Motion of an electron through vacuum fluctuations

By Anirudh Gundhi

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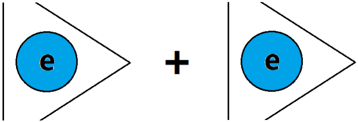

Does an electron in a superposition of two different spatial locations decohere due to zero point fluctuations?

Interaction with the radiation field

1. The Lagrangian for a non-relativistic electron interacting with the radiation field is given by

$$L(t) = \frac{1}{2} m \dot{\mathbf{r}}_e^2 - V_0(\mathbf{r}_e) + \frac{\epsilon_0}{2} \int d^3 r \left(\mathbf{E}_\perp^2(\mathbf{r}) - c^2 \mathbf{B}^2(\mathbf{r}) \right) - e \mathbf{r}_e \mathbf{E}_\perp(\mathbf{r}_e) \,. \quad (1)$$

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2. The goal is to describe the effective dynamics of the charged particle, having taken into account its interaction with the radiation field environment.
3. This can be achieved by deriving the reduced density matrix for the electron.

Density matrix from path integral

1. From the path integral formalism we have

$$\langle x_\mathrm{f}|\, \hat{U}(t;t_\mathrm{i})\,|x_\mathrm{i}\rangle = \int_{\substack{x(t)=x_f,\\ x(t_i)=x_i}} D[x]e^{iS/\hbar}\,,$$

$$\psi(x_f,t) = \int_{x(t)=x_f} D[x]e^{iS/\hbar}\psi(x_i,t_i)\,,$$

$$\langle x_f'|\,\hat{\rho}\,|x_f\rangle = \int_{\substack{x(t)=x_f,\\ x'(t)=x_f'}} D[x,x']e^{\frac{i}{\hbar}(S[x']-S[x])}\psi(x_i')\psi^*(x_i)\,. \tag{2}$$

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2. Separate the environmental degrees of freedom

$$\left\langle x_{\mathbf{f}}';\Pi^{f\prime}\right|\hat{\rho}(t)\left|x_{\mathbf{f}};\Pi^{f}\right\rangle = \int_{\substack{x(t)=x_{\mathbf{f}},\\ x'(t)=x_{\mathbf{f}}'}} D[x,x']e^{\frac{i}{\hbar}(S_{\mathbf{S}}[x']-S_{\mathbf{S}}[x])}\rho_{\mathbf{S}}^{i}\times$$

$$\times\int_{\substack{\Pi(t)=\Pi^f,\\ \Pi'(t)=\Pi^{f\prime}}} D[\mu,\mu']e^{\frac{i}{\hbar}(S_{\mathbf{EM}}[\mu']+S_{\rm int}[x',\Pi']-S_{\mathbf{EM}}[\mu]-S_{\rm int}[x,\Pi])}\rho_{\mathbf{EM}}^{i}\,. \tag{3}$$

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$$S_{\mathbf{IF}}[x, x'] = \frac{1}{2} \int_{t_i}^{t} dt_1 dt_2 \begin{bmatrix} x(t_1) & x'(t_1) \end{bmatrix} \cdot M(t_1; t_2) \cdot \begin{bmatrix} x(t_2) \\ x'(t_2) \end{bmatrix} . \quad (5)$$

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4. with ($\Pi_{\mathsf{E}} := -\Pi/\epsilon_0$)

$$M(t_1;t_2) = \frac{ie^2}{\hbar} \begin{bmatrix} \left\langle \tilde{\mathcal{T}}\{\hat{\Pi}_{\mathsf{E}}(t_1)\hat{\Pi}_{\mathsf{E}}(t_2)\}\right\rangle_0 & -\left\langle \hat{\Pi}_{\mathsf{E}}(t_1)\hat{\Pi}_{\mathsf{E}}(t_2)\right\rangle_0 \\ -\left\langle \hat{\Pi}_{\mathsf{E}}(t_2)\hat{\Pi}_{\mathsf{E}}(t_1)\right\rangle_0 & \left\langle \mathcal{T}\{\hat{\Pi}_{\mathsf{E}}(t_1)\hat{\Pi}_{\mathsf{E}}(t_2)\}\right\rangle_0 \end{bmatrix}\,. \tag{6}$$

