}, \frac{\xi}{x}, \alpha_s(\mu) \right) F_{k/H}^{[\Gamma]}(z, \xi, \Delta^2; \mu)$$

GPD evolution

🍏 The evolution kernels \mathcal{P} are related to the normalisation constants Z as follows:

$$\mathcal{P}_{ik}^{[\Gamma]} \left(\frac{x}{z}, \frac{\xi}{x}, \alpha_s \right) = \lim_{\varepsilon \rightarrow 0} \sum_j \int_{-1}^1 \frac{dy}{|y|} \frac{dZ_{ij}^{[\Gamma]} \left(\frac{x}{y}, \frac{\xi}{x}, \alpha_s, \varepsilon \right)}{d \ln \mu^2} Z_{jk}^{[\Gamma]-1} \left(\frac{y}{z}, \frac{\xi}{y}, \alpha_s, \varepsilon \right)$$

🍏 where the inverse of the renormalisation constant Z^{-1} is defined as:

$$\sum_j \int_{-1}^1 \frac{dw}{|w|} Z_{ij}^{[\Gamma]} \left(\frac{w}{x}, \frac{\xi}{w}, \alpha_s, \varepsilon \right) Z_{jk}^{[\Gamma]-1} \left(\frac{z}{w}, \frac{\xi}{z}, \alpha_s, \varepsilon \right) = \delta_{ik} \delta \left(1 - \frac{z}{x} \right)$$

🍏 If factorisation holds, the evolution kernels \mathcal{P} **must be finite**:

🍏 consider the factorisation of a Compton form factor: $\mathcal{F}(Q) = C(\mu/Q, \alpha_s(\mu)) \otimes F(\mu)$

🍏 Being \mathcal{F} a physical observable, it has to be independent of μ order by order in α_s :

$$C^{-1} \otimes \frac{d\mathcal{F}}{d \ln \mu^2} = 0 = \left[\frac{d \ln C(\mu, \alpha_s(\mu))}{d \ln \mu^2} + \mathcal{P}(\alpha_s(\mu)) \right] \otimes F(\mu)$$

🍏 Since the coefficient function C is finite, so must be \mathcal{P} .

GPD evolution

- 🍏 The finiteness of the evolution kernels \mathcal{P} has important consequences on the structure of the renormalisation constants Z :

$$\mathcal{P} = \frac{d \ln Z}{d \ln \mu^2} = \bar{\beta}(\alpha_s, \varepsilon) \frac{\partial \ln Z}{\partial \alpha_s}$$

- 🍏 but:

$$Z = 1 + \sum_{n=1}^{\infty} \alpha_s^n \sum_{p=1}^n \frac{1}{\varepsilon^p} Z^{[n,p]} = 1 + \sum_{p=1}^{\infty} \frac{1}{\varepsilon^p} \sum_{n=p}^{\infty} \alpha_s^n Z^{[n,p]} = 1 + \sum_{p=1}^{\infty} \frac{1}{\varepsilon^p} Z^{[p]}(\alpha_s)$$

- 🍏 so that:

$$\frac{\partial \ln Z}{\partial \alpha_s} = Z^{-1} \frac{\partial Z}{\partial \alpha_s} = \frac{1}{\varepsilon} \frac{\partial Z^{[1]}}{\partial \alpha_s} + \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

- 🍏 Since $\bar{\beta}(\alpha_s, \varepsilon) = -\varepsilon\alpha_s + \beta(\alpha_s)$, it follows that:

$$\mathcal{P} = -\alpha_s \frac{\partial Z^{[1]}}{\partial \alpha_s} + \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

- 🍏 The evolution kernels are extracted from the **single pole** of the renormalisation constants **to all orders** in α_s .

- 🍏 The finiteness of \mathcal{P} implies that the residual $\mathcal{O}(1/\varepsilon)$ has to be **identically zero**:

- 🍏 higher-order-pole coefficients $Z^{[n]}$, $n > 1$, are related to $Z^{[1]}$ and β .

GPD evolution

🍏 The kernels \mathcal{P} admit the **perturbative expansion**:

$$\mathcal{P}(\alpha_s) = \alpha_s \sum_{n=0}^{\infty} \alpha_s^n \mathcal{P}^{[n]}$$

🍏 At one loop, *i.e.* the leading order, one simply finds:

$$\mathcal{P}^{[0]} = -Z^{[1,1]}$$

🍏 At two loops:

$$\mathcal{P}^{[1]} = -2Z^{[2,1]}$$

🍏 But with the additional constraints that:

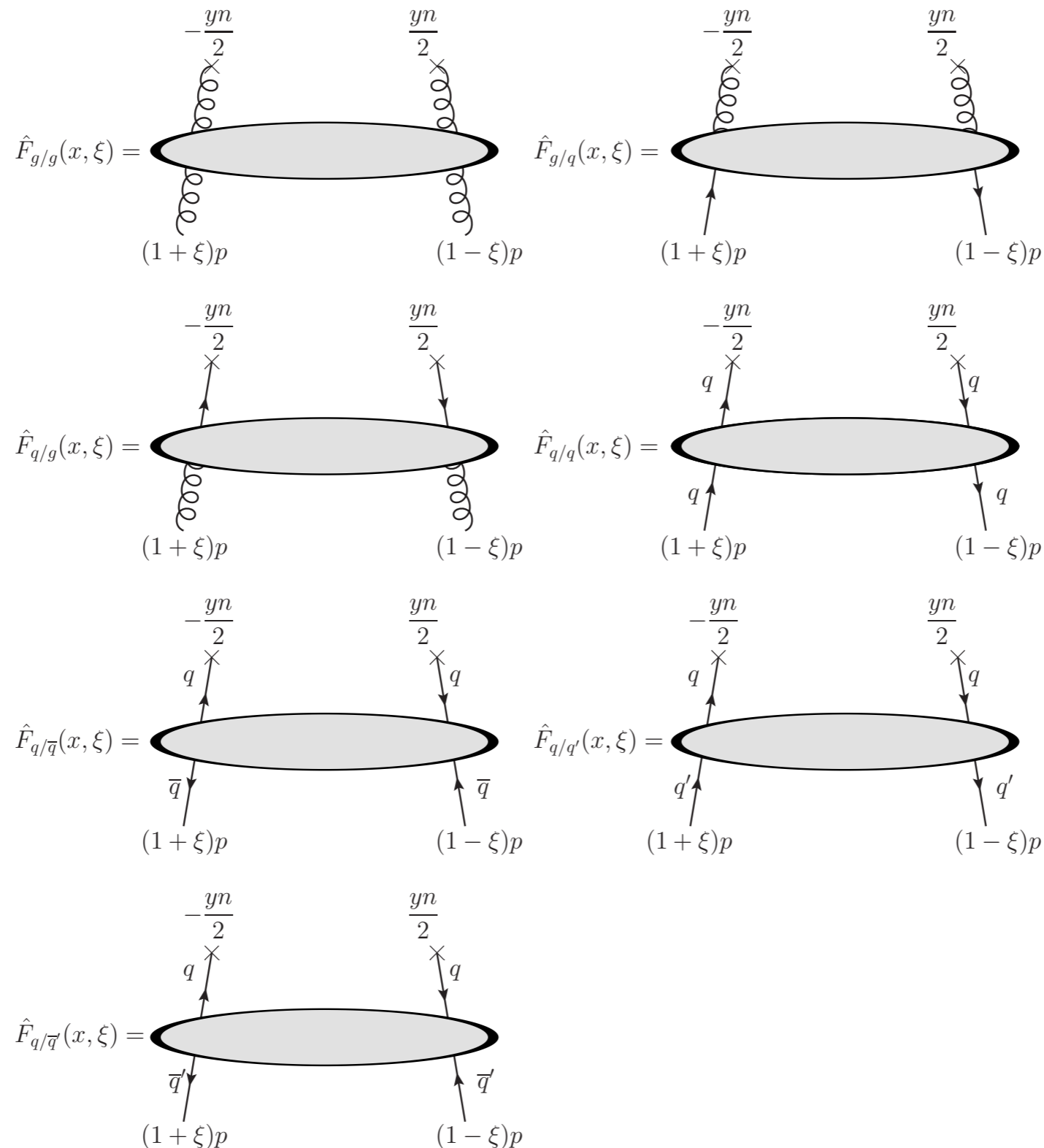
$$Z^{[2,2]} = \frac{1}{2}\beta_0 Z^{[1,1]} + \frac{1}{2}Z^{[1,1]} \otimes Z^{[1,1]}$$

🍏 An explicit two-loop calculation must fulfil this identity, thus providing a **strong check of the calculation** itself.

Parton-in-parton GPDs



The renormalisation constants Z are extracted by means of **parton-in-parton** GPDs, *i.e.* GPDs where the *hadronic* states are replaced by *partonic* states.



Dependence on Δ^2 can be neglected at twist-2.

Parton-in-parton GPDs

🍏 In light-cone gauge:

$$\hat{F}_{g/g,q}^{[\Gamma][\Lambda]}(x, \xi) = \frac{(n \cdot p)(x^2 - \xi^2)}{2(N_c^2 - 1)x} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \left\langle (1 - \xi)p, s' \left| A_a^\mu \left(\frac{yn}{2} \right) \Gamma_{g,\mu\nu} A_a^\nu \left(-\frac{yn}{2} \right) \right| (1 + \xi)p, s \right\rangle_{g,q} \Lambda_{s's}$$

$$\hat{F}_{q/g,q,\bar{q},q',\bar{q}'}^{[\Gamma][\Lambda]}(x, \xi) = \frac{1}{2N_c} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \left\langle (1 - \xi)p, s' \left| \bar{\psi}_q^i \left(\frac{yn}{2} \right) \Gamma_q^{ij} \psi_q^j \left(-\frac{yn}{2} \right) \right| (1 + \xi)p, s \right\rangle_{g,q,\bar{q},q',\bar{q}'} \Lambda_{s's}$$

🍏 The projectors $\Lambda_{s's}$ are introduced for *convenience* to project out the physical partonic spin/helicity states that contribute to the amplitude:

$$\begin{aligned} \Lambda_{s's} \bar{u}_{q,s'}((1 - \xi)p) u_{q,s}((1 + \xi)p) &= \Lambda_q = \sqrt{1 - \xi^2} \{ \not{n}, \not{n} \gamma^5, i \sigma^{\mu\nu} P_\nu \gamma^5 \} \\ \Lambda_{s's} e_{s'}^{\mu*}((1 - \xi)p) e_s^\nu((1 + \xi)p) &= \Lambda_g^{\mu\nu} = \{ -g_T^{\mu\nu}, -i \varepsilon_T^{\mu\nu}, -R^\mu R^\nu - L^\mu L^\nu \} \end{aligned} \in \{U, L, T\}$$

🍏 These quark-in-quark combinations:

$$\hat{F}_{q/q}^{\text{NS},\pm} = (\hat{F}_{q/q} - \hat{F}_{q/q'}) \pm (\hat{F}_{q/\bar{q}} - \hat{F}_{q/\bar{q}'})$$

$$\hat{F}_{q/q}^{\text{NS},V} = \hat{F}_{q/q}^{\text{NS},-} + n_f (\hat{F}_{q/q'} - \hat{F}_{q/\bar{q}'})$$

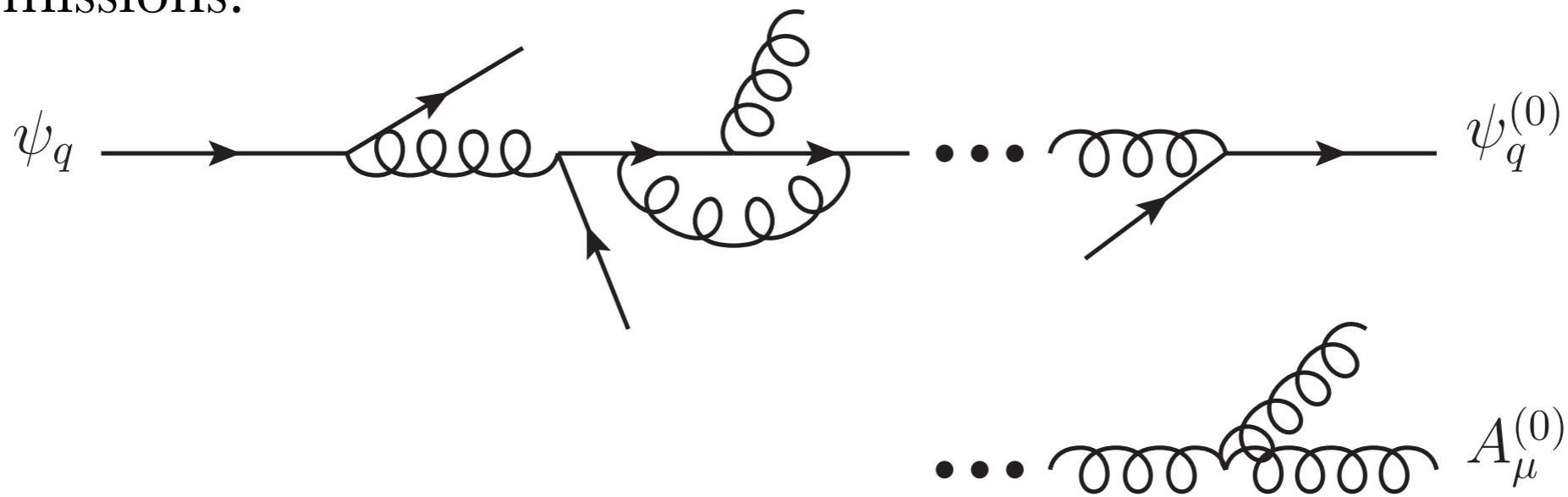
$$\hat{F}_{q/q}^{\text{SG}} = \hat{F}_{q/q}^{\text{NS},+} + n_f (\hat{F}_{q/q'} + \hat{F}_{q/\bar{q}'})$$

🍏 are particularly convenient when implementing the evolution.

Parton-in-parton GPDs

🍏 The partonic fields that appear in the operator definition of the GPD correlators are **interacting fields**.

🍏 Interacting fields reduce to **free fields** after an arbitrary number of *real* and *virtual* emissions:



🍏 Additional radiation gives rise to **perturbative corrections** and the **need for renormalisation**.

🍏 Free partonic fields eventually **annihilate** the appropriate partonic states:

$$\psi_q^{(0)}(x)|k, s\rangle_q = e^{-ik \cdot x} u_{q,s}(k)|0\rangle$$

$$\psi_{\bar{q}}^{(0)}(x)|k, s\rangle_{\bar{q}} = e^{ik \cdot x} v_{q,s}(k)|0\rangle$$

$$A_a^{(0),j}(x)|k, s\rangle_g = e^{-ik \cdot x} e_{a,s}^j(k)|0\rangle$$

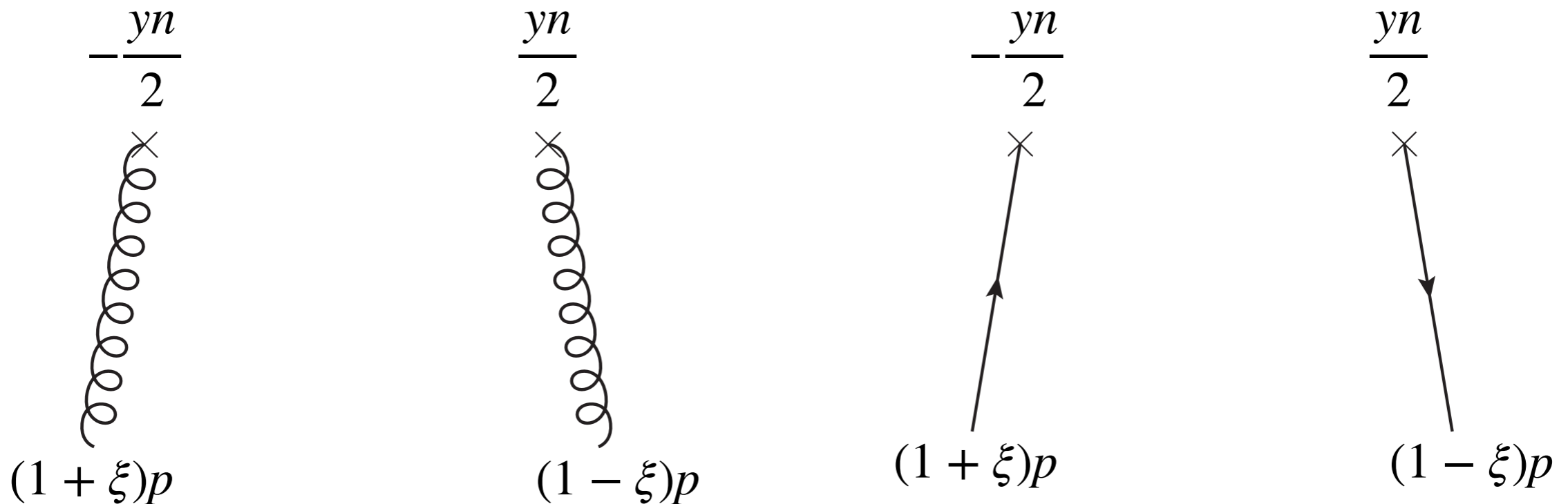
🍏 All other combinations give zero.

Parton-in-parton GPDs at LO

🍏 At $\mathcal{O}(1)$:

$$\psi_q(x) = \psi_q^{(0)}(x) \quad A_a^j(x) = A_a^{(0),j}(x)$$

🍏 One immediately finds that the only non-zero GPDs are g/g and q/q :



$$F_{g/g}^{[U][U],[0]}(x, \xi) = F_{g/g}^{[L][L],[0]}(x, \xi) = F_{g/g}^{[T][T],[0]}(x, \xi) = (1 - \xi^2) \delta(1 - x)$$

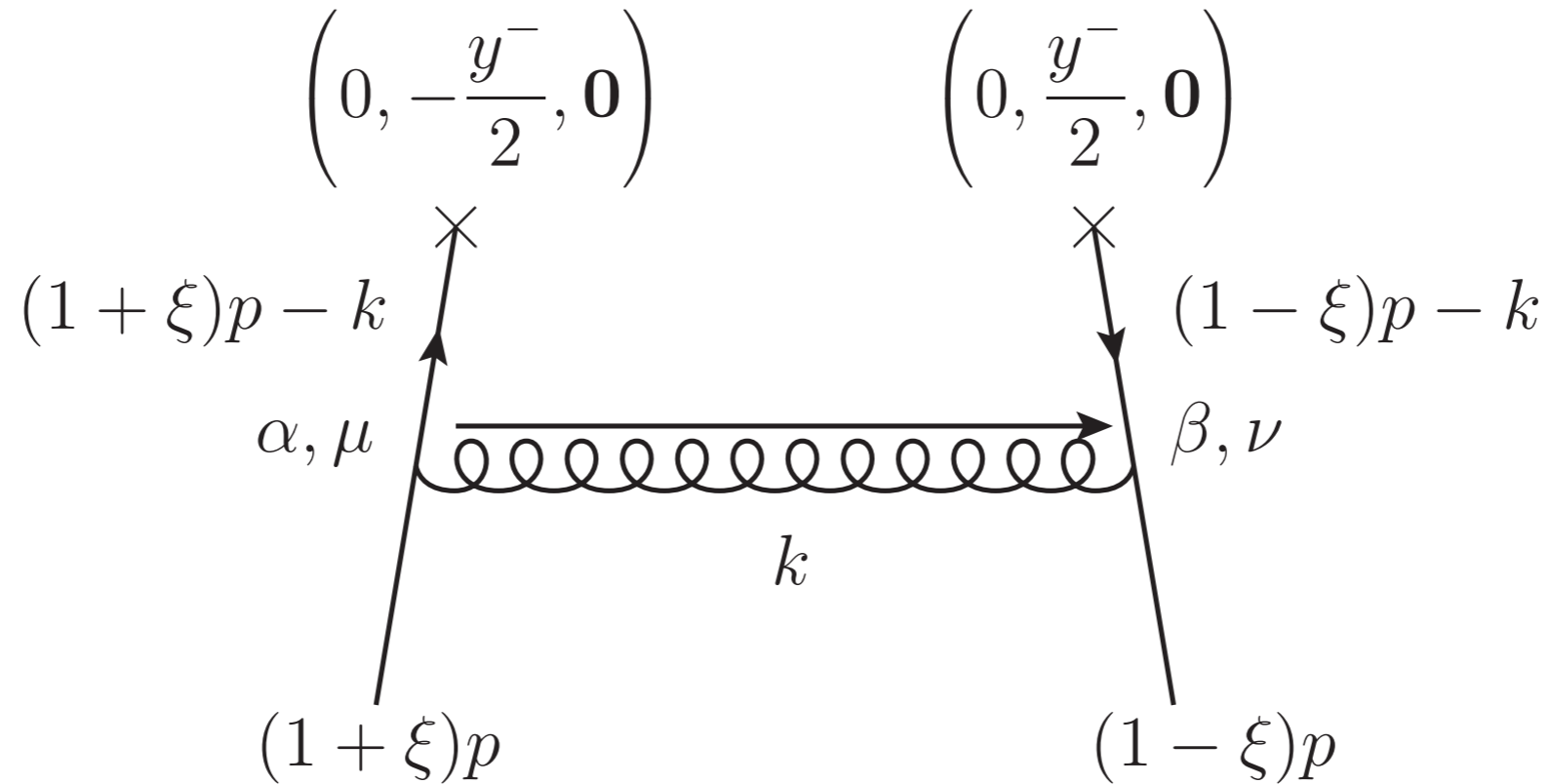
$$F_{q/q}^{[U][U],[0]}(x, \xi) = F_{q/q}^{[L][L],[0]}(x, \xi) = F_{q/q}^{[T][T],[0]}(x, \xi) = \sqrt{1 - \xi^2} \delta(1 - x)$$

🍏 No divergences at this order and thus **no need for renormalisation.**

🍏 This calculation sets the **normalisation** of GPDs.

Parton-in-parton GPDs at NLO

🍏 At $\mathcal{O}(\alpha_s)$ for the q/q channel one has to compute *one single* “real” diagram:



🍏 This produces:

$$\frac{\alpha_s}{4\pi} \hat{F}_{q/q}^{[\Gamma][\Lambda],[1],\text{real}}(x, \xi, \varepsilon) = \sqrt{1-\xi^2} \int_{-\infty}^{\infty} \frac{dy_-}{2\pi} e^{i(1-x)p_+ y_-} \text{Tr} \left[R_{qq}^{[\Gamma]}(y_-, \xi, \varepsilon) \Lambda_q \right]$$

🍏 with:

$$R_{qq}^{[\Gamma]}(y_-, \xi, \varepsilon) = \frac{\alpha_s}{4\pi} iC_F \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{2-2\varepsilon}} e^{-ik_+ y_-} \frac{\gamma^\mu [(1+\xi)\not{p} - \not{k}] \Gamma_q [(1-\xi)\not{p} - \not{k}] \gamma^\nu d_{\mu\nu}(k)}{[((1+\xi)p - k)^2 + i0][((1-\xi)p - k)^2 + i0]}$$

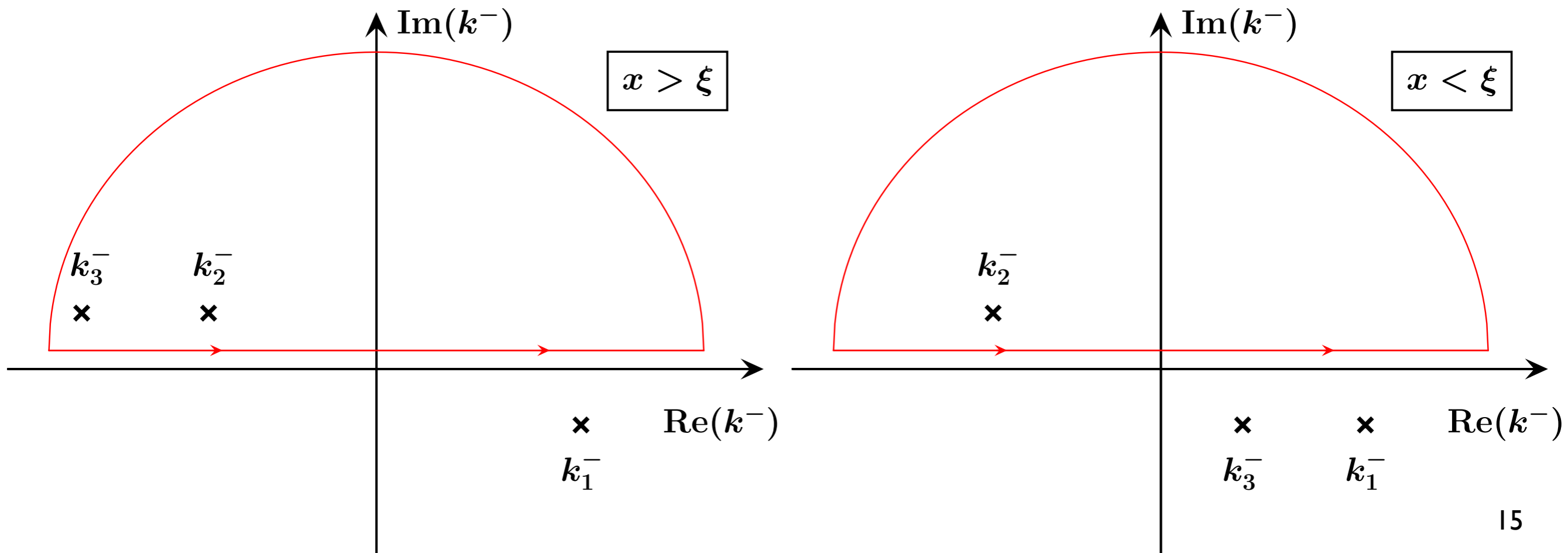
Parton-in-parton GPDs at NLO

- After the trivial integration over k^+ and the evaluation of contractions and traces, one finds:

$$\hat{F}_{q/q}^{[\Gamma][\Lambda],[1],\text{real}}(x, \xi, \varepsilon) = \int \frac{d^{2-2\varepsilon} \mathbf{k}_T}{(2\pi)^{2-2\varepsilon}} \mathbf{k}_T^2 \int_{-\infty}^{+\infty} dk^- \frac{A(x, \xi) + B(x, \xi) p^+ k^- / \mathbf{k}_T^2}{(k^- - k_1^-)(k^- - k_2^-)(k^- - k_3^-)}$$

$$k_1^- = \frac{\mathbf{k}_T^2}{2(1-x)p^+} - i(1-x)\eta \quad k_2^- = -\frac{\mathbf{k}_T^2}{2(x+\xi)p^+} + i(x+\xi)\eta \quad k_3^- = -\frac{\mathbf{k}_T^2}{2(x-\xi)p^+} + i(x-\xi)\eta$$

- Assuming $x, \xi > 0$, the pole structure depends on the sign of $x - \xi$:



Parton-in-parton GPDs at NLO

🍏 The final result looks like this:

$$\hat{F}_{q/q}^{[\Gamma][\Lambda],[1],\text{real}}(x, \xi, \varepsilon) = \sqrt{1 - \xi^2} \theta(1-x) \left[\theta(x + \xi) p_{q/q}^\Gamma \left(x, \frac{\xi}{x} \right) + \theta(x - \xi) p_{q/q}^\Gamma \left(x, -\frac{\xi}{x} \right) \right] \mu^{2\varepsilon} S_\varepsilon \int_0^\infty \frac{dk_T^2}{k_T^{2+2\varepsilon}}$$

🍏 Strictly speaking:

$$\int_0^\infty \frac{dk_T^2}{k_T^{2+2\varepsilon}} = \frac{1}{\varepsilon_{\text{UV}}} - \frac{1}{\varepsilon_{\text{IR}}} = 0 \quad \Rightarrow \quad \hat{F}_{q/q}^{[\Gamma][\Lambda],[1],\text{real}}(x, \xi, \varepsilon) = 0$$

🍏 We are only concerned with the **UV part**: the IR one has to cancel against the partonic cross section when computing a physical observable (**IR safety**).

$$\hat{F}_{q/q}^{[\Gamma][\Lambda],[1],\text{real}}(x, \xi, \varepsilon) = \sqrt{1 - \xi^2} \theta(1-x) \left[\theta(x + \xi) p_{q/q}^\Gamma \left(x, \frac{\xi}{x} \right) + \theta(x - \xi) p_{q/q}^\Gamma \left(x, -\frac{\xi}{x} \right) \right] \frac{\mu^{2\varepsilon}}{\varepsilon} + \text{IR}$$

Evolution kernels at one loop

🍏 The **virtual** contribution (common to all polarisations) is computed as:

$$\frac{1}{2} \left[\begin{array}{c} \frac{yn}{2} \\ \times \\ (1+\xi)p - k \\ \uparrow \\ \text{loop} \\ \downarrow \\ k \\ (1+\xi)p \end{array} + \begin{array}{c} \frac{yn}{2} \\ \times \\ \downarrow \\ (1-\xi)p \end{array} + \begin{array}{c} \frac{yn}{2} \\ \times \\ \uparrow \\ (1+\xi)p \end{array} + \begin{array}{c} \frac{yn}{2} \\ \times \\ \text{loop} \\ \downarrow \\ k \\ (1-\xi)p - k \\ \downarrow \\ (1-\xi)p \end{array} \right]$$

🍏 The final result is:

$$\begin{aligned} \hat{F}_{q/q}^{[\Gamma][\Lambda],[1]}(x, \xi, \varepsilon) &= \sqrt{1-\xi^2} \left\{ \theta(1-x) \left[\theta(x+\xi) p_{q/q}^{\Gamma} \left(x, \frac{\xi}{x} \right) + \theta(x-\xi) p_{q/q}^{\Gamma} \left(x, -\frac{\xi}{x} \right) \right] \right. \\ &\quad \left. + \delta(1-x) C_F \left[\frac{3}{2} - \ln \left(\left| 1 - \frac{\xi^2}{x^2} \right| \right) - 2 \int_0^1 \frac{dz}{1-z} \right] \right\} \frac{\mu^{2\varepsilon}}{\bar{\varepsilon}} + \text{IR} \end{aligned}$$

🍏 The resulting coefficient of the single pole of UV renormalisation constant and thus the corresponding evolution kernel is:

$$\begin{aligned} \mathcal{P}_{qq}^{[\Gamma],[0]}(y, \kappa) = -Z_{qq}^{[\Gamma],[1,1]}(y, \kappa) &= \theta(1-y) \left[\theta(1+\kappa) p_{q/q}^{\Gamma}(y, \kappa) + \theta(1-\kappa) p_{q/q}^{\Gamma}(y, -\kappa) \right] \\ &\quad + \delta(1-y) C_F \left[\frac{3}{2} - \ln(|1-\kappa^2|) - 2 \int_0^1 \frac{dz}{1-z} \right] \quad \kappa = \frac{\xi}{x} \end{aligned}$$

Evolution kernels at one loop

🍏 The general structure is for **all channels**:

$$\mathcal{P}_{ij}^{[\Gamma],[0]}(y, \kappa) = \theta(1-y) \left[\theta(1+\kappa) p_{i/j}^{\Gamma}(y, \kappa) + \theta(1-\kappa) p_{i/j}^{\Gamma}(y, -\kappa) \right] \\ + \delta_{ij} \delta(1-y) C_i \left[K_i - \ln(|1-\kappa^2|) - 2 \int_0^1 \frac{dz}{1-z} \right] \quad \kappa = \frac{\xi}{x}$$

🍏 with $C_q = C_F$ and $C_g = C_A$, and:

$$K_q = \frac{3}{2} \quad K_g = \frac{11C_A - 4n_f T_R}{6C_A}$$

🍏 In [[Eur. Phys. J. C 82 \(2022\) 10, 888](#)] we have computed the full set of p_{ij}^U :

$$p_{q/q}^U \left(x, \frac{\xi}{x} \right) = C_F \frac{(x+\xi)(1-x+2\xi)}{\xi(1+\xi)(1-x)}$$

$$p_{q/g}^U \left(x, \frac{\xi}{x} \right) = T_R \frac{(x+\xi)(1-2x+\xi)}{\xi(1+\xi)(1-\xi^2)}$$

$$p_{g/q}^U \left(x, \frac{\xi}{x} \right) = C_F \frac{(x+\xi)(2-x+\xi)}{\xi x(1+\xi)}$$

$$p_{g/g}^U \left(x, \frac{\xi}{x} \right) = -C_A \frac{x^2 - \xi^2}{\xi x(1-\xi^2)} \left[1 - \frac{2\xi}{1-x} - \frac{2(1+x^2)}{(x-\xi)(1+\xi)} \right]$$

Evolution kernels at one loop

🍏 In collaboration with *R. F. del Castillo, M. G. Echevarria, O. del Río, and S. Rodini*, we have computed these functions also in the **longitudinally polarised** case:

$$p_{q/q}^L \left(x, \frac{\xi}{x} \right) = -C_F \frac{(x + \xi)(x - 1 - 2\xi)}{(1 + \xi)\xi(1 - x)}$$

$$p_{q/g}^L \left(x, \frac{\xi}{x} \right) = -2n_f T_R \frac{x + \xi}{\xi(1 + \xi)^2}$$

$$p_{g/q}^L \left(x, \frac{\xi}{x} \right) = C_F \frac{(x + \xi)^2}{x\xi(1 + \xi)}$$

$$p_{g/g}^L \left(x, \frac{\xi}{x} \right) = \frac{C_A(\xi + x) (-\xi^2(2\xi + 1) + \xi + (\xi - 3)x^2 + (\xi^2 + 3)x)}{(1 - \xi^2)\xi(1 + \xi)(1 - x)x}$$

🍏 and in the **transversely polarised** case:

$$p_{q/q}^T \left(x, \frac{\xi}{x} \right) = 2C_F \frac{x + \xi}{(1 + \xi)(1 - x)}$$

$$p_{q/g}^T \left(x, \frac{\xi}{x} \right) = p_{g/q}^T \left(x, \frac{\xi}{x} \right) = 0$$

$$p_{g/g}^T \left(x, \frac{\xi}{x} \right) = 2C_A \frac{(x + \xi)^2}{(1 + \xi)^2(1 - x)x}$$

🍏 Paper in preparation.