Time evolution of the reduced density matrix

The time evolution of the reduced density matrix takes the standard basis independent form

$$
\begin{aligned}
\delta_t \hat{\rho}_r = & -\frac{i}{\hbar}\left[\hat{H}_s, \hat{\rho}_r\right] \\
& -\frac{1}{\hbar}\int_0^{t-t_i} d\tau \mathcal{N}(t; t-\tau)\left[\hat{x}, \left[\hat{x}_{H_s}(-\tau), \hat{\rho}_r(t)\right]\right] \\
& +\frac{i}{2\hbar}\int_0^{t-t_i} d\tau \mathcal{D}(t; t-\tau)\left[\hat{x}, \left\{\hat{x}_{H_s}(-\tau), \hat{\rho}_r(t)\right\}\right] .
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\mathcal{N}(t_1; t_2) :=& \frac{e^2}{2\hbar}\left\langle \left\{\hat{\Pi}_{\mathsf{E}}(t_1), \hat{\Pi}_{\mathsf{E}}(t_2)\right\}\right\rangle_0 , \\
\mathcal{D}(t_1; t_2) :=& \frac{ie^2}{\hbar}\left\langle \left[\hat{\Pi}_{\mathsf{E}}(t_1), \hat{\Pi}_{\mathsf{E}}(t_2)\right]\right\rangle_0 \theta(t_1 - t_2) .
\end{aligned} \tag{8}
$$

Kernels

1. From standard quantization of the radiation field we have

$$\hat{\Pi}_{\mathbf{E}}(\mathbf{r},t) = i\left(\frac{\hbar c}{2\epsilon_0(2\pi)^3}\right)^{\frac{1}{2}} \int d^3k\sqrt{k}\sum_{\varepsilon}\hat{a}_{\varepsilon}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\varepsilon^x_{\mathbf{k}} + \mathrm{c.c}\,. \qquad (9)$$

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2. Using the expression above we can calculate the two-point correlations

$$\begin{aligned}
&\langle\hat{\Pi}_{\mathbf{E}}(x(t_1),t_1)\hat{\Pi}_{\mathbf{E}}(x(t_2),t_2)\rangle_0 \\
&= \frac{-i\hbar c}{2\epsilon_0 4\pi^2}\hat{\Box}\left\{\frac{1}{r}\int_0^{\infty} dk e^{-i\omega_k\tau}\left(e^{ikr}-e^{-ikr}\right) e^{-\omega_k/\omega_{\max}}\right\} \\
&= \frac{\hbar c}{\pi^2\epsilon_0}\frac{1}{\left(r^2-c^2(\tau-i\epsilon)^2\right)^2}\,, \qquad \epsilon := \frac{1}{\omega_{\max}}\,. \qquad (10)
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3. where

$$\tau := t_1 - t_2\,, \qquad r := |x(t_1)-x(t_2)|\,, \qquad \hat{\Box} := -\frac{1}{c^2}\partial_\tau^2 + \partial_r^2\,. \qquad (11)$$

Kernels

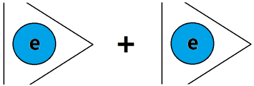
Noise kernel $\mathcal{N}$

$$\mathcal{N}(\tau) = \frac{e^2}{\pi^2 \epsilon_0 c^3} \frac{\left(\epsilon^4 - 6\epsilon^2\tau^2 + \tau^4\right)}{\left(\epsilon^2 + \tau^2\right)^4} \,. \tag{12}$$

Dissipation kernel $\mathcal{D}$

$$\mathcal{D}(\tau) = \frac{e^2}{3\pi\epsilon_0 c^3} \theta(\tau) \frac{d^3}{d\tau^3} \delta_\epsilon(\tau) \,, \qquad \delta_\epsilon(\tau) = \frac{1}{\pi} \frac{d}{d\tau} \tan^{-1}(\tau/\epsilon) \,. \tag{13}$$

Decoherence



$$\rho_r(x', x, t) = \exp\left(-\frac{(x'-x)^2}{\hbar}\mathcal{N}_2(t)\right)\rho_r(x', x, 0)\,, \tag{14}$$

where

$$\mathcal{N}_2(t) = \int_0^t d\tau \int_0^\tau d\tau' \mathcal{N}(\tau')\,. \tag{15}$$