Evolution equations

Defining the **anti-quark** distributions as:

$$F_{\bar{q}/H}^{[U,T]}(x, \xi, \Delta^2; \mu) = -F_{q/H}^{[U,T]}(-x, \xi, \Delta^2; \mu)$$

$$F_{\bar{q}/H}^{[L]}(x, \xi, \Delta^2; \mu) = +F_{q/H}^{[L]}(-x, \xi, \Delta^2; \mu)$$

one can construct **non-singlet** and **singlet** combinations:

$$F^{[\Gamma],-} = F_{q/H}^{[\Gamma]} - F_{\bar{q}/H}^{[\Gamma]} \quad F^{[\Gamma],+} = \left(\begin{array}{c} \sum_{q=1}^{n_f} F_{q/H}^{[\Gamma]} + F_{\bar{q}/H}^{[\Gamma]} \\ F_{g/H}^{[\Gamma]} \end{array} \right)$$

The evolution equations **decouple** and can be written in a **DGLAP-like** fashion:

$$\frac{dF^{[\Gamma],\pm}(x, \xi, \mu)}{d \ln \mu^2} = \frac{\alpha_s(\mu)}{4\pi} \int_x^\infty \frac{dy}{y} \mathcal{P}^{[\Gamma]\pm,[0]} \left(y, \frac{\xi}{x} \right) F^{[\Gamma],\pm} \left(\frac{x}{y}, \xi, \mu \right)$$

$$\mathcal{P}^{[\Gamma]\pm,[0]} \left(y, \frac{\xi}{x} \right) = \theta(1-y) \mathcal{P}_1^{[\Gamma]\pm,[0]} \left(y, \frac{\xi}{x} \right) + \theta(\xi-x) \mathcal{P}_2^{[\Gamma]\pm,[0]} \left(y, \frac{\xi}{x} \right)$$

DGLAP region

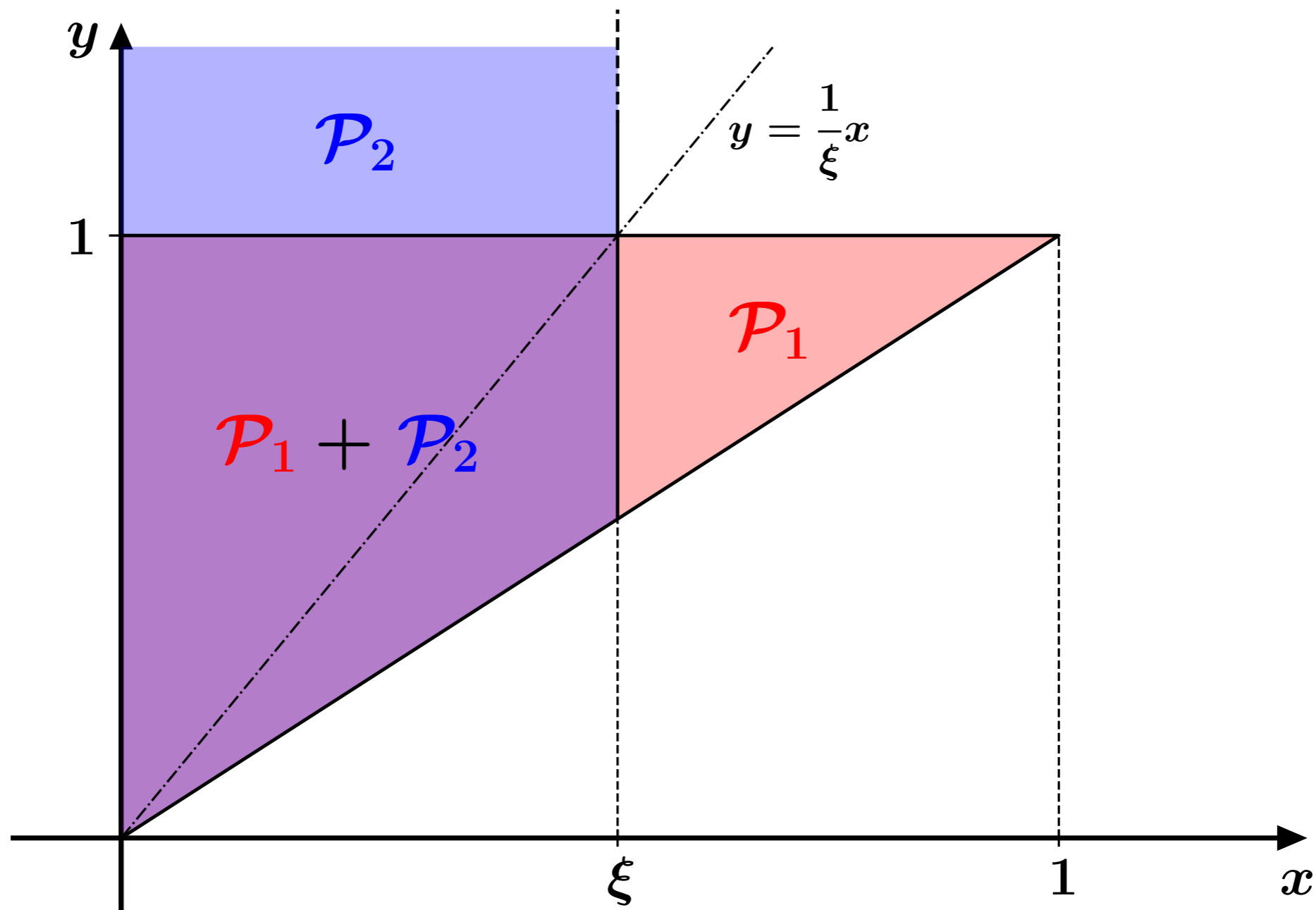
ERBL contribution

$\mathcal{P}_{1,2}^{[\Gamma]\pm,[0]}$ are appropriate combinations of the functions p_{ij}^Γ presented before.

Properties of the kernels

$$\frac{dF^{[\Gamma],\pm}(x, \xi, \mu)}{d \ln \mu^2} = \frac{\alpha_s(\mu)}{4\pi} \int_x^\infty \frac{dy}{y} \mathcal{P}^{[\Gamma]\pm,[0]} \left(y, \frac{\xi}{x} \right) F^{[\Gamma],\pm} \left(\frac{x}{y}, \xi, \mu \right)$$

$$\mathcal{P}^{[\Gamma]\pm,[0]} \left(y, \frac{\xi}{x} \right) = \theta(1-y) \mathcal{P}_1^{[\Gamma]\pm,[0]} \left(y, \frac{\xi}{x} \right) + \theta(\xi-x) \mathcal{P}_2^{[\Gamma]\pm,[0]} \left(y, \frac{\xi}{x} \right)$$



Properties of the kernels

$$\mathcal{P}^{[\Gamma]\pm,[0]}(y, \kappa) = \theta(1-y) \mathcal{P}_1^{[\Gamma]\pm,[0]}(y, \kappa) + \theta(\kappa-1) \mathcal{P}_2^{[\Gamma]\pm,[0]}(y, \kappa) \quad \kappa = \frac{\xi}{x}$$

🍏 In the limit $\kappa \rightarrow 0$ the **DGLAP** splitting functions are recovered:

$$\lim_{\kappa \rightarrow 0} \mathcal{P}^{[\Gamma]\pm,[0]}(y, \kappa) = \theta(1-y) P^{[\Gamma]\pm,[0]}(y)$$

🍏 In the limit $\kappa \rightarrow 1/x$ the **ERBL** non-singlet kernel in the unpolarised case is recovered:
 e.g. [Mikhailov, Radyushkin, *Nucl.Phys.B* 254 (1985) 89-126]
 or [Blümlein, Geyer, Robaschik, *Phys.Lett.B* 406 (1997) 161-170]

$$\frac{1}{2u-1} \mathcal{P}^{[U]-,[0]} \left(\frac{2t-1}{2u-1}, \frac{1}{2t-1} \right) = C_F \left[\theta(u-t) \left(\frac{t-1}{u} + \frac{1}{u-t} \right) - \theta(t-u) \left(\frac{t}{1-u} + \frac{1}{u-t} \right) \right]_+$$

$$\text{with } [f(t, u)]_+ \equiv f(t, u) - \delta(u-t) \int_0^1 du' f(t, u')$$

We have also derived singlet and non-singlet ERBL kernels for the other polarisations.

🍏 **Continuity** of GPDs at the crossover point $x = \xi$ ($\kappa = 1$) guaranteed:

$$\lim_{\kappa \rightarrow 1} \mathcal{P}_1^{[\Gamma]\pm,[0]}(y, \kappa) = \text{finite} \quad \mathcal{P}_2^{[\Gamma]\pm,[0]}(y, \kappa) \propto (1-\kappa)$$

Properties of the kernels

$$\mathcal{P}^{[\Gamma]\pm,[0]}(y, \kappa) = \theta(1-y) \mathcal{P}_1^{[\Gamma]\pm,[0]}(y, \kappa) + \theta(\kappa-1) \mathcal{P}_2^{[\Gamma]\pm,[0]}(y, \kappa) \quad \kappa = \frac{\xi}{x}$$

🍏 Valence **sum rule** (polynomiality of the first moment of the **unpolarised non-singlet**):

$$\int_0^1 dx F^{[U],-}(x, \xi, \Delta^2; \mu) = \text{FF}(\Delta^2) \quad \Rightarrow \quad \int_0^1 dz \left[\mathcal{P}_1^{[U]-,[0]} \left(z, \frac{\xi}{yz} \right) + \frac{\xi}{y} \mathcal{P}_2^{[U]-,[0]} \left(\frac{z\xi}{y}, \frac{1}{z} \right) \right] = 0$$

🍏 As consequence of the **Ji's sum rule** one also finds: [\[Ji, Phys. Rev. Lett. 78 \(1997\) 610-613\]](#)

$$\int_0^1 dx x \left[F_q^{[U],+}(x, \xi, \Delta^2; \mu) + F_g^{[U],+}(x, \xi, \Delta^2; \mu) \right] = \text{constant in } \xi \text{ and } \mu$$

🍏 that leads to:

$$\int_0^1 dz z \left[\mathcal{P}_{1,qq}^{[U]+,[0]} \left(z, \frac{\xi}{yz} \right) + \mathcal{P}_{1,gq}^{[U]+,[0]} \left(z, \frac{\xi}{yz} \right) + \frac{\xi^2}{y^2} \left(\mathcal{P}_{2,qq}^{[U]+,[0]} \left(\frac{z\xi}{y}, \frac{1}{z} \right) + \mathcal{P}_{2,gq}^{[U]+,[0]} \left(\frac{z\xi}{y}, \frac{1}{z} \right) \right) \right] = 0$$

$$\int_0^1 dz z \left[\mathcal{P}_{1,qg}^{[U]+,[0]} \left(z, \frac{\xi}{yz} \right) + \mathcal{P}_{1,gg}^{[U]+,[0]} \left(z, \frac{\xi}{yz} \right) + \frac{\xi^2}{y^2} \left(\mathcal{P}_{2,qg}^{[U]+,[0]} \left(\frac{z\xi}{y}, \frac{1}{z} \right) + \mathcal{P}_{2,gg}^{[U]+,[0]} \left(\frac{z\xi}{y}, \frac{1}{z} \right) \right) \right] = 0$$

🍏 These identities were analytically verified in [\[Eur. Phys. J. C 82 \(2022\) 10, 888\]](#).

Properties of the kernels

$$\mathcal{P}^{[\Gamma]\pm,[0]}(y, \kappa) = \theta(1-y) \mathcal{P}_1^{[\Gamma]\pm,[0]}(y, \kappa) + \theta(\kappa-1) \mathcal{P}_2^{[\Gamma]\pm,[0]}(y, \kappa) \quad \kappa = \frac{\xi}{x}$$

🍏 The ξ -independence of the **1st moment of longitudinally polarised** GPDs implies:

$$\int_0^1 dz \left[\mathcal{P}_{1,ij}^{L,+,[0]} \left(z, \frac{\xi}{yz} \right) + \frac{\xi}{y} \mathcal{P}_{2,ij}^{L,+,[0]} \left(\frac{z\xi}{y}, \frac{1}{z} \right) \right] = \text{constant in } \xi$$

🍏 This is true and we also find that the q/q and q/g channels are identically zero, *i.e.* the first moment of $F_{q/H}^{[L],+}$ is **scale independent**:

🍏 physical observable connected with the anti-symmetric part of the EMT.

🍏 The ξ -independence of the **2nd moment of longitudinally polarised** GPDs implies:

$$\int_0^1 dz z \left[\mathcal{P}_1^{L,-,[0]} \left(z, \frac{\xi}{yz} \right) + \frac{\xi^2}{y^2} \mathcal{P}_2^{L,-,[0]} \left(\frac{z\xi}{y}, \frac{1}{z} \right) \right] = \text{constant in } \xi$$

🍏 Similar arguments apply to **transversely pol.** GPDs and lead to the verified constraints:

$$\int_0^1 dz \left[\mathcal{P}_1^{T,-,[0]} \left(z, \frac{\xi}{yz} \right) + \frac{\xi}{y} \mathcal{P}_2^{T,-,[0]} \left(\frac{z\xi}{y}, \frac{1}{z} \right) \right] = \text{constant in } \xi$$

$$\int_0^1 dz z \left[\mathcal{P}_{1,qq}^{T,+,[0]} \left(z, \frac{\xi}{yz} \right) + \frac{\xi^2}{y^2} \mathcal{P}_{2,qq}^{T,+,[0]} \left(\frac{z\xi}{y}, \frac{1}{z} \right) \right] = \text{constant in } \xi$$

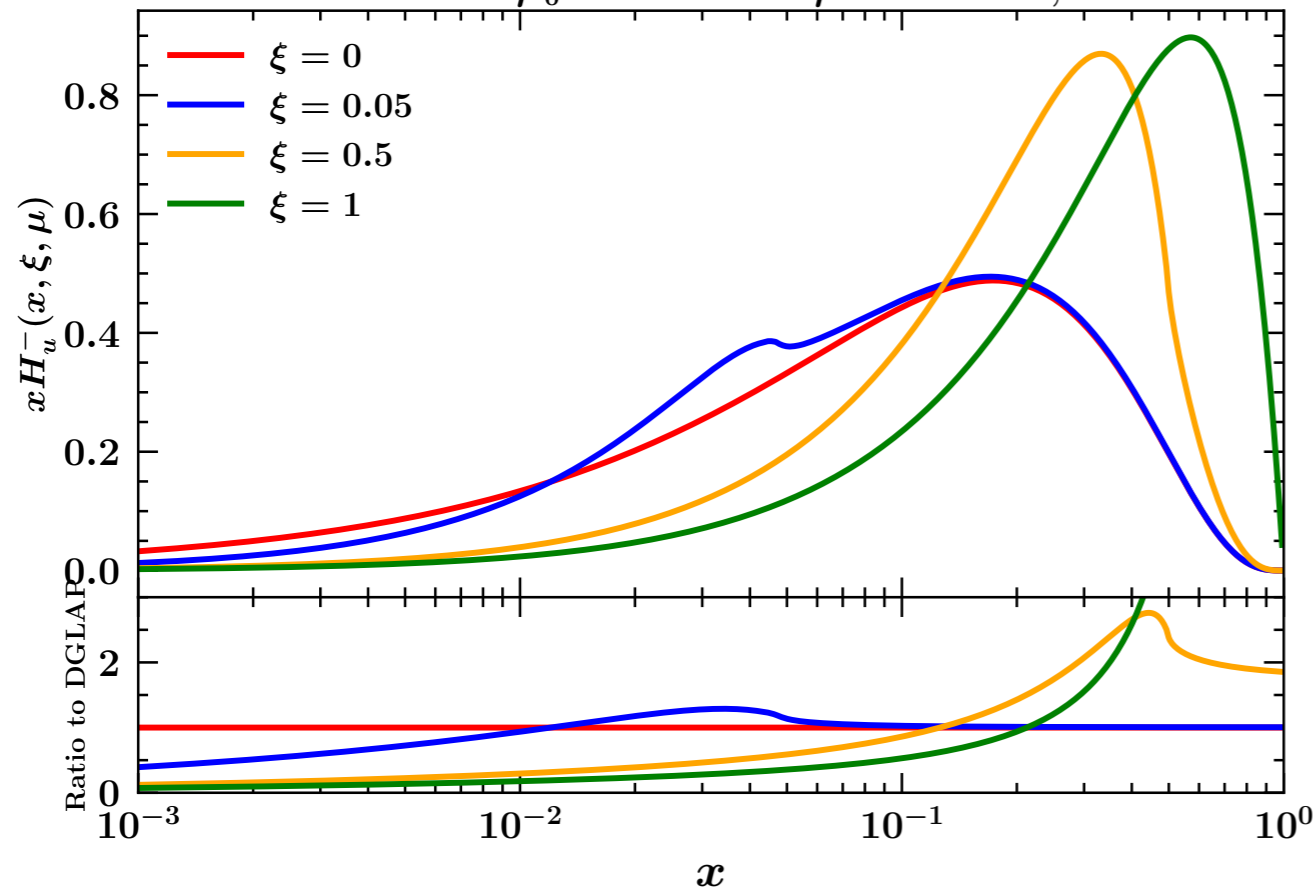
$$\int_0^1 dz z \left[\mathcal{P}_{1,gg}^{T,+,[0]} \left(z, \frac{\xi}{yz} \right) + \frac{\xi^2}{y^2} \mathcal{P}_{2,gg}^{T,+,[0]} \left(\frac{z\xi}{y}, \frac{1}{z} \right) \right] = \text{constant in } \xi$$

Numerical setup

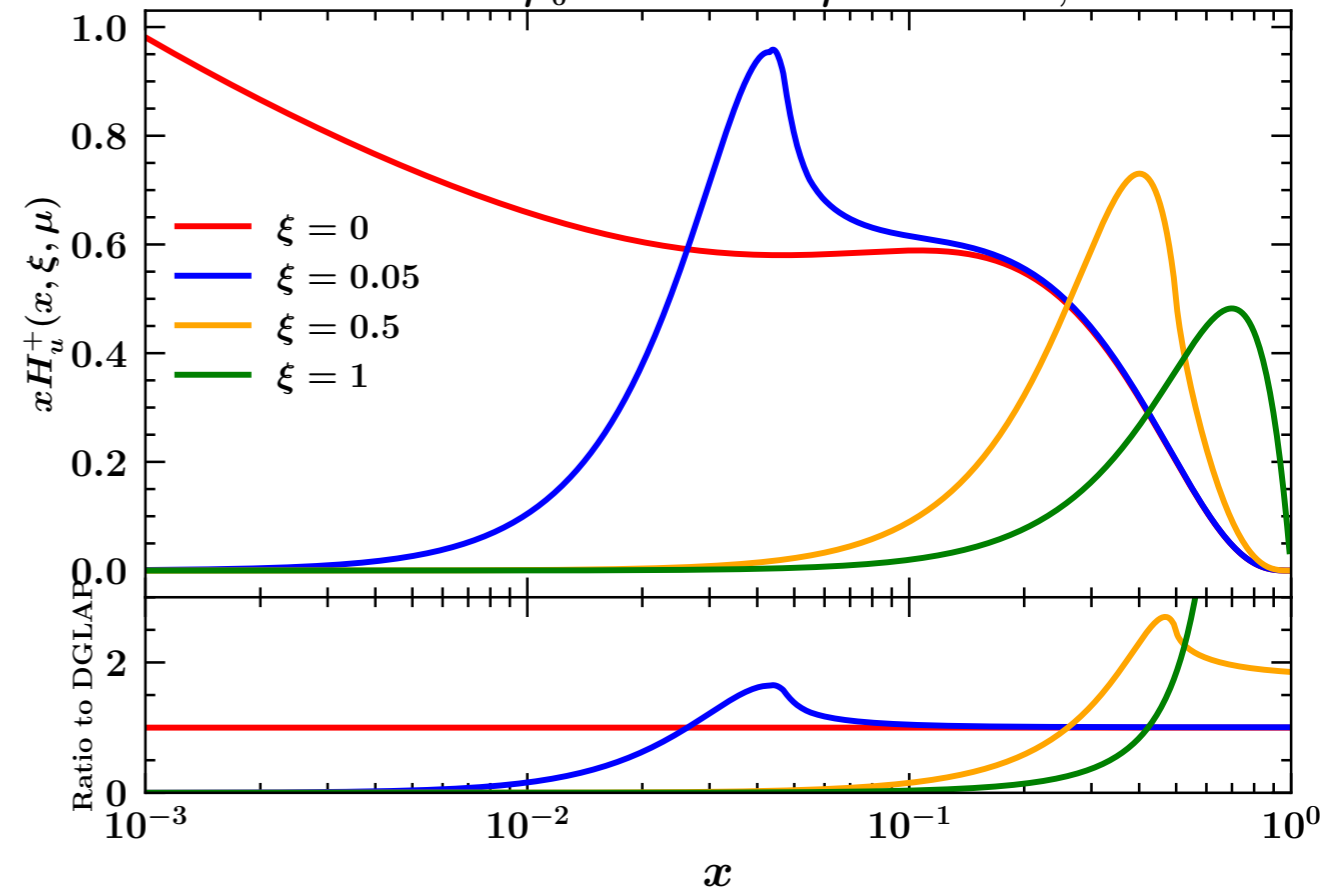
- 🍏 The evolution kernels for *all polarisations* are now implemented in **APFEL++** and will soon be available through **PARTONS** allowing for LO GPD evolution in momentum space.
- 🍏 We achieved a stable numerical implementation over the full range $0 \leq \xi \leq 1$:
 - 🍏 numerical check that both the **DGLAP** and **ERBL** limits are recovered,
 - 🍏 numerical check of **polynomiality** conservation.
- 🍏 Numerical tests use the *realistic* Goloskokov-Kroll (GK) model for proton GPDs [[Eur. Phys. J. C 53 \(2008\) 367-384](#)] as implemented in **PARTONS** as an initial-scale set of distributions:
 - 🍏 we consistently used $H_{i/H}$ for unpolarised, $\widetilde{H}_{i/H}$ for longitudinally polarised, and $H_{i/H}^T$ for transversely polarised evolution.
 - 🍏 GPDs are evolved from 2 to 10 GeV in the **variable-flavour-number scheme**, *i.e.* accounting for heavy-quark-threshold crossing, at $\Delta^2 = -0.1 \text{ GeV}^2$.

Evolution and DGLAP limit [U]

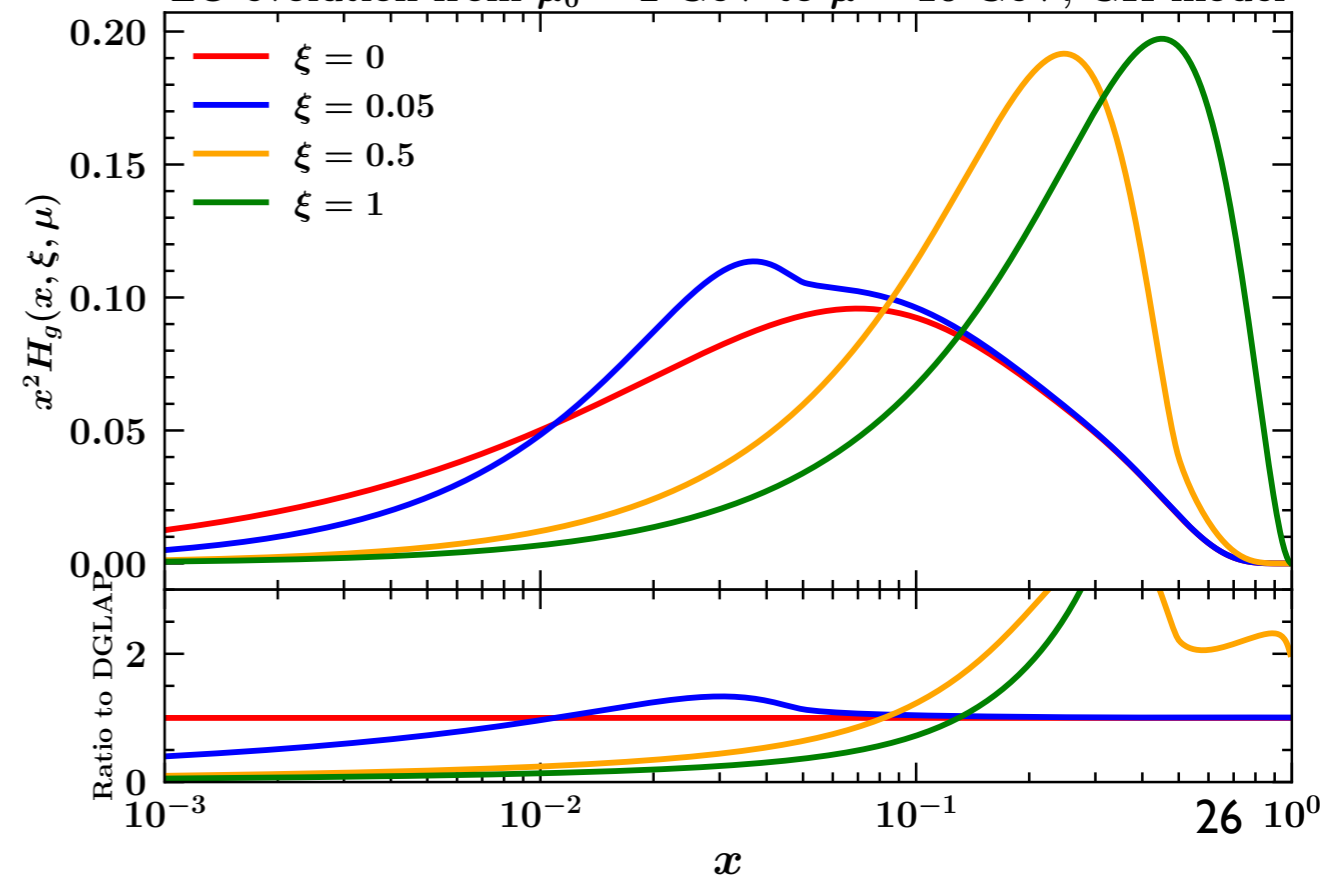
LO evolution from $\mu_0 = 2$ GeV to $\mu = 10$ GeV, GK model



LO evolution from $\mu_0 = 2$ GeV to $\mu = 10$ GeV, GK model




LO evolution from $\mu_0 = 2$ GeV to $\mu = 10$ GeV, GK model



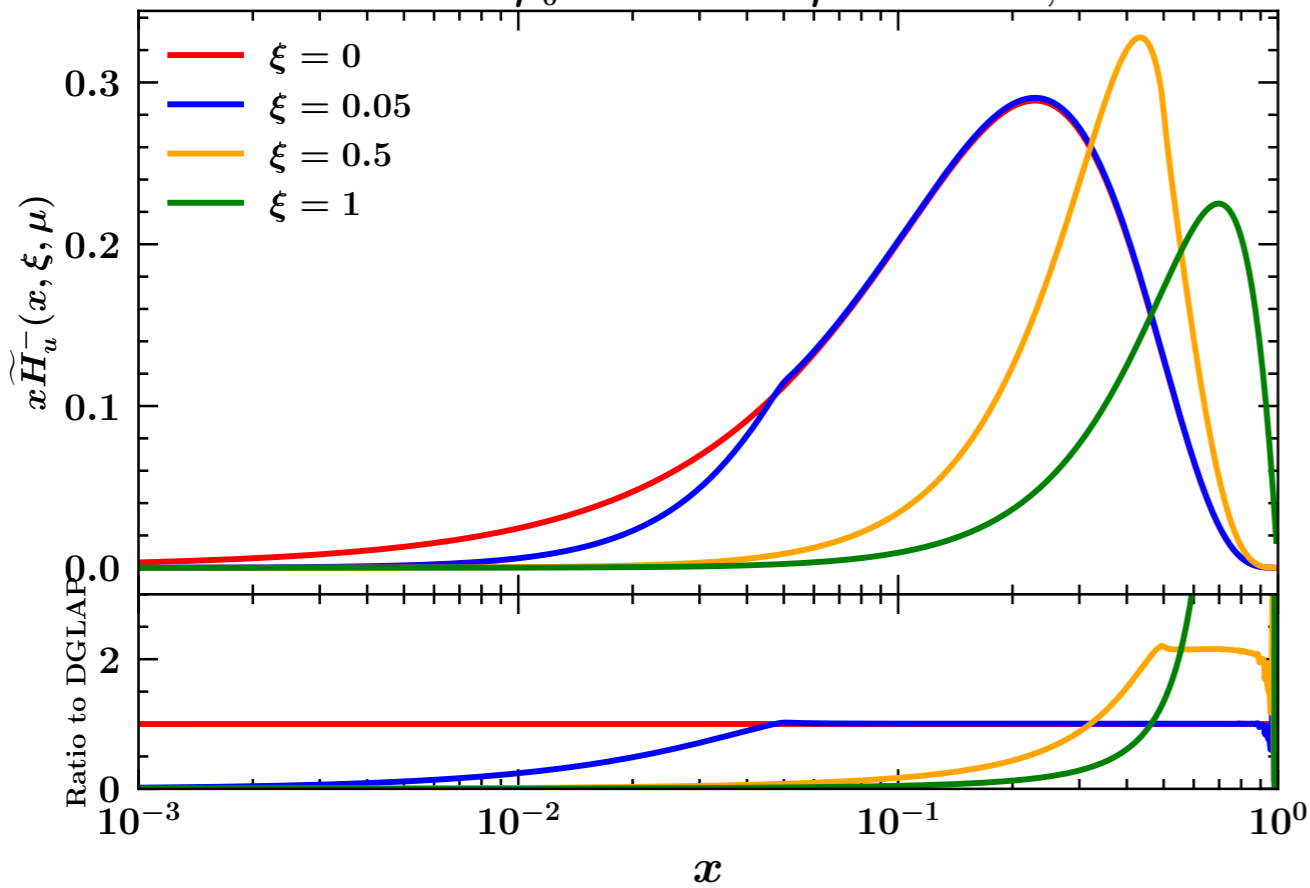
 **DGLAP limit** reproduced within 10^{-5} relative accuracy.

 GPD evolution may significantly deviate from DGLAP evolution.

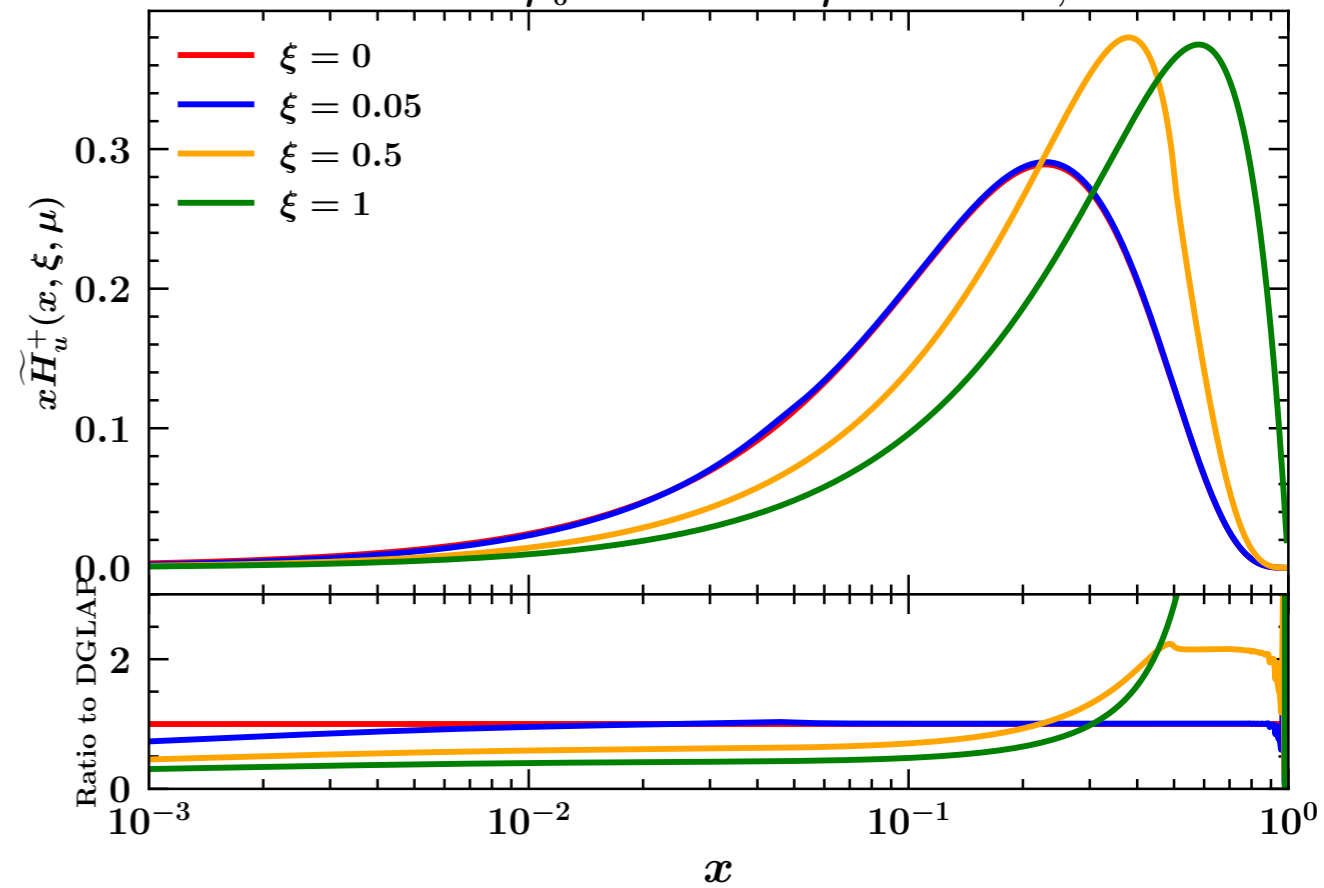
 The evolution generates a cusp at $x = \xi$ but the distribution remains **continuous** at this point.