Decoherence

At large times, the length scales over which the superposition can be maintained ($l_x(t)$) scales with the cutoff

$$\rho_r(x', x, t) = \exp\left\{ -\left(\frac{x'-x}{l_x}\right)^2 \right\} \rho_r(x', x, 0)\,, \tag{16}$$

$$l_x(t) = \sqrt{\frac{3\pi c^2}{2\alpha\omega_{\mathrm{max}}^2} \cdot \frac{(t^2+\epsilon^2)^2}{t^4+3t^2\epsilon^2}} \overset{t \gg \epsilon}{=} \sqrt{\frac{3\pi}{2\alpha}}\frac{c}{\omega_{\mathrm{max}}}\,. \tag{17}$$

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Decoherence

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1. Should we have a small value of the cutoff, as suggested by Barone and Caldeira in Phys. Rev. A 43, 57 (1991))?
2. Or does the density matrix describe false decoherence, as suggested by Unruh in Relativistic Quantum Measurement and Decoherence (False Loss of Coherence, pp. 125–140)?

False decoherence

1. We consider the situation in which the charged particle interacts with vacuum fluctuations for a finite period of time

$$e' = ef(t)\,, \qquad f(0) = f(T) = 0\,, \qquad f(0 < t < T) = 1\,. \tag{20}$$

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2. The Noise kernel transforms as $\mathcal{N} \to \tilde{\mathcal{N}}$

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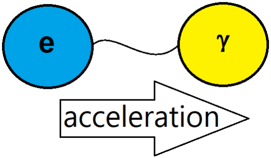
3. Coherence is not lost irreversibly since the coherence length becomes infinite after switching off the interaction

$$\tilde{l}_x(T) = \left(\frac{2}{f^2(0) + f^2(T)}\right)^{\frac{1}{2}} \left(\frac{3\pi}{2\alpha}\right)^{\frac{1}{2}} \frac{1}{k_{\mathrm{max}}}\,. \tag{22}$$

Radiation reaction



1. Neutral particle $m\ddot{x} = -\frac{\partial}{\partial x}V(x) = \mathrm{F}_{\mathrm{ext}}$.

2. Charged particle $m_{\mathrm{R}}\ddot{x} = \mathrm{F}_{\mathrm{ext}} + \frac{2\alpha\hbar}{3c^2}\dddot{x}$.

Features of the Abraham-Lorentz equation

$$m_R \ddot{x} = \mathrm{F}_{\mathrm{ext}} + \frac{2\alpha\hbar}{3c^2}\dddot{x} \ .$$

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3. Does that mean that there is no radiation emitted for uniform acceleration? What about Larmor's formula?
4. $\int_{x_1}^{x_2} \frac{2\alpha\hbar}{3c^2}\dddot{x}\, dx = \int_{t_1}^{t_2} \frac{2\alpha\hbar}{3c^2}\dot{a}\cdot v dt = \underbrace{-\int_{t_1}^{t_2} \frac{2\alpha\hbar}{3c^2} a \cdot a dt}_{\text{Larmor's formula}} + \underbrace{a_{t_2} v_{t_2} - a_{t_1} v_{t_1}}_{\text{Schott term}} \ .$

The problem

1. Classical mechanics

$$m_R \ddot{x} = -\frac{\partial}{\partial x} V(x) + \frac{2\alpha\hbar}{3c^2} \dddot{x} \,. \tag{23}$$

2. The runaway solution $(V(x) = 0)$

$$m_R \ddot{x} = m_R \tau_0 \dddot{x} \implies a(t) = a(t_0) \exp(t/\tau_0) \,. \tag{24}$$

The QM Abraham-Lorentz equation

1. Quantum mechanics

$$m_{\mathrm{R}} \frac{d^2}{dt^2}\langle \hat{x} \rangle = -\langle \hat{V}_{,x} \rangle - \frac{2\alpha\hbar}{3c^2} \mathrm{Tr}\left(\frac{i}{\hbar^3} \hat{\rho}_r(t) \left[\hat{\mathrm{H}}_s, \left[\hat{\mathrm{H}}_s, \left[\hat{\mathrm{H}}_s, \hat{x} \right] \right] \right] \right) . \qquad (25)$$

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2. No runaways ($\hat{V}(x) = 0$)

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3. When $\hat{V}(x) \neq 0$ the two solutions match exactly

$$m_R \frac{d^2}{dt^2}\langle \hat{x} \rangle = -\langle \hat{V}(x)_{,x} \rangle + \frac{2\alpha\hbar}{3c^2}\frac{d^3}{dt^3}\langle \hat{x} \rangle\,. \tag{27}$$

The end

Thanks!