Evolution and DGLAP limit [L]

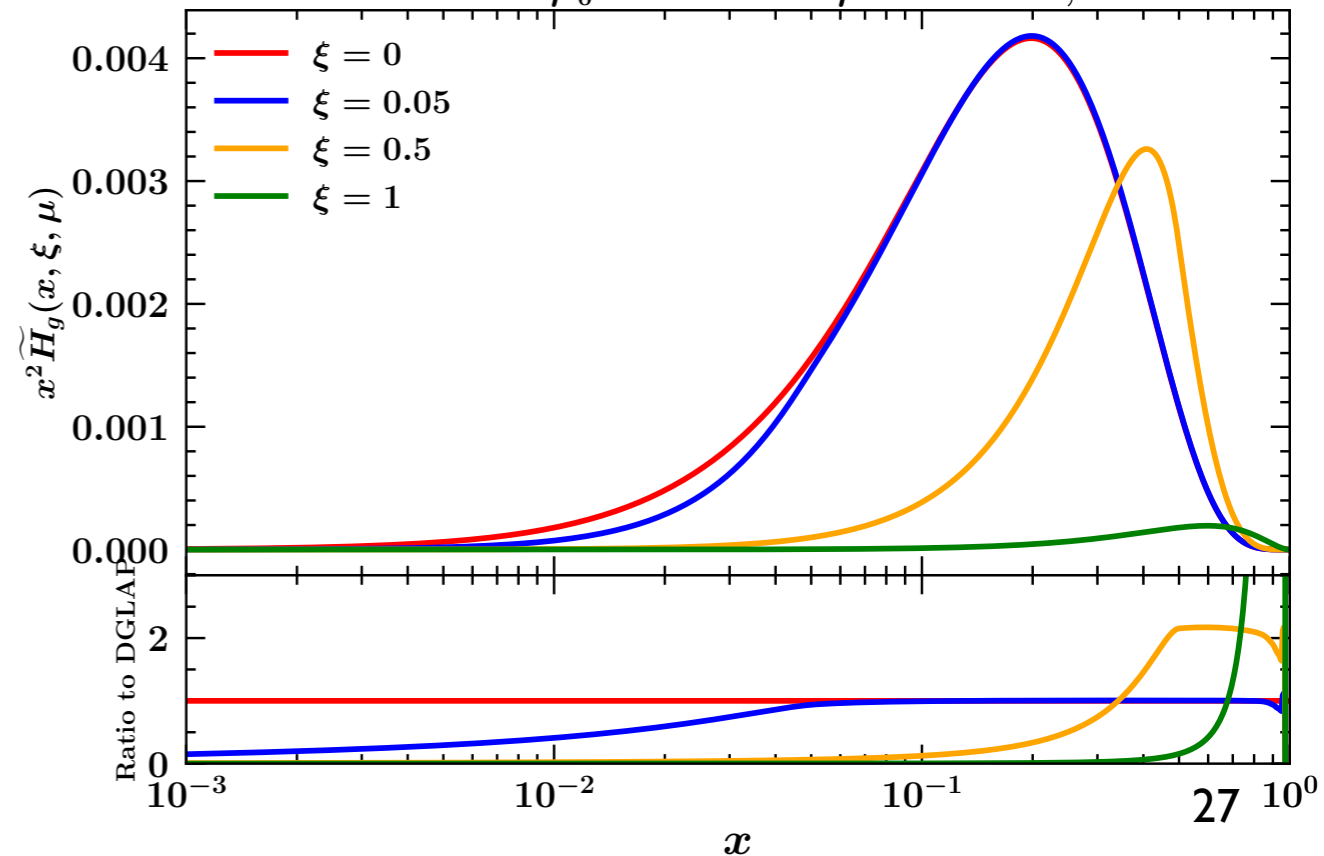
LO evolution from $\mu_0 = 2$ GeV to $\mu = 10$ GeV, GK model



LO evolution from $\mu_0 = 2$ GeV to $\mu = 10$ GeV, GK model



LO evolution from $\mu_0 = 2$ GeV to $\mu = 10$ GeV, GK model



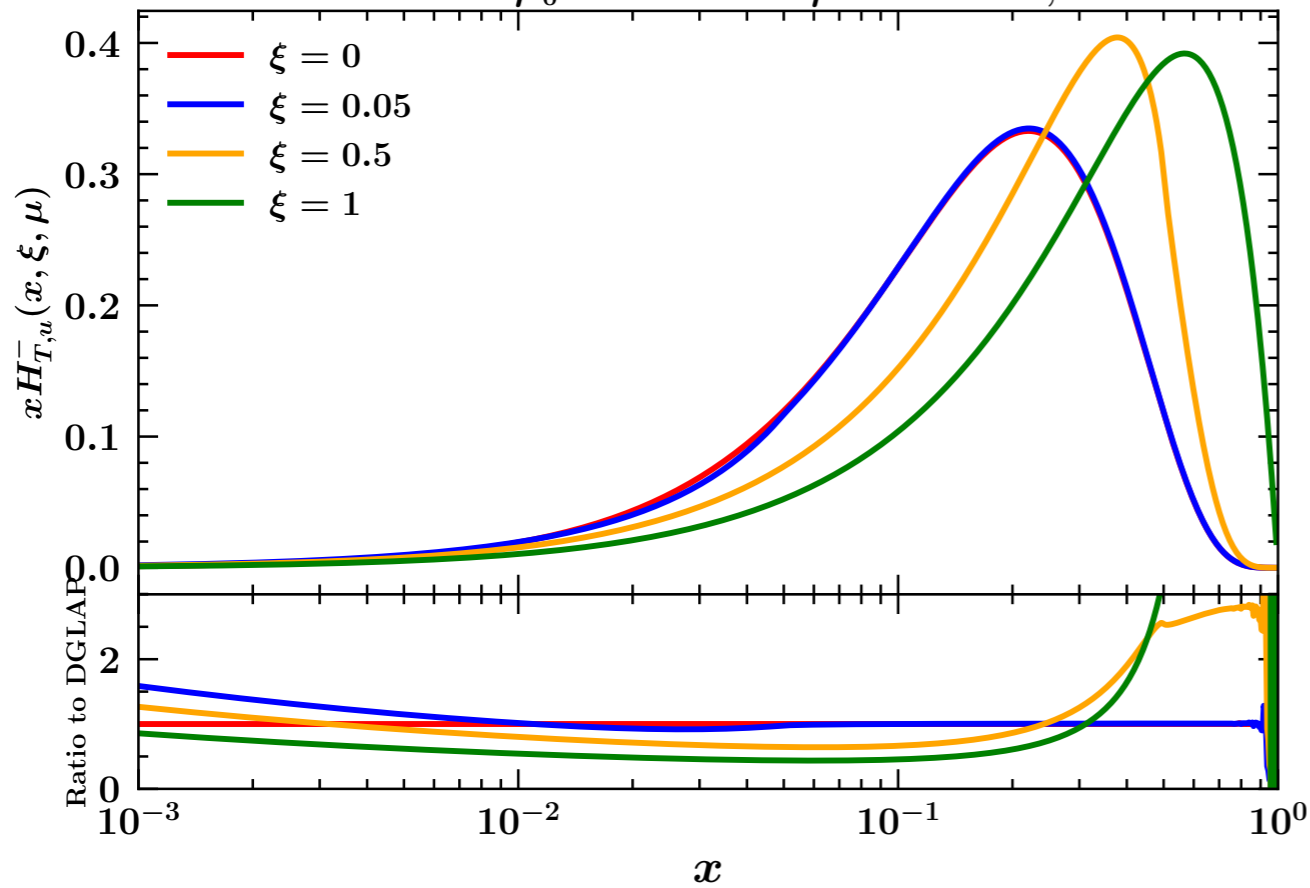
🍎 **DGLAP limit** reproduced within 10^{-5} relative accuracy.

🍎 GPD evolution may significantly deviate from DGLAP evolution.

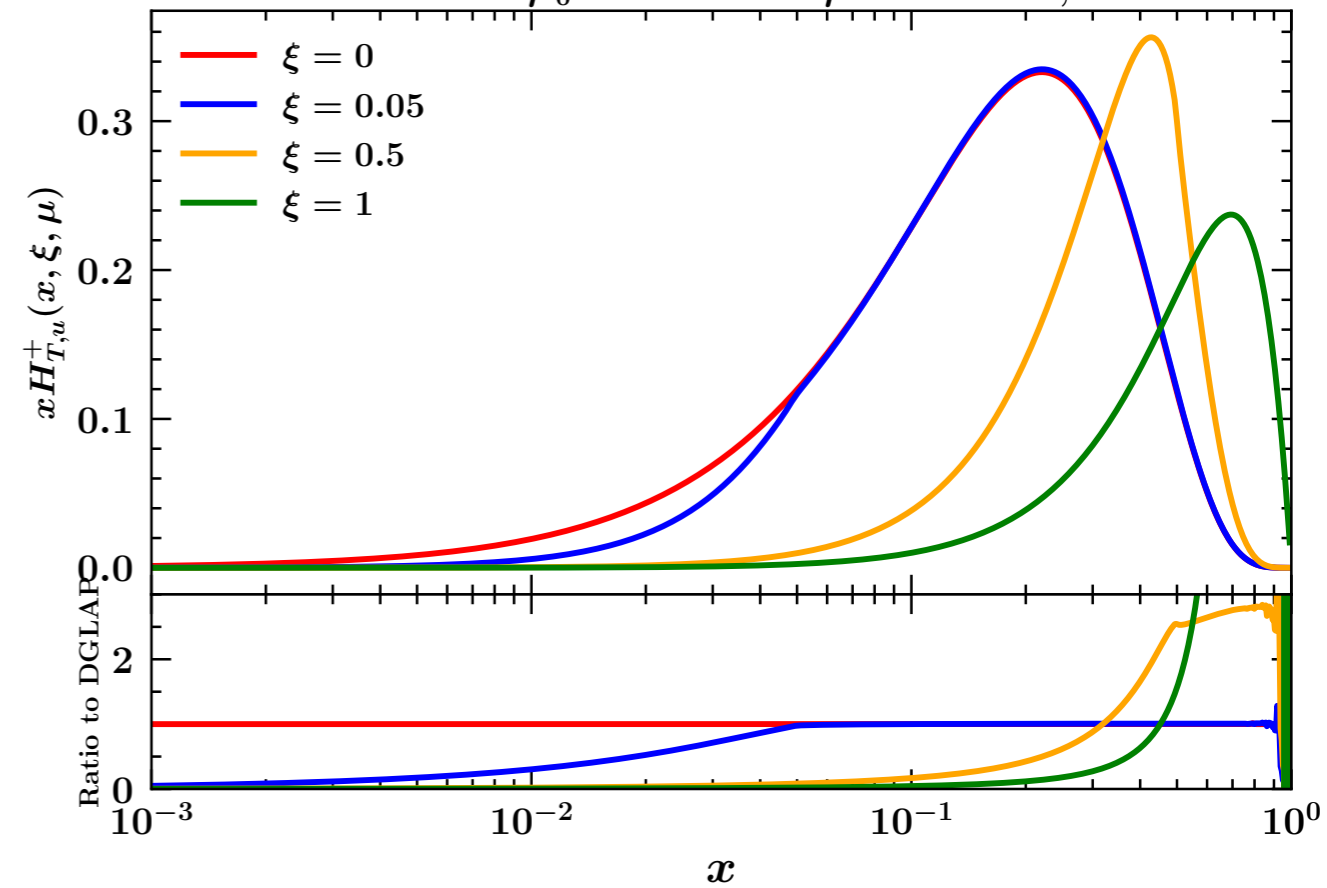
🍎 The evolution generates a cusp at $x = \xi$ but the distribution remains **continuous** at this point.

Evolution and DGLAP limit [T]

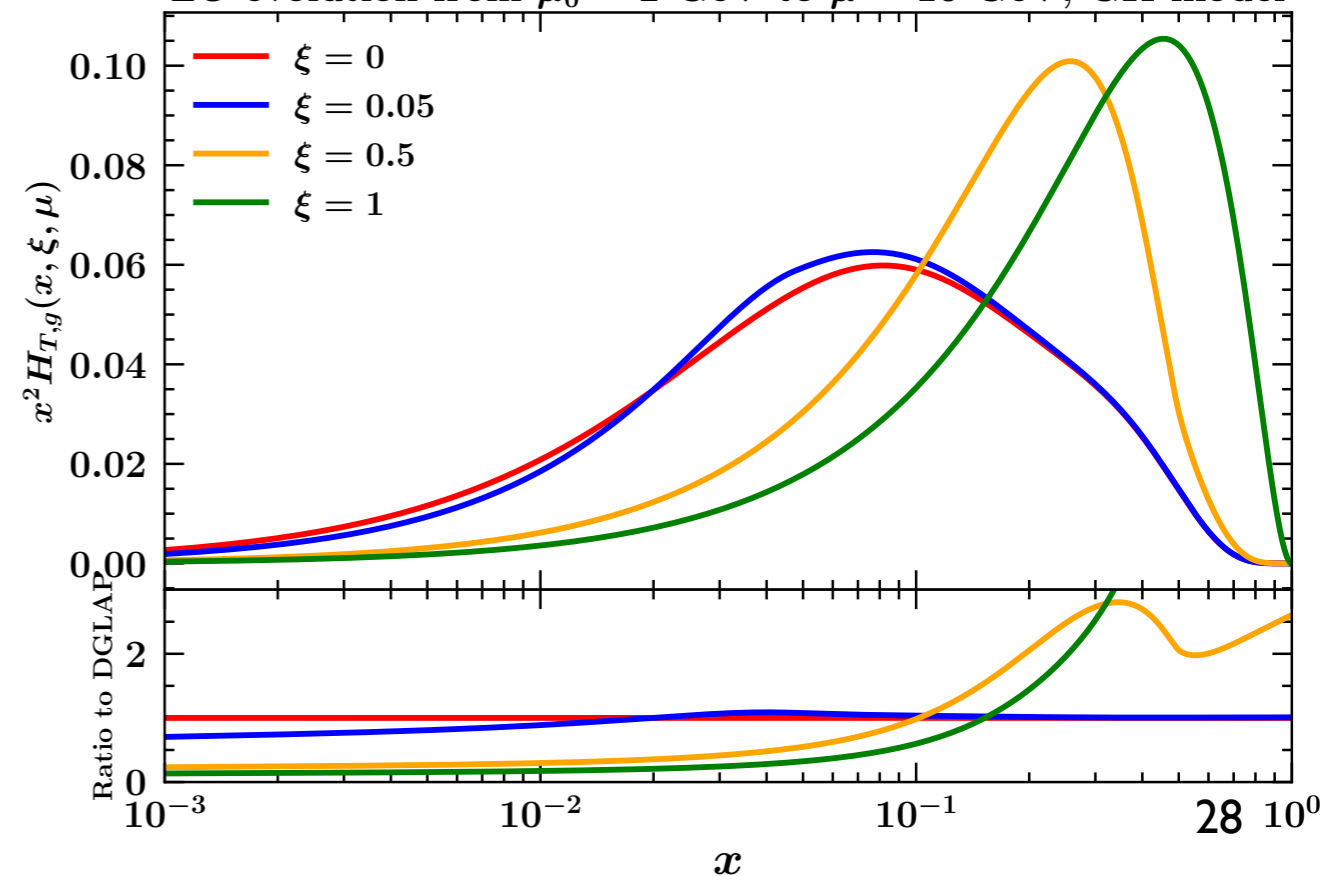
LO evolution from $\mu_0 = 2$ GeV to $\mu = 10$ GeV, GK model



LO evolution from $\mu_0 = 2$ GeV to $\mu = 10$ GeV, GK model




LO evolution from $\mu_0 = 2$ GeV to $\mu = 10$ GeV, GK model



 **DGLAP limit** reproduced within 10^{-5} relative accuracy.

 GPD evolution may significantly deviate from DGLAP evolution.

 The evolution generates a cusp at $x = \xi$ but the distribution remains **continuous** at this point.

Polynomiality

🍏 GPD evolution should preserve **polynomiality**.

[Xiang-Dong Ji, *J.Phys.G* 24 (1998) 1181-1205] [A.V. Radyushkin, *Phys.Lett.B* 449 (1999) 81-88]

🍏 The following relations for the Mellin moments must hold at **all scales**:

$$\int_0^1 dx x^{2n} F_q^{[\Gamma]-}(x, \xi, \mu) = \sum_{k=0}^n A_k^{[\Gamma]}(\mu) \xi^{2k}$$

$$\int_0^1 dx x^{2n+1} F_q^{[\Gamma]+}(x, \xi, \mu) = \sum_{k=0}^{n(+1)} B_k^{[\Gamma]}(\mu) \xi^{2k}$$

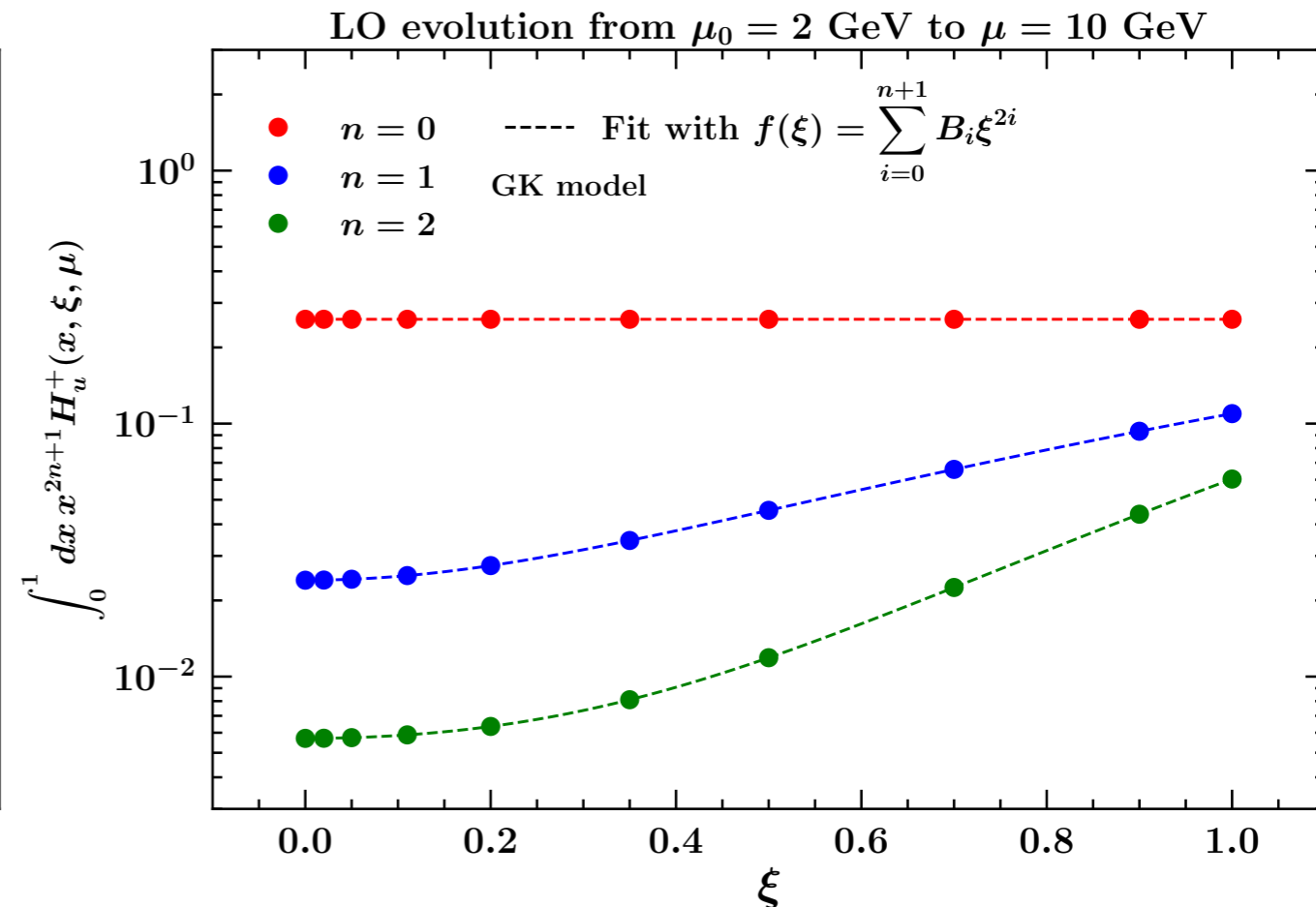
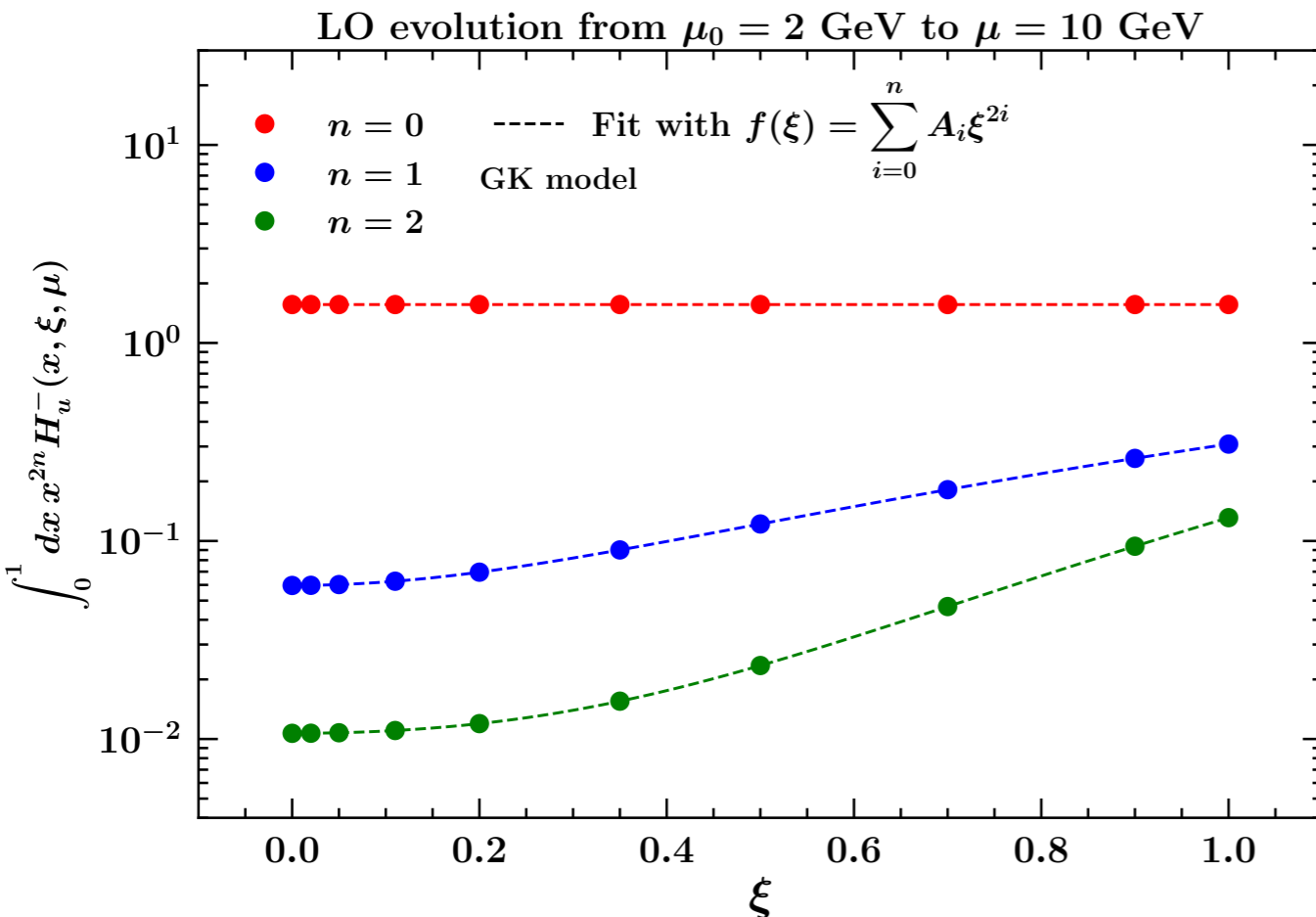
🍏 Polynomiality predicts that the first moment ($n = 0$) of the *non-singlet* distribution is **constant** in ξ .

🍏 The coefficient of the ξ^{2n+2} term of the *singlet* (D-term), only allowed in the unpolarised case, is absent in the GK model and is *not* generated by evolution:

🍏 also the first moment of the singlet is expected to be **constant** in ξ .

🍏 For larger values of n , one can just **fit** the behaviour in ξ and check that it follows the **expected power law in ξ** .

Polynomiality [U]



First moment for both singlet and non-singlet is indeed **constant** in ξ :

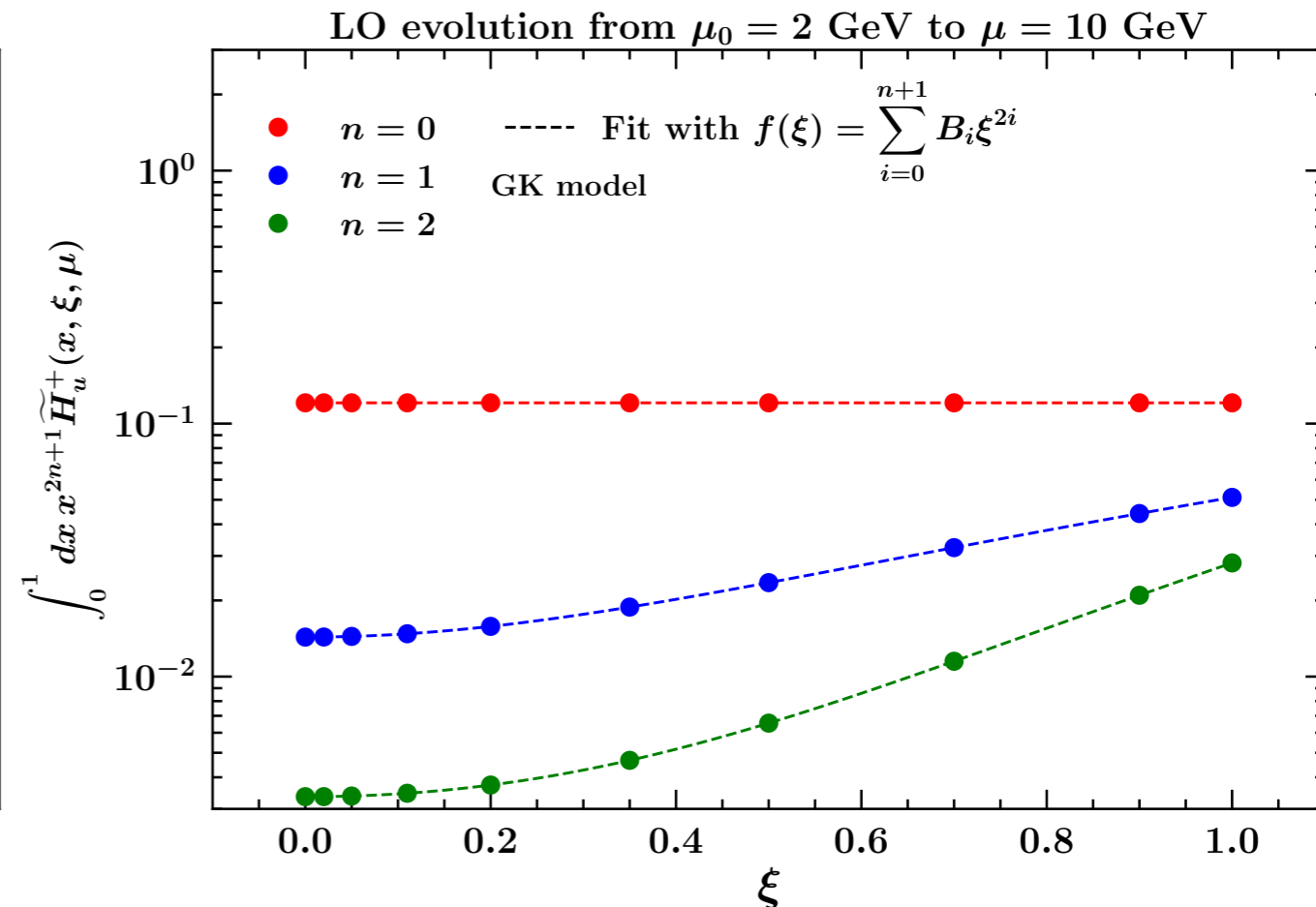
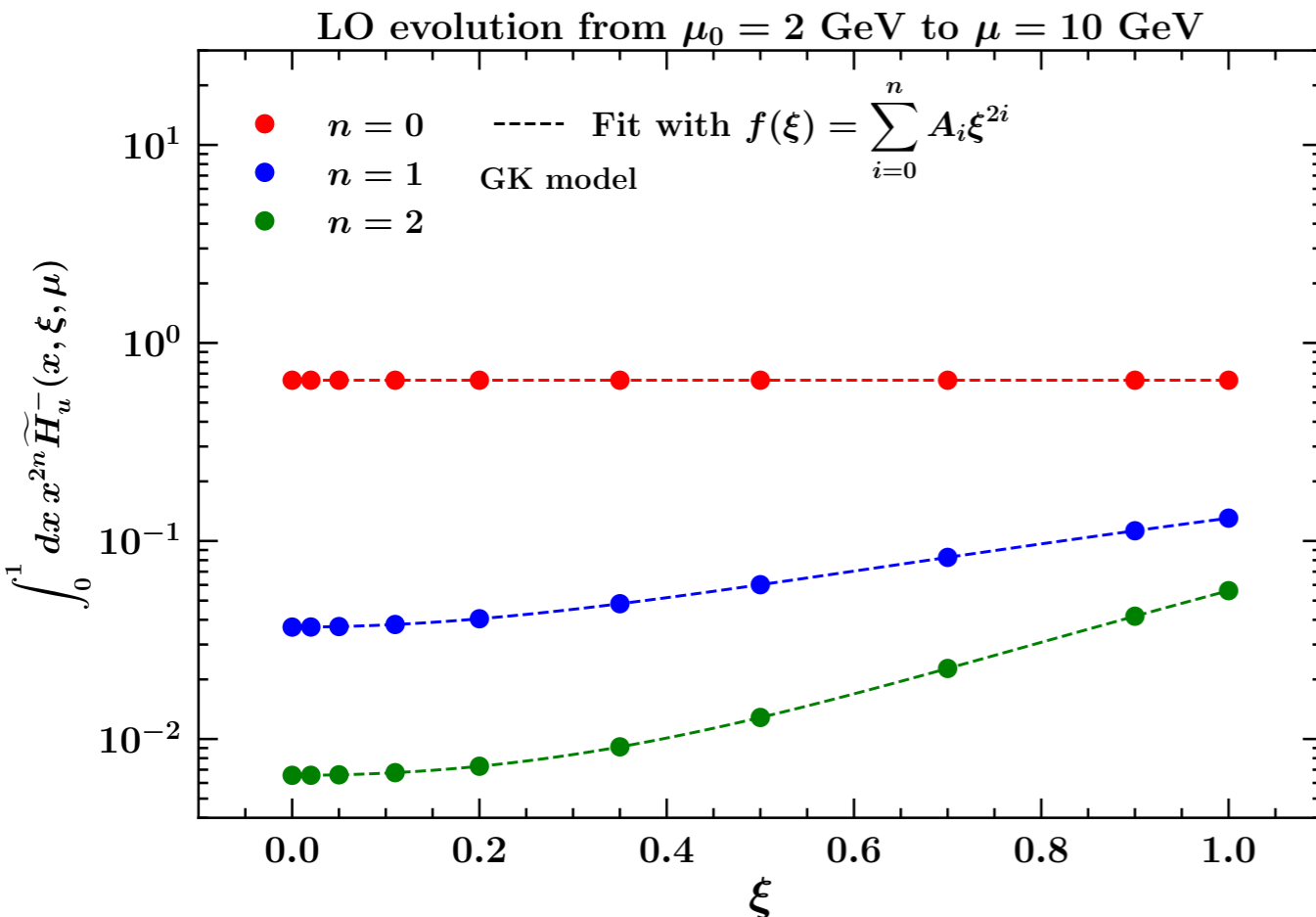
this was expected and the expectation is very nicely fulfilled.

Second and third moments follow the expected law:

including odd-power terms in the fit gives coefficients very close to zero.

B_{n+1} in the singlet is consistently found to be compatible with zero (no D-term)₃₀

Polynomiality [L]



🍏 **First moment** for both singlet and non-singlet is indeed **constant** in ξ :

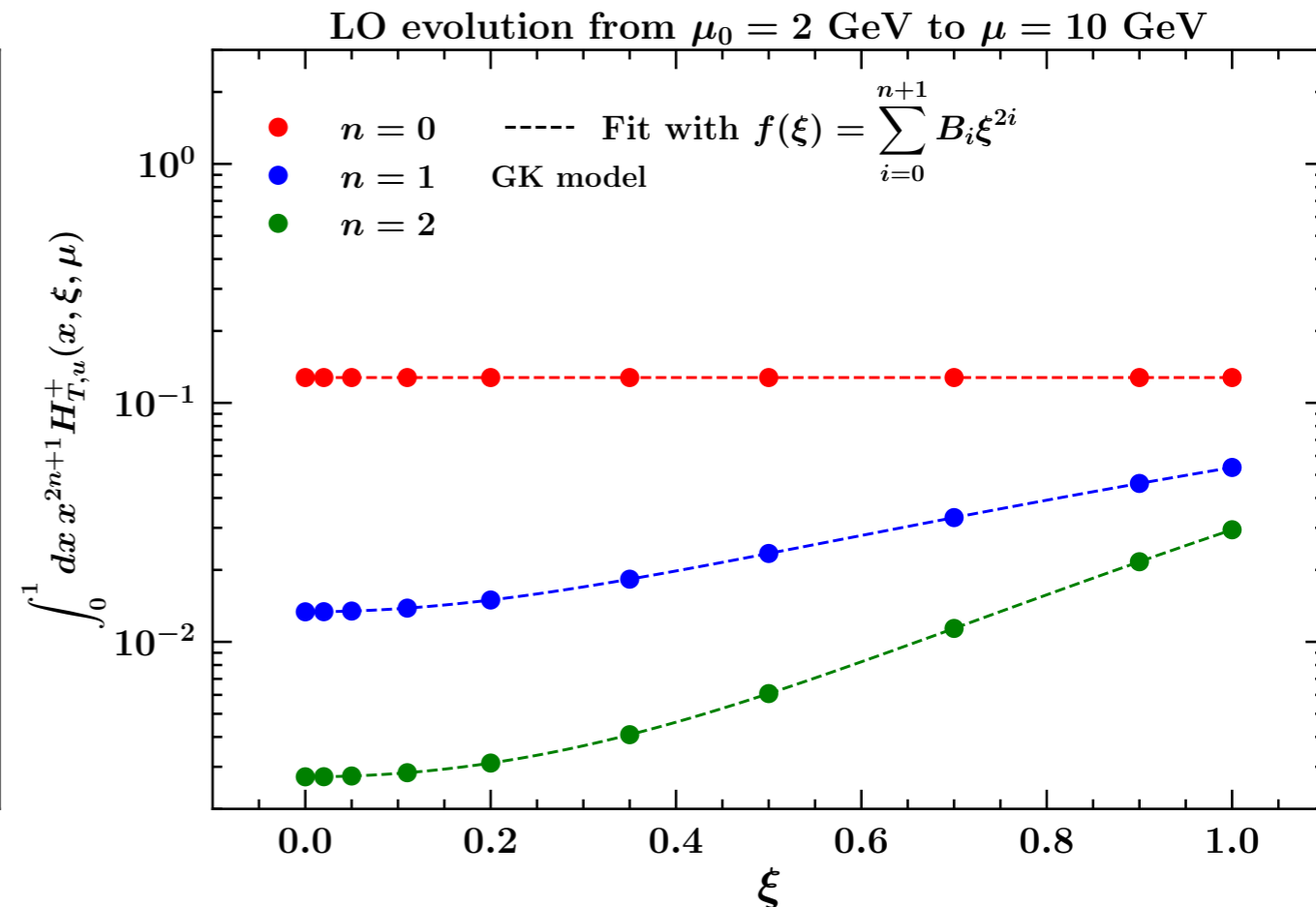
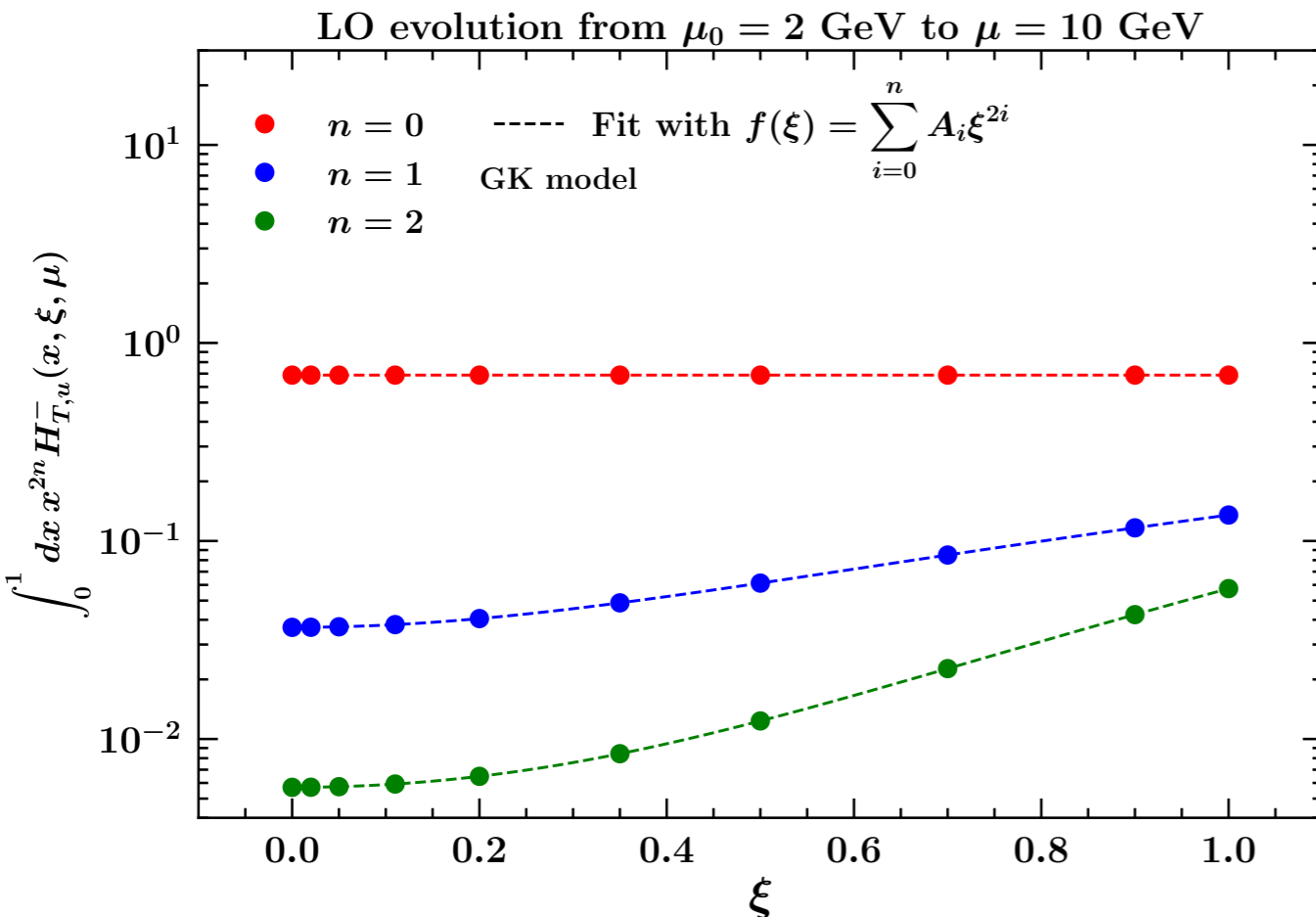
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Polynomiality [T]



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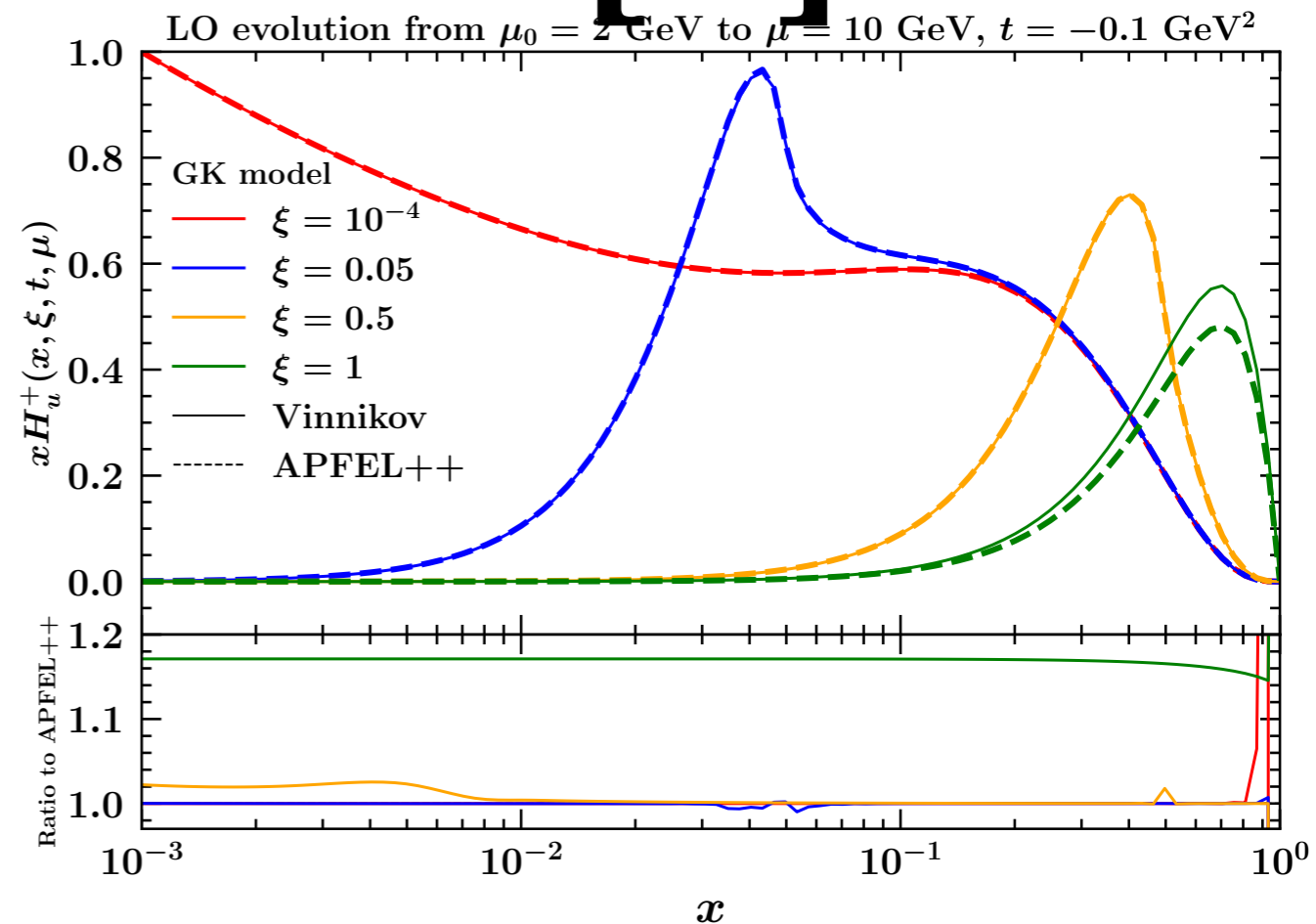
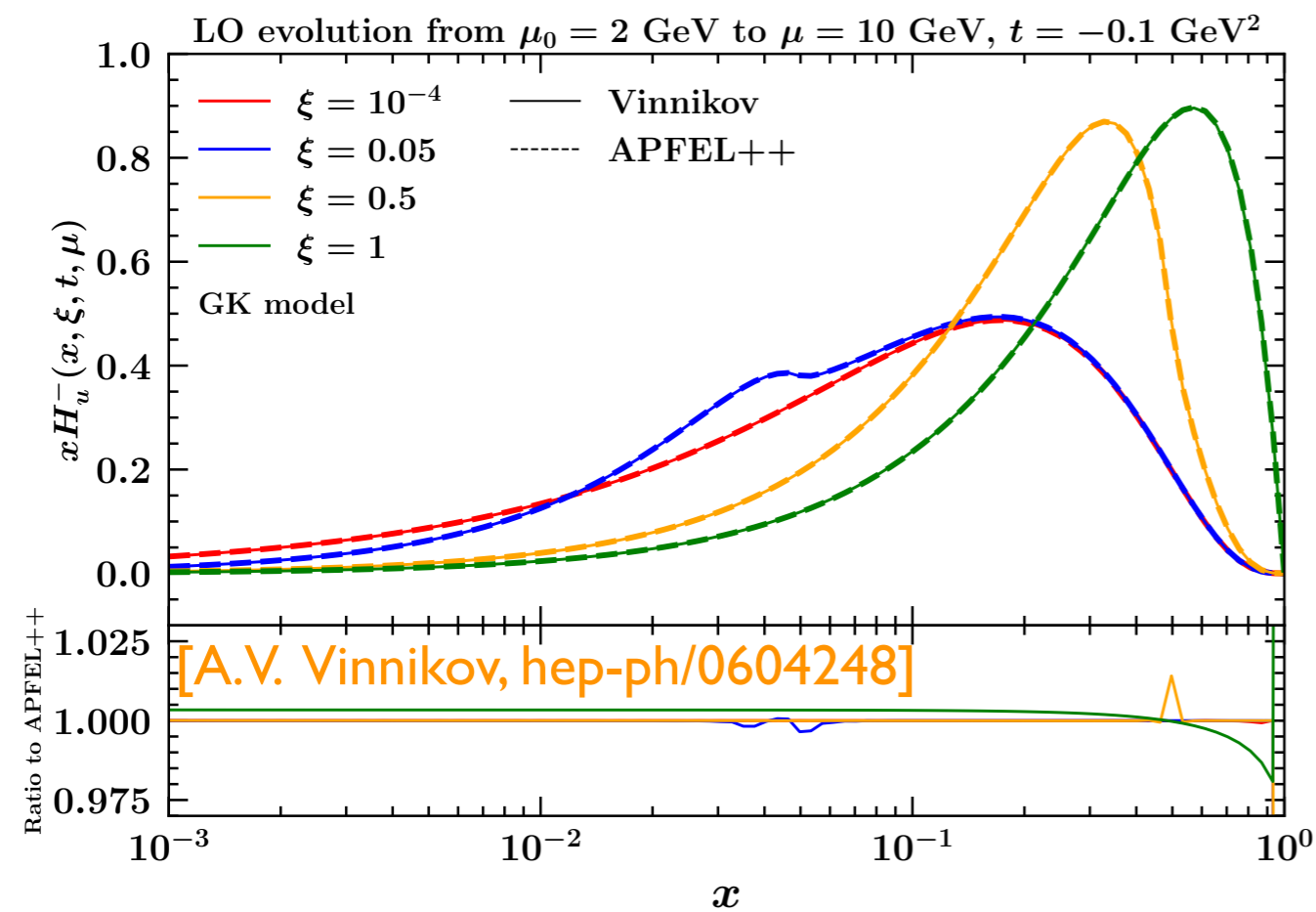
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APFEL++ vs. Vinnikov [U]



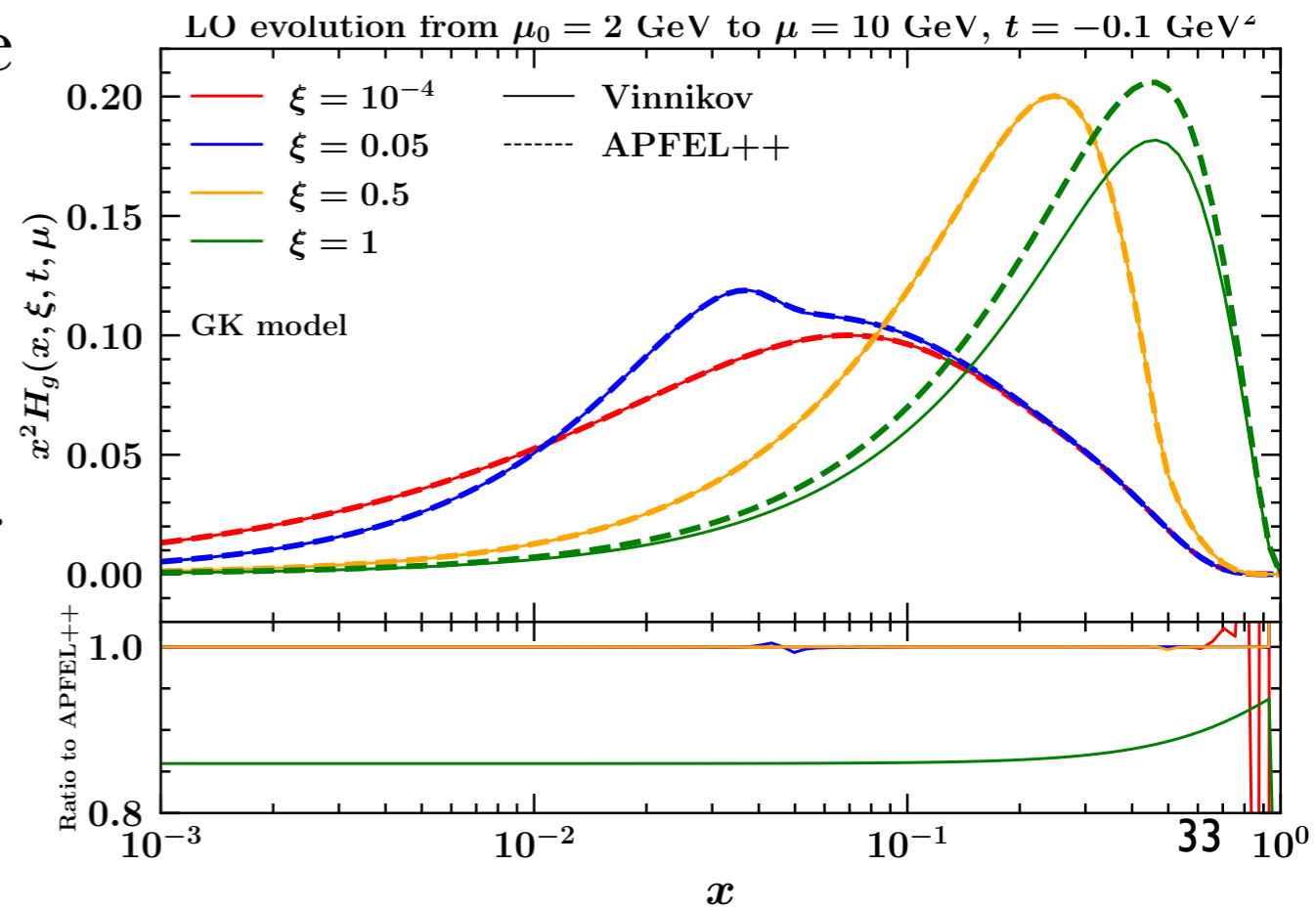
🍏 **Excellent agreement** between the two code for $\xi \lesssim 0.6$.

🍏 Agreement deteriorates for $\xi \gtrsim 0.6$:

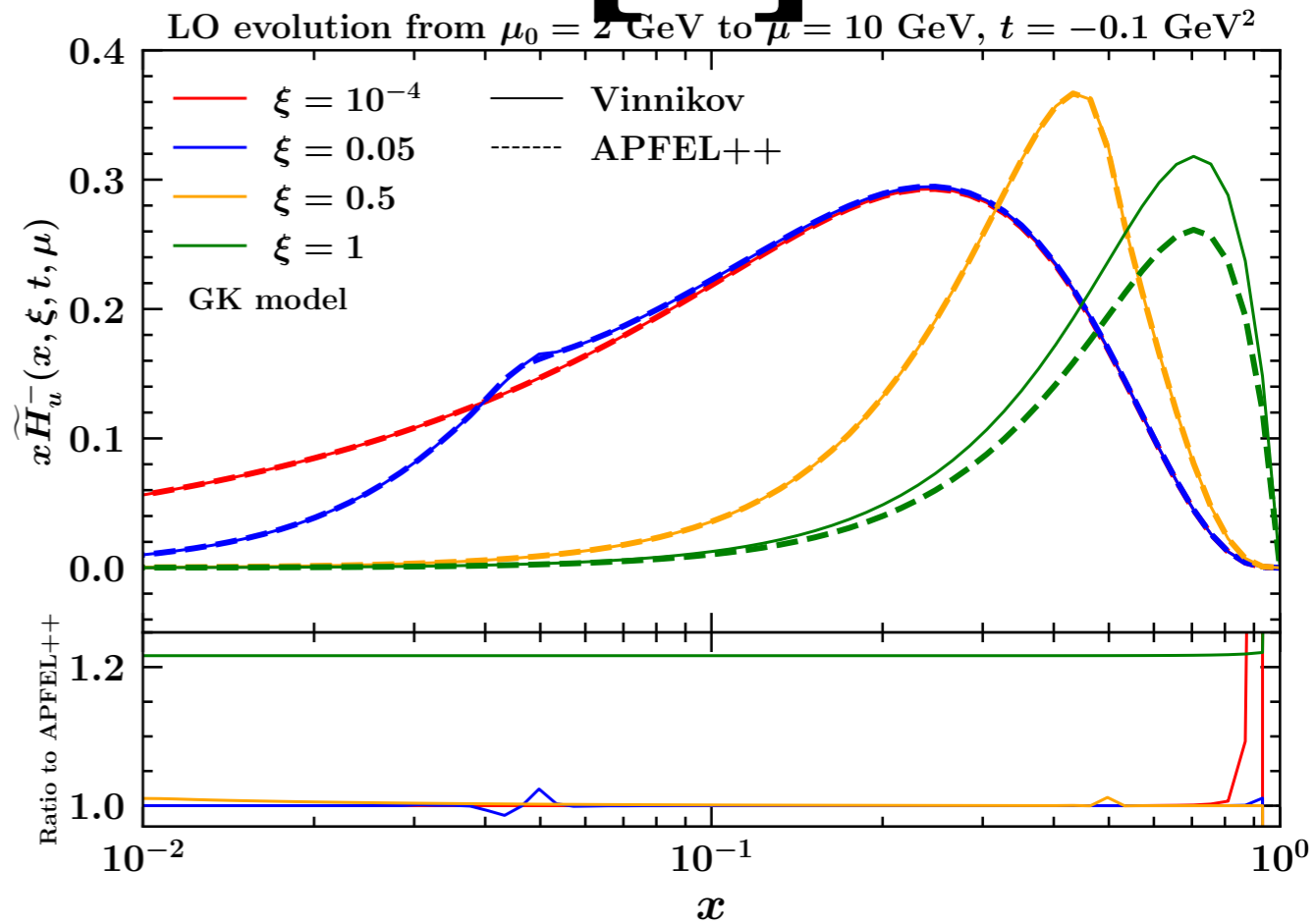
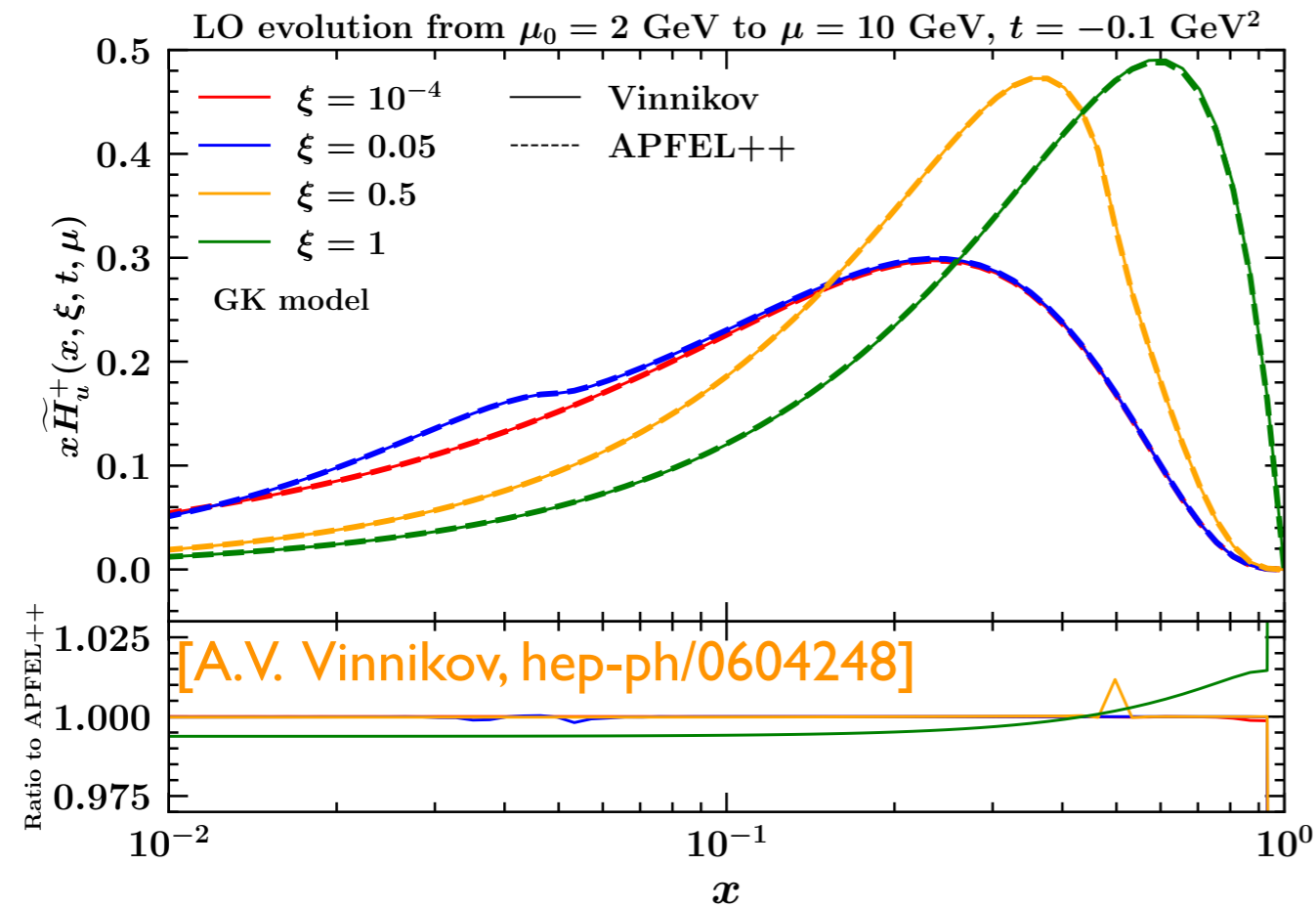
🍏 discrepancy larger for the singlets ($\sim 20\%$) than for the non-singlet ($\sim 1\%$).

🍏 possible numerical instabilities of Vinnikov's code?

🍏 Inability to check the ERBL limit.



APFEL++ vs. Vinnikov [L]



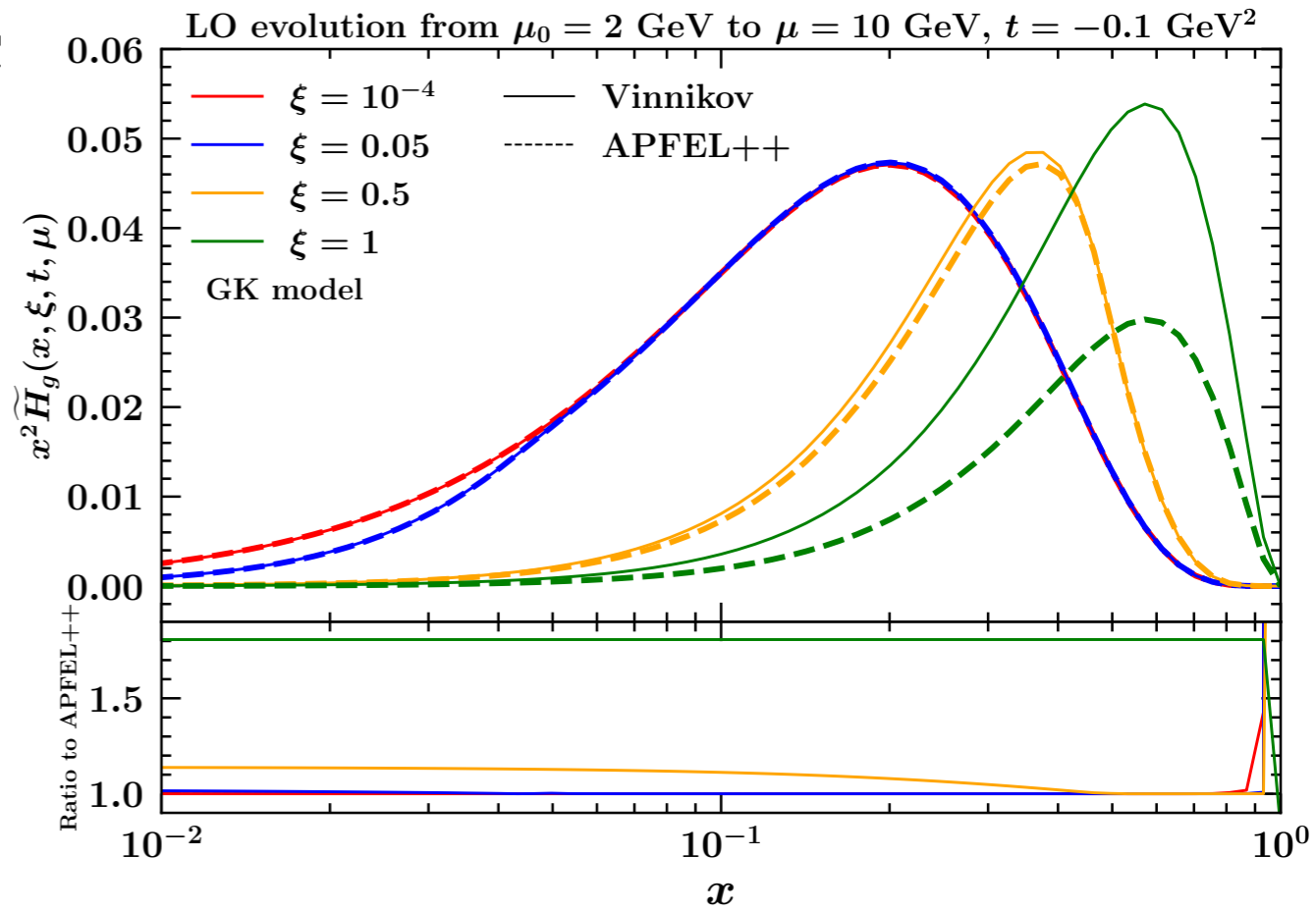
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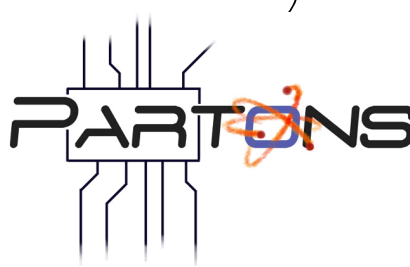
🍏 possible numerical instabilities of Vinnikov's code?

🍏 Inability to check the ERBL limit.



Conclusions and outlook

🍏 We have **revisited LO GPD evolution** in momentum space:

- 🍏 *Ab-initio* calculation of the LO unpolarised splitting kernels based on Feynman diagrams in light-cone gauge for **all twist-2 operators**.
- 🍏 GPD evolution equations recasted in a DGLAP-like form convenient for implementation.
- 🍏 Various analytical properties of the kernels highlighted and numerically checked.
- 🍏 DGLAP (and ERBL) limit correctly recovered within excellent accuracy.
- 🍏 Evolution conserves polynomiality (and agrees with conformal-space evolution).
- 🍏 the code (**APFEL++**) will be made public and available within  **PARTONS**
<https://github.com/vbertone/apfelxx>

<http://partons.cea.fr/partons/doc/html/index.html>

🍏 **Next steps:**

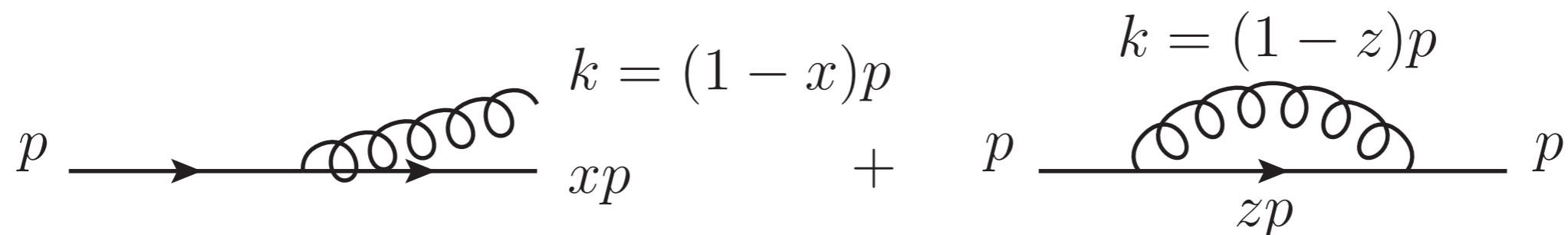
- 🍏 **middle term:** benchmark of public evolution codes (discussion already started),
- 🍏 **longer term:** calculation and implementation of NLO corrections (already on the way).

Back up

GPD definition

$$\mathcal{D}^{\mu\nu}(k) = \frac{id^{\mu\nu}(k)}{k^2 + i0}, \quad d^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{(nk)_{\text{Reg}}}.$$

- 🍏 The linear (eikonal) propagator $(nk)^{-1}$ needs to be **regularised**:
 - 🍏 it separately gives rise to non-integrable end-point singularity in real-emission graphs and to plain divergences in virtual graphs,
 - 🍏 the two **cancel** giving an **integrable** result.



$$\frac{1}{(nk)} \sim \frac{1}{1-x} + \delta(1-x) \int \frac{dk}{(nk)} \sim \delta(1-x) \int \frac{dz}{1-z}$$

$$\sim \left(\frac{1}{1-x} \right)_+$$

The ERBL limit

🍏 The limit $\xi \rightarrow 1$ ($\kappa \rightarrow 1/x$) we should reproduce the **ERBL equation**.

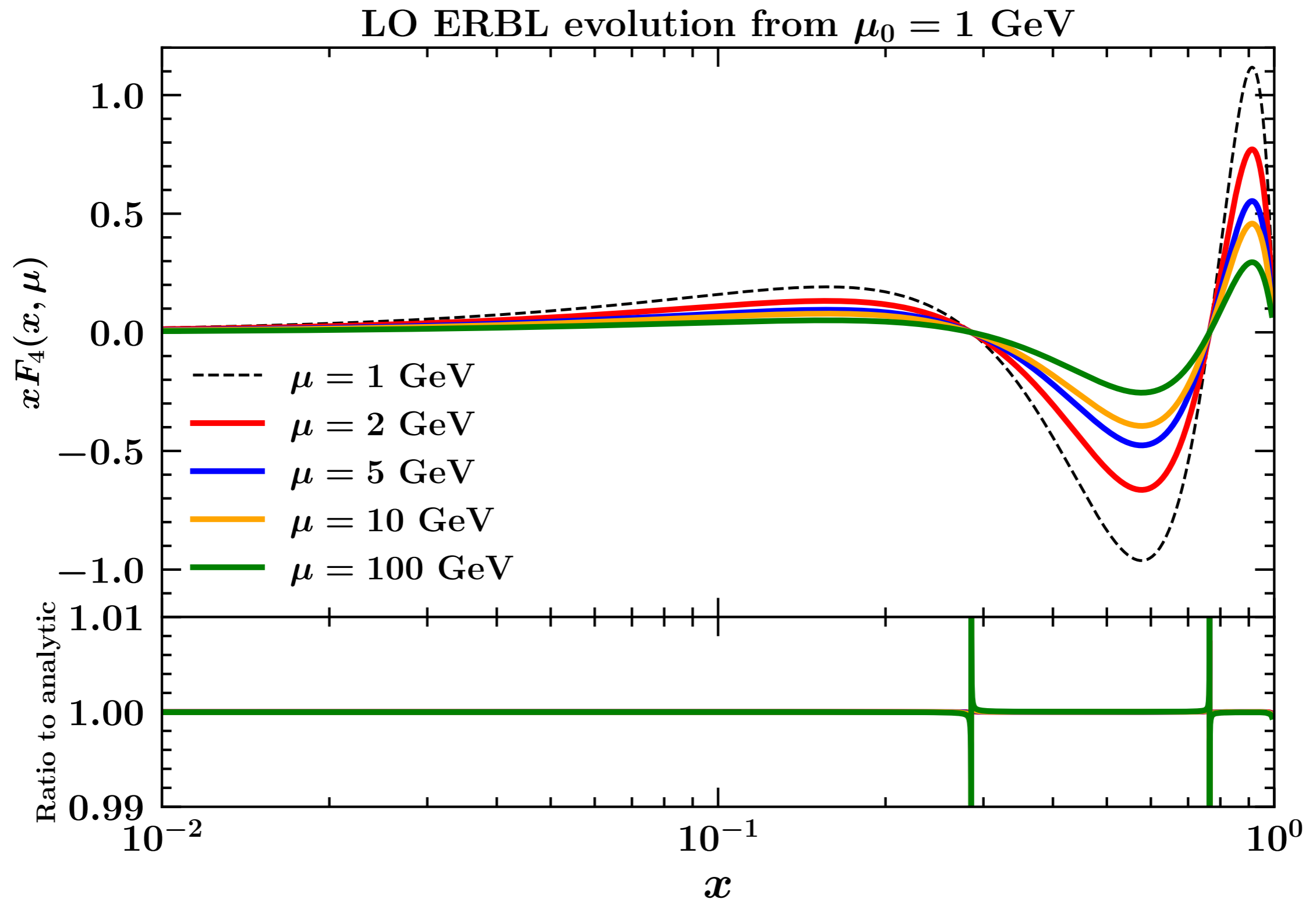
🍏 It is well known that in this limit **Gegenbauer polynomials** decouple upon LO evolution, such that:

$$F_{2n}(x, \mu_0) = (1 - x^2) C_{2n}^{(3/2)}(x) \quad \Rightarrow \quad F_{2n}(x, \mu) = \exp \left[\frac{V_{2n}^{[0]}}{4\pi} \int_{\mu_0}^{\mu} d \ln \mu^2 \alpha_s(\mu) \right] F_{2n}(x, \mu_0)$$

🍏 where the kernels $V_{2n}^{[0]}$ can be read off, for example, from [Brodsky, Lepage, *Phys.Rev.D* 22 (1980) 2157] or [Efremov, Radyushkin, *Phys.Lett.B* 94 (1980) 245-250].

🍏 We have compared this expectation with the numerical results for GPD evolution by setting $\kappa = 1/x$ and using a Gegenbauer polynomial as an initial-scale GPD.

The ERBL limit



🍏 **ERBL limit** reproduced within less than 10^{-5} relative accuracy,

🍏 Same accuracy for **higher-degree** Gegenbauer polynomials.

Conformal-space evolution

🍏 In order to check that LO GPD evolution ($\xi \neq 0$) in conformal space is diagonal in a **realistic** case, we have considered the RDDA:

$$H_q(x, \xi, \mu_0) = \int_{\Omega} d\beta d\alpha \delta(x - \beta - \xi\alpha) q(|\beta|) \pi(\beta, \alpha)$$

with:

$$q(x) = \frac{35}{32} x^{-1/2} (1-x)^3, \quad \pi(\beta, \alpha) = \frac{3}{4} \frac{((1-|\beta|)^2 - \alpha^2)}{(1-|\beta|)^3}$$

We have evolved the 4th moment:

$$C_4^-(\xi, \mu) = \xi^4 \int_{-1}^1 dx C_4^{(3/2)}\left(\frac{x}{\xi}\right) H_q(x, \xi, \mu)$$

from $\mu_0 = 1$ GeV using the (analytic) conformal-space evolution and the (numerical) momentum-space evolution.

we found excellent agreement.

